1 Manual solution to minimum failing example

1.1 Notation

- Δt: time step, set to 1 to simplify the minimm failing example.
- Δx: cell width, set to 1 to simpilfy the minimum failing example.
- $u_{i,j}$: Evolving variable at cell i, time j. $u$is arranged on a 1-D grid with 2 cells.
- $u_{i-,j}$: Face value of left (west) face of cell i
- $u_{i+,j}$: Face value of right (east) face of cell i
- u_g: ghost cell on the right end of the mesh
- α_{i-} or α_{i+}: The αweight to use on u_i when calculating the face value u_{i-} or u_{i+} for a convection term
- v: Convection strength on faces: $[0, 1, 1]$

1.2 Problem definition

We want to solve

$$\frac{du}{dt} = -\nabla \cdot (vu)$$

subject to a boundary condition that

$$u_{0-} = 1, u_{1+} = 1$$

using the implicit method, for one time step, stepping from $t = 0$ to $t = 1$ for simplicity.

The implicit method discretization using $\Delta t = 1$ is:

$$\begin{aligned} u_{0,1} - F_{0,1} = u_{0,0} \\ u_{1,1} - F_{1,1} = u_{1,0} \end{aligned}$$

where F is the contribution of the convection term to the linear system.

$$\rightarrow \begin{aligned} u_{0,1} = F_{0,1} + u_{0,0} \\ u_{1,1} = F_{1,1} + u_{1,0} \end{aligned}$$

Expanding F to define the contribution of the convection term, using $\Delta x = 1$ to keep the expression simple, we get:

$$\begin{aligned} u_{0,1} = v_0 u_{0-,1} - v_1 u_{0+,1} + u_{0,0} \\ u_{1,1} = v_1 u_{1-,1} - v_2 u_{1+,1} + u_{1,0} \end{aligned}$$

Filling in concrete values for v we obtain:

$$u_{0,1} = -u_{0+,1} + u_{0,0}$$
$$u_{1,1} = u_{1-,1} - u_{1+,1} + u_{1,0}$$

Filling in $u_{1+,1} \equiv 1$ from the boundary condition:

$$u_{0,1} = -u_{0+,1} + u_{0,0}$$
$$u_{1,1} = u_{1-,1} - 1 + u_{1,0}$$

Using α weighting to calculate internal face values:

$$u_{0,1} = -\alpha_{0+} u_{0,1} + (1 - \alpha_{0+})\, u_{1,1} + u_{0,0}$$
$$u_{1,1} = \alpha_{1-} u_{1,1} + (1 - \alpha_{1-})\, u_{0,1} - 1 + u_{1,0}$$

Substituting in $\alpha_{0+} = 1$ and $\alpha_{1-} = 0$we get:

$$u_{0,1} = -u_{0,1} + u_{0,0}$$
$$u_{1,1} = u_{0,1} - 1 + u_{1,0}$$

Solving this, we get

$$u_{0,1} = \frac{u_{0,0}}{2}$$
$$u_{1,1} = u_{0,1} - 1 + u_{1,0}$$

Starting from initial conditions of $u = [0, 0]$ this gives $[0, -1]$ but FiPy returns $[0, 0]$.

(I do obtain [0, 0] however if I calculated $u_{1+,1}$ above using α weighting on a ghost cell, though that's probably not the only valid way to obtain [0, 0])