On Analysis

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April 25, 2021

Abstract

This article aims to answer two questions, *what is analysis?* and *why do analysis?* These are the notes from Terence Tao's book **Analysis I**.

What is analysis?

Analysis is the rigourous study of a subject. In *Real Analysis* we study the qualitative and quantitave behaviour of real numbers, sequences and series of real numbers, and real valued functions.

In studying real numbers, we are concerned in answering the following questions:

1. What is a real number?
2. Is there a largest real number?
3. After 0 what is the next real number? [Smallest positive real number]
4. Can a real number be cut into infinitely many pieces?
5. Why does 2 has a square root where as -2 does not?
6. If real and rational numbers are infinite, why are there more reals?
7. How do you take the limit of a sequence of real numbers?
8. Which sequences have limit, which don't?
9. If a sequence can be stopped from escaping to infinity, does it settle down and converge?
10. Can infinitely many real numbers can be added and still get a finite real number?
11. Can infinitely many rationals be added and still get a non-rational number?
12. If the elements of an infinite sum are rearranged, is the sum still the same?

13. What does it mean for a function to be continuous?
14. What does it mean for a function to be differentiable?
15. What does it mean for a function to be integrable?
16. What does it mean for a function to be bounded?
17. Can multiple functions be added infinitely?
18. Can you take limits of sequences of functions?
19. Can you differentiate an infinite series of functions?
20. Can you integrate an infinite series of functions?
21. If $f(0) = 3$ and $f(1) = 5$, does the function take all the values between 3 and 5 within its domain $[0, 1]$ or $(0, 1)$.

Real Analysis is the foundation for calculus. Though we have worked on hundreds of problems in differential and integral calculus, by doing analysis we can understand what is really happening under the hood.

Why do analysis?

There is a philosophical satisfaction in knowing why things work. Also, one can get into trouble if the rules of calculus are applied without blindly without knowing where they came from and what the limits of their applicability are. To prove the above claim, consider the following examples.

Example 1: Division by zero

Consider the cancellation law $ac = bc \Rightarrow a = b$, the cancellation doesn't work if $c = 0$. (Why?)

To answer the question, consider the identity $1 \cdot 0 = 2 \cdot 0$. The identity is true, however if the cancellation law is applied blindly, we cancel out the zeroes and end up with $1 = 2$ which is absurd.

In this case the division by 0 is obvious, but it can be more hidden in other cases.

Example 2: Divergent series

Consider the geometric series,

$$S = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots$$

Multiply the series by 2 and we get

$$2S = 2 + 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots = 2\text{+S}$$

and $S = 2$.

Now apply the same trick to the series

$$S = 1 + 2 + 4 + 8 + \cdots$$
$$2S = 2 + 4 + 8 + 16 + \cdots$$

Adding 1 to both sides

$$2S + 1 = 1 + 2 + 4 + 8 + 16 + \cdots = S$$

and $S = -1$, which is absurd. So, why does the trick work for one series and not for another?

Now consider another series

$$S = 1 - 1 + 1 - 1 + \cdots$$

We rearrane the series in three different ways and get three different results. Which one is correct?

$$S = 1 - (1 + 1 - 1 + 1 - 1) \cdots = 1 - S$$

gives $\frac{1}{2}$ as the sum of the series.

Consider another arrangement

$$S = (1 - 1) + (1 - 1) + (1 - 1) \cdots$$

which gives 0 as the sum of the series.

Yet another arrangement

$$S = 1 + (-1 + 1) + (-1 + 1) + (-1 + 1) \cdots$$

gives 1 as the result of the series.

Example 3: Divergent sequences

Let x be a real number and let L be the limit

$$L = \lim_{n \to \infty} x^n$$

Changing variables $n = m + 1$ we have

$$L = \lim_{m+1 \to \infty} x^{m+1} = \lim_{m+1 \to \infty} x \cdot x^m = x \cdot \lim_{m+1 \to \infty} x^m$$

But if $m + 1 \to \infty$ then $m \to \infty$, thus we have

$$\lim_{m+1 \to \infty} x^m = \lim_{m \to \infty} x^m = \lim_{n \to \infty} x^n = L$$

and thus $x \cdot L = L$.

At this point we cancel the L's and conclude $x = 1$. Or because of our previous experience of blind cancellation problems we can safely assume that $L = 0$. In particular, we seem to have shown

$$\lim_{n \to \infty} x^n = 0$$

for all $x \neq 1$.

But is this a legitimate conclusion? It proves to be absurd if we go for specific values of x. For example for x = 2 our result states $1, 2, 4, 8, \cdots$ converges to 0. And for x = 1 $1, -1, 1, -1, \cdots$ converges 0. Which of these conclusions are right?

Analysis helps in understanding these kinds of problems. For more examples refer *Analysis I* by *T. Tao*.