EE5110: PROBABILITY FOUNDATIONS FOR ELECTRICAL ENGINEERS

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1 Math Review I

1.1 Set Theory

1. A set is a collection of distinct objects.
   - e.g., $\{\text{Head}, \text{Tail}\}, \mathcal{R}, (0,1) = \{x \in \mathcal{R} : 0 < x \leq <\}$
2. If Ω is any set, we write $\omega \in \Omega$ to indicate that ω is a member of the set Ω. If ω is not a member of Ω, we write $\omega \notin \Omega$.
   - e.g., $\text{Head} \in \{\text{Head}, \text{Tail}\}, 2 \notin (0,1)$
3. The set which contains no element will be called the empty set, denoted as ϕ.
4. If A and B are sets, and if every element of A is an element of B, we say that A is a subset of B, and write $A \subset B$ or $B \supset A$.
   - e.g., $[0,1] \subset \mathcal{R}$
   - $\omega \in \Omega$ and $\{\omega\} \subset \Omega$
   - If $A \subset B$ and $B \subset A$, we write $A = B$. Otherwise $A \neq B$.
5. The complement of a set A with respect to a set Ω is defined as
$$A^c = \{\omega : \omega \in \Omega \text{ and } \omega \notin A\}$$
   - e.g., $\{\text{Head}\}^c = \{\text{Tail}\}$
6. The union of sets A and B is defined as
$$A \cup B = \{\omega : \omega \in A \text{ or } \omega \in B\}$$
7. The intersection of sets A and B is defined as
$$A \cap B = \{\omega : \omega \in A \text{ and } \omega \in B\}$$
   - A and B are said to be mutually exclusive (or disjoint) if $A \cap B = \phi$. For example, $(0,1) \cap (2,3) = \phi$.
8. Let A, B, and C be subsets of Ω. Then,
   - Inverse: $(A^c)^c = A, \Omega^c = \emptyset$
   - Commutativity: $A \cup B = B \cup A, A \cap B = B \cap A$

- Associativity: $A \cup (B \cup C) = (A \cup B) \cup C, A \cap (B \cap C) = (A \cap B) \cap C$
- Distribution: $A \cup (B \cap C) = (A \cup B) \cap (A \cup C), A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
- De Morgan's laws: $(A \cup B)^c = A^c \cap B^c, (A \cap B)^c = A^c \cup B^c$
- $A \cup B = A \iff B \subseteq A$ and $A \cap B = A \iff A \subseteq B$

9. Let $\{A_i\}$ for $i \in \mathcal{I}$ be any collection of sets (even infinite).

$$\bigcup_{i \in \mathcal{I}} A_i = \{\omega : \omega \in A_i \text{ for some } i \in \mathcal{I}\}$$

$$\bigcap_{i \in \mathcal{I}} A_i = \{\omega : \omega \in A_i \text{ for all } i \in \mathcal{I}\}$$

   - $\bigcup_{n=1}^{\infty}(-n, n) = \mathcal{R}$, $\bigcap_{n=1}^{\infty}(-\frac{1}{n}, \frac{1}{n}) = \{0\}$
   - $\{A_i\}$ are said to be disjoint if $A_i \cap A_j = \phi$ for all $i, j \in \mathcal{I}, i \neq j$.

10. $A_1, A_2, \cdots$ are said to be increasing if

$$A_1 \subseteq A_2 \subseteq A_3 \subseteq \cdots$$

and decreasing if

$$A_1 \supseteq A_2 \supseteq A_3 \supseteq \cdots$$

1.2 Cardinality of Sets

1. Let A and B be two sets. f is said to be a function from A to B, denoted by $f : A \to B$, if for each element x in A, there is associated an element of B which we denote by $f(x)$. The set A is called the domain of f and the set of all values of f, denoted by $f(A)$, is called the range of f.

2. Let $f : A \to B$. We say that f maps A onto B if $f(A) = B$. For example, $f : \mathcal{R} \to \mathcal{R}$ defined as $f(x) = 2x + 1$ is onto.

3. Let $f : A \to B$. We say that f is one-to-one if $f(x_i) \neq f(x_j)$ whenever $x_i \neq x_j$ (where $x_i, x_j \in A$). For example, $f : \mathcal{R} \to \mathcal{R}$ defined as $f(x) = 2x + 1$ is one-to-one.

4. Let $f : A \to B$ be one-to-one and onto. Then, we say that A and B have the same cardinal number or that A and B are equivalent, and we write $A \sim B$ (A is equivalent to B). For example, $f(x) = 2x + 1$ is one-to-one and onto while $f(x) = x^2$ is neither one-to-one nor onto.

5. For any positive integer n, let J_n denote the set whose elements are the integers $1, 2, \cdots, n$ and let J denote the set consisting of all positive integers. For any set A, we say,

   (a) A is finite if $A \sim J_n$ for some n (the empty set is also considered to be finite).

(b) A is infinite if A is not finite.

(c) A is countable if $A \sim J$.

(d) A is uncountable if A is neither finite nor countable.

6. $\mathcal{Z}$ and $\mathcal{Q}$ are countable, and $\mathcal{R}$ is uncountable.

7. Two finite sets A and B are equivalent if and only if A and B contain the same number of elements.

8. An infinite set can be equivalent to one of its proper subsets. For example, the set of all integers $\mathcal{Z}$ is equivalent to the set of all positive integers $\mathcal{Z}^+$.

9. Let $\{A_n, n = 1, 2, \cdots\}$ be a sequence of countable sets and let

$$A = \bigcup_{n=1}^{\infty} A_n$$

Then, A is countable.

10. Let Ω denote the set of all sequences whose elements are the digits 0 and 1. Then, Ω is uncountable.

1.3 Sequences and Series

- A sequence $\{x_n\}$ in $\mathcal{R}^k$ is said to converge to a $x \in \mathcal{R}^k$ if, for every $\epsilon > 0$, there is an integer N_ϵ such that for all $n \geq N_\epsilon$, $\|x_n - x\| < \epsilon$. In this case, we write

$$\lim_{n\to\infty} x_n = x$$

  - $\frac{(-1)^n}{n} \to 0$ as $n \to \infty$
  - $(-1)^n$ does not converge

- Let $\{x_n\}$ and $\{y_n\}$ be convergent sequences in $\mathcal{R}^k$. Let $c \in \mathcal{R}^k$. Then,

$$\lim_{n\to\infty}(x_n + y_n) = \lim_{n\to\infty} x_n + \lim_{n\to\infty} y_n$$

$$\lim_{n\to\infty}(c + x_n) = c + \lim_{n\to\infty} x_n$$

$$\lim_{n\to\infty}(x_n y_n) = (\lim_{n\to\infty} x_n)(\lim_{n\to\infty} y_n)$$

$$\lim_{n\to\infty}(c x_n) = c \lim_{n\to\infty} x_n$$

$$\lim_{n\to\infty}\left(\frac{1}{x_n}\right) = \frac{1}{\lim_{n\to\infty} x_n}, \text{ if } \lim_{n\to\infty} x_n \neq 0$$

- Limits of some useful sequences:

$$\lim_{n\to\infty} \frac{1}{n^p} = 0, \quad \text{if } p > 0$$

$$\lim_{n\to\infty} p^{\frac{1}{n}} = 1, \quad \text{if } p > 0$$

$$\lim_{n\to\infty} p^n = 0, \quad \text{if } |p| < 1$$

- Given a sequence $\{x_n\}$ in $\mathcal{R}$, define another sequence $\{s_n\}$ as

$$s_n = \sum_{t=1}^{n} x_t = x_1 + x_2 + \cdots + x_n$$

The limit of the sequence $\{s_n\}$, denoted as

$$\lim_{n\to\infty} s_n = \sum_{t=1}^{\infty} x_t = x_1 + x_2 + \cdots$$

is called an infinite series, or simply a series. The numbers s_n are called the partial sums of the series.

- If $s_n \to s$, we say that the series converges and write

$$\sum_{n=1}^{\infty} x_n = s$$

  – $1 + \frac{1}{2} + \frac{1}{4} + \cdots = 2$ and $1 + 1 + 1 + \cdots = \infty$

- If $\sum_{n=1}^{\infty} |x_n|$ converges, then $\sum_{n=1}^{\infty} x_n$ converges. For example, $1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \cdots$ converges. If $\sum_{n=1}^{\infty} |x_n| = \infty$, then the series does not converge. For example, $1 - 1 + 1 - 1 + \cdots$ does not converge!

- Some useful series:

$$\sum_{n=1}^{\infty} \frac{1}{n} = \infty, \quad \sum_{n=1}^{\infty} \frac{1}{n^2} < \infty$$

$$\sum_{n=1}^{\infty} p^{n-1} = \frac{1}{1-p}, \quad \text{if } 0 \le p < 1$$

$$e^x = \lim_{n\to\infty} \left(1 + \frac{x}{n}\right)^n = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$