Final Exam of Quantum Field Theory

Qiang Gao, qg728

December 26, 2018

Problem # 1. Suppose in the problem we did in class $\pi^0+N \to \pi^0+N$ we had an additional term so the Hamiltonian is

$$\mathscr{H}_{int} = g\bar{\psi}(x)\gamma^5\psi(x)\phi(x) + \lambda\phi^3(x) \tag{1}$$

Where ϕ is the field for the π^0 and λ is a coupling constant. What additional Feynman diagrams would there be? Draw the diagram(s) and write out the matrix element for these.

First, label the momentums

$$\pi^0(k_1) + N(p_1) \to \pi^0(k_2) + N(p_2) \tag{2}$$

and the T-matrix reads:

$$\begin{aligned}
&\langle k_2, p_2|T|k_1, p_1\rangle \\
&= \sum_{n=0}^{\infty} \frac{(-i)^{n-1}}{n!} \int d^4x_1 d^4x_2 \cdots d^4x_{n-1} \langle k_2, p_2|(\mathscr{H}_{int}(x_1)\mathscr{H}_{int}(x_2)\cdots\mathscr{H}_{int}(x_{n-1})\mathscr{H}_{int}(0))_+|k_1, p_1\rangle \\
&= \langle k_2, p_2| : g\bar{\psi}(0)\gamma^5\psi(0)\phi(0) + \lambda\phi^3(0)) : |k_1, p_1\rangle \\
&\quad + \frac{(-i)}{2!} \int d^4x \langle k_2, p_2| : \big(g\bar{\psi}(x)\gamma^5\psi(x)\phi(x) + \lambda\phi^3(x)\big)\big(g\bar{\psi}(0)\gamma^5\psi(0)\phi(0) + \lambda\phi^3(0)\big) : |k_1, p_1\rangle \\
&\quad + O(g^3)
\end{aligned} \tag{3}$$

In the first order of g:

$$\langle k_2, p_2| : g\bar{\psi}(0)\gamma^5\psi(0)\phi(0) + \lambda\phi^3(0) : |k_1, p_1\rangle = 0 \tag{4}$$

For the second order of g, there are four different terms labelled as I, II, III and IV:

$$I = \frac{(-i)}{2!} \int d^4x \langle k_2, p_2|g^2 : \bar{\psi}(x)\gamma^5\psi(x)\phi(x)\bar{\psi}(0)\gamma^5\psi(0)\phi(0) : |k_1, p_1\rangle \tag{5}$$

which is exactly what we have done in the class, and the three extra terms:

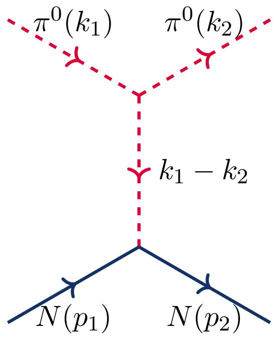
$$II = \frac{(-i)}{2!} \int d^4x \langle k_2, p_2|g\lambda\bar{\psi}(x)\gamma^5\psi(x)\phi(x)\phi^3(0) : |k_1, p_1\rangle \tag{6}$$

$$III = II(x \leftrightarrow 0) \tag{7}$$

and

$$IV = \frac{(-i)}{2!} \int d^4x \langle k_2, p_2|\lambda^2 : \phi^3(x)\phi^3(0) : |k_1, p_1\rangle = 0 \tag{8}$$

So, the non-zero extra terms are just II and III which are both corresponding to Feynman diagram like:



which the matrix element reads:

$$M(II) = \frac{(-i)}{2!} g\lambda \bar{u}^{(s_2)}(p_2)\gamma_5 u^{(s_1)}(p_1) \frac{1}{(k_1 - k_2)^2 - m_\pi^2} \tag{9}$$

and $M(III) = M(II)$ which corresponds to the same diagram shown above.

Problem # 2. We observe three families of particles, ν_e, e, u-quark, d-quark, and ν_μ, μ, c-quark, s-quark, and ν_τ, τ, b-quark, t-quark. There could be other very heavy quarks but we don't think so because they should have light neutrinos associated with them and those neutrinos would contribute to the width of the Z boson through $Z \to \nu\bar{\nu}$ and the Z width has been measured accurately ($\pm 0.0023 GeV$). But suppose there were a scalar particle coupled to the Z through

$$\mathscr{H}_{int} = \frac{e}{4 sin\theta_W cos\theta_W} Z^\alpha (\phi^+ \partial_\alpha \phi - \partial_\alpha \phi^+ \phi) \tag{10}$$

Find the contribution of $Z \to \phi^+ + \phi$ to the total Z width (in GeV). Take $\alpha(M_Z)$ to be 1/128.9 and $sin^2\theta_W = 0.231$. The mass of the Z is 91GeV. First find the result as a function of the mass of the ϕ and then give the numerical result for the mass that makes this contribution maximum.

First, label the momentums:

$$Z(p) \to \phi^+(k_1) + \phi(k_2) \tag{11}$$

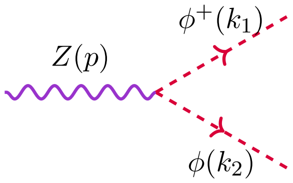
and the T-matrix reads:

$$\begin{aligned} &\langle k_1, k_2 | T | p \rangle \\ &= \sum_{n=0}^{\infty} \frac{(-i)^{n-1}}{n!} \int d^4x_1 d^4x_2 \cdots d^4x_{n-1} \langle k_1, k_2 | (\mathscr{H}_{int}(x_1)\mathscr{H}_{int}(x_2)\cdots\mathscr{H}_{int}(x_{n-1})\mathscr{H}_{int}(0))_+ | p \rangle \end{aligned} \tag{12}$$

The first order:

$$\begin{aligned} &\langle k_1, k_2 | \frac{e}{4 sin\theta_W cos\theta_W} Z^\alpha(0)(\phi^+(0)\partial_\alpha\phi(0) - \partial_\alpha\phi^+(0)\phi(0)) | p \rangle \\ &= \frac{e}{4 sin\theta_W cos\theta_W} \left(\langle k_1, k_2 | Z^\alpha(0)\phi^+(0)\partial_\alpha\phi(0) | p \rangle - \langle k_1, k_2 | Z^\alpha(0)\partial_\alpha\phi^+(0)\phi(0) | p \rangle \right) \\ &= \frac{e}{4 sin\theta_W cos\theta_W} [\epsilon^\alpha(p,\lambda)(ik_{1,\alpha}) - \epsilon^\alpha(p,\lambda)(ik_{2,\alpha})] \\ &= \frac{ie}{4 sin\theta_W cos\theta_W} \epsilon^\alpha(p,\lambda)(k_1 - k_2)_\alpha \\ &= \frac{ie}{4 sin\theta_W cos\theta_W} \epsilon^\alpha(p,\lambda)(2k_1 - p)_\alpha \\ &= \frac{ie}{2 sin\theta_W cos\theta_W} \epsilon^\alpha(p,\lambda) k_{1,\alpha} \end{aligned} \tag{13}$$

where $\epsilon^\alpha(p,\lambda)$ is the polarizations of Z boson and we have used $\partial\phi(0) = \lim_{y\to 0} \partial\phi(y)$ and $\epsilon^\alpha(p,\lambda)p_\alpha = 0$. This corresponds to the following Feynman diagram:



So,

$$
\begin{aligned}
|M|^2 &= \frac{1}{3}\sum_{\lambda=1}^{3}\Big|\frac{ie}{2sin\theta_W cos\theta_W}\epsilon^{\alpha}(p,\lambda)k_{1,\alpha}\Big|^2 \\
&= \frac{e^2}{12sin^2\theta_W cos^2\theta_W}k_{1,\alpha}k_{1,\beta}\sum_{\lambda=1}^{3}\epsilon^{\alpha}(p,\lambda)\epsilon^{*\beta}(p,\lambda) \\
&= \frac{e^2}{12sin^2\theta_W cos^2\theta_W}k_{1,\alpha}k_{1,\beta}(\frac{p^{\alpha}p^{\beta}}{M_Z^2}-g^{\alpha\beta}) \\
&= \frac{e^2}{12sin^2\theta_W cos^2\theta_W}\Big(\frac{(p\cdot k_1)^2}{M_Z^2}-k_1^2\Big)
\end{aligned}
\tag{14}
$$

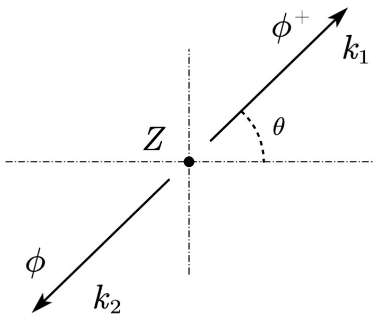
Then, the decay rate is

$$
\begin{aligned}
\frac{1}{\tau} &= \int \frac{(2\pi)^{4-3\times 2}}{2E_Z}\cdot\frac{d^3k_1 d^3k_2}{2E_1\cdot 2E_2}\delta^4(k_1+k_2-p)|M|^2 \\
&= \int \frac{(2\pi)^{-2}}{2E_Z}\cdot\frac{d^3k_1}{2E_1}\delta((p-k_1)^2-m_\phi^2)|M|^2
\end{aligned}
\tag{15}
$$

where we have applied $d^2k_2/2E_2 = d^4k_2\delta(k_2^2-m_\phi^2)$ and integrated over k_2, and using $(p-k_1)^2-m_\phi^2 = M_Z^2-2p\cdot k_1$ we have

$$
\frac{1}{\tau} = \frac{e^2}{12sin^2\theta_W cos^2\theta_W}\int\frac{(2\pi)^{-2}}{2E_Z}\cdot\frac{d^3k_1}{2E_1}\delta(M_Z^2-2p\cdot k_1)\Big(\frac{(p\cdot k_1)^2}{M_Z^2}-m_\phi^2\Big) \tag{16}
$$

Now evaluating this in the center of mass frame:



in which

$$
\begin{aligned}
p &= (M_Z,0,0,0) \\
k_1 &= (E_1,|\vec{k_1}|cos\theta,0,|\vec{k_1}|sin\theta)
\end{aligned}
\tag{17}
$$

$\Rightarrow$

$$
\begin{aligned}
&E_1^2 = |\vec{k_1}|^2+m_\phi^2; \quad p\cdot k_1 = M_Z E_1; \\
&\delta(M_Z^2-2p\cdot k_1) = \frac{1}{2M_Z}\delta(\frac{M_Z}{2}-E_1); \\
&d^3k_1 = d\Omega|\vec{k_1}|^2 d|\vec{k_1}| = d\Omega|\vec{k_1}|E_1 dE_1
\end{aligned}
\tag{18}
$$

Inserting those into the integration, we obtain

$$\begin{aligned}
\frac{1}{\tau} &= \frac{e^2}{12sin^2\theta_W cos^2\theta_W}\int \frac{(2\pi)^{-2}}{2M_Z}\cdot\frac{d\Omega|\vec{k_1}|E_1 dE_1}{2E_1}\frac{1}{2M_Z}\delta(\frac{M_Z}{2}-E_1)\Big(\frac{(M_Z E_1)^2}{M_Z^2}-m_\phi^2\Big) \\
&= \frac{e^2}{384\pi^2 sin^2\theta_W cos^2\theta_W M_Z^2}\int d\Omega\int|\vec{k_1}|dE_1\delta(\frac{M_Z}{2}-E_1)\Big(E_1^2-m_\phi^2\Big) \\
&= \frac{e^2}{96\pi sin^2\theta_W cos^2\theta_W M_Z^2}\int dE_1\sqrt{E_1^2-m_\phi^2}\delta(\frac{M_Z}{2}-E_1)\Big(E_1^2-m_\phi^2\Big) \\
&= \frac{e^2}{96\pi sin^2\theta_W cos^2\theta_W M_Z^2}\big[(M_Z^2/4-m_\phi^2)^{3/2}\big] \\
&= \frac{\alpha}{24 sin^2\theta_W cos^2\theta_W M_Z^2}\big[(M_Z^2/4-m_\phi^2)^{3/2}\big]
\end{aligned} \tag{19}$$

When $m_\phi = 0$, $1/\tau$ has its maximum:

$$\frac{1}{\tau} = \frac{\alpha M_Z}{192sin^2\theta_W cos^2\theta_W} = \frac{91GeV}{192\times 128.9\times 0.231\times(1-0.231)} = 0.0207GeV \tag{20}$$

Problem # 3. (a) Draw all the (lowest order, nonzero) Feynman diagrams for

$$\gamma + e^- \to e^- + \mu^+ + \mu^- \tag{21}$$

Label the momenta. (b)Now draw all the Feynman diagrams for

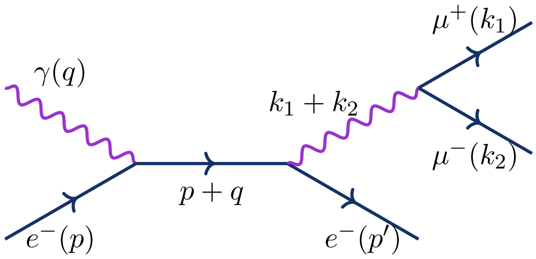
$$\gamma + e^- \to e^- + e^+ + e^- \tag{22}$$

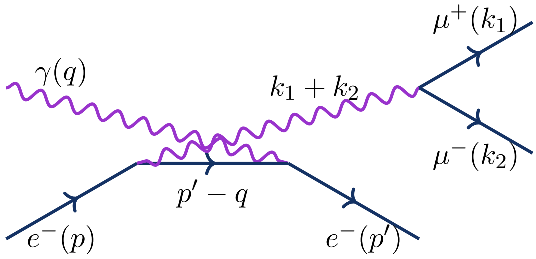
Label the momenta and give the relative signs between the diagrams. (c)Use (b) to write an analytic expression for the matrix element. (d)Use gauge invariance to check your result for (c).

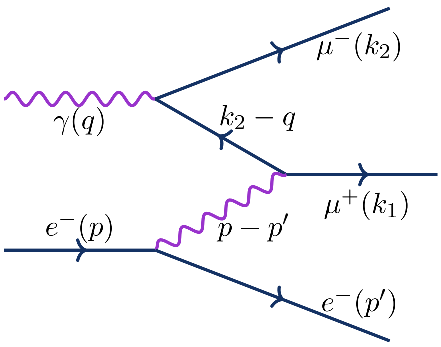
(a)First, label the momenta:

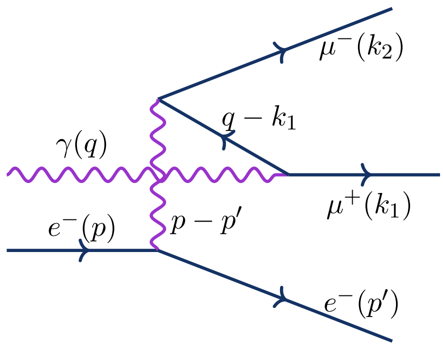
$$\gamma(q) + e^-(p) \to e^-(p') + \mu^+(k_1) + \mu^-(k_2) \tag{23}$$

The lowest order, non-zero Feynman diagrams are the following:





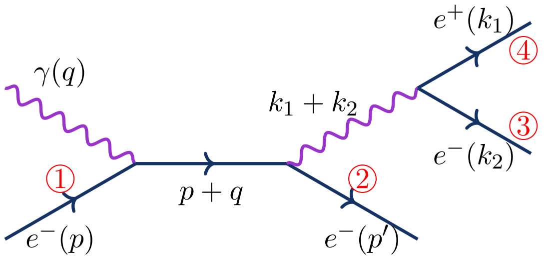




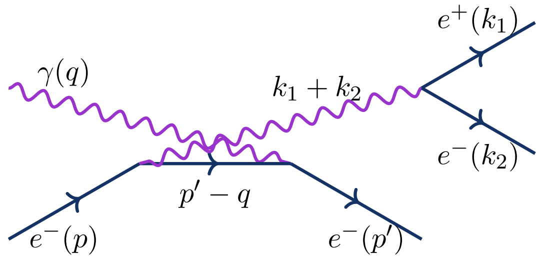
(b)Label the momenta:

$$\gamma(q) + e^-(p) \to e^-(p') + e^+(k_1) + e^-(k_2) \tag{24}$$

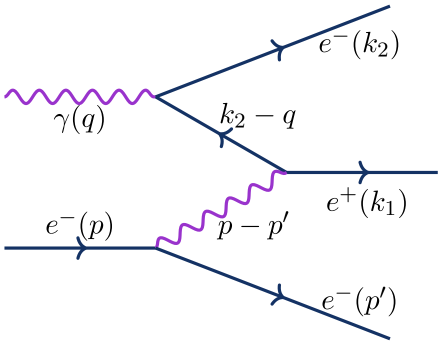
Then, the lowest order, non-zero Feynman diagrams are the following:



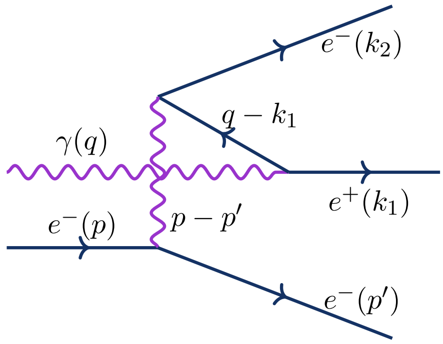
I



II

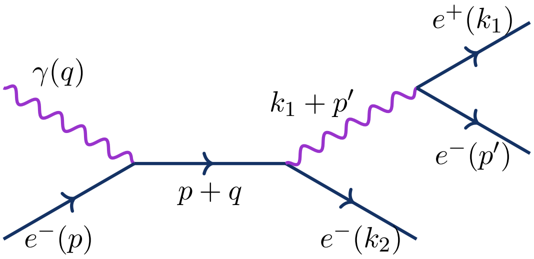


III

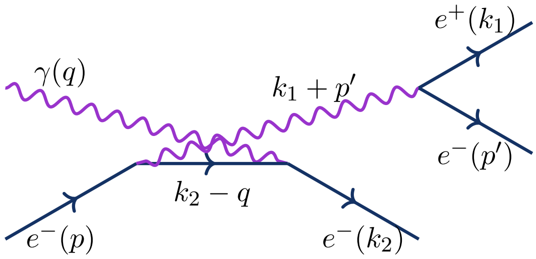


IV

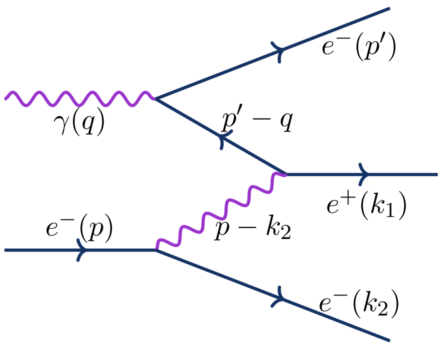
However, in the final state, we have a pair of identical particles ($e^-(p')$ and $e^-(k_2)$). So we can always exchange those two electron lines (plus a minus sign) to get other non-zero Feynman diagrams:



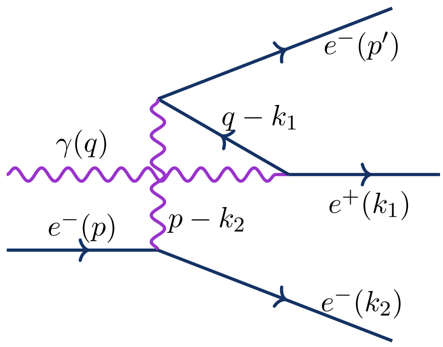
V



VI



VII



VIII

Now, we can use

"exchanging two fermion lines of same kind gains a minus sign"
"exchanging two boson lines of same kind gains a plus sign"

to determine the relative sign between those diagrams. We will use diagram ***I*** as reference diagram and label different lines with different numbers on it to keep track when exchanging lines.

I with exchanging ① with ④ $\Rightarrow$ $(-)$***VIII***
I with exchanging ② with ③ $\Rightarrow$ $(-)$***V***
I doing the above exchanges at the same time $\Rightarrow$ $(+)$***IV***

and exchanging two photon lines to get the other four diagrams but with no sign changing. So, finally we have:

$$M = \boldsymbol{I} + \boldsymbol{II} + \boldsymbol{III} + \boldsymbol{IV} - \boldsymbol{V} - \boldsymbol{VI} - \boldsymbol{VII} - \boldsymbol{VIII} \tag{25}$$

(c) Analytic expressions:

$$M(\boldsymbol{I}) = \frac{-ie^2}{(k_1+k_2)^2}\bar{u}(p')\gamma_\beta(\not{p}+\not{q}-m)^{-1}\not{\epsilon}(q)u(p)\bar{u}(k_2)\gamma^\beta v(k_1) \tag{26}$$

$$M(\boldsymbol{II}) = \frac{-ie^2}{(k_1+k_2)^2}\bar{u}(p')\gamma_\beta(\not{p}'-\not{q}-m)^{-1}\not{\epsilon}(q)u(p)\bar{u}(k_2)\gamma^\beta v(k_1) \tag{27}$$

$$M(\boldsymbol{III}) = \frac{-ie^2}{(p-p')^2}\bar{u}(k_2)\gamma_\beta(\not{k}_2-\not{q}-m)^{-1}\not{\epsilon}(q)v(k_1)\bar{u}(p')\gamma^\beta u(p) \tag{28}$$

$$M(\boldsymbol{IV}) = \frac{-ie^2}{(p-p')^2}\bar{u}(k_2)\gamma_\beta(\not{q}-\not{k}_1-m)^{-1}\not{\epsilon}(q)v(k_1)\bar{u}(p')\gamma^\beta u(p) \tag{29}$$

$$M(\boldsymbol{V}) = \frac{-ie^2}{(k_1+p')^2}\bar{u}(k_2)\gamma_\beta(\not{p}+\not{q}-m)^{-1}\not{\epsilon}(q)u(p)\bar{u}(p')\gamma^\beta v(k_1) \tag{30}$$

$$M(\boldsymbol{VI}) = \frac{-ie^2}{(k_1+p')^2}\bar{u}(k_2)\gamma_\beta(\not{k}_2-\not{q}-m)^{-1}\not{\epsilon}(q)u(p)\bar{u}(p')\gamma^\beta v(k_1) \tag{31}$$

$$M(\boldsymbol{VII}) = \frac{-ie^2}{(p-k_2)^2}\bar{u}(p')\gamma_\beta(\not{p}'-\not{q}-m)^{-1}\not{\epsilon}(q)v(k_1)\bar{u}(k_2)\gamma^\beta u(p) \tag{32}$$

$$M(\boldsymbol{VIII}) = \frac{-ie^2}{(p-k_2)^2}\bar{u}(p')\gamma_\beta(\not{q}-\not{k}_1-m)^{-1}\not{\epsilon}(q)v(k_1)\bar{u}(k_2)\gamma^\beta u(p) \tag{33}$$

(d) Under gauge transformation:

$$\epsilon^\mu(q) \to \epsilon'^\mu(q) = \epsilon^\mu(q) + aq^\mu \tag{34}$$

the matrix element should remain unchanged, which means

$$M(\not{\epsilon} \to \not{q}) = 0 \tag{35}$$

Given that:

$$\not{q} = (\not{p}+\not{q}-m) - (\not{p}-m) \quad and \quad (\not{p}-m)u(p) = 0 \tag{36}$$

we have

$$M(\boldsymbol{I}, \not{\epsilon} \to \not{q}) = \frac{-ie^2}{(k_1+k_2)^2}\bar{u}(p')\gamma_\beta\not{q}u(p)\bar{u}(k_2)\gamma^\beta v(k_1) \tag{37}$$

and in the same sense: given that

$$\not{q} = (\not{p}'-m) - (\not{p}'-\not{q}-m) \quad and \quad \bar{u}(p')(\not{p}'-m) = 0 \tag{38}$$

we have

$$M(\boldsymbol{II}, \not{\epsilon} \to \not{q}) = -\frac{-ie^2}{(k_1+k_2)^2}\bar{u}(p')\gamma_\beta\not{q}u(p)\bar{u}(k_2)\gamma^\beta v(k_1) \tag{39}$$

So,

$$M(\boldsymbol{I}, \not{\epsilon} \to \not{q}) + M(\boldsymbol{II}, \not{\epsilon} \to \not{q}) = 0 \tag{40}$$

Similarly, we can also obtain that:

$$M(\boldsymbol{III}, \not{\epsilon} \to \not{q}) + M(\boldsymbol{IV}, \not{\epsilon} \to \not{q}) = 0 \tag{41}$$

$$M(\boldsymbol{V}, \not{\epsilon} \to \not{q}) + M(\boldsymbol{VI}, \not{\epsilon} \to \not{q}) = 0 \tag{42}$$

$$M(\boldsymbol{VII}, \not{\epsilon} \to \not{q}) + M(\boldsymbol{VIII}, \not{\epsilon} \to \not{q}) = 0 \tag{43}$$

So, M is gauge invariant.

The gauge invariance is guaranteed by the exchange of the photon line in this case.