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Research Statement

1. Overview

My research interests lie at the intersection of algebraic, differential, and complex geometry. In the next few paragraphs I will briefly describe key topics from which my research stems. A general principle bridging fields of my interests suggests that a certain algebraic or differential-geometric positivity of the tangent bundle implies strong topological and geometric restrictions on the underlying manifold, and in many cases makes it possible to classify all such manifolds. A prototypical illustration of the stated principle is the uniformization theorem for Riemann surfaces, which, in particular, states that any closed oriented surface admitting a metric of semipositive Gauss curvature is either conformally equivalent to the round sphere $(S^2, g_{\mathrm{round}})$, or is flat and isomorphic to the torus T^2.

In the last decades the aforementioned classification problems for higher-dimensional manifolds, admitting metrics of (semi)positive curvature in an appropriate sense, were successfully resolved with the use of the tools of geometric analysis, and, especially, by applying the Ricci flow. In the world of Kähler manifolds, the Ricci flow satisfies particularly strong existence, convergence and singularity formation properties. Ties between Kähler and algebraic geometry over $\mathbb{C}$ open vast opportunities for relating various questions about the Kähler-Ricci flow to purely algebraic problems.

Unlike the Kähler situation, there are very little efficient tools to study non-Kähler complex manifolds. Thus, given all the success of the Ricci flow, it is reasonable to extend it to arbitrary, not necessarily Kähler, Hermitian manifolds. However, on a general Hermitian manifold (M, g, J) its Ricci curvature $\mathrm{Ric}(g)$ is not invariant under the operator of almost complex structure, therefore the evolved metric is not necessarily Hermitian. This issue raises the following question motivating my research.

Question 1. *Are there geometric flows useful for studying Hermitian manifolds? Do these modifications, similarly to the Ricci and Kähler-Ricci flows, satisfy strong existence, convergence, and regularization properties?*

The main difficulty in approaching this question is that there exists a multitude of, seemingly natural, modifications of the Ricci flow, while the geometric significance of these modified Ricci curvatures and the corresponding 'Einstein' metrics (i.e, scale-static solutions to the flow equation) is not always apparent. This makes the study of long-time existence and convergence for these flows a difficult task. To find a distinguished evolution equation for an Hermitian metric, I further refine Question 1.

Question 2. *Does there exist a modification of the Ricci flow for the Hermitian setting, which preserves various curvature positivity conditions?*

Preservation of curvature positivity, imposed in Question 2 on an evolution equation for an Hermitian metric, is a reasonable restriction, since: (a) in the Riemannian setting it is satisfied by the Ricci flow; (b) it lies in the core of classification and uniformization problems resolved with the use of the Ricci flow.

In my research, I answer Question 2 by finding a distinguished member (see equation (2) below) of Streets-Tian's family of Hermitian curvature flows [ST11]. I study geometric properties of this flow and use it to approach the following conjecture.

Conjecture 3. *Any complex Fano manifold which admits an Hermitian metric with Griffiths non-negative curvature must be isomorphic to a rational homogeneous space.*

Griffiths non-negativity is a version of semipositivity for the Chern curvature, which implies numerical effectiveness of a tangent bundle. Therefore, Conjecture 3 can be thought of as an Hermitian version of an algebro-geometric Campana-Peternell conjecture [CP91]. Applying flow (2) to manifolds equipped with an Hermitian metric of Griffiths non-negative curvature, I have obtained strong evidence supporting Conjecture 3. The key is that this flow satisfies a version of the strong parabolic maximum principle, which forces the zeros of the Chern curvature to have a large symmetry group.

Further study and development of the Hermitian metric flows rises new analytical problems, which, in turn require better understanding of the geometric essence of the objects involved in the corresponding evolution equations.

2. Introduction & Related Research

Let (M,g,J) be a compact complex manifold equipped with an Hermitian metric. Any non-Kähler manifold admits a one-parameter family of Hermitian connections [Gau97], i.e., connections preserving g and J. Among these connections the Chern and the Bismut connections are of a special interest. Given a general complex Hermitian manifold, it is natural to consider curvatures of Hermitian connections instead of the Ricci form of the Levi-Civita connection to define an Hermitian version of the Ricci flow. Let ∇ be the Chern connection on TM, $T \in \Lambda^2(T^*M) \otimes TM$ its torsion, and $\Omega_{i\bar{j}k}{}^{l} \in \Lambda^{1,1}(\mathrm{End}(TM))$ the corresponding Chern curvature tensor. Let $\Theta(g) \in \mathrm{Sym}^2(T^*M)$ be any J-invariant tensor, associated to g. We are seeking for a modification of the Ricci flow of the form

$$\partial_t g = -\Theta(g)$$

such that (a) it coincides with the Kähler-Ricci flow, if g is a Kähler metric; (b) operator $-\Theta\colon \mathrm{Sym}^2(T^*M) \to \mathrm{Sym}^2(T^*M)$ is weakly elliptic.

There has been a large body of works in this direction. In [Gil11] Gill introduced **Chern-Ricci flow** by taking $\Theta(g)_{i\bar{j}} = (\mathrm{tr}_{\mathrm{End}}\,\Omega)_{i\bar{j}} = \Omega_{i\bar{j}k}{}^{k}$. As in the Kähler case, the study of this flow can be reduced to the analysis of a parabolic complex Monge-Ampère equation for a scalar potential. Properties of this flow and the relevant estimates were studied by Sherman, Tosatti and Weinkove in a series of papers [SW13, TW13, TW15]. In particular, in [TW15] the authors found the maximal existence time in terms of the initial cohomological data. It should be noted, however, that the Chern-Ricci flow does not represent an answer to Question 2, since, by an explicit computation of Yang [Yan16], it does not preserve Griffiths non-negativity on Hopf surfaces.

In [ST11] Streets and Tian considered a whole family of flows, which they call **Hermitian curvature flows**. The corresponding $\Theta(g)$ uses the other trace of Ω, and equals $(\mathrm{tr}_g\Omega - Q(T))_{i\bar{j}} = g^{m\bar{n}}\Omega_{m\bar{n}i\bar{j}} - Q(T)_{i\bar{j}}$, where $Q(T)$ is an arbitrary quadratic term in torsion T of type (1,1).

$$\partial g_t = -g^{m\bar{n}}\Omega_{m\bar{n}i\bar{j}} + Q(T)_{i\bar{j}}. \tag{1}$$

For various Q these flows have very different properties and could be applied in diverse settings. For $Q = 0$ the flow was also proposed by Liu and Yang [LY12], and has Hermitian-Einstein metrics on TM as its scale-stationary solutions. In [ST10] Streets and Tian introduced a **pluriclosed flow** — a particular member of family (1), which preserves the set of pluriclosed metrics, i.e., metrics satisfying $\partial\bar{\partial}\omega_g = 0$, where $\omega_g = g(J\cdot,\cdot)$. Recently many papers [AS17, Str16a, Str16b, Str17, ST12] addressed the properties of pluriclosed flow and new connections with the generalized geometry has been found.

3. Completed Work

3.1. **Research on the Hermitian curvature flow (HCF).** My first paper on the subject [Ust16a] aims at finding a member of a general family of Hermitian curvature flows (1), which would preserve Griffiths positivity of Ω. Recall, that Chern curvature Ω of (M,g,J) is Griffiths positive, if $g(\Omega(\xi,\overline{\xi})\eta,\overline{\eta}) > 0$ for any nonzero $\xi,\eta \in T^{1,0}M$. This is an important curvature positivity notion, since it implies ampleness of $T^{1,0}M$.

In [Ust16a] the following Hermitian flow is introduced.

$$\partial_t g = -g^{m\overline{n}}\Omega_{m\overline{n}i\overline{j}} - \frac{1}{2}g^{m\overline{n}}g^{p\overline{s}}T_{mp\overline{j}}T_{\overline{n}\overline{s}i}. \tag{2}$$

Flow (2) belongs to the family of Hermitian curvature flows (1). In what follows I call flow (2) the HCF (Hermitian curvature flow). Particular choice of quadratic torsion term $Q(T)$ for the HCF is motivated by a very special structure of the corresponding evolution equation for Ω. It takes a form $\partial_t\Omega = \widetilde{\Delta}\Omega + \varphi(\Omega)$, where $\widetilde{\Delta}$ is a Laplacian of a certain connection $\widetilde{\nabla}$ on the space of curvature tensors, and $\varphi(\Omega)$ is the zero-order term. The main result of [Ust16a] is that Griffiths non-negative curvature tensors evolved by the equation $\partial_t\Omega = \widetilde{\Delta}\Omega + \varphi(\Omega)$ satisfy a version of the strong maximum principle. The form of term $\varphi(\Omega)$ plays the key role in this argument, based on the ideas of Bando [Ban84] and Mok [Mok88], who proved a similar statement in the Kähler setting.

Theorem 4 ([Ust16a])**.** *Let $g(t), t \in [0,\tau)$ be the solution to the HCF on a compact complex Hermitian manifold (M,g_0,J). Assume that the Chern curvature Ω^{g_0} at the initial moment $t = 0$ is Griffiths non-negative (resp. positive). Then for $t \in [0,\tau)$ the Chern curvature $\Omega(t) = \Omega^{g(t)}$ remains Griffiths non-negative (resp. positive).*

If, moreover, the Chern curvature Ω^{g_0} is Griffiths positive at least at one point $x \in M$, then for any $t \in (0;\tau)$ the Chern curvature is Griffiths positive everywhere on M.

Theorem 4 can be used to prove a slight generalization of Farnkel conjecture. Original Frankel conjecture states that the only manifolds admitting Kähler metric of positive Griffiths curvature are projective space $\mathbb{P}^n$. It was proved by Siu and Yau [SY80] by studying harmonic maps $S^2 \to M$. A prove of Frankel conjecture based on the Kähler-Ricci flow was found by Chen, Sun and Tian [CST09]. Theorem 4 implies the following slight generalization of Frankel's conjecture.

Theorem 5 ([Ust16a])**.** *Let (M,g_0,J) be a compact complex n-dimensional Hermitian manifold such that (a) its Chern curvature Ω^{g_0} is Griffiths non-negative; (b) Ω^{g_0} is Griffiths positive at some point $x_0 \in M$. Then $T^{1,0}M$ is ample and M is biholomorphic to the projective space $\mathbb{P}^n$.*

Motivated by Conjecture 3, in paper [Ust17a] I studied the HCF on arbitrary complex homogeneous manifolds $M = G/H$. It was proved that the finite-dimensional set of metrics induced from a metric on $\mathrm{Lie}(G)$ via the evaluation map $\mathrm{ev}\colon \mathrm{Lie}(G) \to T^{1,0}M$ is preserved by the HCF. These are precisely the known metrics of non-negative Griffiths curvature on G/H.

Theorem 6 ([Ust17a])**.** *Let $M = G/H$ be a complex homogeneous manifold equipped with an induced Hermitian metric $g_0 = \mathrm{ev}_*(h_0^{-1})$, where h_0 is an Hermitian metric on $\mathrm{Lie}(G)^*$. Let $h(t)$ be the solution to the ODE $\partial_t h = h^\#$. Then $g(t) = \mathrm{ev}_*(h(t)^{-1})$ solves the HCF on (M,g_0,J).*

Here $h \mapsto h^\#$ is a quadratic operator on $\mathrm{Sym}^{1,1}(\mathrm{Lie}(G))$ introduced by Hamilton [Ham86]. This is a surprising result for two reasons: (a) a priori there is no reason for the set of induced metrics to be invariant under a metric flow; (b) ODE for $h(t)$ turns out to be independent of H. Theorem 6 implies that in the case G is semisimple the Hermitian metrics corresponding to the Killing form of Lie(G) induces a scale-static metric on $M = G/H$. Conjecturally any induced metric on G/H with simple G pinches towards a metric defined by a Killing form.

In the same paper it was observed [Ust17a, Cor. 3.2] that in local coordinates evolution equation for $g^{-1} = g^{i\bar{j}}$ takes a very particular form:

$$\partial_t g^{i\bar{j}} = g^{m\bar{n}} \partial_m \partial_n g^{i\bar{j}} - \partial_m g^{i\bar{n}} \partial_{\bar{n}} g^{m\bar{j}}.$$

The main feature of this equation is that it involves only inverses of the metric g and not the metric itself. The HCF is the only member of the family of Hermitian curvature flows (1) satisfying this property.

A far-reaching generalization of Theorem 4 is proved in [Ust17b]. In the Ricci flow case the corresponding result was established by Wilking [Wil13]. By rising the last two indices of $\Omega_{i\bar{j}k\bar{l}}$, the Chern curvature can be interpreted as a section of $\mathrm{Sym}^{1,1}(\mathrm{End}(T^{1,0}M))$. Let $S \subset \mathrm{End}(T^{1,0}M)$ be an $\mathrm{Ad}GL(T^{1,0}M)$-invariant subset. We say that (M, g, J) has an S-non-negative curvature, if $\langle \Omega, s \otimes \bar{s} \rangle \geqslant 0$ for any $s \in S$. Main result of [Ust17b] states:

Theorem 7 ([Ust17b])**.** *Let $g(t), t \in [0, \tau)$ be the solution to equation (2) on a compact complex Hermitian manifold (M, g_0, J). Assume that the Chern curvature Ω^{g_0} at the initial moment $t = 0$ is S-non-negative. Then for $t \in [0, \tau)$ the Chern curvature $\Omega(t) = \Omega^{g(t)}$ remains S-non-negative.*

Theorem 7 implies the preservation of Dual-Nakano non-negativity, Griffiths non-negativity, and non-negativity holomorphic orthogonal bisectional curvature. The proof of this theorem is based on an adaptation of Hamilton's maximum principle applied to the evolution equation for Ω. For $S = \{\mathrm{Id}\}$ we get a scalar quantity $\langle \Omega, \mathrm{Id} \otimes \overline{\mathrm{Id}} \rangle = \widehat{\mathrm{sc}} := \Omega_{i\bar{j}k\bar{l}} g^{i\bar{l}} g^{k\bar{j}}$, which satisfies a notably nice evolution equation under the HCF:

$$\partial_t \widehat{\mathrm{sc}} = \Delta \widehat{\mathrm{sc}} + \frac{1}{2} |\mathrm{div}\, T|^2 + |S^{(3)}|^2,$$

where $(\mathrm{div}\, T)_{jk} = \nabla_i T^i_{jk}$, $S^{(3)}_{k\bar{j}} = \Omega_{i\bar{j}k\bar{l}} g^{i\bar{l}}$. The evolution equation for $\widehat{\mathrm{sc}}$ serves as a basis for the following result.

Theorem 8 ([Ust17b])**.** *If a compact complex manifold (M, J) admits a periodic, e.g, stationary, solution to the HCF (2), then $\mathrm{div}\, T = S^{(3)} \equiv 0$ and the canonical bundle $K_{\widetilde{M}}$ of the universal cover of M is holomorphically trivial.*

This theorem is a tautology for the Kähler-Ricci flow, however it is not satisfied by many Hermitian flows, which have been discussed in the introduction. Miraculously, the vanishing of the right hand side of equation (2) implies the vanishing of $\mathrm{div}\, T$ and $S^{(3)}$.

S-non-negativity under the HCF also satisfies an extension of Brendle and Schoen's strong maximum principle [BS08].

Theorem 9 ([Ust17b])**.** *Let $g(t), t \in [0, \tau)$ be the solution to equation (2) on a compact complex Hermitian manifold (M, g_0, J). Assume that the Chern curvature Ω^{g_0} at the initial moment $t = 0$ is S-non-negative. Then for $t > 0$ the zero set $\{s \in S \mid \langle \Omega, s \otimes \bar{s} \rangle = 0\}$ is invariant under the parallel transport of the connection $\widehat{\nabla}_X Y = \nabla_X Y - T(X, Y)$.*

Being applied to the Griffiths non-negativity this theorem serves as a strong evidence for Conjecture 3. Indeed, for $t > 0$ the zero set of $\Omega^{g(t)}$ turns out to be invariant under the holonomy group $\mathrm{Hol}(\widehat{\nabla})$, so either $\Omega^{g(t)}$ has 'many' zeros; or $\widehat{\nabla}$ has a 'small' holonomy group. Both alternatives put strong constraints on the geometry of the underlying manifold. In the Kähler situation, $\widehat{\nabla}$ coincides with the Levi-Civita connection, so the exact trade-off between these alternatives is controlled by Berger's classification of Riemannian holonomy groups, see the papers of Gu and Zhang [Gu09, GZ10].

3.2. **Other completed research.** Prior to my recent research on Hermitian curvature flows I worked in **toric topology** — the topic at the intersection of combinatorics, convex geometry and algebraic topology. Papers [Ust09, Ust11] investigated the toral rank conjecture on **moment-angle-complexes** — certain spaces admitting an action of a large torus $(S^1)^m$. In these papers I proved the conjecture on moment-angle-complexes by relating it to some the combinatorial properties of the underlying discrete objects. Topology of the moment-angle-complexes was used in [Ust16b] to prove a result in the enumerative combinatorics of graphs. In [Ust12, Ust14b] I studied the relation between the toral rank conjecture and the Horrocks' conjecture in commutative algebra. In joint works with Panov and Verbitsky [PU12, Ust14a, PUV16] we constructed a large family of non-Kähler manifolds by producing equivariant compactifications of the complex-analytic groups $\mathbb{C}^m/\Gamma$, $\Gamma \simeq \mathbb{Z}^n$. Special structure of these manifolds allowed to prove many results about their complex geometry, in particular, under additional assumptions, we constructed transverse Kähler foliations, found algebraic dimension of the manifolds and computed Dolbeault cohomology. In a joint work with Solomadin [SU16] we studied sequential equivariant blow-ups on toric varieties and used them to construct multiplicative generators of the unitary cobordism ring.

4. Future Questions

Currently, studies of metric flows in Hermitian setting are gaining increasing amount of interest and are forming a new actively developing field. The main perspective of my research is that these flows could (and should) be applied to approach various uniformization problems in complex geometry. Given results of [Ust16a, Ust17a, Ust17b], one can argue that the HCF suits well for this task.

Since the study of the HCF has been just started, there are still many basic question to answer. Let us list few of them. By Theorem 8 existence of a periodic solution on (M,g,J) puts strict constraints on the geometry of manifold M. This motivates the following problem.

Problem 10. *Does there exist a non-trivial, i.e., non-stationary, periodic solution to the HCF on some (M,g,J)?*

Streets and Tian [ST11] have proved, that if $\tau_{\max}$ is the maximal time such that there exists a solution to the HCF on $[0,\tau_{\max})$, then $\limsup_{t\to\tau_{\max}} \max\{|\Omega|,|T|,|\nabla T|\} = +\infty$. This basic blow-up is quite impractical to use for analyzing the long-time existence of the flow. So it is important to answer the following question.

Problem 11. *Is it possible to prove existence of the HCF up to time $\tau_{\max}$ by controlling less geometric quantities, than the full norms of Ω, T, ∇T?*

Important analytical tool in studying Ricci flow is the monotonicity of $\mathcal{F}$ and $\mathcal{W}$ functionals. It would be very helpful to have such functionals for the HCF, or its modifications.

Problem 12. *Find analogues of $\mathcal{F}$ and $\mathcal{W}$ functionals for the HCF.*

Theorem 9 suggests that geometry of the HCF has strong ties with the properties of the connection $\widehat{\nabla}$ and its holonomy group. It would be interesting to address differential-geometric questions concerning $\mathrm{Hol}(\widehat{\nabla})$.

Problem 13. *Does there exists Berger-type classification of holonomy groups $\mathrm{Hol}(\widehat{\nabla})$? What can be said about manifold (M,g,J), if $\mathrm{Hol}(\widehat{\nabla})$ is reducible? 'small'?*

While analysis of a long-time behavior of an Hermitian flow on a general Hermitian manifold might be challenging, sometimes it is reasonable to focus on Hermitian metrics satisfying some partial 'integrability condition', e.g., pluriclosed metrics ($\partial\bar{\partial}\omega_g = 0$), Gauduchon metrics ($\partial\bar{\partial}\omega_g^{n-1} = 0$), or balanced metrics ($d^*\omega_g = 0$).

Problem 14. *Does there exist any partial metric integrability condition, preserved by the HCF?*

Another specific direction concerns study of the HCF on complex surfaces. Many computations substantially simplify in $\dim_{\mathbb{C}} = 2$ and make it easier to study long-time behavior.

Problem 15. *Investigate the HCF on complex surfaces.*

Purportedly, if we understand singularity formation under the HCF in complex dimension 2 well enough, we might try using this flow to develop an Hermitian version of **analytical minimal model** program (see [ST17]), and study class VII surfaces, by analyzing the blow-up behavior of the HCF. Another possible direction of research concerns questions on the existence of integrable complex structures. Supposedly, with a complete understanding of limiting behavior of Hermitian flows, we may be able to rule out existence of complex structures on certain topological manifolds.

Solutions to the problems stated above will build a foundation for further applications of the HCF.

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