SDR MODEL, CALIBRATION AND UNCERTAINTY ANALYSIS

1. SDR Model

We consider the software-defined radio (SDR) a two-port network that is terminated by a detector, as shown in Fig. 1. Such detector can be regarded a closely-matched load ($\Gamma_{load} = 0$). The goal is to establish a relation between the output wave b_2 and the input wave a_0. For radiometric measurements, we are more concerned about the ensemble average of the products of the wave and its conjugate. More specifically, $\langle |b_2|^2 \rangle$ indicates the noise power measured by the SDR and $\langle |a_0|^2 \rangle$ is the noise power presented by the noise source. Note that $\langle |a_0|^2 \rangle$ correspond to the power delivered to a matched load or a matched receiver, which is not the desired available power. More on this topic is shown below in Section 1.1.

Typical wave analysis can be applied here, namely

$$\mathbf{b} = \mathbf{S}\mathbf{a} + \mathbf{x}. \tag{1}$$

In addition, we also have the wave relationship governed by the port reflection at Port 1 and 2.

$$a_1 = \Gamma_s b_1 + a_0, \tag{2a}$$

$$a_2 = \Gamma_{load} b_2. \tag{2b}$$

Substitution of (2) in (1) leads to

$$b_1 = S_{11}(\Gamma_s b_1 + a_0) + x_1, \tag{3a}$$

$$b_2 = S_{21}(\Gamma_s b_1 + a_0) + x_2, . \tag{3b}$$

Solving b_1 from (3a) and feed the expression into (3b), we obtain

$$b_2 = \frac{S_{21}}{1 - S_{11}\Gamma_s} a_0 + \frac{S_{21}\Gamma_s}{1 - S_{11}\Gamma_s} x_1 + x_2. \tag{4}$$

We then can derive the ensemble average of the wave products; i.e. $\langle b_2 b_2^* \rangle$ or $\langle |b_2|^2 \rangle$. Note that a_0 does NOT correlate with either x_1 or x_2 waves. However, correlation exists between x_1 and x_2 waves.

(5)

$$\langle |b_2|^2 \rangle = \frac{|S_{21}|^2}{|1 - S_{11}\Gamma_s|^2} \langle |a_0|^2 \rangle + \frac{|S_{21}\Gamma_s|^2}{|1 - S_{11}\Gamma_s|^2} \langle |x_1|^2 \rangle + \langle |x_2|^2 \rangle + 2\Re \left\{ \frac{S_{21}\Gamma_s}{1 - S_{11}\Gamma_s} \langle x_1 x_2^* \rangle \right\}.$$

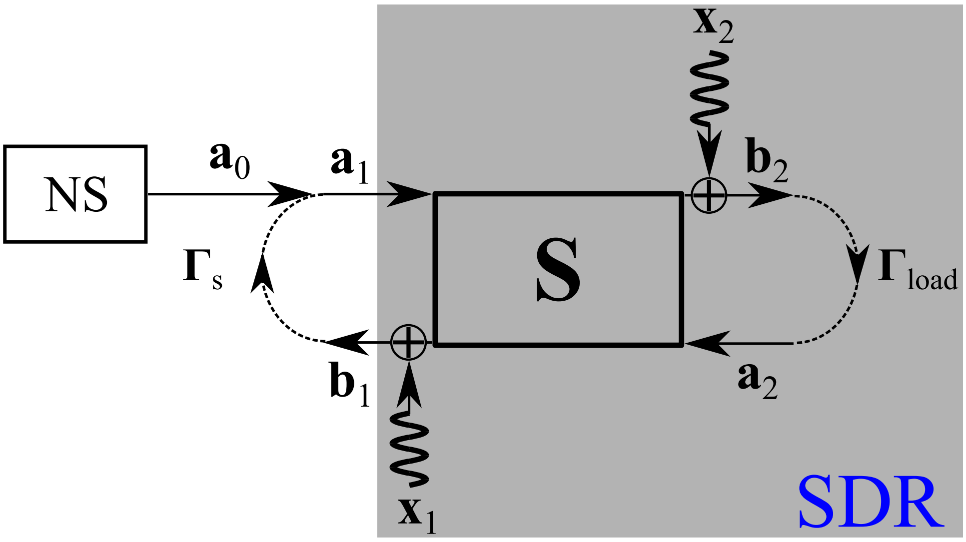

FIGURE 1. Wave modeling of SDR measurements with consideration of intrinsically added noise. The shaded region indicates the SDR receiver.

1.1. **Available Power (Wave).** We also need to clarify one subtle yet important point. The ensemble average $\langle |a_0|^2 \rangle$ is not equivalent to the available noise power of the source, which is a physical quantity of general interest. Referring to Fig. 2, the a and b waves are governed by

$$a = \Gamma_s b + a_0, \tag{6a}$$

$$b = \Gamma_{load} a. \tag{6b}$$

From the above equations, we can solve a as

$$a = \frac{1}{1 - \Gamma_s \Gamma_{load}} a_0. \tag{7}$$

Furthermore, the power delivered to the load is simply

$$P_{del} = |a|^2 - |b|^2 = (1 - |\Gamma_{load}|^2)|a|^2 = \frac{1 - |\Gamma_{load}|^2}{|1 - \Gamma_s \Gamma_{load}|^2} |a_0|^2. \tag{8}$$

The available power is the maximum power that can be delivered to a load. It occurs when the impedance of the load is conjugate matched to the impedance of the source.

$$P_{av} = P_{del}|_{\Gamma_{load} = \Gamma_s^*} = \frac{1}{1 - |\Gamma_s|^2} |a_0|^2. \tag{9}$$

As we can see, the available power is almost always greater than $|a_0|^2$ except when the source is reflectionless. Here, we abuse a bit terminology and introduce the available wave a_{av}.

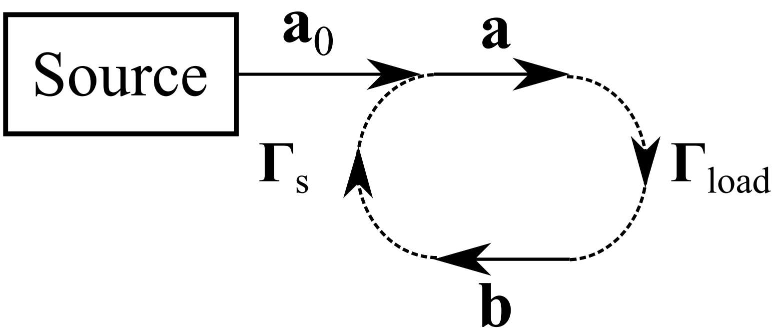

FIGURE 2. Wave generated by a source is delivered to a load. Both source and load present non-ideal reflection.

$$a_{av} = \frac{1}{\sqrt{1-|\Gamma_s|^2}} a_0. \tag{10}$$

2. Uncertainty Due to Unknown x_1

We solve a_0 in (10) and plug its expression in (5).

(11)

$$\begin{aligned}
\langle |b_2|^2 \rangle &= \frac{|S_{21}|^2(1-|\Gamma_s|^2)}{|1-S_{11}\Gamma_s|^2}\langle |a_{av}|^2\rangle + \frac{|S_{21}\Gamma_s|^2}{|1-S_{11}\Gamma_s|^2}\langle |x_1|^2\rangle + \langle |x_2|^2\rangle + 2\Re\left\{\frac{S_{21}\Gamma_s}{1-S_{11}\Gamma_s}\langle x_1 x_2^*\rangle\right\} \\
&= \begin{array}{l} \frac{|S_{21}|^2}{1-|S_{11}|^2}\frac{(1-|\Gamma_s|^2)(1-|S_{11}|^2)}{|1-S_{11}\Gamma_s|^2}\langle |a_{av}|^2\rangle + \langle |x_2|^2\rangle \\ + \frac{|S_{21}\Gamma_s|^2}{|1-S_{11}\Gamma_s|^2}\langle |x_1|^2\rangle + 2\Re\left\{\frac{S_{21}\Gamma_s}{1-S_{11}\Gamma_s}\langle x_1 x_2^*\rangle\right\} \end{array} \\
&= GM\langle |a_{av}|^2\rangle + \langle |x_2|^2\rangle + \frac{|S_{21}\Gamma_s|^2}{|1-S_{11}\Gamma_s|^2}\langle |x_1|^2\rangle + 2\Re\left\{\frac{S_{21}\Gamma_s}{1-S_{11}\Gamma_s}\langle x_1 x_2^*\rangle\right\}
\end{aligned}$$

Here, the gain of the receiver G and the mismatch factor M at the interface follow the convention.

$$G = \frac{|S_{21}|^2}{1-|S_{11}|^2}, \tag{12a}$$

$$M = \frac{(1-|\Gamma_s|^2)(1-|S_{11}|^2)}{|1-S_{11}\Gamma_s|^2}. \tag{12b}$$

Note that S_{11} is approximately the reflection seen at the input of the SDR Γ_{in}, which is directly obtainable from measurements.

We reach the following more familiar radiometer equation for ordinary receiver ($x_1 \ll x_2$) or for a reflectionless noise input ($\Gamma_s = 0$).

$$P_{meas} = GMP_{in} + P_{eqn}, \tag{13}$$

where input noise power $P_{in} = \langle |a_{av}|^2\rangle$ and equivalent noise added by the SDR $P_{eqn} = \langle |x_2|^2\rangle$.

Calibration of the SDR entails connecting two dissimilar devices (e.g. hot and cold) with known P_{in}'s, so that any other DUT measured by the SDR can be quantitatively determined. Three radiometer equations can be formulated below.

$$P_{meas}^{\mathrm{hot}} = GM^{\mathrm{hot}} P_{in}^{\mathrm{hot}} + P_{eqn}, \tag{14a}$$

$$P_{meas}^{\mathrm{cold}} = GM^{\mathrm{cold}} P_{in}^{\mathrm{cold}} + P_{eqn}, \tag{14b}$$

$$P_{meas}^{\mathrm{DUT}} = GM^{\mathrm{DUT}} P_{in}^{\mathrm{DUT}} + P_{eqn}, \tag{14c}$$

From (14), we can solve P_{in}^{DUT} as follows.

$$P_{in}^{\mathrm{DUT}} = \frac{P_{meas}^{\mathrm{DUT}} - P_{meas}^{\mathrm{cold}}}{P_{meas}^{\mathrm{hot}} - P_{meas}^{\mathrm{cold}}} \frac{1}{M^{\mathrm{DUT}}} (M^{\mathrm{hot}} P_{in}^{\mathrm{hot}} - M^{\mathrm{cold}} P_{in}^{\mathrm{cold}}) + \frac{M^{\mathrm{cold}}}{M^{\mathrm{DUT}}} P_{in}^{\mathrm{cold}}. \tag{15}$$

The two unknowns associated with the SDR can also be obtained as byproducts.

$$G = \frac{P_{meas}^{\mathrm{hot}} - P_{meas}^{\mathrm{cold}}}{M^{\mathrm{hot}} P_{in}^{\mathrm{hot}} - M^{\mathrm{cold}} P_{in}^{\mathrm{cold}}}, \tag{16a}$$

$$P_{eqn} = \frac{M^{\mathrm{hot}} P_{in}^{\mathrm{hot}} P_{meas}^{\mathrm{cold}} - M^{\mathrm{cold}} P_{in}^{\mathrm{cold}} P_{meas}^{\mathrm{hot}}}{M^{\mathrm{hot}} P_{in}^{\mathrm{hot}} - M^{\mathrm{cold}} P_{in}^{\mathrm{cold}}}, \tag{16b}$$

Evidently, this approach accounts for the ideal case. For metrology purpose, we need to take into account the residual terms in (11) for measurements of hot, cold and DUT devices. They are

$$\delta P^{\mathrm{hot}} = \frac{|S_{21}\Gamma_s^{\mathrm{hot}}|^2}{|1 - S_{11}\Gamma_s^{\mathrm{hot}}|^2} \langle |x_1|^2 \rangle + 2\Re \left\{ \frac{S_{21}\Gamma_s^{\mathrm{hot}}}{1 - S_{11}\Gamma_s^{\mathrm{hot}}} \langle x_1 x_2^* \rangle \right\}, \tag{17a}$$

$$\delta P^{\mathrm{cold}} = \frac{|S_{21}\Gamma_s^{\mathrm{cold}}|^2}{|1 - S_{11}\Gamma_s^{\mathrm{cold}}|^2} \langle |x_1|^2 \rangle + 2\Re \left\{ \frac{S_{21}\Gamma_s^{\mathrm{cold}}}{1 - S_{11}\Gamma_s^{\mathrm{cold}}} \langle x_1 x_2^* \rangle \right\}, \tag{17b}$$

$$\delta P^{\mathrm{DUT}} = \frac{|S_{21}\Gamma_s^{\mathrm{DUT}}|^2}{|1 - S_{11}\Gamma_s^{\mathrm{DUT}}|^2} \langle |x_1|^2 \rangle + 2\Re \left\{ \frac{S_{21}\Gamma_s^{\mathrm{DUT}}}{1 - S_{11}\Gamma_s^{\mathrm{DUT}}} \langle x_1 x_2^* \rangle \right\}, \tag{17c}$$

From our experience, the intrinsic noise power emanating from the input ($\langle |x_1|^2 \rangle$) is smaller than the intrinsic noise power at the output divided by the gain ($\langle |x_2|^2 \rangle / G$). This knowledge allows us to model the x_1 and x_2 with three random variables; α, θ_1 and θ_2.

$$x_1 = \alpha e^{\jmath\theta_1}, \tag{18a}$$

$$x_2 = \sqrt{P_{eqn}} e^{\jmath\theta_2}, \tag{18b}$$

where α is uniformly distributed in the range $[0, \sqrt{P_{eqn}/G}]$ and both θ_1 and θ_2 are uniformly distributed in the range $[-\pi, \pi]$.

This manipulation allows us to update (17) as follows.

(19a)

$$\delta P^{\mathrm{hot}} = \frac{|\Gamma_s^{\mathrm{hot}}|^2 (1 - |S_{11}|^2)}{|1 - S_{11}\Gamma_s^{\mathrm{hot}}|^2} G\alpha^2 + 2\Re \left\{ \frac{\sqrt{G(1 - |S_{11}|^2)}\Gamma_s^{\mathrm{hot}}}{1 - S_{11}\Gamma_s^{\mathrm{hot}}} \alpha e^{\jmath(\theta_1 + \mathrm{Arg}\{S_{21}\} - \theta_2)} \right\},$$

(19b)
$$\delta P^{\text{cold}} = \frac{|\Gamma_s^{\text{cold}}|^2(1-|S_{11}|^2)}{|1-S_{11}\Gamma_s^{\text{cold}}|^2}G\alpha^2 + 2\Re\left\{\frac{\sqrt{G(1-|S_{11}|^2)}\Gamma_s^{\text{cold}}}{1-S_{11}\Gamma_s^{\text{cold}}}\alpha e^{\jmath(\theta_1+\text{Arg}\{S_{21}\}-\theta_2)}\right\},$$

(19c)
$$\delta P^{\text{DUT}} = \frac{|\Gamma_s^{\text{DUT}}|^2(1-|S_{11}|^2)}{|1-S_{11}\Gamma_s^{\text{DUT}}|^2}G\alpha^2 + 2\Re\left\{\frac{\sqrt{G(1-|S_{11}|^2)}\Gamma_s^{\text{DUT}}}{1-S_{11}\Gamma_s^{\text{DUT}}}\alpha e^{\jmath(\theta_1+\text{Arg}\{S_{21}\}-\theta_2)}\right\},$$

Here S_{21} is also modeled as random variable with a deterministic magnitude $\sqrt{G(1-|S_{11}|^2)}$. Its phase $\text{Arg}\{S_{21}\}$ is uniformly distributed in the range of $[-\pi,\pi]$. Note the linear superposition of the uniformly distributed phase remains uniformly distributed. As such, we can reduce the sum $\theta_1 + \text{Arg}\{S_{21}\} - \theta_2$ to one random angle θ. In the end, (19) can be simplified as

$$\text{(20a)}\quad \delta P^{\text{hot}} = \frac{|\Gamma_s^{\text{hot}}|^2(1-|S_{11}|^2)}{|1-S_{11}\Gamma_s^{\text{hot}}|^2}G\alpha^2 + 2\Re\left\{\frac{\Gamma_s^{\text{hot}}e^{\jmath\theta}}{1-S_{11}\Gamma_s^{\text{hot}}}\right\}\alpha\sqrt{G(1-|S_{11}|^2)},$$

$$\text{(20b)}\quad \delta P^{\text{cold}} = \frac{|\Gamma_s^{\text{cold}}|^2(1-|S_{11}|^2)}{|1-S_{11}\Gamma_s^{\text{cold}}|^2}G\alpha^2 + 2\Re\left\{\frac{\Gamma_s^{\text{cold}}e^{\jmath\theta}}{1-S_{11}\Gamma_s^{\text{cold}}}\right\}\alpha\sqrt{G(1-|S_{11}|^2)},$$

$$\text{(20c)}\quad \delta P^{\text{DUT}} = \frac{|\Gamma_s^{\text{DUT}}|^2(1-|S_{11}|^2)}{|1-S_{11}\Gamma_s^{\text{DUT}}|^2}G\alpha^2 + 2\Re\left\{\frac{\Gamma_s^{\text{DUT}}e^{\jmath\theta}}{1-S_{11}\Gamma_s^{\text{DUT}}}\right\}\alpha\sqrt{G(1-|S_{11}|^2)}.$$

Using the Monte Carlo simulation of α and θ, we can generate a large number of samples $\breve{P}^{\text{hot}}$, $\breve{P}^{\text{cold}}$ and $\breve{P}^{\text{DUT}}$. As a consequence, statistical analysis can be conducted on simulated samples calculated from (15).

3. Uncertainty Due to Nonlinearity (Gain Suppression)

Since the receiver calibration requires measurements of two dissimilar devices in addition to the DUT (unknown spectrum), the gain of the SDR can be slightly different for input signals at different power levels. This effect can be modeled with nonlinearity of the receiver as follows.

$$\text{(21)}\qquad G = \frac{G_0}{1+\beta P_{in}}.$$

where G_0 is ideally the linear gain of the receiver and $G = G_0$ when the nonlinearity is negligible. The parameter β models the nonlinearity effect and $\beta \ll 1$. For adequately small power levels, the nonlinear gain can be approximated by

$$\text{(22)}\qquad G \approx G_0 - \beta P_{in} G_0.$$

The above equation indicates that the gain is suppressed as the input power increases. This model is consistent with the observation in experiments.

In order to account for the uncertainty due to nonlinearity, the immediate goal is to quantify the parameter β so that the gain suppression at different input power levels (P_{in}^{hot}, P_{in}^{cold}, and P_{in}^{DUT}) can be taken into account.

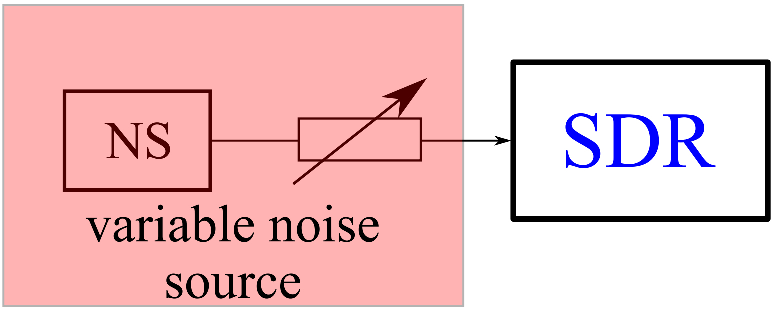

FIGURE 3. Adjustable noise input to SDR receiver to extract non-linearity parameter.

3.1. **Experimental Design to Extract β.** In combination of (13) and (22), we have

$$P_{meas} = -\beta G_0 M P_{in}^2 + G_0 M P_{in} + P_{eqn}. \tag{23}$$

We need to provide a number of known P_{in}'s to the SDR receiver. Three different inputs suffice but more will be better. A practical way will be attaching attenuators of different attenuation levels to a hot noise source as shown in Fig. 3.

The aforementioned procedure describes how to account for the non-ideal noise characteristics of the SDR. Uncertainties regarding to scattering parameters can be taken into consideration by use of the conventional approach, which won't be expanded in this document.