Addition Of Natural Numbers

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Abstract

This document discusses some of the properties of addition of natural numbers. We state and prove some of the lemmas and propositions related to addition of natural numbers.

Introduction

We have been trying to build natural numbers axiomatically, as of now we have the definitiion of natural numbers, increment operation, and a handful of axioms.

Isn't this enough?

With the existing set of definitions we can just increment numbers. How to add two natural numbers, say 3 and 5? We can achieve by incrementing 5 three times or maybe incrementing 3, 5 times.

This is an increment more than adding two to five. Which is one increment more than adding one to five, so on... we can formally rephrase the above discussion recursively.

Definition [*Addition of natural numbers*] Let m be a natural number. To add 0 to m we define $0+m = m$. Now suppose, inductively we have defined how to add n to m. Then we can add n++ to m by defining $(n{+}{+})+m = (n+m){+}{+}$

Now elaborating the earlier definition we can write as follows,

$0 + m = m$,
$1 + m = (0 + +) + m = (0 + m) + + = m + +$
$2 + m = (1 + +) + m = (1 + m) + + = ((0 + +) + m) + + = ((0 + m) + +) + + =$
$(m + +) + +$
so on...

Note: The above definition of addition is assymetric because 5 + 3 is incrementing 3 five times and 3 + 5 is incrementing 5 three times. However, they yeild same results.

Lemma: For any natural number n, $n + 0 = n$

[Note that this is different from $0 + n = n$ which we defined earlier as n and we haven't proved $a + b = b + a$ yet.]

Proof[Proof by induction]
The base case $0 + 0 = 0$ follows since $0 + m = m$ for all natural numbers m and 0 is a natural number. Now suppose inductively we have proved $n + 0 = n$ we wish to prove for $n + +$. $(n + +) + 0 = (n + 0) + +$ (By the definition of addition. And $(n + 0) = n$ by inductive step, hence we have $(n + +) + 0 = (n + 0) + + = n + +$. Thus closes the induction.

Lemma: For any natural numbers n and m, $n + (m + +) = (n + m) + +$

[Note that this is similar to earlier lemma, but we are showing that the definition of the addition operation is commutative as well. Again, since we have not yet showed $a + b = b + a$ we cannot assume $(a + +) + b = a + (b + +) = (a + b) + +$.]

Proof: [Proof by induction]
We induct on n keeping m constant. For the base case $n = 0$ we have $0 + (m + +) = m + +$ and $(0 + m) + + = m + +$ thus LHS is equal to RHS, which proves our base case. Suppose inductively we have proved $n + (m + +) = (n + m) + +$ we wish to show that this is true for n++ as well. i.e we wish to prove
$(n + +) + (m + +) = ((n + +) + m) + + = ((n + m) + +) + +$.
Now consider LHS $(n + +) + (m + +)$ which is equal to $(n + (m + +)) + +$, by the definition of addition and by the inductive step we know $n + (m + +) = (n + m) + +$ and hence we have LHS equal to $((n + m) + +) + +$. And consider the RHS $((n + +) + m) + +$ by the defintion of addition we write $((n + m) + +) + +$ hence we have LHS equal to RHS which closes the induction.

Corollary $n + + = n + 1$

Proof We have by earlier lemma $n + 0 = n$ and $n + +$ can be written as $(n + 0) + +$.
Thus we have $(n + 0) + + = n + (0 + +)$ again by earlier lemma we have $n + (m + +) = (n + m) + +$. We know by definition $0 + + = 1$ hence we have $(n + 0) + + = n + (0 + +) = n + 1$.

Addition is Commutative

Proposition: For any natural numbers n and m, $n + m = m + n$

Proof: [*We prove this by induction*

We induct on n keeping m constant. First the base case, $n = 0$. By the earlier lemma we have $m + 0 = m$ and by definition we have $0 + m = m$ hence we have $0 + m = m + 0$ which proves the base case.

Suppose inductively we have proved $n+m = m+n$. Now we wish to show it is true for $n++$, i.e $(n++)+m = m+(n++)$. Consider the LHS $(n++)+m$, by the definition of addition $(n++)+m=(n+m)++$. Now consider the RHS $m+(n++)$, by the earlier lemma $m+(n++)=(m+n)++$. By the inductive step we know $n+m=m+n$ hence we have the LHS and RHS as $(m+n)++$ which closes the induction.

Addition is associative

Proposition: For any natural numbers a, b, and c we have $(a+b)+c=a+(b+c)$

Proof:

We induct on a, keeping b and c constant. For the base case we have $a=0$ and thus $(0+b)+c=0+(b+c)$ which gives us $b+c=b+c$ which proves our base case.

Suppose we have proved the proposition inductively i.e $(a+b)+c=a+(b+c)$.

Now we wish to prove the proposition for $a++$.

$((a++)+b)+c=(a++)+(b+c)$

Consider the L.H.S $((a++)+b)+c$

$=((a+b)++)+c$

$=((a+b)+c)++$ [By the definition of addition]

Now consider the R.H.S $(a++)+(b+c)$

$=(a+(b+c))++$

And we have by the inductive step and by axiom 4 $((a+b)+c)=(a+(b+c))$. Which closes the induction.

Proposition: *Cancellation Law*

Let a, b, c be natural numbers such that $a+b=a+c$ then $b=c$.

Proof

[Note that we cannot use subtraction or negative numbers to prove this. We prove this by induction, we induct on a, and prove that LHS = RHS]

First let us consider the base case $a=0$, we have $0+b=0+c$ by earlier propositions we have $b=c$, which proves our base case. And suppose inductively we have the the cancellation law for a (i.e $a+b=a+c$ implies $b=c$); now we wish to prove the cancellation law for $a++$.

That is we wish to show $(a++)+b=(a++)+c$ implies $b=c$. By the definition of addition we have $(a++)+b=(a+b)++$ and $(a++)+c=(a+c)++$ and thus $(a+b)++=(a+c)++$. By ***Axiom 4*** we have $a+b=a+c$ and by the inductive step $b=c$, which closes the induction.

Addition and positivity

Definition *Positive Natural Numbers*

A natural number is said to be positive iff it is not equal to 0.

Proposition If a is positive and b is a natural number then $a+b$ is positive.

Proof: [*Proof by induction*]

We induct on b (because we need 0?) keeping a constant. For the base case $b=0$, we have $a+0=a$ (by earlier lemma) which is positive, this proves the base case. Now suppose inductively we have proved $a + b$ is positive. We wish to prove for b++, i.e $a + (b++) = (a + b)++$ is positive. Now by ***Axiom 3*** $a + (b++) = (a + b)++$ cannot be 0, since 0 is not a successor of any natural number. And by inductive step $(a + b)$ is positive and more so $(a + b)++$ is positive, which closes the induction.

Corollary: If a and b are natural numbers such that $a + b = 0$, then $a = 0$ and $b = 0$

Proof: [We cannot use induction in this case, we use the earlier proposition.]

Suppose for the sake of contradiction $a \neq 0$ or $b \neq 0$. If $a \neq 0$ then $a + b$ is positive (by proposition) and if $b \neq 0$ then $a+b$ is again positive, a contradiction in both cases.

Hence if $a + b = 0$ then both a and b are equal to 0. If either one of them is positive then $a + b$ is not equal to 0.

Lemma: Let a be a positive number. Then there exists exactly one natural number b such that $b++ = a$

Proof

For the sake of contradiction let there exist two natural numbers b and c such that $b++ = a$ and $c++ = a$. By axiom 4 we have $b = c$, hence there exists exactly one natural number such that $b++ = a$.

Notion of Order

Definition[*Ordering of the Natural Numbers*]

Let n and m be natural numbers, we say that n is greater than or equal to m, and write $n \geq m$ or $m \leq n$, if we have $n = m + a$ for some natural number a. We say n is strictly greater than m and write $n > m$ or $m < n$ if and only if $n \geq m$ and $n \neq m$

Also note that $n++ > n$ for any n; thus there is no largest natural number n, because next number $n++$ is always larger still.

The definition of ordering of natural numbers is beautiful. Or how else would you say that a number is greater than another?

Proposition [*Basic Properties of order for natural numbers*]

Let $a,\ b,\ c$ be natural numbers then:

1. Order is reflexive: $a \geq a$
2. Order is transitive: if $a \geq b$ and $b \geq c$ then $a \geq c$
3. Order is anti-symmetric: if $a \geq b$ and $b \geq a$ then $a = b$
4. Addition preserves order: if $a \geq b$ iff $a + c \geq b + c$
5. $a < b$ iff $a++ \leq b$
6. $a < b$ iff $b = a + d$ for some positive number d

Order is reflexive: $a \geq a$

By the definition of ordering we have $a \geq a$ iff $a = a + b$ for some natural number b. For $b = 0$ we have $a = a + 0$ and b is a natural number which implies $a \geq a$.

Order is transitive: if $a \geq b$ and $b \geq c$ then $a \geq c$

By the definition of ordering if $a \geq b$ then $a = b + j$ for some natural number j.

And $b \geq c$ then $b = c + k$ for some natural number k.

And we can write $a = b + k$ as $a = c + j + k$ since j and k are natural numbers, by axiom 2, $j + k$ is a natural number.

Hence it follows $a \geq c$

Order is anti-symmetric: if $a \geq b$ and $b \geq a$ then $a = b$

If $a \geq b$, then by definition $a = b + c$ for some natural number c.

If $b \geq a$, then again by definition $b = a + d$ for some natural number d.

By adding the above $a + b = b + c + a + d$ and $a + b = b + a + c + d$ (addition is commutative).

And again $a + b = a + b + c + d$ and we can write the L.H.S as $(a + b) + 0 = (a + b) + c + d$ and by the cancellation law we have $c + d = 0$.

By the earlier proposition if $c + d = 0$, then $c = 0$ and $d = 0$.

Thus we have $a = b + c$ and since $c = 0$ we have $a = b + 0$ or $a = b$. We can show $b = a$ similarly.

Addition preserves order: if $a \geq b$ iff $a + c \geq b + c$

We have to prove that (i) if $a \geq b$ then $a + c \geq b + c$ and (ii) if $a + c \geq b + c$ then $a \geq b$.

Case (i): If $a \geq b$, then by definition $a = b + x$ for some natural number x and $a + c = b + x + c$, $a + c = b + c + x$, and $a + c = (b + c) + x$ which implies $a + c \geq b + c$.

Case (ii): If $a + c \geq b + c$ by definition $a + c = b + c + x$ and $a + c = b + x + c$ and by the cacellation law we have $a = b + x$ for some natural number x.

And thus proves $a \geq b$

Proposition [*Trichotomy of order for natural numbers*] Let a and b be natural numbers, then exactly one of the following statements is true: $a < b$, $a = b$ or $a > b$
[*The proof is tricky, we first prove that we cannot have more than one of the claim is true at the same time and then we prove at least one of the statements is true. This is unconventional (at least for me).*]

Firstly we show that we cannot have more than one statement $a < b$, $a = b$ or $a > b$ holding at the same time. (Note: we do not try to show only one of the claim is true, we can't).

If $a < b$, then by definition $a \neq b$. Again if, $a > b$, then $a \neq b$.
And if $a > b$ and $a < b$ then $a = b$ (By above proposition)
Thus no more than one of the statement is true.

Now we prove that at least one of the statements is true (we use induction for that). We keep b fixed and induct on a.
Consider the base case $a = 0$, if $a = 0$ then $a \leq b$ for all b, because by *Axiom 3*, 0 is not a successor of any natural number. Since b is a natural number we have $b = 0$ or $0 < b$, which proves the base case.

Suppose we have proved the proposition inductively for a, we wish to prove it for $a++$
From the trichotomy for a, there are three cases $a < b$, $a = b$ or $a > b$

1. If $a > b$, then $a++ > b$ because we have by earlier proposition, addition preserves order. i.e $a > b$ then $a++ > b++$ and more so $a++ > b$.

2. If $a = b$, then $a++ > b$ because, we know by the earlier definition $a++ > a$ for every natural number a, and we have $a = b$, hence $a++ > a = b$ or $a++ > b$

3. Now suppose that $a < b$, then by earlier proposition $a++ \leq b$. Thus $a++ < b$ or $a++ = b$ in either case we are done.

This closes the induction.