EE5110: Probability Foundations for Electrical Engineers

July - November 2024, Axioms of Probability Theory

1 References

- See Chapter 1: Foundations, Bruce Hajek for a review of the axioms of the probability theory and for practice problems.
- See Chapter 2: Probability, from Gray and Davisson, for an elementary measure theoretic introduction to probability theory.
- See Chapter 1: Sample Space and Probability, Bertsekas and Tsitsiklis, for a discussion on Bertrand's paradox, Cantor's diagonalization argument and inclusion-exclusion principle.

2 Solved Problems

1. If $A_1, A_2, \cdots$ are events in $\mathcal{F}$, then

$$\mathsf{P}\left(\bigcup_{i=1}^{\infty} A_i\right) \leq \sum_{i=1}^{\infty} \mathsf{P}(A_i)$$

Let $A_1, A_2, \cdots$ be events in $\mathcal{F}$. Define

$$B_1 = A_1, B_2 = A_2 - A_1, B_3 = A_3 - (A_1 \cup A_2)$$

and in general,

$$B_n = A_n - \left(\bigcup_{i=1}^{n-1} A_i\right)$$

We note that that $B_1, B_2, \cdots$ are disjoint sets in $\mathcal{F}$ and

$$\bigcup_{i=1}^{n} A_i = \bigcup_{i=1}^{n} B_i$$

and

$$\bigcup_{i=1}^{\infty} A_i = \bigcup_{i=1}^{\infty} B_i$$

From the countable additivity axiom of the probability measure (p3), we have,

$$\begin{aligned}\mathsf{P}\left(\bigcup_{i=1}^{\infty} A_i\right) &= \mathsf{P}\left(\bigcup_{i=1}^{\infty} B_i\right) \\ &= \sum_{i=1}^{\infty} \mathsf{P}(B_i) \\ &= \lim_{n\to\infty} \sum_{i=1}^{n} \mathsf{P}(B_i)\end{aligned}$$

Now, $B_i \subset A_i$, and from the monotonicity property of the probability measure, $\mathsf{P}(B_i) \leq \mathsf{P}(A_i)$. Upper bounding the partial sum in the previous expression, we get,

$$\begin{aligned}\mathsf{P}\left(\bigcup_{i=1}^{\infty} A_i\right) &= \lim_{n\to\infty} \sum_{i=1}^{n} \mathsf{P}(B_i) \\ &\leq \lim_{n\to\infty} \sum_{i=1}^{n} \mathsf{P}(A_i) \\ &= \sum_{i=1}^{\infty} \mathsf{P}(A_i)\end{aligned}$$

(Result: Let $\{s_n\}$ and $\{r_n\}$ be monotone increasing sequences in $\mathbb{R}$ such that $s_n \leq r_n$. Then, $\lim_{n\to\infty} s_n \leq \lim_{n\to\infty} r_n$.) □

2. Consider a probability space $(\Omega, \mathcal{F}, \mathsf{P})$. Show that the countable additivity axiom implies the following statements.

   (a) Let $B_1, B_2, \cdots$ be events in $\mathcal{F}$. Then,

$$\mathsf{P}\left(\bigcup_{i=1}^{\infty} B_i\right) = \lim_{n\to\infty} \mathsf{P}\left(\bigcup_{i=1}^{n} B_i\right)$$

   (b) Let $C_1 \supset C_2 \supset \cdots$ be events in $\mathcal{F}$. Then,

$$\mathsf{P}\left(\bigcap_{i=1}^{\infty} C_i\right) = \lim_{n\to\infty} \mathsf{P}(C_n)$$

(a) Let $B_1, B_2, \cdots$ be events in $\mathcal{F}$. Define $A_1 = B_1$, $A_2 = B_2 - B_1, \cdots$, $A_n = B_n - (B_1 \cup \cdots \cup B_{n-1}), \cdots$. We note that $A_1, A_2, \cdots$ are a disjoint collection of events in $\mathcal{F}$. Further, we note that

$$\bigcup_{i=1}^{n} B_i = \bigcup_{i=1}^{n} A_i$$

and

$$\bigcup_{i=1}^{\infty} B_i = \bigcup_{i=1}^{\infty} A_i$$

From the countable/finite additivity axiom of the probability measure, we have,

$$\begin{aligned}
\mathsf{P}\left(\bigcup_{i=1}^{\infty} B_i\right) = \mathsf{P}\left(\bigcup_{i=1}^{\infty} A_i\right) &= \sum_{i=1}^{\infty} \mathsf{P}(A_i) \\
&= \lim_{n\to\infty} \sum_{i=1}^{n} \mathsf{P}(A_i) \\
&= \lim_{n\to\infty} \mathsf{P}\left(\bigcup_{i=1}^{n} A_i\right) \\
&= \lim_{n\to\infty} \mathsf{P}\left(\bigcup_{i=1}^{n} B_i\right)
\end{aligned}$$

which is our desired result.

(b) Let $C_1 \supset C_2 \supset \cdots$ be events in $\mathcal{F}$. Define $B_i = C_i^c$ for all $i = 1, 2, \cdots$. Then, $B_1 \subset B_2 \cdots$ are events in $\mathcal{F}$ and

$$\bigcup_{i=1}^{n} B_i = B_n$$

From the continuity property of the probability measure (or, part (a) of the problem), we have

$$\begin{aligned}
\mathsf{P}\left(\bigcup_{i=1}^{\infty} B_i\right) &= \lim_{n\to\infty} \mathsf{P}\left(\bigcup_{i=1}^{n} B_i\right) \\
&= \lim_{n\to\infty} \mathsf{P}\left(B_n\right)
\end{aligned}$$

The probabilities of the complements can be computed as follows.

$$\begin{aligned}
1 - \mathsf{P}\left(\bigcup_{i=1}^{\infty} B_i\right) &= 1 - \lim_{n\to\infty} \mathsf{P}\left(B_n\right) \\
&= \lim_{n\to\infty} 1 - \mathsf{P}\left(B_n\right) \\
&= \lim_{n\to\infty} \mathsf{P}\left(B_n^c\right) \\
&= \lim_{n\to\infty} \mathsf{P}\left(C_n\right)
\end{aligned}$$

Also,

$$1 - \mathsf{P}\left(\bigcup_{i=1}^{\infty} B_i\right) = \mathsf{P}\left(\left(\bigcup_{i=1}^{\infty} B_i\right)^c\right) = \mathsf{P}\left(\bigcap_{i=1}^{\infty} B_i^c\right) = \mathsf{P}\left(\bigcap_{i=1}^{\infty} C_i\right)$$

Thus,

$$\mathsf{P}\left(\bigcap_{i=1}^{\infty} C_i\right) = \lim_{n\to\infty} \mathsf{P}\left(C_n\right)$$

which is our desired result. □

3. Consider the following valid probability space

   - $\Omega = \mathbb{R}$
   - $\mathcal{F} = \mathcal{B}(\mathbb{R})$ (Borel sigma-algebra on $\mathbb{R}$)
   - for $[a, b] \in \mathcal{F}$, define $\mathsf{P}([a, b])$ as

   $$\mathsf{P}([a, b]) = \int_{[a,b]\cap[0,1]} 1 \text{ dx}$$

   Compute $\mathsf{P}(\Omega)$, $\mathsf{P}(\{0.5\})$ and $\mathsf{P}(\mathbb{Q})$.

   $\Omega = \mathbb{R} = (-\infty, \infty)$. We note that $\Omega \in \mathcal{B}(\mathbb{R})$. Hence, $\mathsf{P}(\Omega)$ is well defined.

   $$\mathsf{P}(\Omega) = \mathsf{P}(\mathbb{R}) = \int_{(-\infty,\infty)\cap[0,1]} 1 \text{ dx} = \int_0^1 1 \text{ dx} = 1$$

   Define $A_n = \left[0.5 - \frac{1}{2n}, 0.5 + \frac{1}{2n}\right]$ for $n = 1, 2, \cdots$. Note that $A_1 \supset A_2 \supset \cdots$ are events in $\mathcal{B}(\mathbb{R})$, and,

   $$\bigcap_{i=1}^{\infty} A_i = \bigcap_{i=1}^{\infty} \left[0.5 - \frac{1}{2n}, 0.5 + \frac{1}{2n}\right] = \{0.5\}$$

   ($A = B$ if $A \subset B$ and $B \subset A$)

   The probability of the event A_n can be computed as follows.

   $$\begin{aligned}\mathsf{P}(A_n) = \mathsf{P}\left(\left[0.5 - \frac{1}{2n}, 0.5 + \frac{1}{2n}\right]\right) &= \int_{\left[0.5-\frac{1}{2n},0.5+\frac{1}{2n}\right]\cap[0,1]} 1 \text{ dx} \\ &= \int_{0.5-\frac{1}{2n}}^{0.5+\frac{1}{2n}} 1 \text{ dx} \\ &= \frac{1}{n}\end{aligned}$$

From the monotonicity property of the probability measure, we have,

$$\mathsf{P}(\{0.5\}) = \mathsf{P}\left(\bigcap_{i=1}^{\infty} A_i\right) = \lim_{n\to\infty} \mathsf{P}(A_n) = \lim_{n\to\infty} \frac{1}{n} = 0$$

which is our desired result. (In fact, we can show that $\mathsf{P}(\{x\}) = 0$ for all $x \in \mathcal{R}$.)

In the previous exercise, we proved that the probability of any singleton set in $\mathbb{R}$ is zero. Now, $\mathbb{Q}$ is a countable set in $\mathcal{B}(\mathbb{R})$ which can be written as a countable union of disjoint singleton sets in $\mathcal{B}(\mathbb{R})$. Using the countable additivity axiom, we have,

$$\mathsf{P}(\mathbb{Q}) = \mathsf{P}\left(\bigcup_{i=1}^{\infty}\{q_i\}\right) = \sum_{i=1}^{\infty} \mathsf{P}(\{q_i\}) = 0$$

which is our desired result. The result implies that there are infinite (countable) events in $\mathcal{F}$ with zero probability. We can also show that $\mathsf{P}(\mathbb{P}) = 1 - \mathsf{P}(\mathbb{Q}) = 1$.

2.1 Examples of Important Probability Distributions

- **Bernoulli** $(p : 0 \le p \le 1)$: $(\{0,1\}, 2^{\Omega}, \mathsf{P})$ where

$$\mathsf{P}(\{0\}) = 1 - p, \text{ and } \mathsf{P}(\{1\}) = p$$

- **Uniform** $(N : N \in \mathbb{N})$: $(\{1,2,\cdots,N\}, 2^{\Omega}, \mathsf{P})$ where

$$\mathsf{P}(\{i\}) = \frac{1}{N} \text{ for all } \; i = 1,2,\cdots,N$$

- **Geometric** $(p : 0 < p \le 1)$: $(\mathbb{N}, 2^{\Omega}, \mathsf{P})$ where

$$\mathsf{P}(\{i\}) = (1-p)^{i-1}p \text{ for all } \; i \in \mathbb{N}$$

- **Uniform** $(a, b : a < b \in \mathbb{R})$: $(\mathbb{R}, \mathcal{B}(\mathbb{R}), \mathsf{P})$ where

$$\mathsf{P}([x,y]) = \int_{[x,y]\cap[a,b]} \frac{1}{b-a} \, \mathrm{du}$$

- **Exponential** $(\lambda : \lambda > 0)$: $(\mathbb{R}, \mathcal{B}(\mathbb{R}), \mathsf{P})$ where

$$\mathsf{P}([x,y]) = \int_{[x,y]\cap[0,\infty)} \lambda e^{-\lambda u} \, \mathrm{du}$$

- **Gaussian** $(\mu, \sigma^2 : \mu, \sigma \in \mathbb{R})$: $(\mathbb{R}, \mathcal{B}(\mathbb{R}), \mathsf{P})$ where

$$\mathsf{P}([x,y]) = \int_x^y \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(u-\mu)^2}{2\sigma^2}} \, \mathrm{du}$$

3 Practice Problems

1. (Gray-Davisson) Given that the discrete sample space Ω has n elements, show that the power set of Ω consists of 2^n elements.

2. Consider a sample space Ω (Ω may be finite, countable or uncountable). Show that the power set of Ω (set of all subsets of Ω) is an event space (i.e., the power set is non-empty, closed under complement and countable union).

3. Consider a discrete sample space Ω (Ω may be finite or countable). Show that the sigma-algebra generated by the singleton sets is the power set of Ω (i.e., the smallest sigma-algebra containing all the singleton sets is the power set of Ω).

4. Consider the Borel sigma-algebra on $\mathbb{R}$, $\mathcal{B}(\mathbb{R})$, i.e., a sigma-algebra generated by intervals (a, b) where $-\infty < a < b < \infty$. Show that

   (a) $\{a\} \in \mathcal{B}(\mathbb{R})$

   (b) $[a, b) \in \mathcal{B}(\mathbb{R})$ for $-\infty < a < b \leq \infty$

   (c) $(a, b] \in \mathcal{B}(\mathbb{R})$ for $-\infty \leq a < b < \infty$

   (d) $[a, b] \in \mathcal{B}(\mathbb{R})$ for $-\infty \leq a < b \leq \infty$

   (e) $\bigcup_{i=1}^{\infty} (a_i, b_i) \in \mathcal{B}(\mathbb{R})$ for $-\infty < a_i < b_i < \infty$ for all $i = 1, 2, \cdots$

   (f) $\bigcap_{i=1}^{\infty} (a_i, b_i) \in \mathcal{B}(\mathbb{R})$ for $-\infty < a_i < b_i < \infty$ for all $i = 1, 2, \cdots$

   (g) $\mathbb{Q}, \mathbb{P}, \mathbb{N}, \mathbb{Z}, \mathbb{R} \in \mathcal{B}(\mathbb{R})$

5. (Stark & Woods) What is the smallest sigma-algebra containing the events A and B?

6. (Sheldon Ross) Let E,F and G be three events. Find expressions for the events so that, of E,F and G,

   (a) only E occurs

   (b) both E and G, but not F, occurs

   (c) at least one of the event occurs

   (d) at most two of the event occurs

   (e) none of the event occurs

7. (Gray-Davisson) If $G \subset F$, prove that $\mathsf{P}(F - G) = \mathsf{P}(F) - \mathsf{P}(G)$.

8. (Gray-Davisson) Let $\{F_i\}$ be a countable partition in $\mathcal{F}$ (i.e., $\{F_i\}$ are disjoint and $\bigcup_{i=1}^{\infty} F_i = \Omega$). Show that for any $A \in \mathcal{F}$,

$$\mathsf{P}(A) = \sum_{i=1}^{\infty} \mathsf{P}(A \cap F_i)$$

9. (Gray-Davisson) Let F, G be events in $\mathcal{F}$. Show that if $\mathsf{P}(F) \geq 1 - \delta$ and $\mathsf{P}(G) \geq 1 - \delta$, then $\mathsf{P}(F \cap G) \geq 1 - 2\delta$. (In other words, if two events have probability nearly one, then their intersection has probability nearly one.)

10. (Bertsekas-Tsitsiklis) Out of the students in a class, 60% are geniuses, 70% love chocolate, and 40% fall into both categories. Determine the probability that a randomly selected student is neither a genius nor a chocolate lover.

11. (Hajek) What is $\mathsf{P}(ABC)$ if $\mathsf{P}(A) = \mathsf{P}(B) = \mathsf{P}(C) = 0.5$, $\mathsf{P}(A \cup B) = 0.55, \mathsf{P}(A \cup C) = 0.7$, $\mathsf{P}(BC) = 0.3$ and $\mathsf{P}(ABC) = 2\mathsf{P}(ABC^c)$?

12. (Grimmett-Stirzaker) Let A and B be events with probabilities $\mathsf{P}(A) = \frac{3}{4}$ and $\mathsf{P}(B) = \frac{1}{3}$. Show that $\frac{1}{12} \leq \mathsf{P}(A \cap B) \leq \frac{1}{3}$. Find corresponding bounds for $\mathsf{P}(A \cup B)$.

13. (Inclusion-Exclusion Principle) If $A_1, A_2, \cdots, A_n$ are events in $\mathcal{F}$, then

$$\mathsf{P}\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{i=1}^{n} \mathsf{P}(A_i) - \sum_{1 \leq i < j \leq n} \mathsf{P}(A_i A_j) + \cdots + (-1)^{n-1}\mathsf{P}(A_1 \cdots A_n)$$

14. Show that the standardized Gaussian probability density function, $f(x)$, integrates to one.

$$f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}, \quad x \in \mathbb{R}$$

3.1 Associated Problems

1. (Gray-Davisson) Describe the sigma-field of subsets of $\mathbb{R}$ generated by the points or singleton sets. Does this sigma-field contain intervals of the form (a, b) for $-\infty < a < b < \infty$?

2. (Bertsekas-Tsitsiklis) Show that the unit interval $[0, 1]$ is uncountable.

3. (Gray-Davisson) Consider the uniform probability measure on $[0, 1]$. Show that there exists an uncountable set in $[0, 1]$ with probability zero. (Hint: see Cantor's set)

4. An algebra is a collection of subsets closed under complement and finite unions. Show that the algebra made of singleton sets of $\mathbb{N}$ does not contain the set Even $= \{2, 4, 6, \cdots\}$. (The result implies that countable unions are necessary to define events of interest.)