EE5110: Probability Foundations for Electrical Engineers

July - Nov 2024, Math Review - I

1 References

- See Appendix A: Preliminaries, from Gray and Davisson for a review of set theory, set algebra and related problems.
- "Principles of Mathematical Analysis", Walter Rudin, is an excellent reference for analysis (real number system, sequences, series, continuity, integration).
- You may use online references for the basic concepts (e.g., Wikipedia).

2 Solved Problems

1. Show that $\sum_{n=1}^{\infty} p^{n-1} = \frac{1}{1-p}$ for $|p| < 1$.

   $\sum_{n=1}^{\infty} p^{n-1}$ is a series. Hence, we need to compute the partial sum sequence $\{s_n\}$ and find the limit of the partial sum sequence.

   Consider the partial sum $s_n = \sum_{i=1}^{n} p^{i-1}$. Multiplying $(1-p)$ on both the sides, we get

$$\begin{aligned} s_n(1-p) &= (\sum_{i=1}^{n} p^{i-1})(1-p) \\ &= (1 + p + p^2 + \cdots + p^{n-1})(1-p) \\ &= 1 - p^n \end{aligned}$$

   We can rearrange the terms when $p \neq 1$ to get

$$s_n = \frac{1-p^n}{1-p}$$

   Let us now find the limit of the partial sum sequence $\{s_n\}$.

$$\lim_{n\to\infty} s_n = \lim_{n\to\infty} \frac{1-p^n}{1-p} = \frac{1}{1-p} - \lim_{n\to\infty} \frac{p^n}{1-p} = \frac{1}{1-p} - \frac{1}{1-p} \lim_{n\to\infty} p^n$$

   We know that $\lim_{n\to\infty} p^n = 0$ for $|p| < 1$. Hence,

$$\lim_{n\to\infty} s_n = \frac{1}{1-p}$$

   for $|p| < 1$. Hence, $\sum_{n=1}^{\infty} p^{n-1} = \frac{1}{1-p}$ for $|p| < 1$.

2. Let $x_n \to x$ and $y_n \to y$ be convergent sequences in $\mathbb{R}$. Show that $x_n + y_n \to x + y$.

   Given $x_n \to x$. Then, for any $\epsilon_1 > 0$, we have an $N^x_{\epsilon_1}$ such that $\|x_n - x\| < \epsilon_1$ for all $n > N^x_{\epsilon_1}$. Also, we are given that $y_n \to y$. Then, for any $\epsilon_2 > 0$, we have an $N^y_{\epsilon_2}$ such that $\|y_n - y\| < \epsilon_2$ for all $n > N^y_{\epsilon_2}$.

   Define a new sequence $\{z_n\}$ where $z_n = x_n + y_n$ for all n. To show that $z_n \to x + y$,we need to find, for any $\epsilon > 0$, a N^z_ϵ such that $\|z_n - (x + y)\| < \epsilon$ for all $n > N^z_\epsilon$.

   Given $\epsilon > 0$, define $\epsilon_1 = \frac{\epsilon}{2}$ and $\epsilon_2 = \frac{\epsilon}{2}$. Now, consider $N^z_\epsilon = \max(N^x_{\epsilon_1}, N^y_{\epsilon_2})$ for the appropriately defined ϵ_1 and ϵ_2.

$$\begin{aligned}\|z_n - (x + y)\| &= \|x_n + y_n - x - y\| \\ &= \|(x_n - x) + (y_n - y)\| \\ &\leq \|x_n - x\| + \|y_n - y\|\end{aligned}$$

   where the last inequality follows from triangle inequality. We know that for any $n > N^z_\epsilon = max(N^x_{\frac{\epsilon}{2}}, N^y_{\frac{\epsilon}{2}})$, $\|x_n - x\| < \frac{\epsilon}{2}$ and $\|y_n - y\| < \frac{\epsilon}{2}$. Hence,

$$\|z_n - (x + y)\| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

   Hence, $z_n \to x + y$.

3. Let $A_1, A_2, \cdots$ be sets in Ω. Define for any $n = 1, 2, \cdots$

$$\begin{aligned} B_1 &= A_1 \\ B_2 &= A_2 - A_1, \\ &\vdots \\ B_n &= A_n - \cup_{i=1}^{n-1} A_i \\ &\vdots \end{aligned}$$

   Show that $\{B_n\}$ are disjoint and

$$\bigcup_{n=1}^{\infty} A_n = \bigcup_{n=1}^{\infty} B_n$$

   We note that $B_1 = A_1$, $B_2 = A_2 - A_1 = A_2 A_1^c$, $B_3 = A_3 - (A_1 \cup A_2) = A_3 A_2^c A_1^c$ and so on. In general, $B_n = A_n A^c_{n-1} \dots A^c_1$.

   Consider n and m, integers, such that $n > m$.

$$\begin{aligned} B_n B_m &= (A_n \dots A^c_m A^c_{m-1} \dots A_1) \cap (A_m A^c_{m-1} \dots A_1) \\ &= A_n \dots (A_m A^c_m) \dots A^c_1 && \text{(using associativity)} \\ &= A_n \dots \phi \dots A^c_1 && (AA^c = \phi) \\ &= \phi && (A\phi = \phi) \end{aligned}$$

Thus, $B_n B_m = \phi$, or, B_n and B_m are disjoint for any $n \neq m$. Hence, the collection of sets are $B_1, B_2, \cdots$ disjoint.

We will first show that $\cup_{i=1}^{n} B_i = \cup_{i=1}^{n} A_i$. We will prove the result using mathematical induction.

We know $B_1 = A_1$. Hence, the result is true for $n = 1$. Suppose that the result is true for $n-1$, i.e., $\cup_{i=1}^{n-1} B_i = \cup_{i=1}^{n-1} A_i$. We will now show that the result is true for n, i.e., $\cup_{i=1}^{n} B_i = \cup_{i=1}^{n} A_i$.

Consider $\cup_{i=1}^{n} B_i$.

$$
\begin{aligned}
\cup_{i=1}^{n} B_i &= B_n \cup \left(\cup_{i=1}^{n-1} B_i\right) \\
&= B_n \cup \left(\cup_{i=1}^{n-1} A_i\right) && \text{(by induction)} \\
&= (A_n A_{n-1}^c \ldots A_1^c) \cup \left(\cup_{i=1}^{n-1} A_i\right) && \text{(by definition of } B_n\text{)} \\
&= (A_n \cap (A_{n-1} \cup A_{n-2} \cdots \cup A_1)^c) \cup \left(\cup_{i=1}^{n-1} A_i\right) && \text{(by DeMorgan's law)} \\
&= \left(A_n \cap \left(\cup_{i=1}^{n-1} A_i\right)^c\right) \cup \left(\cup_{i=1}^{n-1} A_i\right) \\
&= \left(A_n \cup \left(\cup_{i=1}^{n-1} A_i\right)\right) \cap \left(\left(\cup_{i=1}^{n-1} A_i\right)^c \cup \left(\cup_{i=1}^{n-1} A_i\right)\right) && \text{(by distributivity)} \\
&= \cup_{i=1}^{n} A_i \cap (\Omega) && (A \cup A^c = \Omega) \\
&= \cup_{i=1}^{n} A_i && (A \cap \Omega = A)
\end{aligned}
$$

By induction, the result is true for all n.

We have shown that $\cup_{i=1}^{n} B_i = \cup_{i=1}^{n} A_i$ for all $n = 1, 2, \cdots$. But, this does not imply that

$$\bigcup_{i=1}^{\infty} B_i = \bigcup_{i=1}^{\infty} A_i$$

We need to prove the equality of the two sets in the old-fashioned way. Define

$$B = \bigcup_{i=1}^{\infty} B_i, \text{and} \quad A = \bigcup_{i=1}^{\infty} A_i$$

We will show that $A \subset B$ and $B \subset A$.

Consider $\omega \in B$. Then, there exists a n such that $\omega \in B_n$ (by definition). Now, we know that $B_n \subset A_n$ (by definition). Hence, $\omega \in A_n$ as well and trivially, $\omega \in A$ as well. Or, $B \subset A$.

Consider $\omega \in A$. Then, there exists a n such that $\omega \in A_n$ (by definition). Clearly, $\omega \in \cup_{i=1}^{n} A_i$ and so $\omega \in \cup_{i=1}^{n} B_i$ (we proved this result just now). Further, $\omega \in B$ as well (since $\cup_{i=1}^{n} B_i \subset B$). Hence, $A \subset B$.

Since $A \subset B$ and $B \subset A$, we have $A = B$.

3 Practice Problems

1. Show that $\sum_{n=1}^{\infty} \frac{1}{n} = \infty$ (i.e., show that the infinite series diverges to infinity (or) show that the sequence of partial sums increases monotonically to infinity).

2. Let $x_n \to x$ and $y_n \to y$ be convergent sequences in $\mathbb{R}$. Show that $x_n y_n \to xy$ (i.e., given $\epsilon > 0$, find N_ϵ such that $\|x_n y_n - xy\| < \epsilon$ for all $n > N_\epsilon$).

3. Using the identities $(A^c)^c = A$, $A \cup B = B \cup A$ and $(A \cup B)^c = A^c \cap B^c$, show that

   (a) $A \cap B = B \cap A$

   (b) $(A \cap B)^c = A^c \cup B^c$

4. Show that the set of rational numbers, $\mathbb{Q}$, is countable.