Eigenvalue and Eigenvector Analysis

$$\frac{d\left(\Psi_{j,k}^{(m)}\right)_{L,1}}{dt} = \mathrm{A}_{\mathrm{L,L}}\left(\Psi_{j,k}^{(m)}\right)_{L,1}$$

$j = 2, 3, \dots, J;\ k = 1, 2, \dots K + 1$

$\mathrm{A}_{\mathrm{L,L}}$ *is* $L \times L$ *complex matrix*, $\quad L = (J-1)(K+1)$

$$\left(\Psi_{j,k}^{(m)}(t)\right)_{L,1} = \left(X_{j,k}\right)_{L,1} e^{\lambda t}$$

$$\lambda\left(X_{j,k}\right)_{L,1} e^{\lambda t} = \mathrm{A}_{\mathrm{L,L}}\left(X_{j,k}\right)_{L,1} e^{\lambda t}$$

$$\left(\mathrm{A}_{\mathrm{L,L}} - \lambda \mathrm{I}_{\mathrm{L,L}}\right)\left(X_{j,k}\right)_{L,1} = 0, \text{where } \mathrm{I}_{\mathrm{L,L}} \text{ is } L \times L \text{ unit matrix.}$$

- The condition for BOTH $\left(\mathrm{A}_{\mathrm{L,L}} - \lambda \mathrm{I}_{\mathrm{L,L}}\right)\left(X_{j,k}\right)_{L,1} = 0$ and $\left(X_{j,k}\right)_{L,1} \neq 0$ is $\left|\mathrm{A}_{\mathrm{L,L}} - \lambda \mathrm{I}_{\mathrm{L,L}}\right| = 0$;
- In general, the equation $\left|\mathrm{A}_{\mathrm{L,L}} - \lambda \mathrm{I}_{\mathrm{L,L}}\right| = 0$ has L solutions for λ, which are called eigenvalues of $\mathrm{A}_{\mathrm{L,L}}$, denoted as $\lambda^{(n)}, n = 1, 2, \dots, \mathrm{L}$.
- Associated with each of L eigenvalues is a vector, referred to as the eigenvector (or eigenmode) associated with $\lambda^{(n)}$ and denoted as $\left(X_{j,k}^{(n)}\right)_{L,1} \neq 0$, satisfying $\left(\mathrm{A}_{\mathrm{L,L}} - \lambda \mathrm{I}_{\mathrm{L,L}}\right)\left(X_{j,k}^{(n)}\right)_{L,1} = 0$.

$$\lambda^{(n)} = \sigma^{(n)} + \sqrt{-1}\,\omega^{(n)},$$

$$\text{where } \sigma^{(n)} \text{ and } \omega^{(n)} \text{ are all real numbers and called growth rate and frequency of the } n-th \text{ eigenmode.}$$

Eigenmode solutions

$$\frac{d\left(\Psi_{j,k}^{(m)}\right)_{L,1}}{dt} = \mathrm{A}_{\mathrm{L,L}}\left(\Psi_{j,k}^{(m)}\right)_{L,1}$$

$j = 2, 3, \ldots, J;\ k = 1, 2, \ldots K + 1$

$\mathrm{A}_{\mathrm{L,L}}$ *is* $L \times L$ *complex matrix,* $\quad L = (J-1)(K+1)$

$$\left(\Psi_{j,k}^{(m)}(t)\right)_{L,1} = \left(X_{j,k}^{(n)}\right)_{L,1} e^{\lambda^{(n)}t} = \left(X_{j,k}^{(n)}\right)_{L,1} e^{\sigma^{(n)}t} e^{\sqrt{-1}\omega^{(n)}t}, \mathrm{n} = 1, 2, \ldots, \mathrm{L}$$

Governing equations at 3D grids: $\frac{dQ_{i,j,k}}{\partial t} + \bar{u}_{j,k}\left(\frac{\partial Q}{\partial x}\right)_{i,j,k} + \left(\frac{\partial \psi'}{\partial x}\right)_{i,j,k}\left(\frac{\partial \bar{Q}}{\partial y}\right)_{j,k} = 0$

The solution associated with zonal wave m at 3D grids:

$$(\psi'_{i,j,k})_{L\times I,1} = \mathrm{Re}\{\left(\Psi_{j,k}^{(m)}(t)\right)_{L,1} e^{\sqrt{-1}\frac{2\pi m}{L_x}x_i}\}$$

$$\psi'_{i,j,k} = \mathrm{Re}\left\{\left(X_{j,k}^{(n)}\right)_{L,1} e^{\sqrt{-1}(\frac{2\pi m}{L_x}x_i + \omega^{(n)}t)}\right\} e^{\sigma^{(n)}t}, \mathrm{n} = 1, 2, \ldots, \mathrm{L}$$

$\omega^{(n)} > 0$: *Westward Propagation*; $\omega^{(n)} < 0$ *Eastward propagation*

Validation of the numerical model

$$\frac{d\left(\Psi_{j,k}^{(m)}\right)_{L,1}}{\partial t} = (\mathrm{B_{L,L}})^{-1}(\mathrm{C_{L,L}}+\mathrm{D_{L,L}})\left(\Psi_{j,k}^{(m)}\right)_{L,1} = \mathrm{A_{L,L}}\left(\Psi_{j,k}^{(m)}\right)_{L,1}$$

$$Simplest\ background\ flow:\ \bar{U}_{j,k} = 10\,\frac{m}{s};\ \left(\frac{\partial \bar{Q}}{\partial y}\right)_{j,k} = \begin{cases} \beta\ for\ 1 < k < K+1. \\ 0\ for\ k = 1\ or\ k = K+1 \end{cases}$$

$$\lambda^{(n)} = \sigma^{(n)} + \sqrt{-1}\,\omega^{(n)},$$

where $\sigma^{(n)}$ and $\omega^{(n)}$ *are all real numbers and called growth rate and frequency of the* $n-th$ *eigenmode.*

Analytical Solution:

$$\sigma = 0, \omega = -\left\{\left(\frac{2\pi m}{L_x}\right)\bar{U} - \frac{\left(\frac{2\pi m}{L_x}\right)\beta}{\left(\frac{2\pi m}{L_x}\right)^2 + \left(\frac{2\pi m_y}{L_y}\right)^2 + \left(\frac{2\pi m_z}{NH/f_0}\right)^2}\right\},$$

Eigenmode solutions and general solutions

$$\frac{d\left(\Psi_{j,k}^{(m)}\right)_{L,1}}{dt} = \mathrm{A}_{\mathrm{L,L}}\left(\Psi_{j,k}^{(m)}\right)_{L,1}$$

$$\left(\Psi_{j,k}^{(m)}(t)\right)_{L,1} = \left(X_{j,k}^{(n)}\right)_{L,1} e^{\sigma^{(n)}t} e^{\sqrt{-1}\omega^{(n)}t}, \mathrm{n} = 1, 2, \dots, \mathrm{L}$$

General solutions:

$$\psi'_{i,j,k}(t) = \sum_{m=1\ \&\ n=1}^{m=M\ and\ n=L} \alpha_{\mathrm{m,n}}(\mathrm{t}=0)\mathrm{Re}\left\{\left(X_{j,k}^{(n)}\right)_{L,1} e^{\sqrt{-1}(\frac{2\pi m}{L_x}x_i+\omega^{(n)}t)}\right\} e^{\sigma^{(n)}t}$$

where, $\alpha_{\mathrm{m,n}}(\mathrm{t}=0)$ *corresponds to the projection coefficient of perturbation stremfunction* field at $\mathrm{t} = 0$ (i.e., initial condition), $\psi'_{i,j,k}(t=0)$.

For t > a few days (say a week),

$\psi'_{i,j,k}(t) = \alpha_{\mathrm{m,n}}(\mathrm{t}=0)\mathrm{Re}\left\{\left(X_{j,k}^{(n_0)}\right)_{L,1} e^{\sqrt{-1}(\frac{2\pi m_0}{L_x}x_i+\omega^{(n_0)}t)}\right\} e^{\sigma^{(n_0)}t}$ will be the dominant pattern, where $\sigma^{(n_0)}(m_0)$ is the largest for all m and n.

Identifying important eigen solutions

For t > a few days (say a week),

$\psi'_{i,j,k}(t) = \alpha_{\mathrm{m,n}}(\mathrm{t}=0)\mathrm{Re}\left\{\left(X_{j,k}^{(n_0)}\right)_{L,1} e^{\sqrt{-1}(\frac{2\pi m_0}{L_x}x_i+\omega^{(n_0)}t)}\right\} e^{\sigma^{(n_0)}t}$ will be the dominant pattern, where $\sigma^{(n_0)}(m_0)$ is the largest for all m and n.

$$\lambda^{(n)} = \sigma^{(n)} + \sqrt{-1}\omega^{(n)},$$

where $\sigma^{(n)}$ and $\omega^{(n)}$ *are all real numbers and called growth rate and frequency of the* $n-th$ *eigenmode.*

- After solve for eigenvalues and their associated eigenvectors of the matrix $A_{L,L}$ associated with zonal wavenumber m (m = 6-9 for synoptic waves, m = 1-3 for planetary waves, and m > 10 for short waves), we sort the real part of these L eigenvalues and their associated eigenvectors in "descend" order so that $\sigma^{(n=1)}$ is the largest, $\left(X_{j,k}^{(n_0)}\right)_{L,1}$ corresponds to the mode that intensifies the most (is therefore most likely to be observed);
- $\sigma^{(n=L)}$ is the smallest (and therefore it is most likely negative) and $\left(X_{j,k}^{(n=L)}\right)_{L,1}$ corresponds to the mode that decays the fastest (and is therefore most likely not to be observed). There is limited merit to study those modes with $\sigma^{(n)}<0$ for explaining OBS.

Diagnostic analysis

$$\psi'_{i,j,k}(t) = \mathrm{Re}\left\{\left(X^{(n)}_{j,k}\right)_{L,1} e^{\sqrt{-1}\left(\frac{2\pi m}{L_x}x_i+\omega^{(n)}t\right)}\right\} e^{\sigma^{(n)}t}$$

- For a linear system, the amplitude of its solutions is arbitrary, meaning that if X is a solution of the linear system, αX is also a solution, where α is a real number.
- Therefore, we just ignore the factor $e^{\sigma^{(n)}t}$ when studying eigen solutions, as long as the information of growth rate is provided (to readers), or is retained for reference.

$$growth\ rate \sigma^{(n)}\ and\ e - foldind\ time = 1/\sigma^{(n)}$$

- For those eigenmodes whose $\omega^{(n)} = 0$, its spatial pattern will remain identical in time (i.e., stationary). For those eigenmodes whose $\omega^{(n)} \neq 0$, its future spatial pattern will be just shifted towards the east ($-\omega^{(n)} > 0$) or the west ($-\omega^{(n)} < 0$).
- Therefore, we need to include the factor $e^{\sqrt{-1}\omega^{(n)}t}$ when studying temporal evolution of eigen solution (e.g., plotting Hovmöller diagram). However, we can ignore the factor $e^{\sqrt{-1}\omega^{(n)}t}$ when focusing on 3D pattern of eigen solutions, as long as the information of propagation speed and direction is provided (to readers),or is retained for reference.

$$Phase\ speed = -\frac{\omega^{(n)}}{2\pi m/L_x} \text{ and } Period = \frac{2\pi}{|\omega^{(n)}|}$$

Normalizing eigensolution

$$\psi'_{i,j,k}(t) = \mathrm{Re}\left\{\left(X_{j,k}^{(n)}\right)_{L,1} e^{\sqrt{-1}\left(\frac{2\pi m}{L_x}x_i + \omega^{(n)}t\right)}\right\} e^{\sigma^{(n)}t}$$

- For a linear system, the amplitude of its solutions is arbitrary, meaning that if X is a solution of the linear system, αX is also a solution, where α is a real number.

- In order to make easy comparison among different solutions of the linear model (e.g., solutions for different eigenvalues or solutions obtained with different values of the zonal wavenumber m), we can normalize the eigenvector solutions produced by MATLAB.

$$\vec{X}^{(n)} = \left(X_{j,k}^{(n)}\right)_{L,1}$$

$$E_n = \left\|\vec{X}^{(n)}\right\| = \frac{1}{L}\vec{X}^{(n)} \bullet \overrightarrow{X^*}^{(n)}, L = (J-1)(K+1)$$

$$\vec{X}^{(n)} = \frac{100}{f_0}\frac{\vec{X}^{(n)}}{\sqrt{E_n}}$$

The factor of $\frac{100}{f_0}$ ensures that amplitude of geopotential height anomalies in the order of 10 meter for all modes.

3D geopotential height field

$$\psi'_{i,j,k}(t=0) = \mathrm{Re}\left\{\left(X_{j,k}^{(n)}\right)_{L,1} e^{\sqrt{-1}\frac{2\pi m}{L_x}x_i}\right\}$$

$$h'(x,y,z,t) = \frac{f_0}{g}\psi'(x,y,z,t)$$

$$[j,k] = l2jk(l), l = 1, 2\ \ldots L$$

$$for\ j = 2, 3, \ldots, J; k = 1,2, \ldots, K+1$$

$$h'_{i,j,k}(t=0) = \frac{f_0}{g}\mathrm{Re}\left\{\left(X_l^{(n)}\right)_{L,1} e^{\sqrt{-1}\frac{2\pi m}{L_x}x_i}\right\}$$

$$x_i = (i-1)\Delta x, i = 1, 2, \ldots 360; \Delta x = Lx/360$$

$$for\ j = 1 and\ j = J+1; k = 1,2, \ldots, K+1; i = 1, 2, \ldots 360$$

$$h'_{i,j,k}(t=0) = 0$$

3D meridional wind field

$$\psi'_{i,j,k}(t=0) = \mathrm{Re}\left\{\left(X_{j,k}^{(n)}\right)_{L,1} e^{\sqrt{-1}\frac{2\pi m}{L_x}x_i}\right\}$$

$$[j,k] = l2jk(l), l = 1, 2\ \ldots L$$

$$for\ j = 2, 3, \ldots, J; k = 1,2, \ldots, K+1$$

$$v'_{i,j,k}(t=0) = \left(\frac{\partial\psi'}{\partial x}\right)_{i,j,k} = \mathrm{Re}\left\{\sqrt{-1}\frac{2\pi m}{L_x}\left(X_l^{(n)}\right)_{L,1} e^{\sqrt{-1}\frac{2\pi m}{L_x}x_i}\right\}$$

$$x_i = (i-1)\Delta x, i = 1, 2, \ldots 360; \Delta x = Lx/360$$

$$for\ j = 1 and\ j = J+1; k = 1,2, \ldots, K+1; i = 1, 2, \ldots 360$$

$$v'_{i,j,k}(t=0) = 0$$

3D zonal wind field

$$u'_{i,j,k}(t=0) = -\left(\frac{\partial \psi'}{\partial y}\right)_{i,j,k}$$

$$[j,k] = l2jk(l), l = 1, 2 \ \dots L$$

$$for\ j = 3, 4, \dots, J-1; k = 1,2, \dots, K+1$$
$$lnh = jk2l(j+1,k), lsh = jk2l(j-1,k)$$
$$u'_{i,j,k}(t=0) = -\frac{1}{2\Delta y}\mathrm{Re}\left\{(X_{lnh}^{(n)} - X_{lsh}^{(n)})e^{\sqrt{-1}\frac{2\pi m}{L_x}x_i}\right\}$$
$$x_i = (i-1)\Delta x, i = 1, 2, \dots 360; \Delta x = Lx/360$$

$$for\ j = 2, k = 1,2, \dots, K+1$$
$$lnh = jk2l(j+1,k)$$
$$u'_{i,j,k}(t=0) = -\frac{1}{2\Delta y}\mathrm{Re}\left\{(X_{lnh}^{(n)} - 0)e^{\sqrt{-1}\frac{2\pi m}{L_x}x_i}\right\}$$

$$for\ j = J, k = 1,2, \dots, K+1$$
$$lsh = jk2l(j-1,k)$$
$$u'_{i,j,k}(t=0) = -\frac{1}{2\Delta y}\mathrm{Re}\left\{(0 - X_{lsh}^{(n)})e^{\sqrt{-1}\frac{2\pi m}{L_x}x_i}\right\}$$

$$for\ j = 1, all\ i\ and\ all\ k:$$
$$lj2 = jk2l(2,k)$$
$$u'_{i,j,k}(t=0) = -\frac{1}{\Delta y}\mathrm{Re}\left\{(X_{lj2}^{(n)} - 0)e^{\sqrt{-1}\frac{2\pi m}{L_x}x_i}\right\}$$

$$for\ j = J+1, all\ i\ and\ all\ k:$$
$$ljj = jk2l(j,k)$$
$$u'_{i,j,k}(t=0) = -\frac{1}{\Delta y}\mathrm{Re}\left\{(0 - X_{ljj}^{(n)})e^{\sqrt{-1}\frac{2\pi m}{L_x}x_i}\right\}$$

3D temperature field

$$T' = \frac{f_0 H}{R} \frac{\partial \psi'}{\partial z}$$

$$for\ j = 1\ and\ j = J + 1; k = 1,2, \dots, K + 1; i = 1, 2, \dots 360$$
$$T'_{i,j,k}(t = 0) = 0$$

$$[j, k] = l2jk(l), l = 1, 2\ \dots L$$

$$for\ j = 2, 3, \dots, J; k = 2,3, \dots, K$$
$$lup = jk2l(j, k + 1), ldw = jk2l(j, k - 1)$$
$$T'_{i,j,k}(t = 0) = \frac{f_0 H}{R_{gas}} \frac{1}{2\Delta Z} \mathrm{Re}\left\{(X^{(n)}_{lup} - X^{(n)}_{ldw}) e^{\sqrt{-1}\frac{2\pi m}{L_x} x_i}\right\}$$
$$x_i = (i - 1)\Delta x, i = 1, 2, \dots 360; \Delta x = Lx/360$$

$$for\ j = 2, 3, \dots, J; k = 1$$
$$lup = jk2l(j, k + 1), ldw = jk2l(j, k)$$
$$T'_{i,j,k}(t = 0) = \frac{f_0 H}{R_{gas}} \frac{1}{2\Delta Z} \mathrm{Re}\left\{(X^{(n)}_{lup} - X^{(n)}_{ldw}) e^{\sqrt{-1}\frac{2\pi m}{L_x} x_i}\right\}$$

$$for\ j = 2, 3, \dots, J; k = KK + 1$$
$$lup = jk2l(j, k), ldw = jk2l(j, k - 1)$$
$$T'_{i,j,k}(t = 0) = \frac{f_0 H}{R_{gas}} \frac{1}{2\Delta Z} \mathrm{Re}\left\{(X^{(n)}_{lup} - X^{(n)}_{ldw}) e^{\sqrt{-1}\frac{2\pi m}{L_x} x_i}\right\}$$