Vertical motion equation

$$\left(\Psi_{j,k}^{(m)}(t)\right)_{L,1} = \left(X_{j,k}\right)_{L,1} e^{\lambda t}$$

$j = 2, 3, \dots, J; k = 1, 2, \dots K+1; L = (J-1)(K+1)$

$$\nabla^2 w + \frac{f_0^2}{N^2}\frac{\partial^2 w}{\partial Z^2} = \frac{f_0}{N^2}(F_1 + F_2 + F_3)$$

$Meridional\ (background) temperature\ \ advection\text{:}\ F_1 = 2\frac{\partial \bar{u}}{\partial z}\frac{\partial}{\partial x}\nabla^2\psi'$

$Vertical\ differential\ meridional\ (background)\ vorticity\ advection\text{:}\ F_2 = -2\frac{\partial^2 \bar{u}}{\partial y^2}\frac{\partial^2 \psi'}{\partial z \partial x}$

$Vertical\ differential\ meridional\ planetary\ vorticity\ advection\text{:}\ F_3 = \beta\frac{\partial^2 \psi'}{\partial z \partial x}$

The boundary conditions for w:

$x\text{:}\ w(x = 0, y, Z, t) = w(x = L_x, y, Z, t);$

$Z\text{:}\ w(x, y, Z = 0, t) = w(x, y, Z, = H, t) = 0;$

y: *no condition (i.e., part of the solution)*

Numerical scheme for solving for vertical motion equation

$$\nabla^2 w + \frac{f_0^2}{N^2}\frac{\partial^2 w}{\partial Z^2} = \frac{f_0^2}{N^2}(F_1 + F_2 + F_3)$$

The boundary conditions for w:

$x: w(x = 0, y, Z, t) = w(x = L_x, y, Z, t);$

$Z: w(x, y, Z = 0, t) = w(x, y, Z, = H, t) = 0;$

y: *no condition (i.e., part of the solution)*

Symbolically, we can write the vertical motion equation at 2D grids for each zonal wave m as

$$G_{LW,LW}\left(w_{j,k}\right)_{LW,1} = \frac{f_0}{N^2}\{\left(F1_{j,k}\right)_{LW,1} + \left(F2_{j,k}\right)_{LW,1} + \left(F3_{j,k}\right)_{LW,1}\}$$

$$j = 1, 2, 3, \dots, J, J + 1; k = 2, 3, \dots, K; LW = (J + 1)(K - 1)$$

From grid-vector to column-vector:

$$l(j, k) = j + (k - 2)(J + 1)$$

From column-vector to grid-vector:

$$k(l) = int\left\{\frac{l - 1}{(J + 1)}\right\} + 2;$$

$$j(l) = l - (k - 2)(J + 1)$$

Numerical scheme for calculating $\nabla^2 w + \frac{f_0^2}{N^2}\frac{\partial^2 w}{\partial Z^2}$

$$w_{i,j,k} = \mathrm{Re}\{W^{(m)}(y_j, Z_k)e^{\sqrt{-1}\frac{2\pi m}{L_x}x_i}\}$$

$$(EW)_{j,k} = (\nabla^2 w)_{j,k} + \frac{f_0^2}{N^2}(\frac{\partial^2 w}{\partial Z^2})_{j,k}$$

$$j = 1, 2, 3, \dots, J, J+1; k = 2, 3, \dots, K; LW = (J+1)(K-1)$$

$$for\ 1 < j < J\ and\ all\ k:$$

$$(EW)_{j,k} = -\left(\frac{2\pi m}{L_x}\right)^2 W_{j,k}^{(m)} + \frac{W_{j+1,k}^{(m)} - 2W_{j,k}^{(m)} + \Psi_{j-1,k}^{(m)}}{\Delta y^2} + \frac{f_0^2}{N^2}\frac{W_{j,k+1}^{(m)} - 2W_{j,k}^{(m)} + W_{j,k-1}^{(m)}}{\Delta Z^2}$$

$$for\ j = 1\ and\ all\ k:$$

$$(EW)_{1,k} = -\left(\frac{2\pi m}{L_x}\right)^2 W_{1,k}^{(m)} + \frac{W_{3,k}^{(m)} - 2W_{2,k}^{(m)} + W_{1,k}^{(m)}}{\Delta y^2} + \frac{f_0^2}{N^2}\frac{W_{1,k+1}^{(m)} - 2W_{1,k}^{(m)} + W_{1,k-1}^{(m)}}{\Delta Z^2}$$

$$for\ j = J+1\ and\ all\ k:$$

$$(EW)_{J+1,k} = -\left(\frac{2\pi m}{L_x}\right)^2 W_{J+1,k}^{(m)} + \frac{W_{J+1,k}^{(m)} - 2W_{J,k}^{(m)} + \Psi_{J-1,k}^{(m)}}{\Delta y^2} + \frac{f_0^2}{N^2}\frac{W_{J+1,k+1}^{(m)} - 2W_{J+1,k}^{(m)} + W_{J+1,k-1}^{(m)}}{\Delta Z^2}$$

Algorithm for obtaining matrix $G_{LW,LW}$ numerically

$$LW = (J+1)\times(K-1)$$

Step 1. Define a column vector of length L, $\vec{x}_n$, in which all elements are zero except the one at the nth column, which equals 1, $1 \leq n \leq LW$.

$$\vec{x}_n = \begin{pmatrix} x_1 = 0 \\ \vdots \\ x_{n-1} = 0 \\ x_n = 1 \\ x_{n+1} = 0 \\ \vdots \\ x_{LW} = 0 \end{pmatrix}_{LW,1}$$

Step 2. Applying the conversion from a column-vector to a grid vector, $k(lw), and\ j(lw),$ convert the column vector $\vec{x}_n$ to its grid − vector counterpart, denoted as $(x_{j,k}^{(n)})$.

Step 3. Repeat Step 1 thorough Step 2 from n = 1, n = 2, …, n = LW, we will obtain all LW columns of $G_{LW,LW}$.

Algorithm for obtaining F1, F2, and F3

$$LW = (J+1)\times(K-1)$$

$$G_{LW,LW}\left(w_{j,k}\right)_{LW,1} = \frac{f_0}{N^2}\left\{\left(F1_{j,k}\right)_{LW,1} + \left(F2_{j,k}\right)_{LW,1} + \left(F3_{j,k}\right)_{LW,1}\right\}$$

$$\left(w_{j,k}\right)_{LW,1} = \frac{f_0}{N^2}\left(G_{LW,LW}\right)^{-1}\left\{\left(F1_{j,k}\right)_{LW,1} + \left(F2_{j,k}\right)_{LW,1} + \left(F3_{j,k}\right)_{LW,1}\right\}$$

$$j = 1,2,3,\ldots,J,J+1;\ k = 2,3,\ldots,K;\ LW = (J+1)(K-1)$$

$$\mathit{Meridional\ (background) temperature\ \ advection}:\ F_1 = 2\,\frac{\partial\bar{u}}{\partial z}\frac{\partial}{\partial x}\nabla^2\psi'$$

$$\mathit{Vertical\ differential\ meridional\ (background)\ vorticity\ advection}:\ F_2 = -2\,\frac{\partial^2\bar{u}}{\partial y^2}\frac{\partial^2\psi'}{\partial z\partial x}$$

$$\mathit{Vertical\ differential\ meridional\ planetary\ vorticity\ advection}:\ F_3 = \beta\,\frac{\partial^2\psi'}{\partial z\partial x}$$

Algorithm for obtaining F1, F2, and F3

$$LW = (J+1)\times(K-1)$$

$$F_3 = \beta\,\frac{\partial^2\psi'}{\partial z\partial x}$$

$$j = 1, 2, 3, \dots, J, J+1;\ k = 2, 3, \dots, K;\ LW = (J+1)(K-1)$$

$$for\ 1 < j < J+1\ and\ all\ k:$$

$$\left(F3_{j,k}\right) = \sqrt{-1}\,\frac{2\pi m}{L_x}\beta\,\left(\frac{\Psi_{j,k+1}^{(m)} - \Psi_{j,k-1}^{(m)}}{2\Delta z}\right)$$

$$for\ j = 1\ and\ all\ k:$$

$$\left(F3_{j,k}\right) = 0$$

$$for\ j = J+1\ and\ all\ k:$$

$$\left(F3_{j,k}\right) = 0$$

Algorithm for obtaining F1, F2, and F3

$$LW = (J+1)\times(K-1)$$

$$F_2 = -2\frac{\partial^2\bar{u}}{\partial y^2}\frac{\partial^2\psi'}{\partial z\partial x}$$

$$j = 1, 2, 3, \ldots, J, J+1; k = 2, 3, \ldots, K; LW = (J+1)(K-1)$$

$$for\ 1 < j < J+1\ and\ all\ k:$$

$$(F2_{j,k}) = -2\sqrt{-1}\,\frac{2\pi m}{L_x}\frac{(\bar{u}_{j+1,k} - 2\bar{u}_{j,k} + \bar{u}_{j-1,k})}{\Delta y^2}\left(\frac{\Psi^{(m)}_{j,k+1} - \Psi^{(m)}_{j,k-1}}{2\Delta z}\right)$$

$$for\ j = 1\ and\ all\ k:$$

$$(F2_{j,k}) = 0$$

$$for\ j = J+1\ and\ all\ k:$$

$$(F2_{j,k}) = 0$$

Algorithm for obtaining F1, F2, and F3

$$F_1 = 2\frac{\partial \bar{u}}{\partial z}\frac{\partial}{\partial x}\nabla^2\psi'$$

$j = 1, 2, 3, \ldots, J, J+1; k = 2, 3, \ldots, K;$

$LW = (J+1)\times(K-1)$

$$for\ 2 < j < J\ and\ all\ k:$$

$$\left(F1_{j,k}\right) = 2\sqrt{-1}\,\frac{2\pi m}{L_x}\frac{\left(\bar{u}_{j,k+1} - \bar{u}_{j,k-1}\right)}{2\Delta z}\left(-\left(\frac{2\pi m}{L_x}\right)^2\Psi_{j,k}^{(m)} + \frac{\Psi_{j+1,k}^{(m)} - 2\Psi_{j,k}^{(m)} + \Psi_{j-1,k}^{(m)}}{\Delta y^2}\right)$$

$$for\ j = 1: \left(F1_{j,k}\right) = 2\sqrt{-1}\,\frac{2\pi m}{L_x}\frac{\left(\bar{u}_{j,k+1} - \bar{u}_{j,k-1}\right)}{2\Delta z}\left(\frac{\Psi_{3,k}^{(m)} - 2\Psi_{2,k}^{(m)} + 0}{\Delta y^2}\right)$$

$$for\ j = 2: \left(F1_{j,k}\right) = 2\sqrt{-1}\,\frac{2\pi m}{L_x}\frac{\left(\bar{u}_{j,k+1} - \bar{u}_{j,k-1}\right)}{2\Delta z}\left(-\left(\frac{2\pi m}{L_x}\right)^2\Psi_{j,k}^{(m)} + \frac{\Psi_{3,k}^{(m)} - 2\Psi_{2,k}^{(m)} + 0}{\Delta y^2}\right)$$

$$for\ j = J: \left(F1_{j,k}\right) = 2\sqrt{-1}\,\frac{2\pi m}{L_x}\frac{\left(\bar{u}_{j,k+1} - \bar{u}_{j,k-1}\right)}{2\Delta z}\left(\frac{0 - 2\Psi_{J,k}^{(m)} + \Psi_{J-2,k}^{(m)}}{\Delta y^2}\right)$$

$$for\ j = J+1: \left(F1_{j,k}\right) = 2\sqrt{-1}\,\frac{2\pi m}{L_x}\frac{\left(\bar{u}_{j,k+1} - \bar{u}_{j,k-1}\right)}{2\Delta z}\left(\frac{0 - 2\Psi_{j-1,k}^{(m)} + \Psi_{j-2,k}^{(m)}}{\Delta y^2}\right)$$

3D vertical motion field

$$w_{i,j,k}(t=0) = \mathrm{Re}\left\{\left(W_{j,k}^{(n)}\right)_{LW,1} e^{\sqrt{-1}\frac{2\pi m}{L_x}x_i}\right\}$$

$$[j,k] = lW2jk(l), l = 1, 2 \ \ldots LW$$

$$for\ j = 1, 3, \ldots, J+1; k = 2, \ldots, K$$

$$w_{i,j,k}(t=0) = \mathrm{Re}\left\{\left(W_{l}^{(n)}\right)_{LW,1} e^{\sqrt{-1}\frac{2\pi m}{L_x}x_i}\right\}$$

$$x_i = (i-1)\Delta x, i = 1, 2, \ldots 360; \Delta x = Lx/360$$

$$for\ k = 1 and\ k = K+1; j = 1,2, \ldots, J+1; i = 1, 2, \ldots 360$$

$$w_{i,j,k}(t=0) = 0$$