EE5110: Probability Foundations for Electrical Engineers

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1 References

- See Chapter 2: Discrete random variables, Bertsekas and Tsitsiklis, for a review of discrete random variables.

2 Solved Problems

1. Consider a probability space $(\Omega, \mathcal{F}, \mathsf{P})$. A function $X : \Omega \to \mathbb{R}$ is a real random variable on the probability space if, for all $x \in \mathbb{R}$,

$$\{\omega : X(\omega) \in (-\infty, x]\} \in \mathcal{F}$$

(i.e., it is sufficient to validate the definition for a generating set instead of all subsets in $\mathcal{B}(\mathbb{R})$.)

We know that $\{(-\infty, x] : x \in \mathbb{R}\}$ is a generating set for the Borel sigma-algebra of $\mathbb{R}$, i.e., any $B \in \mathcal{B}(\mathbb{R})$ is present in the sigma-algebra generated by $\{(-\infty, x] : x \in \mathbb{R}\}$.

Suppose that, for all $x \in \mathbb{R}$,

$$A_x = \{\omega : X(\omega) \in (-\infty, x]\} \in \mathcal{F}$$

Then,

$$\{\omega : X(\omega) \in (-\infty, x]^c\} = A_x^c \in \mathcal{F}$$

as well. Further, if $x_1, x_2, \cdots$ are real numbers, then

$$\left\{\omega : X(\omega) \in \bigcup_{i=1}^{\infty}(-\infty, x_i]\right\} = \bigcup_{i=1}^{\infty}\{\omega : X(\omega) \in (-\infty, x_i]\} \in \mathcal{F}$$

We can now extend the argument to other subsets to prove our result. $\square$

2. Consider a probability space $(\Omega, \mathcal{F}, \mathsf{P})$. Let $X : \Omega \to \mathbb{R}$ be a real random variable. Then, the inverse images of X inspires a sub sigma-algebra of $\mathcal{F}$.

Define $\mathcal{A} = \{A : A = \{\omega : X(\omega) \in B\}$ for some $B \in \mathcal{B}(\mathbb{R})\}$. $\mathcal{A}$ is the set of all inverse images of $B \in \mathcal{B}(\mathbb{R})$. We claim that $\mathcal{A}$ is a sub sigma-algebra of $\mathcal{F}$.

Given X is a real random variable on $(\Omega, \mathcal{F}, \mathsf{P})$. Hence, $X^{-1}(B) \in \mathcal{F}$ for all $B \in \mathcal{B}(\mathbb{R})$. So, $\mathcal{A} \subset \mathcal{F}$.

We know that $X^{-1}(\mathbb{R}) = \{\omega : X(\omega) \in \mathbb{R}\} = \Omega$. Hence, $\Omega \in \mathcal{A}$, or $\mathcal{A}$ is non-empty.

Let $A \in \mathcal{A}$. Then, $A = \{\omega : X(\omega) \in B\}$ for some $B \in \mathcal{B}(\mathbb{R})$. Now, $A^c = \{\omega : X(\omega) \notin B\} = \{\omega : X(\omega) \in B^c\}$. But $B^c \in \mathcal{B}(\mathbb{R})$ whenever $B \in \mathcal{B}(\mathbb{R})$. Hence, $A^c \in \mathcal{A}$ as well.

Let $A_1, A_2, \cdots,$ be in $\mathcal{A}$. Then, there exists $B_1, B_2, \cdots,$ events in $\mathcal{B}(\mathbb{R})$ such that $A_i = \{\omega : X(\omega) \in B_i\}$ for all i. Now,

$$\bigcup_{i=1}^{\infty} A_i = \bigcup_{i=1}^{\infty} \{\omega : X(\omega) \in B_i\} = \{\omega : X(\omega) \in \bigcup_{i=1}^{\infty} B_i\}$$

But, $\bigcup_{i=1}^{\infty} B_i \in \mathcal{B}(\mathbb{R})$ (since $\mathcal{B}(\mathbb{R})$ is a sigma-algebra). Hence, $\bigcup_{i=1}^{\infty} A_i$, which is the corresponding inverse image of a Borel set, is in $\mathcal{A}$.

Thus, $\mathcal{A}$ is non-empty and closed under complement and countable union. Hence, $\mathcal{A}$ is a sub sigma-algebra of $\mathcal{F}$. $\square$

3. Consider a probability space $(\Omega, \mathcal{F}, \mathsf{P})$. Let $X : \Omega \to \mathbb{R}$ be a real random variable. Define, for $B \in \mathcal{B}(\mathbb{R})$,

$$\mathsf{P}_X(B) = \mathsf{P}(X \in B)$$

Then, $(\Omega_X = \mathbb{R}, \mathcal{F}_X = \mathcal{B}(\mathbb{R}), \mathsf{P}_X)$ is a valid probability space.

Clearly, $(\Omega_X, \mathcal{F}_X)$ is a valid measurable space. We only need to show that P_X is a valid probability measure on $(\Omega_X, \mathcal{F}_X)$.

$\mathsf{P}_X(B) = \mathsf{P}(X \in B)$ is defined for all $B \in \mathcal{B}(\mathbb{R})$. Hence, P_X is a function from $\mathcal{F}_X$ to $[0, 1]$.

Clearly, P_X is non-negative (by definition).

$\mathsf{P}_X(\mathbb{R}) = \mathsf{P}(X \in \mathbb{R}) = 1$. Hence, P_X is normalized.

Let $B_1, B_2, \cdots$ be disjoint events in $\mathcal{F}_X$. Then, $X^{-1}(B_1), X^{-1}(B_2), \cdots$ are disjoint events in $\mathcal{F}$ as well. Now,

$$\mathsf{P}_X\left(\bigcup_{i=1}^{\infty} B_i\right) = \mathsf{P}\left(X \in \bigcup_{i=1}^{\infty} B_i\right) = \mathsf{P}\left(\bigcup_{i=1}^{\infty} \{X \in B_i\}\right) = \sum_{i=1}^{\infty} \mathsf{P}(X \in B_i) = \sum_{i=1}^{\infty} \mathsf{P}_X(B_i)$$

Hence, P_X is countably additive as well.

Thus, P_X is a valid probability measure on $(\Omega_X, \mathcal{F}_X)$. $\square$

4. Let X be a discrete random variable on a probability space $(\Omega, \mathcal{F}, \mathsf{P})$. Suppose $\mathsf{P}(X \in \{x_1, x_2, \cdots\}) = 1$, and let the probability mass function (p.m.f.) of X be $\{p_X(x_i)\}$. Then,

   (a) $p_X(x_i) \geq 0$

   (b) $\sum_{i=1}^{\infty} p_X(x_i) = 1$

(c) for $B \in \mathcal{B}(\mathbb{R}), \mathsf{P}_X(B) = \sum_{\{i: x_i \in B\}} p_X(x_i)$

$p_X(x_i) = \mathsf{P}_X(\{x_i\}) = \mathsf{P}(X = x_i) = \mathsf{P}(\{\omega : X(\omega) = x_i\}) \geq 0$, since probabilities are non-negative.

We know that $\mathsf{P}(X \in \{x_1, x_2, \cdots\}) = 1$. Further,

$$\{\omega : X(\omega) \in \{x_1, x_2, \cdots\}\} = \bigcup_{i=1}^{\infty} \{\omega : X(\omega) = x_i\}$$

So,

$$1 = \mathsf{P}(X \in \{x_1, x_2, \cdots\}) = \mathsf{P}\left(\bigcup_{i=1}^{\infty} \{\omega : X(\omega) = x_i\}\right) = \sum_{i=1}^{\infty} \mathsf{P}(X = x_i) = \sum_{i=1}^{\infty} p_X(x_i)$$

as the events are disjoint.

For any $B \in \mathcal{B}(\mathbb{R})$,

$$\mathsf{P}_X(B) = \mathsf{P}_X(B\{x_1, x_2, \cdots\}) + \mathsf{P}_X(B\{x_1, x_2, \cdots\}^c)$$

from total probability theorem and as P_X is a probability measure. Now,

$$\mathsf{P}_X(B\{x_1, x_2, \cdots\}^c) \leq \mathsf{P}_X(\{x_1, x_2, \cdots\}^c) = 1 - \mathsf{P}_X(\{x_1, x_2, \cdots\}) = 0$$

Then,

$$\mathsf{P}_X(B) = \mathsf{P}_X(B\{x_1, x_2, \cdots\}) = \sum_{\{i: x_i \in B\}} \mathsf{P}_X(\{x_i\}) = \sum_{\{i: x_i \in B\}} p_X(x_i)$$

$\square$

5. Let X be a random variable on a probability space $(\Omega, \mathcal{F}, \mathsf{P})$. Let $A \in \mathcal{F}$ such that $\mathsf{P}(A) > 0$. Then, the conditional p.m.f. of X given A, $\{p_{X|A}(\cdot)\}$ is a valid p.m.f.

   The conditional p.m.f. of X given A, $\{p_{X|A}(x_i)\}$, is defined as

   $$p_{X|A}(x_i) = \frac{\mathsf{P}(\{\omega : X(\omega) = x_i\} \cap A)}{\mathsf{P}(A)}$$

   Clearly, $p_{X|A}(x_i) \geq 0$.

   Now,

   $$\sum_{i=1}^{\infty} p_{X|A}(x_i) = \sum_{i=1}^{\infty} \frac{\mathsf{P}(\{\omega : X(\omega) = x_i\} \cap A)}{\mathsf{P}(A)} = \frac{\sum_{i=1}^{\infty} \mathsf{P}(\{\omega : X(\omega) = x_i\} \cap A)}{\mathsf{P}(A)}$$

   Now,

   $$\mathsf{P}(A) = \mathsf{P}\left(A \cap \bigcup_{i=1}^{\infty} \{\omega : X(\omega) = x_i\}\right) + \mathsf{P}\left(A \cap \left(\bigcup_{i=1}^{\infty} \{\omega : X(\omega) = x_i\}\right)^c\right)$$

Using countable additivity, we have,

$$\mathsf{P}\left(A \cap \bigcup_{i=1}^{\infty}\{\omega : X(\omega) = x_i\}\right) = \sum_{i=1}^{\infty} \mathsf{P}(A \cap \{\omega : X(\omega) = x_i\})$$

Also,

$$\mathsf{P}\left(A \cap \left(\bigcup_{i=1}^{\infty}\{\omega : X(\omega) = x_i\}\right)^c\right) \leq \mathsf{P}\left(\left(\bigcup_{i=1}^{\infty}\{\omega : X(\omega) = x_i\}\right)^c\right) = 0$$

Combining the above results, we have,

$$\sum_{i=1}^{\infty} p_{X|A}(x_i) = \frac{\sum_{i=1}^{\infty} \mathsf{P}(\{\omega : X(\omega) = x_i\} \cap A)}{\mathsf{P}(A)} = 1$$

$\square$

2.1 Examples of Important Discrete Random Variables

1. **Bernoulli** ($p : 0 \leq p \leq 1$): $(\{0, 1\}, 2^{\Omega_X}, \mathsf{P}_X)$ where

$$p_X(0) = 1 - p, \text{ and } p_X(1) = p$$

2. **Uniform** ($N : N \in \mathbb{N}$): $(\{1, 2, \cdots, N\}, 2^{\Omega_X}, \mathsf{P}_X)$ where

$$p_X(i) = \frac{1}{N} \text{ for all } \ i = 1, 2, \cdots, N$$

3. **Geometric** ($p : 0 < p \leq 1$): $(\mathbb{N}, 2^{\Omega_X}, \mathsf{P}_X)$ where

$$p_X(i) = (1 - p)^{i-1} p \text{ for all } \ i \in \mathbb{N}$$

4. **Binomial** ($N, p : N \in \mathbb{N}, 0 \leq p \leq 1$): $(\{0, 1, 2, \cdots, N\}, 2^{\Omega_X}, \mathsf{P}_X)$ where

$$p_X(k) = \binom{N}{k} p^k (1 - p)^{N-k} \text{ for all } \ k \in \{0, 1, 2, \cdots, N\}$$

5. **Poisson** ($\lambda : \lambda > 0$): $(\{0, 1, 2, \cdots\}, 2^{\Omega_X}, \mathsf{P}_X)$ where

$$p_X(k) = \frac{\lambda^k e^{-\lambda}}{k!} \text{ for all } \ k \in \{0, 1, 2, \cdots\}$$

3 Practice Problems

1. Consider a discrete probability space $(\Omega, \mathcal{F}, \mathsf{P})$ where $\mathcal{F}$ is the power set of Ω. Show that any $X : \Omega \to \mathbb{R}$ is a real, discrete, random variable. (Consider $(\Omega_X, \mathcal{F}_X, \mathsf{P}_X)$. Then, any function $f : \Omega_X \to \mathbb{R}$ induces a real, discrete, random variable $Y : \Omega \to \mathbb{R}$ if $Y = f(X) = f(X(\omega))$.)

2. Find conditions on $K, \epsilon \in \mathbb{R}$ such that $\{\frac{K}{n^{1+\epsilon}} : n = 1, 2, \cdots\}$ corresponds to a valid probability mass function of some discrete random variable with $\Omega_X = \mathbb{N}$.

3. Consider the uniform probability measure on $[0, 1]$. Find a function $X : [0, 1] \to \mathbb{R}$ such that X is a binomial random variable with parameters $(n = 2, p = \frac{1}{2})$.

4. Describe the distribution of a Poisson random variable and show that the p.m.f. sums up to one.

5. (Gray and Davisson) Use the properties of probability measures to prove the following facts about cumulative distribution functions, $F_X(x) = \mathsf{P}_X((-\infty, x])$ of a random variable X.

   (a) $F_X(-\infty) = 0$, $F_X(\infty) = 1$

   (b) $F_X(\cdot)$ is monotone non-decreasing, i.e., if $x_1 \leq x_2$, then $F_X(x_1) \leq F_X(x_2)$

   (c) $F_X(\cdot)$ is right-continuous, i.e., $\lim_{n\to\infty} F_X(x + \frac{1}{n}) = F_X(x)$ for all $x \in \mathbb{R}$.

6. (Grimmett and Stirzaker) Let X be a random variable on a given probability space, and let $a \in \mathbb{R}$. Show that aX is a real random variable.

7. Let $\{p_i : i \in \mathbb{N}\}$ and $\{q_i : i \in \mathbb{N}\}$ denote two valid p.m.f.s. Identify conditions on $\alpha \in \mathbb{R}$ such that $\{\alpha p_i + (1 - \alpha) q_i : i \in \mathbb{N}\}$ is a valid p.m.f.

8. (Poisson approximation) Consider the p.m.f. of a binomial random variable with parameters n and p. Show that asymptotically, as $n \to \infty, p \to 0$, while np is fixed at a given value λ, this p.m.f. approaches the p.m.f. of a Poisson random variable with parameter λ.

3.1 Associated Problems

1. Consider a valid probability space $(\mathbb{R}, \mathcal{B}(\mathbb{R}), \mathsf{P})$. Then, show that any monotone $X : \mathbb{R} \to \mathbb{R}$ is a real random variable.

2. Consider a valid probability space $(\Omega, \mathcal{F}, \mathsf{P})$. Let $A \in \mathcal{F}$ and 1_A denote the indicator random variable corresponding to the event A. Find the sub sigma-algebra of $\mathcal{F}$ corresponding to the inverse images of the random variable 1_A.

3. Show that the Geometric random variable is the only discrete random variable with the memoryless property, i.e., $\mathsf{P}(X > k+l|X > k) = \mathsf{P}(X > l)$ for $k, l \in \mathbb{N}$.

4. Write a Python program to generate a discrete random variable with known p.m.f. using a uniform random variable between $[0, 1]$.