Homework 3 - Report

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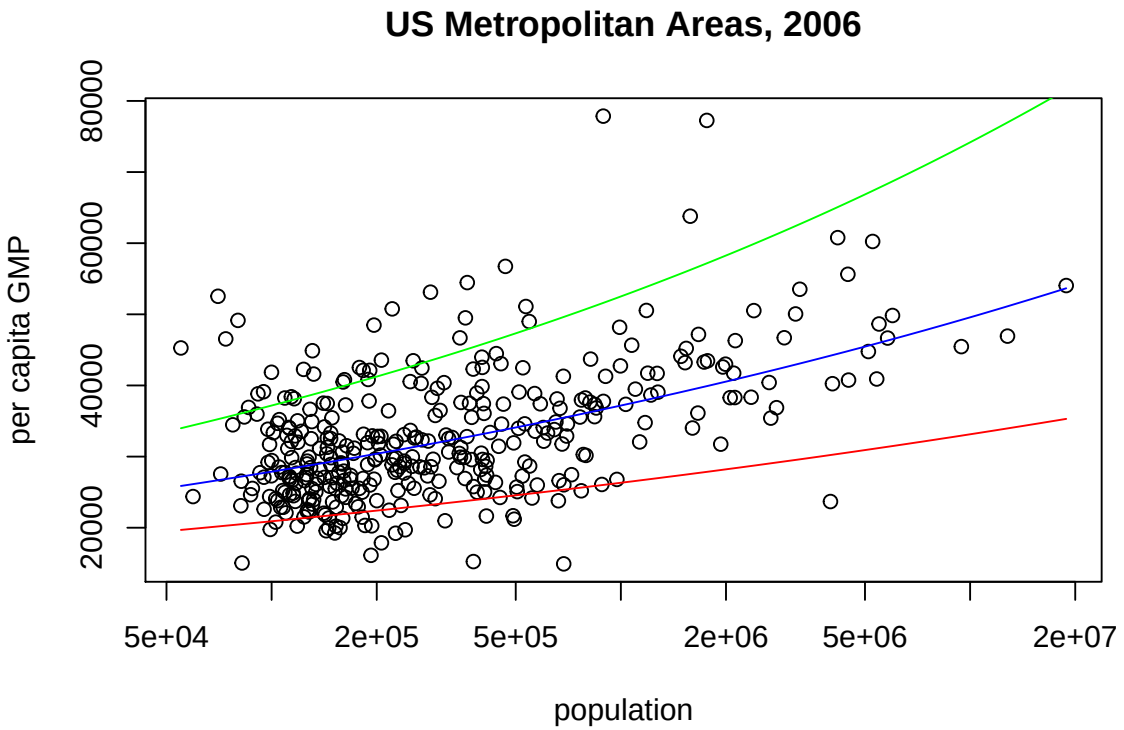
You will estimate the power-law scaling model, and its uncertainty, using the data alluded to in lecture, available in the file `gmp.dat` from lecture, which contains data for 2006.

```
gmp <- read.table("../data/gmp.dat")
gmp$pop <- round(gmp$gmp / gmp$pcgmp)
```

1. First, plot the data as in lecture, with per capita GMP on the y-axis and population on the x-axis. Add the curve function with the default values provided in lecture. Add two more curves corresponding to $a = 0.1$ and $a = 0.15$; use the `col` option to give each curve a different color (of your choice).

```
plot(gmp$pcgmp ~ gmp$pop, log = "x",
     xlab = "population", ylab = "per capita GMP",
     main = "US Metropolitan Areas, 2006")
curve(6611*x^(1/8), add = TRUE, col = "blue")
curve(6611*x^(0.1), add = TRUE, col = "red")
curve(6611*x^(0.15), add = TRUE, col = "green")
```



2. Write a function, called `mse()`, which calculates the mean squared error of the model on a given data set. `mse()` should take three arguments: a numeric vector of length two, the first component standing for y_0 and the second for a; a numerical vector containing the values of N; and a numerical vector containing the values of Y. The function should return a single numerical value. The latter two arguments should have as the default values the columns `pop` and `pcgmp` (respectively) from the `gmp` data frame from lecture. Your function may not use `for()` or any other loop. Check that, with the default data, you get the following values.

```
> mse(c(6611,0.15))
[1] 207057513
> mse(c(5000,0.10))
[1] 298459915
```

```
mse <- function(x, N = gmp$pop, Y = gmp$pcgmp) {
  return(sum((Y - x[1]*N^x[2]) ^ 2) / length(N))
}
mse(c(6611, 0.15))
```

```
## [1] 207057513
```

```
mse(c(5000, 0.10))
```

```
## [1] 298459914
```

3. R has several built-in functions for optimization, which we will meet as we go through the course. One of the simplest is `nlm()`, or non-linear minimization. `nlm()` takes two required arguments: a function, and a starting value for that function. Run `nlm()` three times with your function `mse()` and three starting value pairs for y_0 and a as in

   ```
   nlm(mse, c(y0=6611,a=1/8))
   ```

   What do the quantities `minimum` and `estimate` represent? What values does it return for these?

   Ans: The `minimum` is the minimum of `mse(c(y0, a))` which the function can find, and the `estimate` is a vector with the optimal solution.

```
nlm(mse, c(y0=6611,a=1/8))
```

```
## Warning in nlm(mse, c(y0 = 6611, a = 1/8)): NA/Inf replaced by maximum positive
## value

## Warning in nlm(mse, c(y0 = 6611, a = 1/8)): NA/Inf replaced by maximum positive
## value

## Warning in nlm(mse, c(y0 = 6611, a = 1/8)): NA/Inf replaced by maximum positive
## value

## Warning in nlm(mse, c(y0 = 6611, a = 1/8)): NA/Inf replaced by maximum positive
## value

## Warning in nlm(mse, c(y0 = 6611, a = 1/8)): NA/Inf replaced by maximum positive
## value

## Warning in nlm(mse, c(y0 = 6611, a = 1/8)): NA/Inf replaced by maximum positive
## value

## $minimum
## [1] 61857060
##
```

```
## $estimate
## [1] 6611.0000000    0.1263177
##
## $gradient
## [1] 50.048639 -9.976327
##
## $code
## [1] 2
##
## $iterations
## [1] 3
```

```r
nlm(mse, c(y0=6611,a=0.1))
```

```
## Warning in nlm(mse, c(y0 = 6611, a = 0.1)): NA/Inf replaced by maximum positive
## value

## Warning in nlm(mse, c(y0 = 6611, a = 0.1)): NA/Inf replaced by maximum positive
## value

## Warning in nlm(mse, c(y0 = 6611, a = 0.1)): NA/Inf replaced by maximum positive
## value

## Warning in nlm(mse, c(y0 = 6611, a = 0.1)): NA/Inf replaced by maximum positive
## value

## Warning in nlm(mse, c(y0 = 6611, a = 0.1)): NA/Inf replaced by maximum positive
## value

## Warning in nlm(mse, c(y0 = 6611, a = 0.1)): NA/Inf replaced by maximum positive
## value

## $minimum
## [1] 61857060
##
## $estimate
## [1] 6611.0000003    0.1263177
##
## $gradient
## [1]   50.04683 -166.46087
##
## $code
## [1] 2
##
## $iterations
## [1] 6
```

```r
nlm(mse, c(y0=6611,a=0.15))
```

```
## $minimum
## [1] 61857060
##
## $estimate
## [1] 6610.9999997    0.1263182
##
## $gradient
```

```
## [1]   51.76354 -210.18952
##
## $code
## [1] 2
##
## $iterations
## [1] 7
```

We can find that all the three command return the same `minimum`, which is 61857060, and `estimate`, which is 6610.9999997 and 0.1263182.

4. Using `nlm()`, and the `mse()` function you wrote, write a function, `plm()`, which estimates the parameters y_0 and a of the model by minimizing the mean squared error. It should take the following arguments: an initial guess for y_0; an initial guess for a; a vector containing the N values; a vector containing the Y values. All arguments except the initial guesses should have suitable default values. It should return a list with the following components: the final guess for y_0; the final guess for a; the final value of the MSE. Your function must call those you wrote in earlier questions (it should not repeat their code), and the appropriate arguments to `plm()` should be passed on to them.

```
plm <- function(y0, a, N = gmp$pop, Y = gmp$pcgmp) {
  res <- nlm(mse, c(y0, a), N, Y)
  return(c(res$estimate[1], res$estimate[2], res$minimum))
}
```

What parameter estimate do you get when starting from $y_0 = 6611$ and $a = 0.15$? From $y_0 = 5000$ and $a = 0.10$? If these are not the same, why do they differ? Which estimate has the lower MSE?

```
plm(6611, 0.15)
```

```
## [1] 6.611000e+03 1.263182e-01 6.185706e+07
```

```
plm(5000, 0.10)
```

```
## Warning in nlm(mse, c(y0, a), N, Y): NA/Inf replaced by maximum positive value

## Warning in nlm(mse, c(y0, a), N, Y): NA/Inf replaced by maximum positive value

## Warning in nlm(mse, c(y0, a), N, Y): NA/Inf replaced by maximum positive value

## Warning in nlm(mse, c(y0, a), N, Y): NA/Inf replaced by maximum positive value

## Warning in nlm(mse, c(y0, a), N, Y): NA/Inf replaced by maximum positive value

## Warning in nlm(mse, c(y0, a), N, Y): NA/Inf replaced by maximum positive value

## [1] 5.000000e+03 1.475913e-01 6.252148e+07
```

The parameter estimates are different. Because different initial value may lead to different solution, which may be local optimal solution, rather than global optimal solution.

5. *Convince yourself the jackknife can work.*

    a. Calculate the mean per-capita GMP across cities, and the standard error of this mean, using the built-in functions `mean()` and `sd()`, and the formula for the standard error of the mean you learned in your intro. stats. class (or looked up on Wikipedia…).

Ans: Given i.i.d. sample $X_1, \cdots, X_n$, the variance of the mean $\bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$ can be computed as below:

$$Var(\bar{X}) = \frac{1}{n^2} Var(\sum_{i=1}^{n} X_i) = \frac{1}{n^2} \sum_{i=1}^{n} Var(X_i) = \frac{1}{n} Var(X)$$

Then the standard error of the mean is $\sqrt{\frac{1}{n}Var(X)} = \frac{SD(X)}{\sqrt{n}}$.

```
n <- length(gmp$pcgmp)
mean(gmp$pcgmp)
```

```
## [1] 32922.53
```

```
sd(gmp$pcgmp) / sqrt(n)
```

```
## [1] 481.9195
```

b. Write a function which takes in an integer `i`, and calculate the mean per-capita GMP for every city *except* city number `i`.

```
mean_except <- function(i) {
  return(mean(gmp$pcgmp[-i]))
}
```

c. Using this function, create a vector, `jackknifed.means`, which has the mean per-capita GMP where every city is held out in turn. (You may use a `for` loop or `sapply()`.)

```
jackknifed.means <- c()
for (city in 1:n) {
  jackknifed.means <- c(jackknifed.means, mean_except(city))
}
```

d. Using the vector `jackknifed.means`, calculate the jack-knife approximation to the standard error of the mean. How well does it match your answer from part (a)?

```
sqrt(((n-1)^2/n) * var(jackknifed.means))
```

```
## [1] 481.9195
```

We can find that the jack-knife approximation to the standard error of the mean matches the real standard error very well.

6. Write a function, `plm.jackknife()`, to calculate jackknife standard errors for the parameters y_0 and a. It should take the same arguments as `plm()`, and return standard errors for both parameters. This function should call your `plm()` function repeatedly. What standard errors do you get for the two parameters?

7. The file `gmp-2013.dat` contains measurements for for 2013. Load it, and use `plm()` and `plm.jackknife` to estimate the parameters of the model for 2013, and their standard errors. Have the parameters of the model changed significantly?