Note of Quantum Optics

1 场量子化

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1 算符的对易子

$$[a, a^\dagger] = 1 \tag{1}$$
$$[a, (a^\dagger)^n] = n(a^\dagger)^{n-1} \tag{2}$$
$$[a^\dagger, a^n] = -na^{n-1}. \tag{3}$$

于是, 对于函数 $f(a)$ 及 $f(a^\dagger)$

$$\begin{aligned}[a^\dagger, f(a)] &= \sum_{k=0}^{\infty} \frac{a_k}{k!}[a^\dagger, a^k] &(4a)\\ &= -\sum_{k=0}^{\infty} \frac{a_k}{k!}\frac{\mathrm{d}}{\mathrm{d}a}a^k &(4b)\\ &= -\frac{\mathrm{d}}{\mathrm{d}a}f(a), &(4c)\end{aligned}$$

以及

$$[a, f(a^\dagger)] = \frac{\mathrm{d}}{\mathrm{d}a^\dagger}f(a^\dagger). \tag{5}$$

2 位移算符

$$D(\alpha) = \exp(\alpha a^\dagger - \alpha^* a) = e^{-|\alpha|^2/2}e^{\alpha a^\dagger}e^{-\alpha^* a}. \tag{6}$$

若将其写作 e^{iH}, 易知其 H 厄米, 于是可知 $D(\alpha)$ 幺正, 即

$$D^\dagger(\alpha) = D^{-1}(\alpha) = D(-\alpha). \tag{7}$$

下面考察 $D^\dagger(\alpha)aD(\alpha)$

$$\begin{aligned} D^\dagger(\alpha)aD(\alpha) &= D(-\alpha)e^{-|\alpha|^2/2}(ae^{\alpha a^\dagger})e^{-\alpha^* a} && \text{(8a)} \\ &= D(-\alpha)e^{-|\alpha|^2/2}(e^{\alpha a^\dagger}a + \alpha e^{\alpha a^\dagger})e^{-\alpha^* a} && \text{(8b)} \\ &= D(-\alpha)D(\alpha)(a+\alpha) && \text{(8c)} \\ &= a + \alpha, && \text{(8d)} \end{aligned}$$

其中第二行通过 a 与函数 $e^{\alpha a^\dagger}$ 的对易关系得到. 上式两边取厄米共轭即得

$$D^\dagger(\alpha)a^\dagger D(\alpha) = a^\dagger + \alpha^* \tag{9}$$

3 相干态的 Quadrature 不确定度

将湮灭算符 a 的实部和虚部

$$a = \frac{X_1 + iX_2}{2} \tag{10}$$

于是有

$$\begin{aligned} X_1 &= a^\dagger + a, && \text{(11a)} \\ X_2 &= i(a^\dagger - a). && \text{(11b)} \end{aligned}$$

易证

$$[X_1, X_2] = 2i. \tag{12}$$

下面计算相干态的 ΔX_1 与 ΔX_2.

$$\langle\alpha|X_1|\alpha\rangle^2 = \langle\alpha|(a^\dagger + a)|\alpha\rangle^2 = (\alpha + \alpha^*)^2 \tag{13}$$

$$\langle\alpha|(a^\dagger + a)^2|\alpha\rangle = \langle\alpha|a^{\dagger 2} + a^2 + 2a^\dagger a + 1|\alpha\rangle = (\alpha + \alpha^*)^2 + 1 \tag{14}$$

故 $(\Delta X_1)^2 = 1$, ΔX_2 同理.

4 压缩算符

压缩算符形式为

$$S(\epsilon) = \exp\left[\frac{1}{2}(\epsilon^* a^2 - \epsilon a^{\dagger 2})\right] \tag{15}$$

其中 $\epsilon = re^{i2\phi}$. 其指数项乘 i 后为厄米算符, 故压缩算符幺正, 有

$$S^\dagger(\epsilon) = S^{-1}(\epsilon) = S(-\epsilon). \tag{16}$$

接下来计算 $S^\dagger(\epsilon)aS(\epsilon)$, 使用如下结论: 对于算符 $\hat{A}$ 和 $\hat{B}$ 有

$$e^{\hat{A}}\hat{B}e^{-\hat{A}} = \sum_{k=0}^{\infty}\frac{\hat{C}_k}{k!}, \tag{17}$$

其中,

$$\hat{C}_0 = \hat{B},\quad \hat{C}_1 = [\hat{A}, \hat{C}_0],\quad \hat{C}_2 = [\hat{A}, \hat{C}_1], \dots \tag{18}$$

这里我们取 $\hat{A} = (\epsilon a^{\dagger 2} - \epsilon^* a^2)/2$, $\hat{B} = a$, 于是

$$\begin{aligned} \hat{C}_0 = a \quad &, \quad \hat{C}_1 = -e^{i2\phi} r a^\dagger \\ \hat{C}_2 = r^2 a \quad &, \quad \hat{C}_3 = -e^{i2\phi} r^3 a^\dagger \\ \cdots \quad &, \quad \cdots \end{aligned}$$

故

$$\begin{aligned} S^\dagger(\epsilon)aS(\epsilon) &= \sum_{k=0}^{\infty}\frac{r^{2k}}{(2k)!}\cdot a - e^{i2\phi}\sum_{k=0}^{\infty}\frac{r^{2k+1}}{(2k+1)!}\cdot a^\dagger && \text{(19a)} \\ &= a\cosh r - a^\dagger e^{i2\phi}\sinh r && \text{(19b)} \end{aligned}$$

两边取厄米共轭, 得

$$S^\dagger(\epsilon)a^\dagger S(\epsilon) = a^\dagger\cosh r - ae^{-i2\phi}\sinh r. \tag{20}$$

在此基础上, 我们取相空间中坐标旋转, 即引入

$$\hat{b} = \frac{Y_1 + iY_2}{2} = e^{-i\phi}\frac{X_1 + iX_2}{2} = e^{-i\phi}a, \tag{21}$$

于是

$$\begin{aligned} S^\dagger(\epsilon)bS(\epsilon) &= e^{-i\phi}(a\cosh r - a^\dagger e^{i2\phi}\sinh r) && \text{(22a)} \\ &= b\cosh r - b^\dagger\sinh r && \text{(22b)} \\ &= \frac{e^{-r}Y_1 + ie^{r}Y_2}{2}, && \text{(22c)} \end{aligned}$$

可见, 压缩算符将 Y_1 压缩, 将 Y_1 放大. 压缩量与 $r = \epsilon$ 密切相关, 称之为压缩系数.

5 压缩态的性质

构造压缩态 $|\alpha, \epsilon\rangle$ 如下

$$|\alpha, \epsilon\rangle = D(\alpha)S(\epsilon)|0\rangle, \tag{23}$$

将其看成对真空态先压缩再利用平移算符平移得到. 该量子态有如下性质

$$\langle X_1 + iX_2\rangle = \langle Y_1 + iY_2\rangle e^{i\phi} = 2\alpha, \tag{24a}$$
$$\Delta Y_1 = e^{-r}, \qquad \Delta Y_2 = e^{r}, \tag{24b}$$
$$\langle N\rangle = |\alpha|^2 + \sinh^2 r, \tag{24c}$$
$$(\Delta N)^2 = |\alpha\cosh r - \alpha^* e^{i2\phi}\sinh r|^2 + 2\cosh^2 r\sinh^2 r. \tag{24d}$$

证明如下:

(a)

$$\langle a\rangle = \langle 0|S^\dagger(a+\alpha)S|0\rangle \tag{25a}$$
$$= \alpha + \langle 0|S^\dagger aS|0\rangle(=0). \tag{25b}$$

(b) 易证, $D^\dagger Y_1 D = Y_1 + c(\alpha)$, 其中 $c(\alpha)$ 为与 α 取值相关的常数. 于是

$$\langle Y_1\rangle = \langle 0|S^\dagger[Y_1 + c(\alpha)]S|0\rangle \tag{26a}$$
$$= c(\alpha). \tag{26b}$$

$$\langle Y_1^2\rangle = \langle 0|S^\dagger[Y_1 + c(\alpha)]^2 S|0\rangle \tag{27a}$$
$$= c(\alpha)^2 + 2c(\alpha)e^{-r}\langle 0|Y_1|0\rangle + e^{-2r}\langle 0|Y_1^2|0\rangle \tag{27b}$$
$$= c(\alpha)^2 + e^{-2r}. \tag{27c}$$

故 $(\Delta Y_1)^2 = \langle Y_1^2\rangle - \langle Y_1\rangle^2 = e^{-2r}$, Y_2 同理可证.

(c)

$$\langle N\rangle = \langle 0|S^\dagger D^\dagger a^\dagger a DS|0\rangle \tag{28a}$$
$$= \langle 0|S^\dagger(a^\dagger + \alpha^*)(a+\alpha)S|0\rangle \tag{28b}$$
$$= |\alpha|^2 + \langle 0|S^\dagger a^\dagger aS|0\rangle \tag{28c}$$
$$= |\alpha|^2 + \sinh^2 r\langle 0|aa^\dagger|0\rangle \tag{28d}$$
$$= |\alpha|^2 + \sinh^2 r. \tag{28e}$$

(d)

$$\langle N^2\rangle = \langle 0|S^\dagger D^\dagger (a^\dagger a)^2 DS|0\rangle \tag{29a}$$
$$= \langle 0|S^\dagger[(a^\dagger + \alpha^*)(a+\alpha)]^2 S|0\rangle \tag{29b}$$
$$= \sum_{k=0}^{\infty} |\langle n|S^\dagger(a^\dagger + \alpha^*)(a+\alpha)S|0\rangle|^2 \tag{29c}$$

上式第三行是通过插入 Identity 算符得到的. 易知上式求和号中只有 $n = 0, 1, 2$ 不为 0, 即可将求和上限改为 2,

$$n = 0, \quad |\sim|^2 = \langle N\rangle^2, \tag{30a}$$

$$n = 1, \quad |\sim|^2 = |\alpha \cosh r - \alpha^* e^{i2\phi} \sinh r|^2, \tag{30b}$$

$$n = 2, \quad |\sim|^2 = (\sinh r \cosh r)^2 |\langle 2|a^{\dagger 2}|0\rangle|^2. \tag{30c}$$

得证.

6 位移算符和压缩算符的顺序交换

压缩算符可看做如下过程

$$b = UaU^\dagger = \mu a + \nu a^\dagger, \tag{31}$$

其中 b 要满足对易关系 $[b, b^\dagger] = 1$, 就要求

$$|\mu|^2 - |\nu|^2 = 1. \tag{32}$$

我们只需取 $\mu = \cosh r$, $\nu = e^{i2\phi} \sinh r$, 便有 $U = S(\epsilon)$.

这里, 我们直接将 b 视为另一个光场模式, 用脚标 g 标识. 对于该模式中的相干态, 我们有

$$b|\beta\rangle_g = \beta|\beta\rangle_g, \tag{33}$$

该相干态可由利用 b 给出的位移算符得到, 即

$$|\beta\rangle_g = D_g(\beta)|0\rangle_g, \tag{34}$$

其中

$$D_g(\beta) = \exp(\beta b^\dagger - \beta^* b). \tag{35}$$

值得注意的是, $|\beta\rangle$ 与 $|\beta\rangle_g$ 存在关系, 将 (31) 带入 (33) 有

$$UaU^\dagger|\beta\rangle_g = \beta|\beta\rangle_g \tag{36}$$

即

$$a\left(U^\dagger|\beta\rangle_g\right) = \beta\left(U^\dagger|\beta\rangle_g\right), \tag{37}$$

可见

$$|\beta\rangle = U^\dagger|\beta\rangle_g. \tag{38}$$

现在我们来关注式 (34), 有

$$UD(\beta)|0\rangle = U|\beta\rangle = |\beta\rangle_g = D_g(\beta)U|0\rangle \tag{39}$$

即

$$S(\epsilon)D(\alpha) = D_g(\alpha)S(\epsilon) \tag{40}$$

其中, $D_g(\alpha)$ 是利用模式 b 定义的算符, 我们将 b 用 a 模式表示, 可得

$$D_g(\alpha) = D(\alpha'),\ 其中\ \alpha' = \mu^*\alpha - \nu\alpha^*. \tag{41}$$

最终我们得到结论

$$S(\epsilon)D(\alpha) = D(\alpha')S(\epsilon),\ 其中\ \alpha' = \mu^*\alpha - \nu\alpha^*. \tag{42}$$

7 双模压缩算符

双模压缩态记作 $|\alpha_+, \alpha_-\rangle$, 其定义为

$$|\alpha_+, \alpha_-\rangle = D_+(\alpha_+)D_-(\alpha_-)S(G)|0\rangle, \tag{43}$$

其中 $D_\pm$ 为位移算符, 其定义为

$$D_\pm(\alpha) = \exp(\alpha a_\pm^\dagger - \alpha^* a_\pm), \tag{44}$$

$S(G)$ 为双模压缩算符, 定义为

$$S(G) = \exp(G^* a_+ a_- - G a_+^\dagger a_-^\dagger). \tag{45}$$

其中 $G = re^{i\theta}$.

由 (17), 这里我们取 $\hat{A} = Ga_+^\dagger a_-^\dagger - G^* a_+ a_-$, $\hat{B} = a_\pm$, 于是

$$\begin{aligned} \hat{C}_0 = a_\pm \quad &, \quad \hat{C}_1 = -e^{i\theta} r a_\mp^\dagger \\ \hat{C}_2 = r^2 a_\pm \quad &, \quad \hat{C}_3 = -e^{i\theta} r^3 a_\mp^\dagger \\ \cdots \quad &, \quad \cdots \end{aligned}$$

故

$$S^\dagger(G) a_\pm S(G) = \sum_{k=0}^{\infty} \frac{r^{2k}}{(2k)!} \cdot a_\pm - e^{i\theta} \sum_{k=0}^{\infty} \frac{r^{2k+1}}{(2k+1)!} \cdot a_\mp^\dagger \tag{46a}$$

$$= a_\pm \cosh r - a_\mp^\dagger e^{i\theta} \sinh r \tag{46b}$$

8 双模压缩态的性质

双模压缩态有如下性质

$$\langle a_\pm \rangle = \alpha_\pm, \tag{47a}$$
$$\langle a_\pm a_\pm \rangle = \alpha_\pm^2, \tag{47b}$$
$$\langle a_+ a_- \rangle = \alpha_+ \alpha_- - e^{i\theta} \sinh r \cosh r, \tag{47c}$$
$$\langle a_\pm^\dagger a_\pm \rangle = |\alpha_\pm|^2 + \sinh^2 r. \tag{47d}$$

证明如下:

(a)

$$\langle a_\pm \rangle = \langle 0|S^\dagger D_-^\dagger D_+^\dagger a_\pm D_+ D_- S|0\rangle \tag{48a}$$
$$= \langle 0|S^\dagger (a_\pm + \alpha_\pm) S|0\rangle \tag{48b}$$
$$= \alpha_\pm. \tag{48c}$$

(b)

$$\langle a_\pm a_\pm \rangle = \langle 0|S^\dagger D_-^\dagger D_+^\dagger a_\pm a_\pm D_+ D_- S|0\rangle \tag{49a}$$
$$= \langle 0|S^\dagger (a_\pm + \alpha_\pm)^2 S|0\rangle \tag{49b}$$
$$= \alpha_\pm^2. \tag{49c}$$

(c)

$$\langle a_+ a_- \rangle = \langle 0|S^\dagger D_-^\dagger D_+^\dagger a_+ a_- D_+ D_- S|0\rangle \tag{50a}$$
$$= \langle 0|S^\dagger (a_+ + \alpha_+)(a_- + \alpha_-) S|0\rangle \tag{50b}$$
$$= \alpha_+ \alpha_- + \langle 0|S^\dagger a_+ a_- S|0\rangle \tag{50c}$$
$$= \alpha_+ \alpha_- - e^{i\theta} \cosh r \sinh r. \tag{50d}$$

(d)

$$\langle a_\pm^\dagger a_\pm \rangle = \langle 0|S^\dagger D_-^\dagger D_+^\dagger a_\pm^\dagger a_\pm D_+ D_- S|0\rangle \tag{51a}$$
$$= \langle 0|S^\dagger (a_\pm^\dagger + \alpha_\pm^*)(a_\pm + \alpha_\pm) S|0\rangle \tag{51b}$$
$$= |\alpha_\pm|^2 + \langle 0|S^\dagger a_\pm^\dagger a_\pm S|0\rangle \tag{51c}$$
$$= |\alpha_\pm|^2 + \sinh^2 r. \tag{51d}$$

接着, 我们定义双模压缩态的 quadrature

$$X = \frac{1}{\sqrt{2}}(a_+ + a_+^\dagger + a_- + a_-^\dagger), \tag{52}$$

利用上面四条双模压缩态的性质, 易知

$$\langle X \rangle = \frac{1}{\sqrt{2}}(\alpha_+ + \alpha_+^* + \alpha_- + \alpha_-^*) = \sqrt{2}\mathrm{Re}(\alpha_+ + \alpha_-). \tag{53}$$