1 Manual solution to minimum failing example

1.1 Notation

- Δt: time step, set to 1 to simplify the minimm failing example.
- Δx: cell width, set to 1 to simpilfy the minimum failing example.
- $u_{i,j}$: Evolving variable at cell i, time j. $u$is arranged on a 1-D grid with 2 cells.
- $u_{i-,j}$: Face value of left (west) face of cell i
- $u_{i+,j}$: Face value of right (east) face of cell i
- u_g: ghost cell on the right end of the mesh
- α_{i-} or α_{i+}: The αweight to use on u_i when calculating the face value u_{i-} or u_{i+} for a convection term
- v: Convection strength on faces: $[1, 1, 0]$

1.2 Problem definition

We want to solve

$$\frac{du}{dt} = -\nabla \cdot (vu)$$

subject to a boundary condition that

$$u_{0-} = 1, u_{1+} = 1$$

using the implicit method, for one time step, stepping from $t = 0$ to $t = 1$ for simplicity.

The implicit method discretization using $\Delta t = 1$ is:

$$\begin{aligned} u_{0,1} - F_{0,1} &= u_{0,0} \\ u_{1,1} - F_{1,1} &= u_{1,0} \end{aligned}$$

where F is the contribution of the convection term to the linear system.

$$\rightarrow \begin{aligned} u_{0,1} &= F_{0,1} + u_{0,0} \\ u_{1,1} &= F_{1,1} + u_{1,0} \end{aligned}$$

Expanding F to define the contribution of the convection term, using $\Delta x = 1$ to keep the expression simple, we get:

$$\begin{aligned} u_{0,1} &= v_0 u_{0-,1} - v_1 u_{0+,1} + u_{0,0} \\ u_{1,1} &= v_1 u_{1-,1} - v_2 u_{1+,1} + u_{1,0} \end{aligned}$$

Filling in concrete values for v we obtain:

$$u_{0,1} = u_{0-,1} - u_{0+,1} + u_{0,0}$$
$$u_{1,1} = u_{1-,1} + u_{1,0}$$

Using α weighting to calculate face values:

$$u_{0,1} = \alpha_{0-} u_{0,1} + (1 - \alpha_{0-})\, u_{g,1} - \alpha_{0+} u_{0,1} - (1 - \alpha_{0+})\, u_{1,1} + u_{0,0}$$
$$u_{1,1} = \alpha_{1-} u_{1,1} + (1 - \alpha_{1-})\, u_{0,1} + u_{1,0}$$

The αvalues are (rounding ϵ values away):

$$\alpha_{0-} = 0$$
$$\alpha_{0+} = 1$$
$$\alpha_{1-} = 0$$

Substituting in these concrete values for α we get:

$$u_{0,1} = u_{g,1} - u_{0,1} + u_{0,0}$$
$$u_{1,1} = u_{0,1} + u_{1,0}$$

The ghost cell is defined by the left boundary condition:

$$u_{0-} = 1 \rightarrow \frac{1}{2}\,(u_g + u_0) = 1 \rightarrow u_g = 2 - u_0$$

Substituting in the expression for the ghost cell we get:

$$u_{0,1} = 2 - 2u_{0,1} + u_{0,0}$$
$$u_{1,1} = u_{0,1} + u_{1,0}$$

Solving this, we get

$$u_{0,1} = \tfrac{2+u_{0,0}}{3}$$
$$u_{1,1} = u_{0,1} + u_{1,0}$$

Starting from initial conditions of $u = [0, 0]$ this gives $[2/3, 2/3]$ but FiPy returns $[1/2, 1/2]$.