EE5110: PROBABILITY FOUNDATIONS FOR ELECTRICAL ENGINEERS

JULY - NOV 2024

1 Real Random Variable

1. Motivation
   - in many experiments, sample points are in $\mathbb{R}^k$
   - in other experiments, we can associate numbers to sample points
   - real numbers for sample points permits us to exploit the mathematical structure and do computations
2. Definition of a real random variable
   - Consider a probability space $(\Omega, \mathcal{F}, \mathsf{P})$. A function $X : \Omega \to \mathbb{R}$ is called a real random variable on the probability space if
   $$X^{-1}(B) = \{\omega : X(\omega) \in B\} \in \mathcal{F}$$
   for all $B \in \mathcal{B}(\mathbb{R})$. The probability that the random variable takes value in the set B is defined as
   $$\mathsf{P}_X(B) = \mathsf{P}(X^{-1}(B)) = \mathsf{P}(X \in B) = \mathsf{P}(\{\omega : X(\omega) \in B\})$$
   P_X is called the distribution of r.v. X
   - An equivalent description is $X : \Omega \to \mathbb{R}$ is a real random variable on a probability space if $\{\omega : X(\omega) \leq x\} \in \mathcal{F}$ for all $x \in \mathbb{R}$. In other words, $P(X \leq x)$ must be well-defined.
3. A random variable X inspires a new probability space $(\mathbb{R}, \mathcal{B}(\mathbb{R}), \mathsf{P}_X)$!
   - $\Omega_X = \mathbb{R}$
   - $\mathcal{F} = \mathcal{B}(\mathbb{R})$
   - $\mathsf{P}_X((-\infty, x]) = \mathsf{P}(X \leq x) = \mathsf{P}(\{\omega : X(\omega) \leq x\})$ thus defined is the probability measure on the space; P_X is non-negative, normalized and countably additive!
   - It is common practice to describe the derived probability space directly!
4. Comments
   - $X(\omega) = x$ is the realized outcome of the random variable; event $B \in \mathcal{B}(\mathbb{R})$ is said to occur if $x \in B$ is realized, or if $\omega \in X^{-1}(B)$ occurs.

- a random variable is neither random nor a variable! The randomness is in the original probability space and in the realization of ω. Given ω, $X(\omega)$ is fixed!
- a random variable may be discrete, continuous or a mixture
- common and conceivable functions are often random variables!

5. Example of a random variable
   - let $\Omega = \{H, T\}$, $\mathcal{F}$ be the power set, and let p be the bias of the coin
   - Define $X(H) = 1$ and $X(T) = 0$. Then, X is a real-random variable.
   - What is $\mathsf{P}_X(\{1\}) = \mathsf{P}(X = 1)$ and $\mathsf{P}_X(\{0\}) = \mathsf{P}(X = 0)$?
   - Draw $\mathsf{P}(X \leq x)$ for all $-\infty < x < \infty$.
   - X is an example of a discrete real random variable!

6. A random variable $X : \Omega \to \mathbb{R}$ is called a discrete type random variable if there exists a discrete (finite or countable) set of real numbers $\{x_1, x_2, \cdots\}$ such that
$$\mathsf{P}(X \in \{x_1, x_2, \cdots\}) = 1$$
   In particular, X is discrete if $X(\Omega)$ is discrete!
   - Define $p_X(x_i) = \mathsf{P}_X(\{x_i\}) = \mathsf{P}(X = x_i) = \mathsf{P}(\{\omega : X(\omega) = x_i\})$ for all $i = 1, 2, \cdots$. $\{p_X(x_1), p_X(x_2), \cdots\}$ is called the probability mass function (p.m.f.) of X.
     - (non-negative) $p_X(x_i) \geq 0$
     - (normalized) $\sum_{i=1}^{\infty} p_X(x_i) = 1$
     - (countably additive) for $B \in \mathcal{B}(\mathbb{R})$, $\mathsf{P}_X(B) = \sum_{\{i: x_i \in B\}} p_X(x_i)$
   - $\{x_1, x_2, \cdots, \}$ and p.m.f. $\{p_X(\cdot)\}$ describes the derived probability space completely!

7. Examples of discrete random variables
   - **Bernoulli** ($p : 0 \leq p \leq 1$): $(\{0, 1\}, 2^{\Omega_X}, \mathsf{P}_X)$ where
$$p_X(0) = 1 - p, \text{ and } p_X(1) = p$$
   - **Uniform** ($N : N \in \mathbb{N}$): $(\{1, 2, \cdots, N\}, 2^{\Omega_X}, \mathsf{P}_X)$ where
$$p_X(i) = \frac{1}{N} \text{ for all } \; i = 1, 2, \cdots, N$$
   - **Geometric** ($p : 0 < p \leq 1$): $(\mathbb{N}, 2^{\Omega_X}, \mathsf{P}_X)$ where
$$p_X(i) = (1 - p)^{i-1} p \text{ for all } \; i \in \mathbb{N}$$
   - Indicator random variable (also a Bernoulli random variable)

- Constant random variable (degenerate random variable)

8. Exercise: How would you construct a given discrete random variable from a uniform random variable with finer resolution?

9. Exercise: Compute $\mathsf{P}(X > k)$ for X, a geometric random variable with parameter p.

10. Conditioning a random variable

    - The conditional p.m.f. of a random variable X, conditioned on a event A with $\mathsf{P}(A) > 0$ is defined as

    $$\mathsf{P}(X = x|A) = p_{X|A}(x) = \frac{\mathsf{P}(\{\omega : X(\omega) = x\} \cap A)}{\mathsf{P}(A)}$$

    - $\{p_{X|A}(x)\}$ is a valid probability mass function, i.e., it is non-negative, normalized, and it sums to one.
    - We say that the random variable X is independent of the event A if $p_{X|A}(x) = p_X(x)$ for all x.

11. Exercise: Compute $\mathsf{P}(X > k + l|X > k)$ and $\mathsf{P}(X = k + l|X > k)$ for X, a geometric random variable with parameter p.

12. Functions of a random variable

    - Consider a probability space $(\Omega, \mathcal{F}, \mathsf{P})$ and let $X : \Omega \to \mathbb{R}$ be a random variable.
    - Let $f : \mathbb{R} \to \mathbb{R}$ be such that

    $$f^{-1}(C) = \{x : f(x) \in C\} \in \mathcal{B}(\mathbb{R})$$

    for all $C \in \mathcal{B}(\mathbb{R})$, i.e., f is a nice (measurable) function.
    - Define $Y = f(X)$ or $Y(\omega) = f(X(\omega))$. Then, Y is a real random variable.
    - When X is discrete, Y is discrete. Further,

    $$p_Y(y) = \mathsf{P}_Y(\{y\}) = \mathsf{P}_X(\{x : f(x) = y\}) = \mathsf{P}(\{\omega : f(X(\omega)) = y\})$$

13. Example: Let X be a uniform random variable with parameter $N = 6$. Define $Y = X^2$. Compute $\{p_Y(\cdot)\}$.

14. Example: Consider two independent throws of a six-faced dice. Let X denote the sum of the two throws. Find

    (a) $\mathsf{P}(X = 7)$

    (b) $\mathsf{P}(X = 7|\text{one of the throw is } 6)$

    (c) $\mathsf{P}(X = 7|X > 6)$