Introduction to Natural Numbers

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Abstract

This document tries to answer the questions "*What is the motivation to study the concept of Natural numbers?*" and "*why do we analyse their properties?*"

Why study Natural Numbers?

We have dealt with numbers and manipulated the rules of algebra all our lives. By going back to basics we understand why the rules work at all. For example we know $a(b+c) = ab + ac$ and $ab = ba$. This is seemingly an obvious rule, we will study why this is so obvious. Even though the rules are obvious, the proof is not really easy.

The different number systems are $N \to Z \to Q \to R \to C$, in the order of complexity, where each is used to build the next one.

We will learn how to define the natural numbers ($\mathbf{N}$) using Peano axioms, and use naturals to define integers, and integers are used to define rationals, and finally we use rationals to define reals.

Definition [Informal]

A natural number is any element of the set $N = \{0, 1, 2, 3, ...\}$ The set contains all the numbers starting from 0 and then counting forward indefinitely.

The natural number system can be defined using two fundamental concepts; the number 0, and the increment operation.

With these simple operations we define complicated operations:

addition — repeated increments

multiplication — repeated addition

exponentiation — repeated multiplication

We omit subtraction and division as they are not closed under natural numbers.

On Peano Axioms

To define the natural numbers we use five axioms called the Peano axioms. It is the standard way to define natural numbers. However, this is not the only way to define the natural numbers. Cardinality of finite sets can also be used, but Peano axioms are the standard and widely used method to define natural numbers.

Axiom 1 0 is a natural number.

Axiom 2 If n is a natural number n++ is also a natural number.

Axiom 3 0 is not the successor of any natural number. i.e if n is a natural number n++ != 0.

Axiom 4 Different natural numbers have different successors i.e if n and m are natural numbers and n != m then n++ != m++. Equivalently if n++ = m++ then n = m.

Axiom 5 *[Principle Of Mathematical Induction]* Let p(n) is any property pertaining to a natural number n. Suppose that p(0) is true, and suppose that whenever p(n) is true, p(n++) is also true. Then P(n) is true for every natural number.

Propositions

Proposition: 3 is a natural number

How to prove it? We make use of two Peano axioms *(1 and 2)*

0 is a natural number *(axiom 1)*. By axiom 2, 1 := 0++ is a natural number, 2 := 1++ is a natural number and applying axiom 2 again 3 := 2++ is a natural number. Thus 3 is a natural number.

Discussion: *Are axioms 1 and 2 enough to define natural numbers?* Not exactly, consider the possibility of wrapping around. i.e after 0, 1, 2, 3 what if 3++ = 0. Axiom 1 and 2 does not prevent this possibility. Axiom 3 (see above) prevents this possibility.

Proposition: 4 is not equal to 0

By repeated application of axiom 1 and 2 we have 0, and 4 := ((((0++)++)++)++) i.e 4 := 3++. By axiom 3, 0 is not a successor of any natural number. Which implies $3++ \neq 0$ Or $4 \neq 0$.

Now again repeating the question *"Are axioms 1, 2, and 3 enough to define the natural number system?"*

Answer is still no. This scheme does not rule out the possibility that the natural number stops at an arbitrary number. For example, 3++ = 4, 4++ = 4, 5++ = 4, so on. . . i.e the natural number system hits the ceiling.

Axiom 4 prevents this case from happening.

Proposition: 6 is not equal to 2.

Can axiom 4 alone prove this proposition? 2 := 1++ and 6 := 5++. Since 1 and 5 are natural numbers and different, they have different successors, hence 2 != 6. i.e 1 != 5 hence 2 != 6. This proof has a flaw, we assumed 1 and 5 are different without actually proving that fact. However if we can prove 1 and 5 are different, the proof will be complete.

Consider the alternate proof (*proof by contradiction*). Suppose $6 = 2$, then $5{+}{+} = 1{+}{+}$. Then by axiom 4, $5 = 1$ and thus $4{+}{+} = 0{+}{+}$, thus $4 = 0$. But by our earlier proposition $4 \neq 0$, hence $6 \neq 2$.

Notes

1. The Peano axioms define natural numbers.
2. Each individual number is finite, the set of natural numbers is infinite.
3. ∞ is not one of the natural numbers. Calling ∞ as the largest natural number is wrong.
4. The natural numbers can approach ∞ but never actually reach it. That is, N is infinite but consists of individually finite elements.
5. However, there are other number systems which "admit" infinite numbers:
   (a) Cardinals
   (b) Ordinals
   (c) p-adics