微分几何(H) 期中 2024秋

1. (a) $df(v_p) = \frac{d}{dt}\Big|_{t=0} f\left((2,0,-1)+t(2,-1,3)\right)$

$= \frac{d}{dt}\Big|_{t=0} f(2+2t, -t, -1+3t)$

$= \frac{d}{dt}\Big|_{t=0} e^{2+2t}\cos(-t)$

$= \left(e^{2+2t}\cdot 2\cos(-t) + e^{2+2t}\sin t\right)\Big|_{t=0}$

$= 2e^2$

(也可以用 $df(v_p) = \left(\frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz\right)(v_p)$)

(b) $dx = \cos\theta\, dr + (-r\sin\theta)\, d\theta$

$dy = \sin\theta\, dr + r\cos\theta\, d\theta$

$dz = dz$

因此 $dx\wedge dy\wedge dz = (\cos\theta\, dr - r\sin\theta\, d\theta)\wedge(\sin\theta\, dr + r\cos\theta\, d\theta)\wedge dz$

$= r\, dr\wedge d\theta\wedge dz$.

2. (a) $r'(s) = \left(-\frac{4}{5}\sin s, -\cos s, \frac{3}{5}\sin s\right)$

因此有 $|r'(s)| = 1$，故而 s 为弧长参数

(b) $r''(s) = \left(-\frac{4}{5}\cos s, \sin s, \frac{3}{5}\cos s\right)$

因为 $|r''(s)|=1$，知主法向量 $n(s) = \left(-\frac{4}{5}\cos s, \sin s, \frac{3}{5}\cos s\right)$

从而副法向量 $b(s) = r'(s)\wedge n(s) = \left(-\frac{3}{5}, 0, -\frac{4}{5}\right)$.

(c) 曲率 $\kappa(s) = |r''(s)| = 1$.

因为 $\frac{d}{ds}b(s) = -\tau(s)n(s)$，而 $\frac{d}{ds}b(s)\equiv 0$，知 $\tau(s)\equiv 0$

由空间曲线基本定理，知该曲线为单位圆曲线

扫描全能王 创建

3. (a) 由题意，M_1 可表为

$$r(s,v) = \beta(s) + v(\beta'(s) + (0,0,1))$$
$$= (\cos s, \sin s, 0) + v(-\sin s, \cos s, 1)$$
$$= (\cos s - v\sin s, \sin s + v\cos s, v)$$

此曲面作为点集为 $\{(x,y,z)\in\mathbb{R}^3 \mid x^2+y^2-z^2=1\}$

同样地，M_2 可表为

$$\bar{r}(s,v) = \beta(s) + v(-\beta'(s) + (0,0,1))$$
$$= (\cos s + v\sin s, \sin s - v\cos s, v)$$

其点集仍为 $\{(x,y,z)\in\mathbb{R}^3 \mid x^2+y^2-z^2=1\}$

因此，M_1 和 M_2 作为 $\mathbb{R}^3$ 的点集相等.

(b) 由题意，旋转面 M 可参数化为

$$r(t,\theta) = (g(t), h(t)\cos\theta, h(t)\sin\theta)$$

作参数变换 $\begin{cases} u = h(t) \\ v = \theta \end{cases}$

因为 $\begin{vmatrix} \frac{\partial u}{\partial t} & \frac{\partial u}{\partial \theta} \\ \frac{\partial v}{\partial t} & \frac{\partial v}{\partial \theta} \end{vmatrix} = \begin{vmatrix} h'(t) & 0 \\ 0 & 1 \end{vmatrix} \neq 0$ 故为可容许参数变换

又因 $h'(t)$ 恒不为 0，可设 $h'(t)>0, \forall t$. 即 h 单调

因此反函数 h^{-1} 存在，即 $t = h^{-1}(u)$.

定义函数 $f(u) := g(h^{-1}(u))$，则曲面被重新参数化

为 $r(u,v) = (f(u), u\cos v, u\sin v)$

(c) 计算其切向量如下

$$r_u = (f_u, \cos v, \sin v)$$
$$r_v = (0, -u\sin v, u\cos v)$$

故而 $r_u \wedge r_v = (u, -uf_u\cos v, -uf_u\sin v)$

则其单位法向量 $n = \dfrac{r_u\wedge r_v}{|r_u\wedge r_v|} = \dfrac{u(1, -f_u\cos v, -f_u\sin v)}{\sqrt{u^2f_u^2+u^2}}$

$$= \frac{(1, -f_u\cos v, -f_u\sin v)}{\sqrt{f_u^2+1}}$$

由此可计算第一、第二基本形式如下

$E = 1+f_u^2$, $F = 0$, $G = u^2$

$L = \langle r_{uu}, n\rangle = \langle (f_{uu}, 0, 0), n\rangle = \dfrac{f_{uu}}{\sqrt{f_u^2+1}}$

$M = \langle r_{uv}, n\rangle = \langle (0, -\sin v, \cos v), n\rangle = 0$

$N = \langle r_{vv}, n\rangle = \langle (0, -u\cos v, -u\sin v), n\rangle = \dfrac{uf_u}{\sqrt{f_u^2+1}}$

~~则 Weingarten 变换在基 $\{r_u, r_v\}$ 下的~~

则 Weingarten 变换 W 满足

$$W\begin{pmatrix} r_u \\ r_v\end{pmatrix} = \begin{pmatrix} L & M \\ M & N\end{pmatrix}\begin{pmatrix} E & F \\ F & G\end{pmatrix}^{-1}\begin{pmatrix} r_u \\ r_v\end{pmatrix}$$

$$= \begin{pmatrix} \frac{L}{E} & 0 \\ 0 & \frac{N}{G}\end{pmatrix}\begin{pmatrix} r_u \\ r_v\end{pmatrix}$$

故主曲率 $k_1 = \dfrac{L}{E} = \dfrac{f_{uu}}{(1+f_u^2)^{3/2}}$, $k_2 = \dfrac{N}{G} = \dfrac{f_u}{u\sqrt{f_u^2+1}}$

高斯曲率 $K = \dfrac{f_uf_{uu}}{u(1+f_u^2)^2}$, $H = \dfrac{1}{2}\left[\dfrac{f_{uu}}{(1+f_u^2)^{3/2}} + \dfrac{f_u}{u\sqrt{f_u^2+1}}\right]$.

4. (a) 证明: $r-c$ 为 $\mathbb{R}^3$ 中向量, 必存在函数 $x(s), y(s), z(s)$

使 $r(s)-c = x(s)t(s) + y(s)n(s) + z(s)b(s)$

其中 $\{r; t, n, b\}$ 为曲线的 Frenet 标架

首先 $r(s)-c$ 为球面的径向, 与 $t(s)$ 垂直 (即 $\langle r(s)-c, r(s)-c\rangle \equiv \text{const}$ $\Rightarrow \langle r(s)-c, t(s)\rangle = 0$)

故 $x(s)\equiv 0$.

即 $r(s)-c = y(s)n(s) + z(s)b(s)$ <1>

$\langle 1\rangle$式两边求导得

$$t(s) = y'(s)n(s) + y(s)n'(s) + z'(s)b(s) + z(s)b'(s).$$

由Frenet标架运动方程知

$$t(s) = y'(s)n(s) + y(s)(-k(s)t(s)+\tau(s)b(s)) + z'(s)b(s) + z(s)(-\tau(s)n(s))$$

故有 $-y(s)k(s) \equiv 1$, $y'(s)-\tau(s)z(s)\equiv 0$, $y(s)\tau(s)+z'(s)\equiv 0$

这意味着 $y(s) = -\dfrac{1}{k(s)} = -\rho(s)$

$$z(s) = \frac{y'(s)}{\tau(s)} = -\rho'(s)\sigma(s).$$

即 $r-c = -\rho n - \rho'\sigma b.$

(b) 由(a)提示,我们只需证 $r+\rho n+\rho'\sigma b$ 为常向量.

$$\frac{d}{ds}(r+\rho n+\rho'\sigma b) = t(s) + \rho' n + \rho n' + (\rho'\sigma)'b + \rho'\sigma b'$$

$$= t(s) + \rho' n + \rho(-kt+\tau b) + (\rho'\sigma)'b + \rho'\sigma(-\tau n)$$

$$= (1-\rho k)t + (\rho' - \rho'\sigma\tau)n + (\rho\tau + (\rho'\sigma)')b. \quad \langle 2\rangle$$

易见 $1-\rho k\equiv 0$, $1-\sigma\tau\equiv 0$.

注意到 $\rho^2+(\rho'\sigma)^2\equiv$ 常数,求导得 $2\rho\rho'+2(\rho'\sigma)(\rho'\sigma)'\equiv 0$

因此 $\rho'\neq 0$, 则 $(\rho'\sigma)' = -\dfrac{2\rho\rho'}{2\rho'\sigma} = -\dfrac{\rho}{\sigma} = -\rho\tau$

故而 $\rho\tau+(\rho'\sigma)'\equiv 0$.

代入 $\langle 2\rangle$ 式得 $\frac{d}{ds}(r+\rho n+\rho'\sigma b)\equiv 0$. 即 $r+\rho n+\rho'\sigma b\equiv c$

为常向量。故而 $r-c=-\rho n-\rho'\sigma b$ 模长 $\rho^2+(\rho'\sigma)^2$ 为常数.

即 r 落在一个球面上. □

5. (a) 因 Weigarten 变换的特征值为两个主曲率 k_1, k_2，故其特征多项式为

$$(x-k_1)(x-k_2) = x^2-(k_1+k_2)x+k_1k_2$$
$$= x^2-2Hx+K$$

因此即有 $W^2-2HW+K\,\mathrm{Id}=0$

对 $v, w\in T_pM$，则有 $\langle W^2(v)-2HW(v)+Kv, w\rangle=0$

此即 $\langle W^2(v),w\rangle-2H\,\mathrm{II}(v,w)+K\,\mathrm{I}(v,w)=0$.

(亦可通过表 v, w 为主方向组合直接验证).

(b) $v=v_1r_u+v_2r_v$ 为主方向 当且仅当 $W(v)\parallel v$

而 $W(v)=(v_1\ \ v_2)\,W\begin{pmatrix}r_u\\ r_v\end{pmatrix}$

$$=(v_1\ \ v_2)\begin{pmatrix}L & M\\ M & N\end{pmatrix}\begin{pmatrix}E & F\\ F & G\end{pmatrix}^{-1}\begin{pmatrix}r_u\\ r_v\end{pmatrix}$$

$$=(v_1\ \ v_2)\frac{1}{EG-F^2}\begin{pmatrix}LG-MF & -LF+ME\\ MG-NF & -MF+NE\end{pmatrix}\begin{pmatrix}r_u\\ r_v\end{pmatrix}$$

$$=\frac{1}{EG-F^2}\big(v_1(LG-MF)+v_2(MG-NF),\ v_1(-LF+ME)+v_2(-MF+NE)\big)\begin{pmatrix}r_u\\ r_v\end{pmatrix}$$

$W(v)\parallel v \Rightarrow [v_1(LG-MF)+v_2(MG-NF)]v_2$

$$=v_1[v_1(-LF+ME)+v_2(-MF+NE)]$$

即：$v_1^2(ME-LF)+v_1v_2[EN-LG]+v_2^2(NF-MG)=0$

即 $\begin{vmatrix}v_2^2 & -v_1v_2 & v_1^2\\ E & F & G\\ L & M & N\end{vmatrix}=0.$

6.

(a) 作业题

(b) ~~由定义~~ 因参数化为正交曲率线网，则 $F=M=0$.

由定义 $\Gamma^1_{22} = \frac{1}{2} g^{1\alpha}(g_{\alpha 2,2} + g_{2\alpha,2} - g_{22,\alpha})$

$$= \frac{1}{2} g^{11}(2 g_{12,2} - g_{22,1}) + \frac{1}{2} g^{12}(g_{22,2})$$

因 $\begin{pmatrix} E & 0 \\ 0 & G \end{pmatrix}^{-1} = \begin{pmatrix} \frac{1}{E} & 0 \\ 0 & \frac{1}{G} \end{pmatrix}$ 知 $g^{11} = \frac{1}{E}$，$g^{12} = 0$

$$\Rightarrow \Gamma^1_{22} = \frac{1}{2}\frac{1}{E}(0 - G_u) + \cancel{\frac{1}{2}\frac{1}{G}G_v} = -\frac{G_u}{2E}.$$

(c). 由自然标架运动方程知

$$r_{vv} = \Gamma^\alpha_{22} r_\alpha + b_{22} n = \Gamma^1_{22} r_1 + \Gamma^2_{22} r_2 + b_{22} n$$

故 $r_{vv} \wedge r_v = \Gamma^1_{22}\, r_u \wedge r_v$.

(d). 我们证明 $\Gamma^1_{22} = 0$. 只需 $G_u = 0$

由Codazzi方程 $N_u = H G_u \quad \langle 3 \rangle$

由于 $F=M=0$，知 r_u, r_v 均为主方向，其相应主曲率为 $k_1 = \frac{L}{E}$，$k_2 = \frac{N}{G}$. 由题设 $\frac{N}{G} \equiv 0$. 因 $G \neq 0$ 知 $N \equiv 0$.

代入Codazzi方程 $\langle 3 \rangle$ 有 $HG_u \equiv 0$.

又因 $H = \frac{1}{2}(\frac{L}{E} + \frac{N}{G}) = \frac{1}{2}\frac{L}{E}$，及曲面无脐点从假设有 $H \neq 0$.

这说明 $G_u \equiv 0$. 故而 $\Gamma^1_{22} = 0$. 即 $r_{vv} \wedge r_v = 0$

由(a)知 r_v 方向不变，故 v-线为直线，M 为直纹面.