EE5110: PROBABILITY FOUNDATIONS FOR ELECTRICAL ENGINEERS

JULY - NOV 2024

1 Conditional Probability

1. Consider a random experiment with a probability space $(\Omega, \mathcal{F}, \mathsf{P})$
   - suppose we have partial information about the outcome of the experiment, specifically that $\omega \in A$ for some $A \in \mathcal{F}$
   - how does this information affect the probabilities of the events of interest?
   - can we derive a new probability space to represent the updated experiment?
   - how does the original probability space influence our new model?
2. (Definition) Let $A \in \mathcal{F}$ and $\mathsf{P}(A) > 0$. Then, the conditional probability of $B \in \mathcal{F}$ given the occurence of A is defined as

$$\mathsf{P}(B|A) = \mathsf{P}_A(B) = \frac{\mathsf{P}(BA)}{\mathsf{P}(A)}$$

   - $\mathsf{P}(B|A)$ is called the conditional probability of B given A
   - $\mathsf{P}(B|A)$ is undefined if the conditioning event has zero probability
3. $(\Omega, \mathcal{F}, \mathsf{P}_A)$ is a valid probability space (suited for the derived experiment)!
   - $\mathsf{P}_A(\cdot)$ is non-negative
   - $\mathsf{P}_A(\Omega) = \mathsf{P}_A(A) = 1$
   - P_A is countably additive as P is countably additive
4. Example: You toss a coin first. If the outcome is heads, you toss a four-sided die. If the outcome is tails, you toss a six-sided die. A probability space for the problem is $(\Omega = \{H1, H2, H3, H4, T1, T2, T3, T4, T5, T6\}, \mathcal{F} = 2^{\Omega}, \mathsf{P})$.
   - what is the conditional probability that the outcome is H4, given that the first flip lands on head?
5. **Multiplication Rule**:
   - Let $A_1, A_2 \in \mathcal{F}$ and $\mathsf{P}(A_1) > 0$. Then,

$$\mathsf{P}(A_1A_2) = \mathsf{P}(A_1A_2)\frac{\mathsf{P}(A_1)}{\mathsf{P}(A_1)} = \frac{\mathsf{P}(A_1A_2)}{\mathsf{P}(A_1)}\mathsf{P}(A_1) = \mathsf{P}(A_2|A_1)\mathsf{P}(A_1)$$

- If $A_1, A_2, \cdots, A_n \in \mathcal{F}$ and $\mathsf{P}(A_1) > 0, \mathsf{P}(A_1A_2) > 0, \ldots, \mathsf{P}(A_1A_2A_{n-1}) > 0$, then,

$$\mathsf{P}(A_1A_2 \ldots A_n) = \mathsf{P}(A_1)\mathsf{P}(A_2|A_1) \ldots \mathsf{P}(A_n|A_1A_2 \ldots A_{n-1}) = \prod_{i=1}^{n} \mathsf{P}(A_i|A_1 \ldots A_{i-1})$$

- multiplication rule permits us to describe the probability of events in terms of the probability of conditioning events!
- Example: What is the probability that the outcome of the experiment is H4 (assuming a fair coin and a fair die)? (Hint: $\frac{1}{8}$.)

6. **Total Probability Theorem**:

- For events A and B such that $0 < \mathsf{P}(A) < 1$,

$$\begin{aligned} \mathsf{P}(B) &= \mathsf{P}(B \cap \Omega) = \mathsf{P}(B \cap (A \cup A^c) \\ &= \mathsf{P}((BA) \cup (BA^c)) = \mathsf{P}(BA) + \mathsf{P}(BA^c) \\ &= \mathsf{P}(A)\mathsf{P}(B|A) + \mathsf{P}(A^c)\mathsf{P}(B|A^c) \end{aligned}$$

- Let $\{A_i\}_{i=1}^n$ be a partition of Ω, i.e., $\{A_i\}_{i=1}^n$ are mutually exclusive and $\bigcup_{i=1}^n A_i = \Omega$. Further, suppose that $\mathsf{P}(A_i) > 0$ for all i. Then for $B \in \mathcal{F}$,

$$\mathsf{P}(B) = \sum_{i=1}^{n} \mathsf{P}(A_i)\ \mathsf{P}(B|A_i)$$

- Extend the results for a countably infinite partition.
- Example: What is the probability that the outcome of the second throw is 4? (Hint: $\frac{5}{24}$.)

7. **Bayes' theorem**

- Let $\{A_i\}_{i=1}^n$ be a partition of Ω. Further, suppose that $\mathsf{P}(A_i) > 0$ for all i. Then for $B \in \mathcal{F}$ such that $\mathsf{P}(B) > 0$,

$$\mathsf{P}(A_i|B) = \frac{\mathsf{P}(A_iB)}{\mathsf{P}(B)} = \frac{\mathsf{P}(A_i)\ \mathsf{P}(B|A_i)}{\mathsf{P}(B)} = \frac{\mathsf{P}(A_i)\ \mathsf{P}(B|A_i)}{\sum_i \mathsf{P}(A_i)\ \mathsf{P}(B|A_i)}$$

- Bayes' theorem allows us to identify cause given the effect!
- Example: Given that the outcome of the second throw is 4, what is the probability that the outcome of the first toss is a H? (Hint: $\frac{3}{5}$.)
- Example (false-positive puzzle): A laboratory test is 95% effective in detecting a disease, when it is present. The test also yields a false-positive result for 1% of healthy persons. If 0.5% of the population actually has the disease, what is the probability that a person has the disease given that the test result is positive? (Hint: Answer is 32.3%)
- Example: Review Monty-Hall problem.

2 Independence

1. Definition: Two events A and B are said to be independent if

$$\mathsf{P}(AB) = \mathsf{P}(A)\ \mathsf{P}(B)$$

   If in addition, $\mathsf{P}(A) > 0$, then independence is equivalent to the condition

$$\mathsf{P}(B|A) = \mathsf{P}(B)$$

   If A and B are independent, we denote it as $A \perp B$

2. A and B are said to be dependent, if they are not independent.

3. Independence is a symmetric relation
   - if $A \perp B$, then $B \perp A$; also, $A \perp B^c$

4. If $A \cap B = \phi$, then A and B are dependent (!)

5. Example: Consider two tosses of a coin. Let $\Omega = \{HH, HT, TH, TT\}$, $\mathcal{F} = 2^{\Omega}$. Assume that all elementary outcomes have equal probability of $\frac{1}{4}$. Define $A = \{HH, HT\}, B = \{HH, TH\}$ and $C = \{HT, TH\}$. Identify independent events among A, B and C.

6. The description of a probability space is easier with the notion of independence.
   - Let $\mathcal{F}$ be the sigma-algebra generated by events A and B.
   - We will need $\mathsf{P}(AB)$, $\mathsf{P}(A^cB)$, $\mathsf{P}(AB^c)$ and $\mathsf{P}(A^cB^c)$ to describe the probability measure
   - However, with independence assumption, we will only need $\mathsf{P}(A)$ and $\mathsf{P}(B)$ to describe the probability measure!

7. Definition: The events $A_1, A_2, \cdots A_n$ are said to be (mutually) independent if

$$\mathsf{P}\left(\bigcap_{i \in S} A_i\right) = \prod_{i \in S} \mathsf{P}(A_i)$$

   for every subset S of $\{1, 2, \cdots, n\}$.
   - mutual independence implies pairwise independence, but pairwise independence does not imply mutual independence!
   - Example: In the previous experiment, the events A, B and C are pair-wise independent, but they are not mutually independent.
   - Example: Consider three tosses of a coin. Assume that all elementary outcomes are equally likely. Define $A = \{HHH, HTH, HHT, HTT\}, B = \{HHH, HHT, THH, THT\}$ and , $C = \{HHH, HTH, THH, TTH\}$. Show that A, B and C are mutually independent.

8. The definition of mutual independence can be extended to an infinite collection of events as well!

9. The definition of independence can be extended to conditional probabilities as well. For example, events A and B are said to be conditionally independent given C (assuming $\mathsf{P}(C) > 0$), if

$$\mathsf{P}(AB|C) = \mathsf{P}(A|C)\ \mathsf{P}(B|C)$$

   If in addition, $\mathsf{P}(A|C) > 0$, then conditional independence is equivalent to

$$\mathsf{P}(B|AC) = \mathsf{P}(B|C)$$

   In this case, A and B are said to be independent given C.

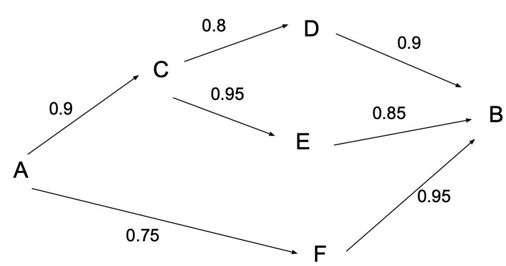
10. Example: Consider two tosses of a coin. Let $\Omega = \{HH, HT, TH, TT\}$, and $\mathcal{F} = 2^{\Omega}$. Assume that all elementary outcomes have equal probability of $\frac{1}{4}$. Define $A = \{HH, HT\}, B = \{HH, TH\}$ and $C = \{HT, TH\}$. Verify that A and B are not conditionally independent given C.

11. Example: Consider n independent tosses of a coin with bias p.
    - The probability of an all head sequence is _____
    - The probability of an all tail sequence is _____
    - The probability of first head and remaining tails is _____
    - The probability of one head and n-1 tails is _____
    - The probability of one sequence with k head and n-k tails is _____
    - The probability of k head and n-k tails is _____

12. Example: Show that if $A_1, A_2, \cdots, A_n$ are independent, then

$$\mathsf{P}\left(\bigcup_{i=1}^{n} A_i\right) = 1 - \prod_{i=1}^{n}(1 - \mathsf{P}(A_i))$$

13. Example: Determine the probability of a path connecting nodes A and B in the given graph. Edge weights represent the probability of the corresponding link existing. Assume that link existences are independent events.


0.8
D
0.9
C
0.95
0.9
0.85
B
E
A
0.95
0.75
F