Problemas Analíticos para la Ecuación de Boltzmann

Analytical issues from the Boltzmann Transport Equation

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• **Classical problem**: Rarefied ideal gases: ***conservative Boltzmann Transport eq.***

• **Energy dissipative phenomena:** Gas of elastic or inelastic interacting systems in the presence of a thermostat with a fixed background temperature θ_b or Rapid granular flow dynamics: (**inelastic hard sphere interactions**): homogeneous cooling states, randomly heated states, shear flows, shockwaves past wedges, etc.

•**(Soft) condensed matter at nano scale; mean field theory of charged transport:** Bose-Einstein condensates models, Boltzmann Poisson charge transport in electro chemistry and materials: hot electron transport and semiconductor modeling.

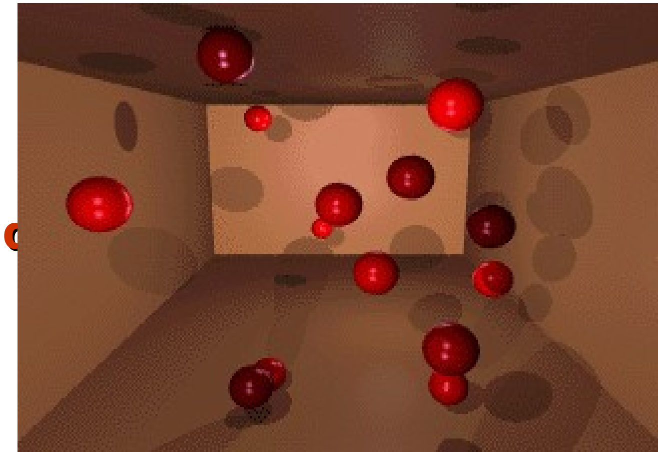
•**Emerging applications from stochastic dynamics and connections to probability theory** for multi-linear Maxwell type interactions : Social networks, Pareto tails for wealth distribution, non-conservative dynamics: opinion dynamic and information percolation models in social dynamics, particle swarms in population dynamics, etc.

y: The classical Boltzmann equation:

volution estimates, exact and best constants

tence and stability for in a certain class of initial c
L^p stability of the initial value problem

ectral-Lagrangian solvers for BTE



Overview

Part I

- **Introduction to classical kinetic equations for elastic and inelastic interactions:**

  The Boltzmann equation for binary elastic and inelastic collisions

  * Description of interactions, collisional frequency and potentials
  * Energy dissipation & heat source mechanisms
  * Revision of Elastic (conservative) vs inelastic (dissipative) theory.

Part II

- **Convolution estimates type for the collisional integrals**:
  - Radial rearrengements methods
  - Connections to Brascamp-Lieb-Luttinger type estimates
  - Young and Hardy-Littlewood-Sobolev type inequalities
  - Exponentially weighted L^∞ estimates
- **Existence and stability of global in time of the Boltzmann equation**
  - $L^\infty \cap L^p$ solutions of the Cauchy problem of the space inhomogeneous problem with initial data near Maxwellian distributions
  - Propagation and moment creation of the space homogeneous solution for large data.

Part III

Some issues of variable hard and soft potential interactions

- Dissipative models for Variable hard potentials with heating sources:

  All moments bounded

  Stretched exponential high energy tails

Spectral - Lagrange solvers for collisional problems

- Deterministic solvers for Dissipative models - The space homogeneous problem
- FFT application - Computations of Self-similar solutions

- Space inhomogeneous problems

Time splitting algorithms

Simulations of boundary value – layers problems

Benchmark simulations

Part I The classical Elastic/Inelastic Boltzmann Transport Equation

for hard spheres in 3-d: (L. Boltzmann 1880's), ***in strong form***:
For $f(t; x; v) = f$ and $f(t; x; v_*) = f_*$ describes the ***evolution of a probability distribution function (pdf)*** of finding a particle *centered at $x \in \mathbb{R}^d$, with velocity $v \in \mathbb{R}^d$, at time $t \in \mathbb{R}_+$, satisfying*

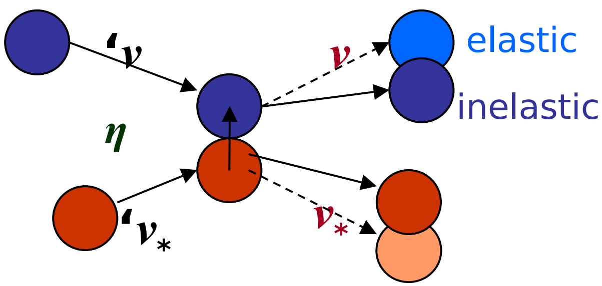
$$f_t + v \cdot \nabla_x f = C\, a^{d-1} G(x|\rho) \int_{\mathbb{R}^d} \int_{S_+^{d-1}} \left[\frac{1}{e J_e}\, 'f\, 'f_* - f\, f_*\right] |u \cdot \eta|\, d\eta\, dv_*$$

$u = v - v^*$:= relative velocity

$|u \cdot \eta|\, d\eta$:= collision rate

$u \cdot \eta = u_\eta$:= impact velocity

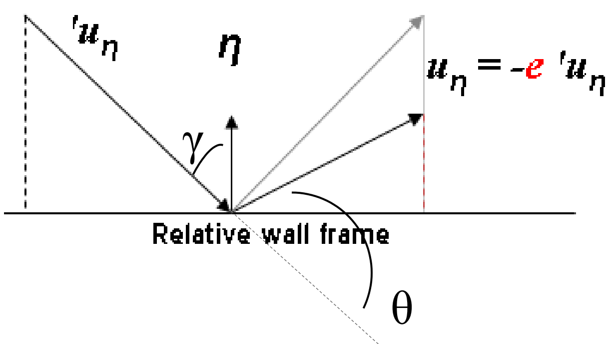
η := impact direction (random in S_+^{d-1})



$'v_*$ and $'v$ are called pre-collisional velocities, and v_* and v are the corresponding post-collisional velocities

$$u \cdot \eta = (v - v_*) \cdot \eta = -e\,('v - 'v_*) \cdot \eta = -e\, 'u \cdot \eta$$

$$u \cdot \eta_\perp = (v - v_*) \cdot \eta_\perp = ('v - 'v_*) \cdot \eta_\perp = 'u \cdot \eta_\perp$$



Inelastic reflection of relative velocities

C = number of particle in the box

a = diameter of the spheres

d = space dimension

e := restitution coefficient : $0 < e \leq 1$

$e = 1$ elastic interaction , $0 < e < 1$ inelatic interaction, ($e=0$ 'sticky' particles)

$$f_t + v \cdot \nabla_x f \;=\; C\, a^{d-1} G(x|\rho) \int_{\mathbb{R}^d} \int_{S_+^{d-1}} [\tfrac{1}{e J_e}\, 'f\, 'f_* - f\, f_*]\, |u \cdot \eta|\, d\eta\, dv_*$$

$\rho(x,t) = \int_{\mathbb{R}^d} f(x,v,t)\, dv$ *:= mass density*

$G(x|\rho)$ *:= statistical correlation function* *(sort of mean field ansatz,i.e. independent of v)*

*= for elastic interactions (**e**=1)*

Main assumptions to be able to write the equation are:

- *Molecular Chaos hypothesis:* *The probability of having the velocities of two interacting spheres are* **uncorrelated before the interaction** $f^{(2)}(t,x,v,y,v_*) = G(x \mid \rho(t,x)) f(t,x,'v) f(t, x + a \cdot \eta, 'v_*)$ $\Rightarrow$ **H-theorem**

*Loss of memory of the **previous** collision*

- *The Boltzmann-Grad limit:* *as* $C \to \infty$; $a \to 0$ *while* $C\, a^{d-1}$ *remains bounded*, i.e. "*state of rarefied gas*" **i.e. enough intersitial space**

- *Binary interactions*: the probability of three particle colliding at the **same** time is **zero**. **May be extended to multi-linear interactions** *(in some special cases)*

- Jacobian of the velocities transformation $\boxed{J_e : (v, v_*) \to (v', v'_*) = \left|\frac{v', v'_*}{v, v_*}\right|.}$

- Revised Enskog theory for inelastic collision mechanism it is assumed that the restitution coefficient is only a function of the impact velocity $e = e(|u \cdot n|)$.

The properties of the map $z \to e(z)$ are (i) $z \mapsto e(z)$ is absolutely continuous and non-increasing.
(ii) $z \mapsto ze(z)$ is non-decreasing.

$$v' = v + \frac{(1+e)}{2} (u \cdot \eta)\, \eta \qquad \text{and} \qquad v'_* = v_* + \frac{(1+e)}{2} (u \cdot \eta)\, \eta$$

The notation for pre-collision perspective uses symbols $'v$, $'v_*$: Then,

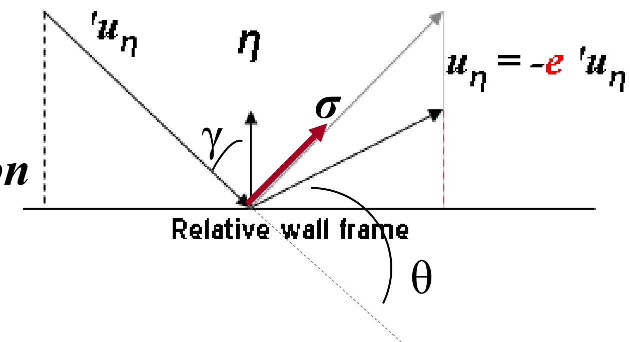
for $'e = e(|\,'u \cdot n|) = 1/e$, the pre-collisional velocities are clearly given by

$$'v = v + \frac{(1+'e)}{2} ('u \cdot \eta)\, \eta \qquad \text{and} \qquad 'v_* = v_* + \frac{(1+'e)}{2} ('u \cdot \eta)\, \eta$$

In addition, the Jacobian of the transformation is then given by

$$J(e(z)) = \left| \frac{\partial v', v'_*}{\partial v, v_*} \right| = e(z) + z e_z(z) = \theta_z(z) = (z\, e(z))_z$$

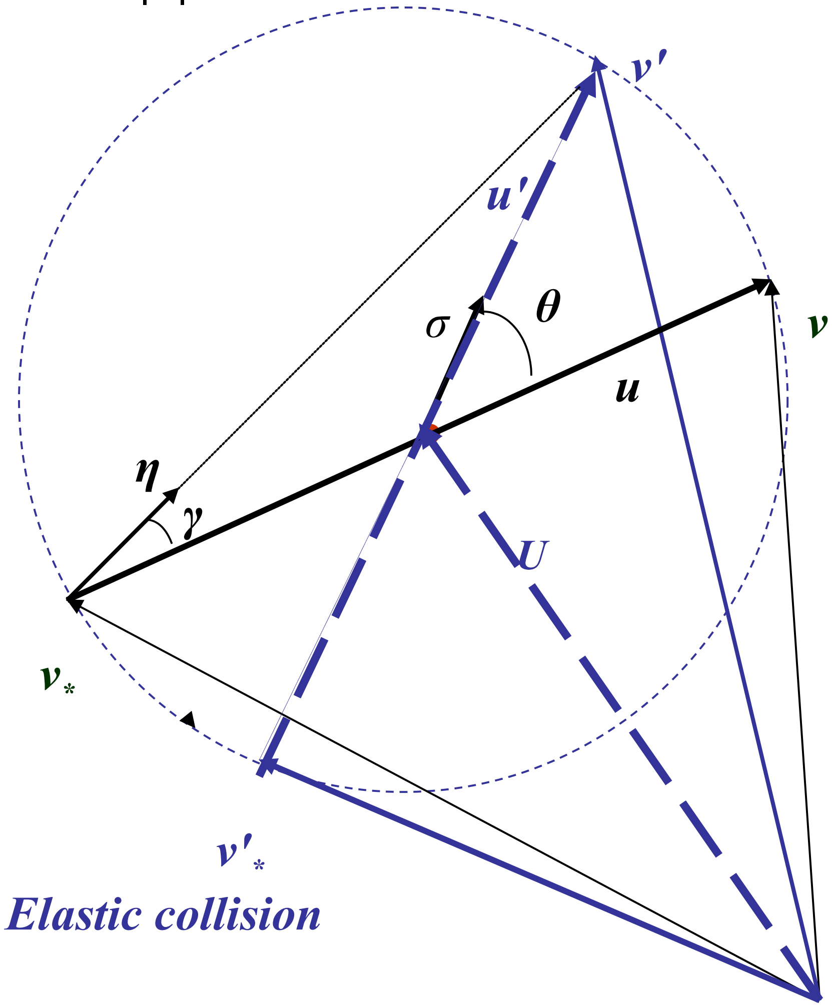
However, for a 'handy' weak formulation we need to write the equation in a different set of coordinates involving $\sigma := u'/|u|$ ***the unit direction of the specular (elastic) reflection of the postcollisional relative velocity, for d=3***



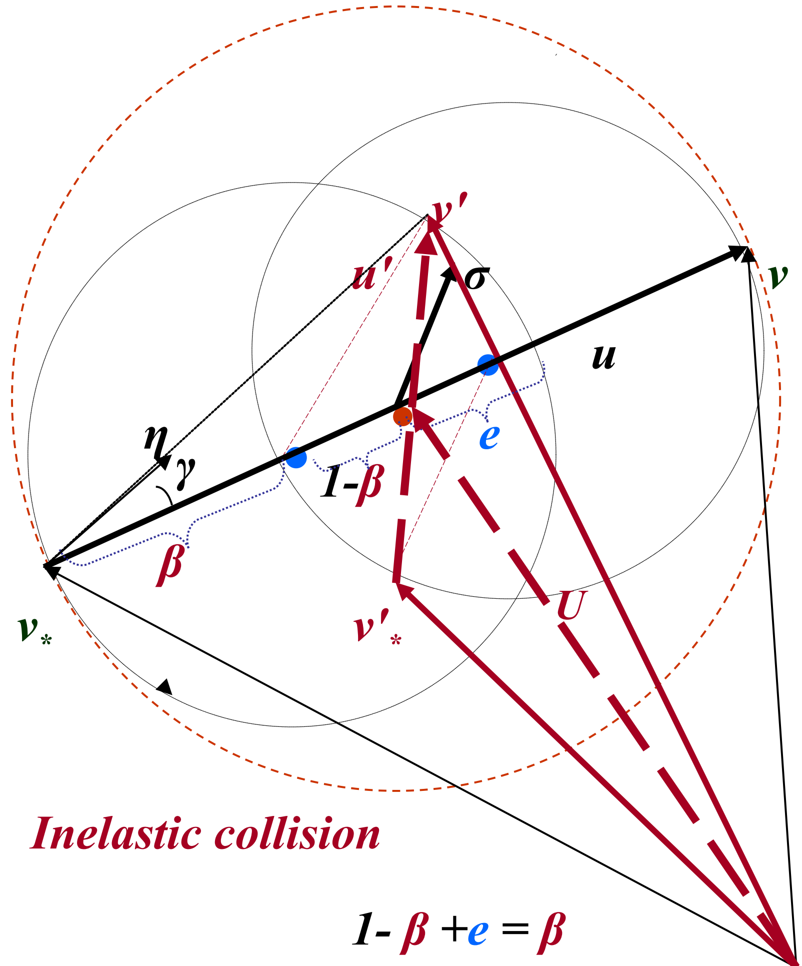
Inelastic reflection of relative velocities

Interchange of velocities during a binary collision or interaction

***σ** = $u^{ref}/|u|$ is the unit vector in the direction of the relative velocity w.r.t. an elastic collision*



Elastic collision



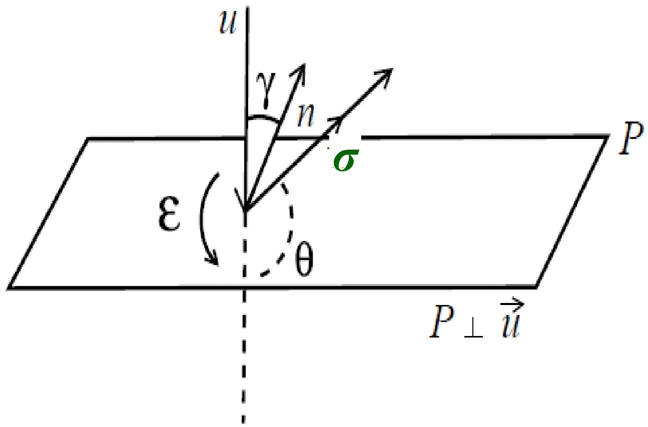
Inelastic collision

$1-\beta+e=\beta$

Center of Mass-Relative velocity coordinates

$$v' = U + \frac{|u'|}{2}\ ; \qquad v'_* = U - \frac{|u'|}{2}\ ; \qquad u' = (1-\beta)u + \beta\,|u|\,\sigma$$

Remark: $\theta \approx 0$ **grazing** and $\theta \approx \pi$ ***head on*** *collisions or interactions*



Goal: *Write the BTE in ($(\mathbf{v} + \mathbf{v}_*)/2$; u) = (center of mass, relative velocity) coordinates.*

Let $u = \mathbf{v} - \mathbf{v}_*$ the relative velocity associated to an ***elastic*** *interaction. Let* ***P*** *be the orthogonal plane to* ***u****.*

Spherical coordinates to represent the ***d****-space spanned by* {***u***; ***P***} *are* {***r***; φ; ε_1; ε_2; …; ε_{d-2}}, *where* ***r*** *= radial coordinates, φ = polar angle, and* {ε_1; ε_2; …; ε_{d-2}}, *the n-2 azimuthal angular variables.*

then $\cos\gamma = \frac{u}{|u|} \cdot \eta$ *with* $\gamma = \frac{\pi - \theta}{2}$, θ = scattering angle $\boxed{|u|\sigma = u - 2(u \cdot \eta)\eta}$

- $0 \le \sin\gamma = b/a \le 1$, with ***b* = impact parameter**, a = diameter of particle
- *Assume* ***scattering effects are symmetric with respect to $\theta = 0$*** $\rightarrow$ $\boxed{0 \le \theta \le \pi \leftrightarrow 0 \le \gamma \le \pi/2}$
- *The unit direction* $\boldsymbol{\sigma}$ *is the* ***specular reflection*** *of* ***u*** *w.r.t.* γ*, that is* $\boxed{|\mathbf{u}|\boldsymbol{\sigma} = \mathbf{u} - 2(\mathbf{u} \cdot \boldsymbol{\eta})\, \boldsymbol{\eta}}$
- Then write the BTE collisional integral with the ***σ-direction $d\eta\, dv_* \rightarrow d\sigma\, dv_*$, η, σ in S^{d-1}***

using the identity

$$\boxed{\frac{1}{|S^{d-1}|\,|u|} \int_{S^{d-1}} (u \cdot \eta)_+ \; g((u \cdot \eta)\eta)\, d\eta = \frac{1}{|S^{d-2}|} \int_{S^{d-1}} g\left(\frac{u - |u|\sigma}{2}\right) d\sigma}$$

So the exchange of coordinates can be performed.

In addition, since $\boldsymbol{d\sigma = |S^{d-2}| \sin^{d-2}\theta\, d\theta}$, *then any function* $b(\frac{\mathbf{u}}{|\mathbf{u}|} \cdot \boldsymbol{\sigma})$ *defined on* $\mathbf{S^{d-1}}$ *satisfies*

$$\boxed{\int_{S^{d-1}} b\left(\frac{\mathbf{u}}{|\mathbf{u}|} \cdot \boldsymbol{\sigma}\right) d\boldsymbol{\sigma} = |S^{d-2}| \int_0^1 b(z)\, (1-z^2)^{(d-3)/2}\, dz} \quad , z = \cos\theta$$

Weak (Maxwell) Formulation: center of mass/ (specular reflected) relative velocity

Due to symmetries of the collisional integral one can obtain (after interchanging the variables of integration)
Both **Elastic/inelastic** formulations: The inelasticity shows only in the exchange of velocities.

$$\left(\frac{\partial}{\partial t} + \nabla_x\right) \int_{\mathbb{R}^d} f(t,x,v)\,\varphi(v)\,dv = \int_{\mathbb{R}^d} Q(f,f)(t,x,v)\,\varphi(v)\,dv$$

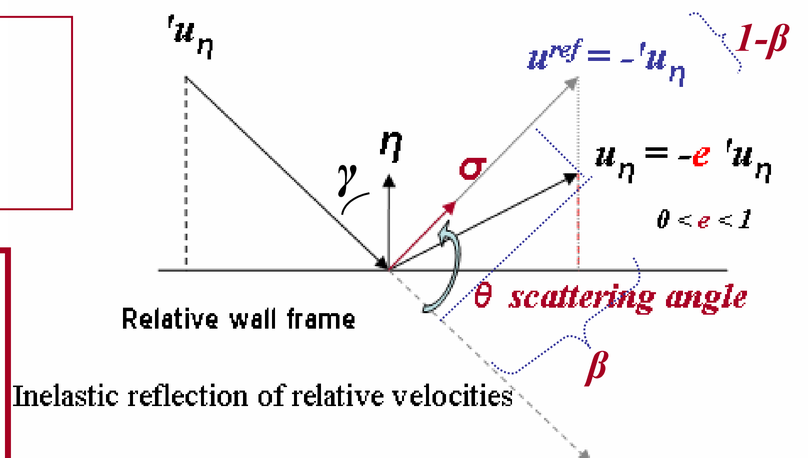
Center of mass-relative velocity coordinates for Q(f; f): *(see ref. [19])*

$$\int_{\mathbb{R}^d} Q(f,f)\,\varphi\,dv = \frac{1}{2}\int_{\mathbb{R}^{2d}}\int_{S_{\perp}^{d-1}} f f_* \left(\varphi' + \varphi'_* - \varphi - \varphi_*\right)|u|^{\gamma} b(\sigma)\,d\sigma\,dv_*\,dv$$

$$v' = \frac{v+v_*}{2} + \frac{1-\beta}{2}u + \frac{\beta}{2}|u|\sigma \qquad \text{or} \qquad u' = (1-\beta)u + \beta|u|\sigma$$

$$v'_* = \frac{v+v_*}{2} - \frac{1-\beta}{2}u - \frac{\beta}{2}|u|\sigma \qquad \text{for } 0 < \beta = \frac{1+e}{2} \leq 1$$

σ = u^ref/|u| is the unit vector in the direction of the relative velocity with respect to an elastic collision



Inelastic reflection of relative velocities

*λ = 0 for **Maxwell Type** (or Maxwell Molecule) models γ = 1 for **hard spheres** models;*
*0< λ <1 for variable **hard potential** models,*
*-d < λ < 0 for variable **soft potential** models.*

***Collisional kernel** or transition probability of interactions is calculated using intramolecular potential laws:*

$$V = r^{-s} \quad \text{with} \quad s \in (2, \infty)$$

$$B_{\beta,\gamma,d}(|u|, \sigma(\theta)) = b_{\beta,\gamma,d}(\sigma(\theta)) \ |u|^{\gamma} \ ,$$ with $b_{\beta,\gamma,d}(\sigma(\theta))$ ***the angular cross section***

which satisfies

$$\boxed{\int_{\sigma \in S^{d-1}} b_{\beta}(\sigma) d\sigma = 1 \quad \textbf{Grad cut-off condition}}$$

In 3 dimensions:

$$\gamma = \frac{s-5}{s-1} \quad \text{and} \qquad b_{\beta,\gamma,d}(\sigma(\theta)) \approx \theta^{-d+3-\nu} \text{ with } \quad \nu = \frac{2}{s-1}$$

- the Grad cut-off assumption is satisfied for variable hard potentials $(s \in (5, \infty))$

In addition, for some extra properties we call for the α-growth condition

$$\boxed{0 < b_{\beta,\gamma,d}(\sigma(\theta))\, \theta^{\alpha(d)} < K}$$

which is satisfied for ***angular cross section function*** $b_{\beta,\gamma,d}(\sigma(\theta))$ ***for α > d-1*** *(in 3-d is for α>2)*

Weak Formulation & fundamental properties of the collisional integral and the equation: Conservation of moments & entropy inequality

x-space homogeneous (*or periodic boundary condition*) problem: Due to symmetries of the collisional integral one can obtain (after interchanging the variables of integration): **Maxwell form of the BTE**

$$\frac{\partial}{\partial t}\int_{I\!R^d} f\,\varphi\,dv = \int_{I\!R^N} Q(f,f)\varphi(v)\,dv =$$

$$\frac{\kappa(t)}{2}\int_{\mathbb{R}^{2d}}\int_{S_+^{d-1}} f f_*\,(\varphi' + \varphi'_* - \varphi - \varphi_*)|u|^\gamma\,\tilde{b}(\sigma)\,d\sigma\,dv_*\,dv$$

Invariant quantities (or observables) - These are statistical moments of the 'pdf'

conservation of mass ρ and momentum J: set $\varphi(v) = 1$ and $\varphi(v) = v$

Using local conservation of momentum on the test function: $\boxed{{}'v + {}'v_* = v + v_*}$

$$\frac{\partial}{\partial t}\int_{I\!R^d} f\,\{1, v_i\}dv = \kappa(t)\int_{I\!R^d} Q(f,f)(v)\{1, v_i\}\,dv = 0, \quad i = 1,2,3.$$

holds, both for the **Elastic** and **Inelastic** cases

Next, set $\varphi(v) = |v|^2 \Rightarrow$ It **conserves energy for** $e = 1$ **– ELASTIC:**

Using local conservation of energy on the test function: $\boxed{|{}'v|^2 + |{}'v_*|^2 = |v|^2 + |v_*|^2}$

➔ $\frac{\partial}{\partial t}\Theta(t) = \kappa(t)\int_{I\!R^d} Q(f,f)(v)\,|v|^2\,dv = 0$ *Conservation of energy*

Recall Boltzmann H-Theorem for **ELASTIC interactions:**

$$\frac{\partial}{\partial t}\int f \log f\, dv = \kappa(t)\int_{I\!R^d} Q(f,f)\log f\, dv =$$

$$\frac{\kappa(t)}{2}\int_{I\!R^d}\int_{I\!R^d}\int_{S_+^{d-1}} (f f_* - f' f'_*)\log\frac{f' f'_*}{f f_*}|u|^\gamma b(\sigma)d\sigma\, dv\, dv_* \le 0$$

Time irreversibility is expressed in this inequality $\Longrightarrow$ **stability**

In addition:

The Boltzmann Theorem: *there are only N+2 collision invariants* $\Longleftrightarrow$

$$\int_{I\!R^N} Q(f,f)\log f\, dv = 0 \iff \log f(\cdot,v) = A + B\cdot v + C|v|^2 \iff$$

$f(\cdot,v) = M_{A,B,C}(v)$ Maxwellian (Gaussian in v-space) parameterized by A, B, C

related the first $N+2$ moments of the initial probability state of $f(0,v) = f_0(v)$

Elastic (conservative) Interactions

Time Irreversibility and relation to Thermodynamics

- **Stability** $\lim_{t\to\infty} \|f(t,v) - M_{A,B,C}\|_{L^1_2} \to 0$ where $\{A,B,C\} \longleftrightarrow$ $\{\rho, u, w\}$, $\rho = \int f_0\, dv$, $\rho u = \int v f_0\, dv$ and $\rho w = \int |v|^2 f_0\, dv$

- **Macroscopic balance equations:** For the space inhomogeneous problem:

Under the ansatz of a Maxwellian state in v-space

$$\boxed{f(t,x,v) = M_{a,b,\mathbf{u}} = a e^{-(b|v-\mathbf{u}|^2)}}$$

where the dependance of (t,x) is only though the parameters $(a,b,\mathbf{u})$:

$\mathbf{u} = \dfrac{\mathbf{J}}{\rho}$ mean velocity and $\Theta = \rho\mathbf{w} = \dfrac{1}{2}\rho\mathbf{u} + \rho\,\mathbf{e}$ kinetic energy, $\mathbf{e} =$ internal energy

choosing $a = \dfrac{3^{3/2}\rho}{(4\pi\, e)^{3/2}}$; $b = \dfrac{3}{4e}$

plus equilibrium constitutive relations : $P = \dfrac{2}{3}\rho\, e$ **pressure.**

$\to$ yields the compressible Euler equations ➔

Elastic (conservative) Interactions: Connections to

Hydrodynamic limits: evolution models of a 'few' statistical moments (mass, momentum and energy)

One obtains the Euler equations:

$$\frac{\partial \rho}{\partial t} + \sum_{i=1}^{3} \frac{\partial}{\partial x_i}(\rho\, \mathbf{u_i}) = 0\,,$$

$$\frac{\partial}{\partial t}(\rho\, \mathbf{u_i}) + \sum_{i=1}^{3} \frac{\partial}{\partial x_i}(\rho\, \mathbf{u_i}\mathbf{u_j} + p) = 0\,, \quad (j = 1, 2, 3)$$

$$\frac{\partial}{\partial t}(\rho(\frac{1}{2}|\mathrm{u}|^2 + e)) = \sum_{i=1}^{3} \frac{\partial}{\partial x_i}(\rho\, \mathbf{u_i}(\frac{1}{2}|\mathbf{u}|^2 + \mathbf{e} + \frac{\mathbf{p}}{\rho})) = 0\,.$$

- **Hydrodynamic limits:** for ϵ-perturbations of Maxwellians plus constitutive relations $\Rightarrow$ $\{A, B, C\}(t, x)$ the corresponding macroscopic system satisfy compressible Euler

or ϵ-Navier-Stokes equations with higher order partial derivatives terms proportional to an $O(\epsilon)$ deviations from Gaussian (Maxwellian) distributions.

Reviewing Inelastic (dissipative) properties: loss of classical hydrodynamics

Set $\varphi(v) = |v|^2$ and using local energy dissipation:

$$\boxed{|v|^2 + |v_*|^2 - |'v|^2 - |'v_*|^2 = -\frac{1-e^2}{4}(1-\nu\cdot\sigma)|v-v_*|^2}$$

INELASTIC Boltzmann collision term:

It dissipates total energy for ***e=e(z)*** < 1 (by Jensen's inequality):

$$\frac{\partial}{\partial t}\Theta(t) = -c_d\frac{(1-e^2)}{4}\kappa(t)\int_{\mathbb{R}^{2d}} f f_* |v-v_*|^{2+\gamma}\, dv_*\, dv \leq -c_d\frac{(1-e^2)}{4}\kappa(t)\Theta(t)^{\frac{\gamma+2}{2}}$$

and there is no classical H-Theorem if ***e*** = constant < 1

$$\int_{\mathbb{R}^d} Q(f,f)\log f\, dv = \frac{1}{2}\int_{\mathbb{R}^{2d}\times S^{d-1}} f f_* \left(\log\frac{f' f'_*}{f f_*} - \frac{f' f'_*}{f f_*} + 1\right)|u|^\gamma b(\sigma)\, d\sigma\, dv\, dv_* + \frac{1-e^2}{2e^2}\int_{\mathbb{R}^{2d}} f f_* |u|^\gamma\, dv\, dv_*.$$

→ **Inelasticity brings loss of micro reversibility**

→but keeps **time irreversibility !!**: That is, there are stationary states and, in some particular cases we can show stability to stationary and self-similar states → However: Existence of **NESS**: Non Equilibrium Statistical States (**stable stationary states are non-Gaussian** pdf's)

→ ***f(v,t)*** $\rightarrow \delta_0$ as ***t*** $\rightarrow \infty$ **to a singular concentrated measure** (unless there is 'source')

→*(Multi-linear Maxwell molecule equations of collisional type and variable hard potentials for collisions with a background thermostat)*

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  - Radial rearrengements methods
  - Connections to Brascamp-Lieb-Luttinger type estimates
  - Young and Hardy-Littlewood-Sobolev type inequalities
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- **Existence and stability of global in time of the Boltzmann equation**
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Consider the Cauchy Boltzmann problem ***(Maxwell, Boltzmann** 1860s-80s);*
***Grad** 1950s; **Cercignani** 60s**; Kaniel Shimbrot** 80's, **Di Perna-Lions** late 80's)*

*Find a function $f(t, x, v) \geq 0$ that solves the equation (written in **strong form**)*

$$\boxed{\frac{\partial f}{\partial t} + v \cdot \nabla f = Q(f,f) \text{ in } (0,+\infty)\times\mathbb{R}^{2n} \quad \text{with} \quad f(0,x,v) = f_0(x,v).}$$

$$Q(f,g) := \int_{\mathbb{R}^n}\int_{S^{n-1}} \{f('v)g('v_*) - f(v)g(v_*)\}\; B(u,\hat{u}\cdot\sigma)\, d\sigma dv_*$$

$$'v = v - (u\cdot\sigma)\,\sigma\ , \quad 'v_* = v_* + (u\cdot\sigma)\,\sigma \ \text{ and } \ u = v - v_*.$$

Conservative interaction
(elastic)

Assumption on the model: *the collision kernel $B(u, \hat{u}\cdot\sigma)$ satisfies*

(i) $B(u, \hat{u}\cdot\sigma) = |u|^{\lambda}\; b(\hat{u}\cdot\sigma)$ *with* $-n < \lambda \leq 1$; *we call* ***soft potentials: $-n < \lambda < 0$***

(i) ***Grad's assumption***: $b(\hat{u}\cdot\sigma) \in L^1(S^{n-1})$, *that is*

$$\boxed{||b||_{L^1(S^{n-1})} = \int_{S^{n-1}} b(\hat{u}\cdot\sigma)d\sigma.}$$

Grad's assumption allows to ***split the collision operator in a gain and a loss part,***

$$Q(f,g) = Q^{+}(f,g) - Q^{-}(f,g) = \textbf{Gain} - \textbf{Loss}$$

But not pointwise bounds *are assumed on* $b(\hat{u}\cdot\sigma)$

The loss operator has the following structure

$Q^{-}(f,g) = f\ \ R(g)$, with **$R(g)$, called the collision frequency, given by**

$$R(g) = \int_{\mathbb{R}^n}\int_{S^{n-1}} g(v_*)\ |u|^{\lambda}\ b(\hat{u}\cdot\sigma)d\sigma dv_*$$

$$= ||b||_{L^1(S^{n-1})}\int_{\mathbb{R}^n} g(v_*)\ |u|^{\lambda}\ dv_* = ||b||_{L^1(S^{n-1})}\ \ g * |v|^{\lambda}.$$

The **loss** bilinear form is a **convolution.**

We shall see also the **gain is a weighted convolution**

Recall: $Q^+(v)$ operator in weak (Maxwell) form, and then it can easily be extended to dissipative (inelastic) collisions

$$\int_{\mathbb{R}^n} Q^+(f,g)(v)\psi(v)\,\mathrm{d}v := \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} f(v)g(v_*)\int_{S^{n-1}} \psi(v')B(|u|,\hat{u}\cdot\omega)\,\mathrm{d}\omega\,\mathrm{d}v_*\,\mathrm{d}v$$

Exchange of velocities in center of mass-relative velocity frame

$$\boxed{u = v - v_* \quad , \quad v' = v - \frac{\beta}{2}(u - |u|\omega) \quad \text{and} \quad v + v_* = v' + v'_*}$$

Energy dissipation parameter or restitution parameters

$\beta : [0,\infty) \to (0,1]$ defined by $\beta(z) := \frac{1+e(z)}{2}$ *with* $z = |u|\sqrt{\frac{1-\hat{u}\cdot\omega}{2}}$

(i) $z \mapsto e(z)$ is absolutely continuous and non-increasing.
(ii) $z \mapsto ze(z)$ is non-decreasing.

Same the collision kernel form $B(|u|,\hat{u}\cdot\omega) = |u|^\lambda b(\hat{u}\cdot\omega)$ with $-n < \lambda.$

With the Grad Cut-off assumption: $\displaystyle\int_{S^{n-1}} b(\hat{u}\cdot\omega)\mathrm{d}\omega < \infty.$

And convolution structure in the loss term: $Q^-(f,g) = f\ \|b\|_{L^1(S^{n-1})}\ g * |v|^\lambda$.

Outline of recent work

Average angular estimates (for the inelastic case as well) by means of radial rearrengement arguments

- ***Young's inequalities for $1 \le p, q, r \le \infty$** (with exact constants) **for Maxwell type and hard potentials $|u|^{\lambda}$ with $0 \le \lambda = 1$***

→ ***Sharp constants for Maxwell type interaction for $(p, q, r) =$** (1,2, 2) and (2,1,2)* **$\lambda = 0$**

- ***Hardy Littlewood Sobolev inequalities , for $1 < p, q, r < \infty$** (with exact constants) **for soft potentials $|u|^{\lambda}$ with $-n \le \lambda < 0$***

- ***Triple Young's inequalities for $1 \le p, q, r, s \le \infty$** (with exact constants) **for radial non-increasing potentials in $L^s(R^d)$***

- ***Existence, uniqueness and regularity estimates for the near vacuum and near (different) Maxwellian solutions for the space inhomogeneous problem (using Kaniel-Shimbrot iteration type solutions) elastic interactions for soft potential and the above estimates.***

- ***L^p stability estimates in the soft potential case, for $1 \le p \le \infty$***

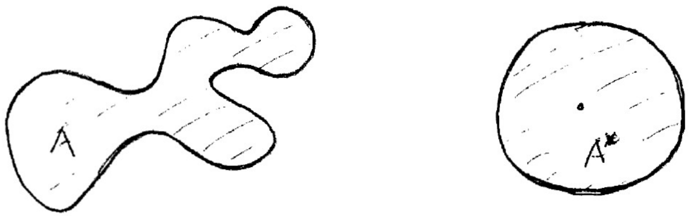
Work in n collaboration with Ricardo Alonso and Emanuel Carneiro

First, some useful concepts of real analysis

1. Radial rearrengements and L^p norms

*Let **A** be a measurable set of finite volume in $\mathbb{R}^n$. Its* **symmetric rearrangement** ***A***** is the open centered ball whose volume agrees with **A**:*

$$A^* = \{x \in \mathbb{R}^n \mid \omega_n |x|^n < \mathrm{Vol}(A)\}$$



*Define the **symmetric decreasing rearrangement f*** of **f** by **symmetrizing** its the level sets,*

$$f^*(x) = \int_0^\infty \mathcal{X}_{\{f(x)>t\}^*}\, dt\,.$$

*Then **f*** is lower semicontinuous (since its level sets are open), and is uniquely determined by the* ***distribution function***
$\mu_f(t) = Vol\{x \mid f(x) > t\}$ ➔ By construction, **f*** is equimeasurable with f, i.e., corresponding level sets of the two functions have the same volume

$$\mu_{f^*}(t) = \mu_f(t)\ ,\ (\text{all } t > 0)\ .$$

Lemma: (**Rearrangement preserves L^p-norms**) *For every nonnegative function **f** in $L^p(\mathbb{R}^n)$,*

$$\|f\|_p = \|f^*\|_p \quad 1 \le p \le \infty,$$

See reference [17]

2. ***Brascamp, Lieb, and Luttinger*** *(1974) showed that functionals of the form*

$$\int_{(\mathbb{R}^k)^m} \prod_{i=1}^{m} f_i \left(\sum_{j=1}^{n} \eta_{ij} x_j \right) dx_1, \ldots dx_n$$

can only increase under a radial rearrangement, where the η_{ij} form an arbitrary real n×m matrix

Moreover, they obtain exact inequality constants

3. ***Beckner (75), Brascamp-Lieb (76, 83) calculation of best/sharp constants for maximizations by radial rearrangements by constructing a family of optimizers.***

1. ***Calculation of Young and Hardy-Littlewood-Sobolev (convolutions) inequalities with exact and best constants – Also extended to multiple Young's ineq.***

Applications to problems in mathematical physics where the solutions are probabilities, i.e. Ornstein-Uhlenbeck; Fokker Plank equations, optimal decay rates to equilibrium, stability estimatesIsoperimetric inequalities, etc.

Recall classical L^p convolution inequalities

Youngs inequality (1912)

Suppose f is in $L^p(R^d)$ and g is in $L^q(R^d)$ and

$$\boxed{\frac{1}{p}+\frac{1}{q}=\frac{1}{r}+1}$$

with $1 \le p, q, r \le \infty$. Then

$$\boxed{\|f*g\|_r \le c_{p,q}\|f\|_p\|g\|_q.}$$

Hardy-Littlewood-Sobolev inequality (1928-38)

Let p, q > 1 and 0 < λ < n be such that ***1/p + 1/q + λ/n = 2.*** *There exists a constant* ***C(n, λ, p)*** *such that*

$$\boxed{\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} f(x)g(y)|x-y|^{-\lambda}\,dxdy \le C(n,\lambda,p)||f||_p||g||_q}$$

for all $f \in L^p(\mathbb{R}^n), g \in L^q(\mathbb{R}^n)$.

The calculation of exact and sharp constants was done was Brascamp-Lieb 76 and Lieb 83 and 90 (see ref [35] and more refs therein). ➔

By interpolation arguments

$$\boxed{\Big\|\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} f(x)g(y)|x-y|^{-\lambda}\,dxdy\Big\|_r \le C(n,\lambda,p)||f||_p||g||_q}$$

$$\boxed{\textit{for}\ \ 1/p + 1/q + \lambda/n = 1 + 1/r}$$

Average angular estimates & weighted Young's inequalities & Hardy Littlewood Sobolev inequalities & sharp constants

R. Alonso and E. Carneiro'08 (to appear in Adv. Math.), and R. Alonso and E. Carneiro, IG, 09 *(refs[1,2]):*
by means of ***radial symmertrization (rearrangement) techniques***

The weak formulation of the gain operator is a ***weighted convolution***

$$\int_{\mathbb{R}^n} Q^+(f,g)(v)\psi(v)dv = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} f(v)g(v-u)\mathcal{P}(\tau_v\mathcal{R}\psi,1)(u)\ |u|^\lambda d$$

Where the weight is an invariant under rotation operator involving

translations and reflections $\tau_v\psi(x) := \psi(x-v)$ and $\mathcal{R}\psi(x) := \psi(-x)$

and the Bobylev's variables and operator

$$\mathcal{P}(\psi,\phi)(u) := \int_{S^{n-1}} \psi(u^-)\phi(u^+)b(\hat{u}\cdot\omega)\,\mathrm{d}\omega\,,$$

$$u^- = \frac{\beta}{2}(u-|u|\sigma) \ \text{ and } \ u^+ = u - u^- = (1-\beta)u + \frac{\beta}{2}(u+|u|\sigma).$$

is invariant under rotations

Bobylev's operator ('75) on Maxwell type interactions λ=0
is the well know identity for the Fourier transform of the Q^+

$$\widehat{Q^+(f,g)} = \mathcal{P}(\hat{f},\hat{g})$$

for $\|f\|_{L^p_k(\mathbb{R}^n)} = \left(\int_{\mathbb{R}^n} |f(v)|^p \left(1+|v|^{pk}\right) dv\right)^{1/p}$ and $B(|u|, \hat{u}\cdot\omega) = |u|^\lambda\, b(\hat{u}\cdot\omega)$,

Young's inequality for variable hard potentials : $0 \leq \lambda \leq 1$

Theorem 1. *Let* $1 \leq p,\ q,\ r \leq \infty$ *with* $\boxed{1/p + 1/q = 1 + 1/r}$ *and* $\lambda \geq 0$. *For* $\alpha \geq 0$, *the bilinear operator* Q^+ *extends to a bounded operator from* $L^p_{\alpha+\lambda}(\mathbb{R}^n) \times L^q_{\alpha+\lambda}(\mathbb{R}^n) \to L^r_\alpha(\mathbb{R}^n)$ *via the estimate*

$$\boxed{\left\|Q^+(f,g)\right\|_{L^r_\alpha(\mathbb{R}^n)} \leq C\, \|f\|_{L^p_{\alpha+\lambda}(\mathbb{R}^n)}\, \|g\|_{L^q_{\alpha+\lambda}(\mathbb{R}^n)}.}$$

$0 \leq \lambda \leq 1$

Hardy-Littlewood-Sobolev type inequality for soft potentials : $-n < \lambda < 0$

Theorem 2. *Let* $1 < p,\ q,\ r < \infty$ *with* $-n < \lambda < 0$ *and* $\boxed{1/p + 1/q = 1 + \lambda/n + 1/r.}$ *Then the bilinear operator* Q^+ *extends to a bounded operator from* $L^p(\mathbb{R}^n) \times L^q(\mathbb{R}^n) \to L^r(\mathbb{R}^n)$ *via the estimate*

$$\boxed{\left\|Q^+(f,g)\right\|_{L^r(\mathbb{R}^n)} \leq C\, \|f\|_{L^p(\mathbb{R}^n)}\, \|g\|_{L^q(\mathbb{R}^n)}.}$$

$-n < \lambda < 0$

- **In both theorems the constant depends on $C = C(n, \alpha, p, q, b, \beta, \lambda)$ are explicit and depend on bounds for $\mathcal{P}(\tau_v \mathcal{R}\psi, 1)(u)$, but generally not sharp.**

- **Only in the cases $\alpha = \lambda = 0$ (Maxwell type interactions), $(p, q, r) = (2, 1, 2)$ and $(p, q, r) = (1, 2, 2)$ we find sharp constants C for the Young's inequality.**

- This theorem exhibits the convolution character of $Q^+(f, g)$: it behaves as $f * g * |u|^\lambda$ in the case of soft potentials.

Sketch of proof: important facts

$$\mathcal{P}(\psi,\phi)(u) := \int_{S^{n-1}} \psi(u^-)\phi(u^+) b(\hat{u}\cdot\omega)\, d\omega\,,$$

1- Radial rearrangement

Radial Symmetrization and the operator $\mathcal{P}$

- $G = SO(n)$ the group of orthonormal rotations in $I\!R^n$.
- The Haar measure $d\mu$ of this compact topological group re-normalized to $\int_G d\mu(R) = 1$.
- The radial symmetrization $f_p^\star$ is defined by

$$f_p^\star(x) = \left(\int_G |f(Rx)|^p\, d\mu(R)\right)^{\frac{1}{p}}\,, \quad \text{if} \quad f \in L^p(I\!R^n) \quad 1 \le p < \infty.$$

and

$$f_\infty^\star(x) = \text{ess sup}_{|y|=|x|} |f(y)|$$

taken over the sphere of radius $|x|$ w.r.t.measure over that sphere

- The rearrangement $f_p^\star$ can be seen as an L^p-average of f over all the rotations $R \in G$:

Let $d\nu$ be a rotationally invariant measure on $I\!R^n$:

$$\int_{I\!R^n} |f(x)|^p\, d\nu(x) = \int_{I\!R^n} |f_p^\star(x)|^p\, d\nu(x) \quad \text{so} \quad \|f\|_{L^p(I\!R^n)} = \|f_p^\star\|_{L^p(I\!R^n)}.$$

2- Radial symmetrization lemma: the the weak formulation of opertor invariant under rotations is maximized on its radial rearrangement
This approach is a non-linear analog to a Brascamp-Lieb-Luttinger type of argument

Lemma 3. *Let $\psi, \varphi, \nu \in C_0(\mathbb{R}^n)$ and $1/p + 1/q + 1/r = 1$, with $1 \le p, q, r \le \infty$*
Then

$$\left| \int_{\mathbb{R}^n} \mathcal{P}(\psi, \varphi)(u)\eta(u)\, du \right| \le \int_{\mathbb{R}^n} \mathcal{P}(\psi_p^\star, \varphi_q^\star)(u)\eta_r^\star(u)\, du.$$

Sketch of proof

- $\mathcal{P}(\psi, \varphi)(Ru) = \mathcal{P}(\psi \circ R, \varphi \circ R)(u)$ for any rotation R.

- $\left| \int_{\mathbb{R}^n} \mathcal{P}(\psi, \varphi)(u)\eta(u)\, du \right| \le \int_{\mathbb{R}^n} \int_{S^{n-1}} |\psi(Ru^-)|\, |\varphi(Ru^+)|\, |\eta(Ru)| b(\hat{u} \cdot \omega)\, \mathrm{d}\omega\, du.$ ind. of R.

- Integration over the group $G = SO(n)$ leads to
$$\left| \int_{\mathbb{R}^n} \mathcal{P}(\psi, \varphi)(u)\eta(u)\, du \right| \le \int_{\mathbb{R}^n} \int_{S^{n-1}} \left(\int_G |\psi(Ru^-)||\varphi(Ru^+)||\eta(Ru)|\, \mathrm{d}\mu(R) \right) b(\hat{u} \cdot \omega)\, \mathrm{d}\omega\, du.$$

- Applying of Hölder's inequality with exponents p, q and r yields
$\int_G |\psi(Ru^-)|\, |\varphi(Ru^+)|\, |\eta(Ru)|\, \mathrm{d}\mu(R) \le \psi_p^\star(u^-)\, \varphi_q^\star(u^+)\, \eta_r^\star(u),$

In particular, for radial function set: $f(x) = \tilde{f}(|x|)$ α corresponds to moments weights

$$\Longrightarrow \quad \int_{\mathbb{R}^n} f(x)^p \, |x|^\alpha \, dx = \left|S^{n-1}\right| \int_0^\infty \tilde{f}(t)^p \, t^{n-1+\alpha} \, dt.$$

and for $d\nu_\alpha(x) = |x|^\alpha dx$, and σ_n^α on $I\!R^+$ by $d\sigma_n^\alpha(t) = t^{n-1+\alpha} dt$ ***set***

$$\Longrightarrow \quad \boxed{||f||_{L^p(\mathbb{R}^n,\, d\nu_\alpha)} = \left|S^{n-1}\right|^{\frac{1}{p}} \, ||\tilde{f}||_{L^p(\mathbb{R}^+,\, d\sigma_n^\alpha)}}$$

- For radial functions $\mathcal{P}$ simplifies to a 1-dimensional integral

$$\begin{aligned}\mathcal{P}(\psi,\varphi)(u) &= \int_{S^{n-1}} \tilde{\psi}\left(|u^-|\right) \tilde{\varphi}\left(|u^+|\right) \, b(\hat{u}\cdot\omega) \, d\omega \\ &= \left|S^{n-2}\right| \int_{-1}^{1} \tilde{\psi}\big(a_1(|u|,s)\big) \tilde{\varphi}\big(a_2(|u|,s)\big) b(s) \, (1-s^2)^{\frac{n-3}{2}} \, ds.\end{aligned}$$

with a_1 and a_2 are defined on $\mathbb{R}^+ \times [-1,1] \to \mathbb{R}^+$ by

$a_1(x,s) = \beta \, x \left(\frac{1-s}{2}\right)^{1/2}$ and $a_2(x,s) = x \left[\left(\frac{1+s}{2}\right) + (1-\beta)^2 \left(\frac{1-s}{2}\right)\right]^{1/2}$

So $\widetilde{\mathcal{P}(\psi,\varphi)}(x) = \left|S^{n-2}\right| \int_{-1}^{1} \tilde{\psi}\left(a_1(x,s)\right) \tilde{\varphi}\left(a_2(x,s)\right) \mathrm{d}\xi_n^b(s)$

where the measure ξ_n^b on $[-1,1]$ is defined as $\mathrm{d}\xi_n^b(s) = \left|S^{n-2}\right| b(s)(1-s^2)^{\frac{n-3}{2}}$

Angular averaging lemma

Lemma *Let* $1 \leq p,q,r \leq \infty$ *with* $1/p + 1/q = 1/r$. *For* $\psi \in L^p(\mathbb{R}^n, \mathrm{d}\sigma_n^\alpha)$ *and* $\varphi \in L^q(\mathbb{R}^n, \mathrm{d}\sigma_n^\alpha)$ *we have*

$$\|\mathcal{P}(\psi,\varphi)\|_{L^r(\mathbb{R}^n,\mathrm{d}\sigma_n^\alpha)} \leq \left\|\widetilde{\mathcal{P}(\psi,\varphi)}\right\|_{L^r(\mathbb{R}^+,\mathrm{d}\sigma_n^\alpha)} \leq C \|\psi\|_{L^p(\mathbb{R}^n,\mathrm{d}\sigma_n^\alpha)} \|\varphi\|_{L^q(\mathbb{R}^n,\mathrm{d}\sigma_n^\alpha)},$$

where the constant C *is given explicitly as a functions of the weight, the inelasticity and the angular integration.*

In the case of constant parameter $\beta = (1+e)/2$, *one can show that* C *is sharp*

$$C(n,\alpha,p,q,b,\beta) = \beta^{-\frac{n+\alpha}{p}} \int_{-1}^{1} \left(\frac{1-s}{2}\right)^{-\frac{n+\alpha}{2p}} \left[\left(\frac{1+s}{2}\right) + (1-\beta)^2 \left(\frac{1-s}{2}\right)\right]^{-\frac{n+\alpha}{2q}} \mathrm{d}\xi_n^b(s)$$

How to show ***C is sharp****?*

The radial symmetrization method generated the "extremal" operator for $x \in \mathbb{R}^+$

$$\widetilde{\mathcal{P}(\psi,\varphi)}(x) = \left|S^{n-2}\right| \int_{-1}^{1} \tilde{\psi}\left(a_1(x,s)\right) \tilde{\varphi}\left(a_2(x,s)\right) \, \mathrm{d}\xi_n^b(s)$$

where the measure ξ_n^b on $[-1,1]$ is defined as $\mathrm{d}\xi_n^b(s) = \left|S^{n-2}\right| b(s)(1-s^2)^{\frac{n-3}{2}}$

Then , define the following bilinear operator for any two bounded and continuous functions $f, g : \mathbb{R}^+ \to \mathbb{R}$,

$$\mathcal{B}(f,g)(x) := \int_{-1}^{1} f\left(a_1(x,s)\right) g\left(a_2(x,s)\right) \mathrm{d}\xi_n^b(s)$$

Following Beckner's approach '75 Brascamp Lieb 76, one can find show C if the "best" constant by finding a pair sequence of functions such the operator acting on them achieves it.

take the sequences $\{\psi_\epsilon\}$ and $\{\varphi_\epsilon\}$ with $\epsilon > 0$ defined by

$$\psi_\epsilon(x) = \begin{cases} \epsilon^{1/p}\, x^{-(n+\alpha-\epsilon)/p} & \text{for } 0 < x < 1, \\ 0 & \text{otherwise.} \end{cases} \qquad \varphi_\epsilon(x) = \begin{cases} \epsilon^{1/q}\, x^{-(n+\alpha-\epsilon)/q} & \text{for } 0 < x < 1 \\ 0 & \text{otherwise.} \end{cases}$$

so $\quad \|\psi_\epsilon\|_{L^p(\mathbb{R}^+,\, \mathrm{d}\sigma_n^\alpha)} = \|\varphi_\epsilon\|_{L^q(\mathbb{R}^+,\, \mathrm{d}\sigma_n^\alpha)} = 1 \quad$ and $\quad \|\mathcal{B}(\psi_\epsilon, \varphi_\epsilon)\|_{L^r(\mathbb{R}^+,\, \mathrm{d}\sigma_n^\alpha)} \to C$

Maxwell type interactions with β constant: the constants are sharp in (1,2, 2) and in (2,1,2)

Corollary: *Let $f \in L^1(\mathbb{R}^n)$ and $g \in L^2(\mathbb{R}^n)$. Then*

$$\|Q^+(f,g)\|_{L^2(\mathbb{R}^n)} = \left\|\widehat{Q^+(f,g)}\right\|_{L^2(\mathbb{R}^n)} = \left\|\mathcal{P}(\hat{f},\hat{g})\right\|_{L^2(\mathbb{R}^n)}$$
$$\leq C_0 \,\|\hat{f}\|_{L^\infty(\mathbb{R}^n)} \,\|\hat{g}\|_{L^2(\mathbb{R}^n)} \leq C_0 \,\|f\|_{L^1(\mathbb{R}^n)} \,\|g\|_{L^2(\mathbb{R}^n)}$$

The constant is given by

$$C_0 = \left|S^{n-2}\right| \int_{-1}^{1} \left[\left(\tfrac{1+s}{2}\right) + (1-\beta)^2 \left(\tfrac{1-s}{2}\right)\right]^{-\frac{n}{4}} \mathrm{d}\xi_n^b(s)\,.$$

Similarly, for $f \in L^2(\mathbb{R}^n)$ and $g \in L^1(\mathbb{R}^n)$ we have

$$\left\|Q^+(f,g)\right\|_{L^2(\mathbb{R}^n)} \leq C_1 \,\|f\|_{L^2(\mathbb{R}^n)} \,\|g\|_{L^1(\mathbb{R}^n)},$$

where

$$C_1 = \left|S^{n-2}\right| \beta^{-\frac{n}{2}} \int_{-1}^{1} \left(\tfrac{1-s}{2}\right)^{-\frac{n}{4}} \mathrm{d}\xi_n^b(s).$$

The constant is achieved by the sequences: $f \geq 0 \quad \|\hat{f}\|_{L^\infty(\mathbb{R}^n)} = \|f\|_{L^1(\mathbb{R}^n)}$ ***So approximate a Dirac in x***

$$\boxed{\widetilde{\hat{f}}_\epsilon(x) = e^{-\pi\epsilon^2 x^2}}$$

and

$$\boxed{\widetilde{\hat{g}}_\epsilon(x) = \begin{cases} \epsilon^{1/2}\, x^{-(n-\epsilon)/2} & \text{for } \; 0 < x < 1 \\ 0 & \text{otherwise.} \end{cases}}$$

(see Alonso and Carneiro, to appear in Adv Math 2009)

Young's inequality for hard potentials for general $1 \leq p, q, r \leq \infty$

The main idea is to establish a connection between the Q^+ and P operators, and then use the knowledge from the previous estimates. For $\alpha = 0 = \lambda$ (Maxwell type interactions) no weighted norms

$$I := \int_{\mathbb{R}^n} Q^+(f,g)(v)\psi(v)\,\mathrm{d}v = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} f(v)g(v-u)\mathcal{P}(\tau_v\mathcal{R}\psi,1)(u)\,\mathrm{d}u\,\mathrm{d}v.$$

The exponents p, q, r in Theorem 1 satisfy $\quad 1/p' + 1/q' + 1/r = 1,$

First introduced by Gustafsson in '88, here is obtain in a sharp form.

Regroup and use Holder's inequality and the angular averaging estimates on $L^{r'/q'}$ to obtain

$$I = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\left(f(v)^{\frac{p}{r}}g(v-u)^{\frac{q}{r}}\right)\left(f(v)^{\frac{p}{q'}}\mathcal{P}(\tau_v\mathcal{R}\psi,1)(u)^{\frac{r'}{q'}}\right)$$
$$\left(g(v-u)^{\frac{q}{p'}}\mathcal{P}(\tau_v\mathcal{R}\psi,1)(u)^{\frac{r'}{p'}}\right)\mathrm{d}u\,\mathrm{d}v \;\leq\; C\,\|f\|_{L^p(\mathbb{R}^n)}\,\|g\|_{L^q(\mathbb{R}^n)}\,\|\psi\|_{L^{r'}(\mathbb{R}^n)},$$

$$\boxed{C = \left|S^{n-2}\right|\left(2^{\frac{n}{r'}}\int_{-1}^{1}\left(\tfrac{1-s}{2}\right)^{-\frac{n}{2r'}}\mathrm{d}\xi_n^b(s)\right)^{\frac{r'}{q'}} \left(\int_{-1}^{1}\left[\left(\tfrac{1+s}{2}\right)+(1-\beta_0)^2\left(\tfrac{1-s}{2}\right)\right]^{-\frac{n}{2r'}}\mathrm{d}\xi_n^b(s)\right)^{\frac{r'}{p'}}}$$

These estimates resemble a Brascamp-Lieb type inequality argument (for a nonlinear weight) with best/exact constants approach to obtain Young's inequality

2- Young's inequality for hard potentials with $|v|^\alpha$ weights with $\alpha + \lambda > 0$:

For $\psi_\alpha(v) = \psi(v)|v|^\alpha$

As in the previous case, by Holder and the unitary transformations

$$\boxed{\begin{aligned}\int_{\mathbb{R}^n} Q^+(f,g)(v)\psi_\alpha(v)\,\mathrm{d}v &= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} f(v)g(v-u)\mathcal{P}(\tau_v\mathcal{R}\psi_\alpha,1)(u)\ |u|^\lambda\,\mathrm{d}u\,\mathrm{d}v \\ &\le 4\ 2^{\alpha/2}\ 2^\lambda\, C\, \|f\|_{L^p_{\alpha+\lambda}(\mathbb{R}^n)}\, \|g\|_{L^q_{\alpha+\lambda}(\mathbb{R}^n)}\, \|\psi\|_{L^{r'}(\mathbb{R}^n)}.\end{aligned}}$$

Then, one obtains

1- $$\big\|Q^+(f,g)(v)|v|^\alpha\big\|_{L^r(\mathbb{R}^n)} \le 2^{\alpha/2}\ 2^{\lambda+2}\, C\, \|f\|_{L^p_{\alpha+\lambda}(\mathbb{R}^n)}\, \|g\|_{L^q_{\alpha+\lambda}(\mathbb{R}^n)}.$$

2- $$\big\|Q^+(f,g)(v)\big\|_{L^r(\mathbb{R}^n)} \le 2^{\lambda+1}\, C\, \|f\|_{L^p_{\alpha+\lambda}(\mathbb{R}^n)}\, \|g\|_{L^q_{\alpha+\lambda}(\mathbb{R}^n)}\,,$$

3- $$\big\|Q^+(f,g)(v)\big\|_{L^r_\alpha(\mathbb{R}^n)} \le 2^{1/r}\ 2^{\alpha/2}\ 2^{\lambda+2}\, C\, \|f\|_{L^p_{\alpha+\lambda}(\mathbb{R}^n)}\, \|g\|_{L^q_{\alpha+\lambda}(\mathbb{R}^n)}\,,$$

all with the same
$$\boxed{C = |S^{n-2}|\left(2^{\frac{n}{r'}}\int_{-1}^{1}\left(\tfrac{1-s}{2}\right)^{-\frac{n}{2r'}}\mathrm{d}\xi_n^b(s)\right)^{\frac{r'}{q'}}\left(\int_{-1}^{1}\left[\left(\tfrac{1+s}{2}\right)+(1-\beta_0)^2\left(\tfrac{1-s}{2}\right)\right]^{-\frac{n}{2r'}}\mathrm{d}\xi_n^b(s)\right)^{\frac{r'}{p'}}}$$

Remark: ***1-*** *Previous L^p estimates by Gustafsson 88, Villani-Mouhot '04* ***for pointwise bounded b(u · σ),*** *I.M.G-Panferov-Villani '03 for* ***(p,1,p)*** *with* ***σ -integrable b(u . σ) in S^{n-1}.***
2*-The dependence on the weight α may have room to improvement. One may expect estimates with polynomial (?) decay in α, like in L^1_α as shown Bobylev,I.M.G, Panferov and recently with Villani (97, 04,08) (also previous work of Wennberg '94, Desvilletes, 96, without decay rates.)*

Hardy-Littlewood-Sobolev inequality for soft potentials $-n < \lambda < 0$:

$$\int_{\mathbb{R}^n} Q^+(f,g)(v)\,\psi(v)\,\mathrm{d}v = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} f(v)g(v-u)\mathcal{P}(\tau_v\mathcal{R}\psi,1)(u)\,|u|^\lambda\,\mathrm{d}u\,\mathrm{d}v$$
$$= \int_{\mathbb{R}^n} f(v)\left(\int_{\mathbb{R}^n} \tau_v\mathcal{R}g(u)\,\mathcal{P}(\tau_v\mathcal{R}\psi,1)(u)\ |u|^\lambda \mathrm{d}u\right)\mathrm{d}v.$$

Also here estimates resemble a ***Brascamp-Lieb*** *type inequality argument (for a nonlinear weight)*

Applying Holder's inequality and then the angular averaging lemma to the inner integral with (p, q, r) = (a,1, a), a to be determined, one obtains $\boxed{1/a + 1/a' = 1}$

$$\int_{\mathbb{R}^n} \tau_v\mathcal{R}g(u)\,\mathcal{P}(\tau_v\mathcal{R}\psi,1)(u)\ |u|^\lambda \mathrm{d}u \le C_1\,\|\tau_v\mathcal{R}\psi\|_{L^a(\mathbb{R}^n,\,\mathrm{d}\nu_\lambda)}\,\|\tau_v\mathcal{R}g\|_{L^{a'}(\mathbb{R}^n,\,\mathrm{d}\nu_\lambda)}$$
$$= C_1\left[\left(|\psi|^a * |u|^\lambda\right)(v)\right]^{1/a}\left[\left(|g|^{a'} * |u|^\lambda\right)(v)\right]^{1/a'}$$

$$\boxed{C_1 = \left|S^{n-2}\right|\,2^{\frac{n+\lambda}{a}}\int_{-1}^{1}\left(\tfrac{1-s}{2}\right)^{-\frac{n+\lambda}{2a}}\ \mathrm{d}\xi_n^b(s)\,.}$$

The choice of integrability exponents allowed to get rid of the integrand singularity at s = −1, producing a uniform control with respect to the inelasticity β.

Is it possible to make such choice of a ?

Indeed, combining with the complete integral above, using triple Holder's inq. yields

$$\int_{\mathbb{R}^n} Q^+(f,g)(v)\,\psi(v)\,\mathrm{d}v \le C_1\ \|f\|_{L^p(\mathbb{R}^n)}\ \left\||\psi|^a * |u|^\lambda\right\|^{1/a}_{L^{b/a}(\mathbb{R}^n)}\ \left\||g|^{a'} * |u|^\lambda\right\|^{1/a'}_{L^{c/a'}(\mathbb{R}^n)}$$

Then: *for* $\boxed{\frac{1}{a}+\frac{1}{a'}=1, \qquad 1\le a\le\infty}$ and $\boxed{\frac{1}{p}+\frac{1}{b}+\frac{1}{c}=1, \quad 1<b,c<\infty}$

$$\int_{\mathbb{R}^n} Q^+(f,g)(v)\,\psi(v)\,\mathrm{d}v \;\le C_1\,\|f\|_{L^p(\mathbb{R}^n)}\,\big\||\psi|^a*|u|^\lambda\big\|^{1/a}_{L^{b/a}(\mathbb{R}^n)}\,\Big\||g|^{a'}*|u|^\lambda\Big\|^{1/a'}_{L^{c/a'}(\mathbb{R}^n)}$$

*Using the classical **Hardy-Littlewood-Sobolev** inequality to obtain (Lieb '83)*

$$\big\||\psi|^a*|u|^\lambda\big\|_{L^{b/a}(\mathbb{R}^n)}\le C_2\,\|\psi\|^a_{L^{ad}(\mathbb{R}^n)} \qquad \boxed{ad=r'}$$

$$\Big\||g|^{a'}*|u|^\lambda\Big\|_{L^{c/a'}(\mathbb{R}^n)}\le C_3\,\|g\|^{a'}_{L^{a'e}(\mathbb{R}^n)} \qquad \boxed{a'e=q}$$

where the exponents satisfy $\boxed{1+\frac{a}{b}=\frac{1}{d}-\frac{\lambda}{n} \quad\text{and}\quad 1+\frac{a'}{c}=\frac{1}{e}-\frac{\lambda}{n}.}$

In fact, it is possible to find 1/a in the non-empty interval

$$\boxed{\max\left\{\frac{1}{r'(2+\frac{\lambda}{n})},\,1-\frac{1}{q(1+\frac{\lambda}{n})}\right\}<\frac{1}{a}<\min\left\{\frac{1}{r'(1+\frac{\lambda}{n})},\,1-\frac{1}{q(2+\frac{\lambda}{n})}\right\}}$$

such that $\boxed{\big\|Q^+(f,g)\big\|_{L^r(\mathbb{R}^n)}\le C\,\|f\|_{L^p(\mathbb{R}^n)}\,\|g\|_{L^q(\mathbb{R}^n)}}$ $\quad C=C_1\,C_2^{1/a}\,C_3^{1/a'}$

for $\boxed{1<p,\ q,\ r<\infty \text{ with } -n<\lambda<0 \text{ and } 1/p+1/q=1+\lambda/n+1/r.}$

Inequalities with Maxwellian weights – fundamental estimates for pointwise exponentially weighted estimates

As an application of these ideas one can also show ***Young type estimates for the non-symmetric Boltzmann collision operator with exponential weights.***

First, for any $a > 0$ and $\gamma \geq 0$ define $\mathcal{M}_\gamma(v) := \exp\left(-a|v|^\gamma\right)$

Theorem 7. *Let $1 \leq p, q, r \leq \infty$ with $1/p + 1/q = 1 + 1/r$. Assume that*

$$B(|u|, \hat{u}\cdot\omega) = |u|^\lambda\, b(\hat{u}\cdot\omega)\,,$$

with $0 \leq \lambda \leq 2$. Then, for non-increasing restitution coefficient such that $e(z) < 1$ for $z \in (0, \infty)$,

$$\left\|Q^+(f,g)\ \mathcal{M}_\lambda^{-1}\right\|_{L^r(\mathbb{R}^n)} \leq C\,\|f\ \mathcal{M}_\lambda^{-1}\|_{L^p(\mathbb{R}^n)}\,\|g\ \mathcal{M}_\lambda^{-1}\|_{L^q(\mathbb{R}^n)}$$

The constant ***C := C(n, λ , p, q,b, β)*** *is computed in the proof and is similar to the one obtained for Young's inequality proof.*

In the important case ***(p, q, r) = (∞, 1, ∞) The constant reduces to***

$$C = C(n,\lambda)\int_{-1}^{1}\left[\left(\tfrac{1+s}{2}\right) + (1-\beta(0))^2\left(\tfrac{1-s}{2}\right)\right]^{-n/2} b_\beta(s)\,\mathrm{d}s, \qquad b_\beta(s) := \left[1 - \left(\tfrac{1+|\vartheta(s)|}{2}\right)^{\lambda/2}\right]^{-1} b(s)\,,$$

with $\quad |\vartheta(s)| = \sqrt{(1-\beta(x))^2 + \beta^2(x) + 2(1-\beta(x))\beta(x)s}\,, \quad and\ x = \sqrt{\frac{1-s}{2}}.$

Proof: an elaborated argument of the pre/post collision exchange of coordinates (see Alonso, Carneiro, G, 09)

Distributional and classical solutions to the Cauchy Boltzmann problem for soft potentials with integrable angular cross section *(Ricardo Alonso & I.M.G., 09 submitted)*

Consider the Cauchy Boltzmann problem:

$$\textbf{(1)}\quad \frac{\partial f}{\partial t} + v \cdot \nabla f = Q(f,f) \ \text{ in } \ (0,+\infty) \times \mathbb{R}^{2n} \qquad f(0,x,v) = f_0(x,v).$$

$$Q(f,g) := \int_{\mathbb{R}^n} \int_{S^{n-1}} \{f('v)g('v_*) - f(v)g(v_*)\} \ B(u, \hat{u}\cdot\sigma)\, d\sigma dv_*$$

$$'v = v - (u\cdot\sigma)\,\sigma\,, \quad 'v_* = v_* + (u\cdot\sigma)\,\sigma \ \text{ and } \ u = v - v_*.$$

$B(u, \hat{u}\cdot\sigma) = |u|^{-\lambda}\ b(\hat{u}\cdot\sigma)$ with ***$0 \le \lambda < n-1$*** *with the* ***Grad's assumption:***

$$\boxed{\|b\|_{L^1(S^{n-1})} = \int_{S^{n-1}} b(\hat{u}\cdot\sigma)d\sigma.}$$

with $\quad Q^-(f,g) = f\ \|b\|_{L^1(S^{n-1})}\ g * |v|^{-\lambda}.$

Notation and spaces: *For* $\quad M_{\alpha,\beta}(x,v) := \exp\left(-\alpha|x|^2 - \beta|v|^2\right)$

Set $\mathcal{M}_{\alpha,\beta} = L^\infty(\mathbb{R}^{2n}, M^{-1}_{\alpha,\beta}).$ *with the norm* $\|f\|_{\alpha,\beta} = \left\|f\ M^{-1}_{\alpha,\beta}\right\|_{L^\infty(\mathbb{R}^{2n})}$ *and*

Set $f^\#(t,x,v) := f(t,x+tv,v)$ *, so problem one reduces to* $\boxed{\dfrac{df^\#}{dt}(t) = Q^\#(f,f)(t) \ \text{ with } \ f(0) = f_0.}$

Definition. A *distributional solution* in $[0,T]$ of problem (1) is a function $f \in W^{1,1}(0,T; L^\infty(\mathbb{R}^{2n}))$ that solves (5) a.e. in $(0,T] \times \mathbb{R}^{2n}$.

Kaniel & Shinbrot iteration '78: define the sequences $\{l_n(t)\}$ and $\{u_n(t)\}$ as the mild solutions to *(also Illner & Shinbrot '83)*

$$\frac{dl_n^\#}{dt}(t) + Q_-^\#(l_n, u_{n-1})(t) = Q_+^\#(l_{n-1}, l_{n-1})(t)$$
$$\frac{du_n^\#}{dt}(t) + Q_-^\#(u_n, l_{n-1})(t) = Q_+^\#(u_{n-1}, u_{n-1})(t)$$

with $0 \le l_n(0) \le f_0 \le u_n(0)$.

which relies in choosing a pair of functions (l_0, u_0) satisfying so called *the beginning condition* in $[0, T]$:

$$u_0^\# \in L^\infty(0, T; \mathcal{M}_{\alpha,\beta}) \quad \textit{and} \qquad 0 \le l_0^\#(t) \le l_1^\#(t) \le u_1^\#(t) \le u_0^\#(t) \ \text{ a.e. in } \ 0 \le t \le T.$$

Theorem: *Let* $\{\mathbf{l}_n(\mathbf{t})\}$ *and* $\{\mathbf{u}_n(\mathbf{t})\}$ *the sequences defined by the mild solutions of the linear system above, such that the beginning condition is satisfied in* $[0, T]$*, then*

(i) The sequences $\{\mathbf{l}_n(\mathbf{t})\}$ *and* $\{\mathbf{u}_n(\mathbf{t})\}$ *are well defined for* $n \ge 1$*. In addition,* $\{\mathbf{l}_n(\mathbf{t})\}$, $\{\mathbf{u}_n(\mathbf{t})\}$ ***are*** *increasing and decreasing sequences respectively, and*

$$\boldsymbol{l^\#_n (t) \ \le \ u^\#_n (t) \qquad \text{a.e. in } 0 \le t \le T.}$$

(ii) If $\mathbf{0} \le \mathbf{l}_n(\mathbf{0}) = \mathbf{f}_0 = \mathbf{u}_n(\mathbf{0})$ *for* $n \ge 1$*, then*

$$\boxed{\lim_{n\to\infty} l_n(t) = \lim_{n\to\infty} u_n(t) = f(t) \ \ \textit{a.e. in } [0, T].}$$

The limit $\boldsymbol{f(t) \in C(0, T; M^\#_{\alpha,\beta})}$ *is the unique distributional solution of the Boltzmann equation in* $[0, T]$ *and fulfills*

$$\boldsymbol{0 \le \ l^\#_0(t) \ \le \ f^\#(t) \ \le \ u^\#_0(t) \quad \text{a.e.} \quad \text{in } [0, T].}$$

Hard and soft potentials case for small initial data

Lemma : *Assume* $-1 \le \lambda < n-1$. *Then, for any* $0 \le s \le t \le T$ *and functions* $f^{\#}, g^{\#}$ *that lie in* $L^\infty(0, T; M^{\#}_{\alpha,\beta})$, **then** *the following inequality holds*

$$\int_s^t \left|Q_+^{\#}(f,g)(\tau)\right| d\tau \le k_{\alpha,\beta} \exp\left(-\alpha|x|^2 - \beta|v|^2\right) \left\|f^{\#}\right\|_{L^\infty(0,T;\mathcal{M}^{\#}_{\alpha,\beta})} \left\|g^{\#}\right\|_{L^\infty(0,T;\mathcal{M}^{\#}_{\alpha,\beta})},$$

with

$$\boxed{k_{\alpha,\beta} = \sqrt{\pi}\,\alpha^{-1/2}\,\|b\|_{L^1(S^{n-1})}\left(\frac{|S^{n-1}|}{n-\lambda-1} + C_n\,\beta^{-n/2}\right)}$$

So the following statement holds: ***Distributional solutions for small initial data:*** *(near vacuum)*

Theorem: *Let* $B(u, \hat{u}\cdot\sigma) = |u|^{-\lambda}\, b(\hat{u}\cdot\sigma)$ *with* $-1 \le \lambda < n-1$ *with the* ***Grad's assumption***
Then, the Boltzmann equation has a unique global distributional solution if

$$\|f_0\|_{\alpha,\beta} \le \frac{1}{4k_{\alpha,\beta}} \quad . \text{ Moreover for any } T \ge 0, \qquad \left\|f^{\#}\right\|_{L^\infty(0,T;\mathcal{M}^{\#}_{\alpha,\beta})} \le C := \frac{1-\sqrt{1-4k_{\alpha,\beta}\,\|f_0\|_{\alpha,\beta}}}{2k_{\alpha,\beta}}.$$

As a consequence, one concludes that the distributional solution f is controlled by a traveling Maxwellian, and that

$$\lim_{t\to\infty} f(t,x,\xi) \to 0 \text{ a.e. in } \mathbb{R}^{2n}.$$

It behaves like the heat equation, as mass spreads as t grows

Distributional solutions near local Maxwellians : *Ricardo Alonso, IMG'08*

Previous work by Toscani '88, Goudon'97, Mischler –Perthame '97

Theorem: *Let* $B(u, \hat{u} \cdot \sigma) = |u|^{-\lambda}\ b(\hat{u} \cdot \sigma)$ *with* $-n < \lambda \leq 0$ *with the* ***Grad's assumption***
In addition, assume that f_0 *is* ε*–close to the local Maxwellian distribution* $M(x, v) = C\, M_{\alpha,\beta}(x - v, v)$ $(0 < \alpha,\ 0 < \beta)$.
Then, for sufficiently small ε *the Boltzmann equation has a unique solution satisfying*

$$C_1(t)\, M_{\alpha_1,\beta_1}(x-(t+1)v, v) \leq f(t, x\text{-}vt, v) \leq C_2(t)\, M_{\alpha_2,\beta_2}(x-(t+1)v, v)$$

for some positive functions $0 < C_1(t) \leq C \leq C_2(t) < \infty$, *and parameters* $0 < \alpha_2 \leq \alpha \leq \alpha_1$ *and* $0 < \beta_2 \leq \beta \leq \beta_1$.

Moreover, the case $\beta = 0$ *(infinite mass) is permitted as long as* $\beta_1 = \beta_2 = 0$.
(this last part extends the result of Mishler & Perthame '97 to soft potentials)

Sketch of proof:

Define the ***distance*** between two Maxwellian distributions $M_i = C_i M_{\alpha i, \beta i}$ for $i = 1, 2$ as

$$d(M_1, M_2) := |C_2 - C_1| + |\alpha_2 - \alpha_1| + |\beta_2 - \beta_1|.$$

Second, we say that f is **ε–*close*** to the Maxwellian distribution $M = C\, M_{\alpha,\beta}$ if there exist Maxwellian distributions M_i $(i = 1, 2)$ such that $d(M_i, M) < \varepsilon$ for some small $\varepsilon > 0$, and $M_1 \leq f \leq M_2$.

Also define

$$\boxed{\phi_{\alpha,\beta}(t,x,v) := \|b\|_{L^1(S^{n-1})} \int_{\mathbb{R}^n} \exp\left(-\alpha\,|x+u|^2 - \beta\,|v - u/t|^2\right) |u|^{-\lambda} du.}$$

and notice that for $-n < \lambda \leq 0$

$$\left\|\phi_{\alpha_2,\beta_2} - \phi_{\alpha_1,\beta_1}\right\|_{L^\infty} \leq C(\min \alpha_i, \min \beta_i)\; d(M_1, M_2),$$

Following the **Kaniel-Shimbrot** procedure, one obtains the following non-linear system of inequations

$$\boxed{\begin{aligned} C_1'(t) + \frac{C_1(t)\, C_2(t)}{t^{n-\lambda}}\, \phi_2 &\leq \frac{C_1^2(t)}{t^{n-\lambda}}\, \phi_1 \\ C_2'(t) + \frac{C_1(t)\, C_2(t)}{t^{n-\lambda}}\, \phi_1 &\geq \frac{C_2^2(t)}{t^{n-\lambda}}\, \phi_2. \end{aligned}}$$

which can be solved for a suitable choice of $C_1(t)$ and $C_2(t)$ satisfying:

$$\boxed{\frac{C_1(t)}{C_1(t_0)} = \frac{C_2(t_0)}{C_2(t)}.}$$

and the initial data for $t_0=1$ satisfying:

$$\boxed{\begin{aligned} &|C_2(1) - k| \leq K_1(C,\alpha,\beta)\; d(M_1, M_2) \leq 2\; K_1(C,\alpha,\beta)\; \epsilon\;, \\ &\exp\left(k\, \frac{\|\phi_1 + \phi_2\|_{L^\infty} + \|\phi_1 - \phi_2\|_{L^\infty}}{n - \lambda - 1}\right) \leq K_2(C,\alpha,\beta)\;, \\ &\qquad \text{and} \quad C_2(1) + k \geq K_3(C,\alpha,\beta), \end{aligned}}$$

Classical solutions

(Different approach from Guo'03, our methods follow some of the those by Boudin & Desvilletes '00, plus new ones)

Definition. A *classical solution* in $[0, T]$ of problem our is a function such that

$$(i)\ \ f(t) \in W^{1,1}(0,T;L^\infty(\mathbb{R}^{2n}))\ , \qquad (ii)\ \ \nabla f \in L^1(0,T;L^p(\mathbb{R}^{2n}))\ \text{for some}\ 1 \leq p,$$

Theorem (Application of HLS inequality to Q^+ for soft potentials): *Let the collision kernel satisfying assumptions* $\lambda < n$ and the Grad cut-off, *then for* $1 < p < \infty$

$$\|Q_+(f,g)\|_{L^p_v(\mathbb{R}^n)} \leq C_1 \, \|f\|_{L^p_v(\mathbb{R}^n)} \ \ \|g\|_{L^\gamma_v(\mathbb{R}^n)} \ \ ,$$
$$\|Q_+(f,g)\|_{L^p_v(\mathbb{R}^n)} \leq C_2 \, \|g\|_{L^p_v(\mathbb{R}^n)} \ \ \|f\|_{L^\gamma_v(\mathbb{R}^n)} \ \ and$$
$$\|Q_-(g,f)\|_{L^p_v(\mathbb{R}^n)} \leq C_3 \, \|f\|_{L^p_v(\mathbb{R}^n)} \, \|g\|_{L^\gamma_v(\mathbb{R}^n)} \ \ ,$$

where $\gamma = n/(n-\lambda)$ *and* $C_i = C(n, \lambda, p, \|b\|_{L^1(S^{n-1})})$ with $i = 1, 2, 3$.

The constants can be explicitly computed and are proportional to

$$C_i \propto \left|S^{n-2}\right| \int_{-1}^{1} \left(\frac{2}{1-s}\right)^{\frac{n-\lambda}{2q}} b(s)(1-s^2)^{\frac{n-3}{2}}\,ds \ \ \text{with}\ \ i = 1,2$$

with parameter $1 < q = q(n, \lambda, p) < \infty$

(*the singularity at $s = 1$ is removed by symmetrazing $b(s)$ when $f = g$*)

Theorem (global regularity near Maxwellian data) *Fix* $0 \le T \le \infty$ *and assume the collision kernel satisfies* $B(u, \hat{u}\cdot\sigma) = |u|^{-\lambda}\ b(\hat{u}\cdot\sigma)$ *with* $\boldsymbol{-1 \le \lambda < n\text{-}1}$ *with the* ***Grad's assumption***.

Also, assume that f_0 *satisfies the smallness assumption or is near to a local Maxwellian.*
In addition, assume that $\nabla f_0 \in L^p(\mathbb{R}^{2n})$ *for some* $1 < p < \infty$.
Then, there is a unique classical solution f *to problem (1) in the interval* $[0, T]$ *satisfying the estimates of these theorems, and*

$$\|\nabla f\|_{L^p(\mathbb{R}^{2n})}(t) \le C\, \|\nabla f_0\|_{L^p(\mathbb{R}^{2n})} \quad \text{for all } t \in [0, T],$$

with constant $C = C(n, p, \lambda, \|b\|_{L^1(S^{n-1})})$.

Proof: set

$$(D_{h,\hat{x}} f)(x) := \frac{f(x + h\hat{x}) - f(x)}{h}, \qquad (\tau_{h,\hat{x}} f)(x) := f(x + h\hat{x}).$$

$$\boxed{p\, |(Df)^{\#}|^{p-1}\, \mathrm{sgn}((Df)^{\#})} \quad : \int \quad \boxed{\frac{d(Df)^{\#}}{dt}(t) = (DQ(f, f))^{\#}(t) = Q^{\#}(Df, f)(t) + Q^{\#}(\tau f, Df)(t).} \quad \Rightarrow$$

$$\frac{d\,\|Df\|_{L^p}^p}{dt} \le p\, C \int_{\mathbb{R}^n} \|Df\|_{L_v^p(\mathbb{R}^n)}^p \left(\|f\|_{L_v^a(\mathbb{R}^n)} + \|\tau f\|_{L_v^a(\mathbb{R}^n)} \right) dx. \quad \text{with} \begin{cases} \|f\|_{L_v^a(\mathbb{R}^n)} \le \dfrac{C}{(1+t)^{n/a}} = \dfrac{C}{(1+t)^{n-\lambda}}, \\ \|\tau f\|_{L_v^a(\mathbb{R}^n)} \le \dfrac{C}{(1+t)^{n-\lambda}}. \end{cases}$$

$\Rightarrow$ By Gronwall inequality

$$\boxed{\|Df\|_{L^p(\mathbb{R}^{2n})}(t) \le \|Df_0\|_{L^p(\mathbb{R}^{2n})} \exp\left(\int_0^t \frac{C}{(1+s)^{n-\lambda}}\, ds \right),}$$

with $\boldsymbol{a = n/(n-\lambda)}$

Velocity regularity

Theorem *Let f be a classical solution in* $[0, T]$ *with* f_0 *satisfying the condition of smallness assumption or is near to a local Maxwellian and* $\nabla_x f_0 \in L^p(R^{2n})$ *for some* $1 < p < \infty$. *In addition assume that* $\nabla_v f_0 \in L^p(\mathrm{R}^{2n})$. *Then, f satisfies the estimate*

$$\|(\nabla_v f)(t)\|_{L^p(\mathbb{R}^{2n})} \leq C\left(\|\nabla_v f_0\|_{L^p(\mathbb{R}^{2n})} + t\,\|\nabla_x f_0\|_{L^p(\mathbb{R}^{2n})}\right),$$

with $C = C(n, p, \lambda, \|b\|_{L^1(S^{n-1})})$ *independent of the time.*

Proof : Take $(D_{h,\hat{v}} f)(v) := \dfrac{f(v + h\hat{v}) - f(v)}{h}$ for a fix $h > 0$ and $\hat{v} \in S^{n-1}$ and the corresp. translation operator and transforming $v_* \to v_* + h\hat{v}$ in the collision operator.

$\boxed{p\,|(Df)|^{p-1}\,\mathrm{sgn}((Df))}$ $: \int$

$$\boxed{\frac{d(Df)}{dt}(t) + v \cdot \nabla(Df)(t) + \hat{v} \cdot \nabla(\tau f)(t) = Q(Df, f)(t) + Q(\tau f, Df)(t).}$$

$$\Longrightarrow \quad \frac{d\,\|Df\|^p_{L^p}}{dt}(t) \leq \frac{p\,C}{(1+t)^{n-\lambda}}\,\|Df\|^p_{L^p(\mathbb{R}^{2n})} + p\,\|Df\|^{p-1}_{L^p(\mathbb{R}^{2n})}\,\|\nabla f\|_{L^p(\mathbb{R}^{2n})}\,.$$

$$\frac{d\,\|Df\|^p_{L^p}}{dt}(t) \le \frac{p\,C}{(1+t)^{n-\lambda}}\,\|Df\|^p_{L^p(\mathbb{R}^{2n})} + p\,\|Df\|^{p-1}_{L^p(\mathbb{R}^{2n})}\,\|\nabla f\|_{L^p(\mathbb{R}^{2n})}\,.$$

Just set

$$X(t) := \|Df\|^p_{L^p(\mathbb{R}^{2n})}(t)$$

then $\boxed{\dfrac{dX(t)}{dt} \le a(t)X(t) + b(t)X^{\frac{p-1}{p}}(t).}$ ***Bernoulli ODE***

with $a(t) = \dfrac{p\,C}{(1+t)^{n-\lambda}}$ and $b(t) = p\,\|(\nabla f)(t)\|^{p-1}_{L^p(\mathbb{R}^{2n})}\,.$

Which is solved by

$$\boxed{X^{\frac{1}{p}}(t) \le X_0^{\frac{1}{p}} \exp\left(\frac{1}{p}\int_0^t a(s)ds\right) + \frac{1}{p}\int_0^t \exp\left(\frac{1}{p}\int_\sigma^t a(s)ds\right) b(\sigma)d\sigma,}$$

Then, by the regularity estimate

$$\|Df\|_{L^p(\mathbb{R}^{2n})}(t) \le \left(\|Df_0\|_{L^p(\mathbb{R}^{2n})} + t\,\|\nabla f_0\|_{L^p(\mathbb{R}^{2n})}\right)\exp\left(\int_0^t \frac{C}{1+s^{n-\lambda}}ds\right).$$

with $0 < \lambda < n-1$

L^p and $M_{\alpha,\beta}$ stability

Set $$\frac{d(f-g)^{\#}}{dt}(t) = Q^{\#}(f,f)(t) - Q^{\#}(g,g)(t) = \frac{1}{2}\left[Q^{\#}(f-g, f+g) - Q^{\#}(f+g, f-g)\right].$$

multiplying by $|(f-g)^{\#}|^{p-1}\operatorname{sgn}((f-g)^{\#})$ with $p > 1$

$$\Longrightarrow \boxed{\frac{d\,\|f-g\|^p_{L^p}}{dt}(t) \le C \int_{\mathbb{R}^n} \|f-g\|^p_{L^p_v(\mathbb{R}^n)}\,\|f+g\|_{L^q_v(\mathbb{R}^n)}\,dx.}$$

Now, since f and g are controlled by traveling Maxwellians one has $\|f+g\|_{L^q_v(\mathbb{R}^n)} \le \dfrac{C}{(1+t)^{n-\lambda}}$.

with $0 < \lambda < n-1$

Theorem *Let f and g distributional solutions of problem associated to the initial datum f_0 and g_0 respectively. Assume that these datum satisfies the condition of theorems for small data or near Maxwellians solutions (**$0 < \lambda < n-1$**). Then, there exist $C > 0$ independent of time such that*

$$\boxed{\|f-g\|_{L^p} \le C\ \|f_0-g_0\|_{L^p} \quad \text{with} \quad 1 < p < \infty.}$$

Our result is for integrable $b(\hat{u}\cdot\sigma)$

Moreover, for f_0 and g_0 sufficiently small in $M_{\alpha,\beta}$

$$\boxed{\left\|(f-g)^{\#}\right\|_{L^\infty(0,T;\mathcal{M}_{\alpha,\beta})} \le C\ \|f_0-g_0\|_{L^\infty(0,T;\mathcal{M}_{\alpha,\beta})}\,.}$$

(For the extension to p=1 and P=∞ see R.Alonso & I.M Gamba [3])

Part III

Some issues of variable hard and soft potential interactions

- Dissipative models for Variable hard potentials with heating sources:

  All moments bounded
  Stretched exponential high energy tails

Spectral - Lagrange solvers for collisional problems

- Deterministic solvers for Dissipative models - The space homogeneous problem
- FFT application - Computations of Self-similar solutions

- Space inhomogeneous problems

Time splitting algorithms
Simulations of boundary value – layers problems
Benchmark simulations

A *general form* *statistical transport* : The space-homogenous BTE with external heating sources Important examples from mathematical physics and social sciences:

$$f_t + v \cdot \nabla_x f \ = Q_{\beta,\gamma,d}(f)(v,t) + \mathcal{G}(f)(v,t)$$

where the interacting integral is written in **weak form** as

$$\int_{v\in\mathbb{R}^d} Q_{\beta,\gamma,d}(f)(\cdot,t)\phi dv = c_d \int_{v,v_*\in\mathbb{R}^{2d};\sigma\in S^{d-1}} f\, f_* \, (\phi(v') - \phi(v)) B_{\beta,\gamma,d}(|u|, \tfrac{u\cdot\sigma}{|u|}) d\sigma dv_* dv$$

The ter $\mathcal{G}(f)(v,t)$

els external heating sources:

- background thermostat (linear collisions),
- thermal bath (diffusion)
- shear flow (friction),
- dynamically scaled long time limits (self-similar solutions).

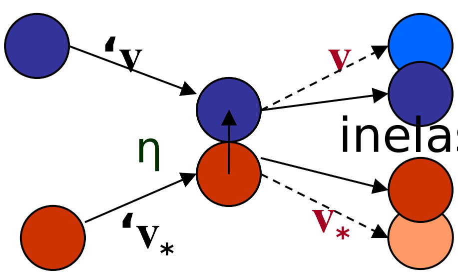
$v' = v + \frac{\beta}{2}(|u|\sigma - u), \qquad v'_* = v_* - \frac{\beta}{2}(|u|\sigma - u)$ interaction law;

$u = v - v_*$ (relative velocity)

$B_{\beta,\gamma,d}(|u|, \sigma(\theta))$ (collisional kernel)

$\cos\theta \ = \ \frac{(u\cdot\sigma)}{|u|}$ cosine of scattering angle,

$\beta = \frac{1+e}{2}, \qquad e =$ restitution coefficient



inelastic collision

$\beta = e = 1$ elastic interaction , $\beta < 1$ dissipative interaction

$J_\beta = \frac{\partial(v,v_*)}{\partial('v,'v_*)}$ post-precollision Jacobian

nelastic Collision $u' = (1-\beta)\, u + \beta\, |u|\, \sigma$, with σ the direction of **elastic post-collisional relative velocity**

Non-Equilibrium Stationary Statistical States

$$(\frac{\partial f}{\partial t}, \varphi)(t) = g(\rho, \theta) \left[\int_{\mathbb{R}^{2d} \times S^{d-1}} f\, f_* [\varphi('v) - \varphi(v)]\, |u|^\gamma\ b_{\gamma,d,\beta}(\frac{u\cdot\sigma}{|u|}) d\sigma dv_* dv \right](t) + (\mathcal{G}(f), \varphi)(t)$$

NESS satisfies : $\boxed{\int_{\mathbb{R}^d} f_\infty(v) \mathcal{M}_\gamma^{-1} dv}$

β	γ	$\mathcal{G}(f)$	$\mathcal{M}_\gamma$ = **NESS** tail asymptotics
$\beta = 1$	$0 \le \gamma \le 1\ (VHP)$	0	$C \exp(-r\,\|v\|^2)$
$\frac{1}{2} \le \beta < 1$	$0 \le \gamma \le 1\ (VHP)$	$\Delta_v f$	$C \exp(-r\|v\|^{\frac{\gamma+2}{2}})$
$\frac{1}{2} \le \beta < 1$	$\gamma = 1\ (HS)$	$\Delta_v f + \tau \nabla \cdot (v f)$	$C \exp(-r\,\|v\|^2)$
$\frac{1}{2} \le \beta < 1$	$\gamma = 1\ (HS)$	$v_2 \frac{\partial f}{\partial v_3}$	at least $C \exp(-r\,\|v\|^1)$
$\frac{1}{2} \le \beta < 1$	$0 < \gamma \le 1\ (VHP)$	$Q(f, M_{aT}) - \mu\ v \cdot \nabla f$ $a = 0$ or 1	$C((1-a) \exp(-r\,\|v\|^\gamma) +$ $+ aC \exp(-r\,\|v\|^2)$
$\frac{1}{2} \le \beta \le 1$	$\gamma = 0$ (MM)	$\theta_b\, Q(f, M_{aT}) - \mu\, v \cdot \nabla f$ $a = 0$ or 1	$(1-a) C (c_1 + c_2 \|v\|^{\mathrm{k}})^{-1} +$ $+ aC \exp(-r\,\|v\|^2)$

for $C = C_{(\gamma,\beta,\theta,d)}$ and $r = r_{(\gamma,\beta,\theta,d)}$. Also C, c_1, c_2 and $\mathbf{k}$ in the last case depend on $\beta, \theta, \theta_b, T, d$

Spectral - Lagrange solvers for collisional problems

Transformations for a efficient Numerical Method

- The difficulty lays in computing the collision integral.
- The crux of the method is the weak form of the collision integral.

Thus for a suitably regular test function $\psi(v)$, the weak form of the collision integral (operator) takes the form (suppressing the time dependence in f):

$$\int_{v\in\mathbb{R}^d} Q(f,f)\psi(v)dv = \int\int\int_{v,v_*,\sigma\in\mathbb{R}^{2d}\times S^{d-1}} ff^* B(|u|,\mu(\sigma))[\psi(v')-\psi(v)]d\sigma dw dv$$

Using $e^{-ik.v}$ for $\psi(v)$ and substituting the definition of v', we get the Fourier transformed collision operator:

$$\hat{Q}(k) = \int\int\int_{v,v_*,\sigma\in\mathbb{R}^{2d}\times S^{d-1}} ff^* B(|u|,\mu(\sigma))[e^{-ik.(v+\frac{\beta}{2}(|u|\sigma-u))} - e^{-ik.v}]d\sigma dv_* dv$$

Substituting the definition of the **Variable Hard Potential (VHP)** collision kernel $B(|u|,\mu(\sigma)) = b_\lambda(\sigma)|u|^\lambda$, get:

$$\widehat{Q}(k) = \int\int\int_{v,v*,\sigma\in\mathbb{R}^{2d}\times S^{d-1}} f f^* b_\lambda(\sigma)|u|^\lambda e^{-ik.v}[e^{-i\frac{\beta}{2}k.(|u|\sigma-u))} - 1]d\sigma dv_* dv$$

With a change of variables $v_* = v - u \Rightarrow dv_* = du$, re-arrangement and re-grouping:

$$\boxed{\widehat{Q}(k) = \int_{y\in\mathbb{R}^d} \widehat{f}(y)\widehat{f}(k-y)\widehat{G}_{\lambda,\beta}(y,k)dy}$$

with $\widehat{G}(y,k) = \mathcal{F}_{u\to y}G(u,k)$ and

$G_{\lambda,\beta}$ depends on the integral of the scattering function b_λ

$$\boxed{G_{\lambda,\beta}(u,k) = \int_{\sigma\in S^{d-1}} b_\lambda(\sigma)|u|^\lambda[e^{-i\frac{\beta}{2}k.(|u|\sigma-u)} - 1]d\sigma}$$

is an operator invariant under rotations in (y,k): **it has an expansion on a basis of d-dimensional spherical harmonics**

d-dimensional Spherical Harmonics: orthogonal set of d-dimensional polynomials that are harmonic functions on the S^{d-1}-sphere

$$\hat{Q}_{\lambda,\beta}[f](k) = \int_{u\in\mathbb{R}^d} \hat{G}_{\lambda,\beta}(y,k)\hat{f}(y)\hat{f}(y-k)dy$$

with either $\boxed{G_{\lambda,\beta}(u,k) = \int_{\sigma\in S^{d-1}} b_\lambda(\sigma)|u|^\lambda[e^{-i\frac{\beta}{2}k.(|u|\sigma-u)} - 1]d\sigma}$ **anisotropic**

or $\boxed{G_{\lambda,\beta}(u,k) = b_\lambda c_d|u|^\lambda\{e^{i\frac{\beta}{2}k.u}\mathrm{sinc}(\frac{\beta|u||k|}{2}) - 1\}}$ **isotropic case.**

To get back the collision integral, one takes the inverse Fourier Transform of $\hat{Q}_{\lambda,\beta}[f](k)$

$$Q_{\lambda,\beta}[f,f](v) = \check{\hat{Q}}_{\lambda,\beta}[f](k) = \int_{k\in\mathbb{R}^d}\{\int_{y\in\mathbb{R}^d} \hat{G}_{\lambda,\beta}(y,k)\hat{f}(y)\hat{f}(y-k)]dy\check{\}}e^{ik\cdot v}$$

- computational cost: FFTW of $f(v)f(v-u)$ for each u, with respect to v; multiplying this result with $G_{\lambda,\beta}(u,k)$ for each u and k. Take Inverse FFTW with respect to k: Total # operations $\boxed{d\,O(N^{2d}\log N) + O(N^{2d})}$.

Collision Integral Algorithm

[1] $(O(N^3 log(N)))$ $\hat{f}(\zeta_{\mathbf{m}}) = \mathrm{FFT}_{\mathbf{v_k}\to\zeta_{\mathbf{m}}}[f(\mathbf{v_k})]$

[2] $(O(N^3))$ For $\zeta_{\mathbf{m}} \in C_u,$ Do

[2.1] $\hat{Q}(\zeta_{\mathbf{m}}) = 0$

[2.2] $(O(N^3))$ For $\xi_{\mathbf{l}} \in C_u,$ Do

[2.2.1] $g(\xi_{\mathbf{l}}) = \hat{f}(\xi_{\mathbf{l}}) \times \hat{f}(\zeta_{\mathbf{m}} - \xi_{\mathbf{l}})$

[2.2.2] $\hat{Q}(\zeta_{\mathbf{m}}) = \hat{Q}(\zeta_{\mathbf{m}}) + \bar{G}_{\mathbf{l},\mathbf{m}} \times \omega[\mathbf{l}] \times g(\xi_{\mathbf{l}})$

[2.2]* End Do

[2]* End Do

[3] $(O(N^3 log(N)))$ $Q(\mathbf{v}_k) = \mathrm{IFFT}_{\zeta_{\mathbf{m}}\to\mathbf{v_k}}[\hat{Q}(\zeta_{\mathbf{m}})]$

- Further reduction can be done by using a classical Carlemann integral representation: *Bobylev, Rjasanov 99, Rjasanov, Ibrahimov 02, Filbet, Mouhot,Pareschi'07 to reduced the number of operations in a factor of* N *harmonic modes* $\times M^{d-1}$ *spherical angular discretizations to* $\boxed{O(M^{d-1}N^d \log N) + O(M^{d-1}N^d)}$ *(Example: isotropic hard sphere for* $d = 3$ *or Maxwell type interactions in for* $d = 2$.*)*

Discrete version of the conservation scheme

$M = N^d =$ the total number of Fourier modes. For elastic collisions, $\mathbf{a} \in I\!R^m$, m=number of conserved moments (*collision invariants*)

$\tilde{\mathbf{Q}} = (\tilde{Q}_1, \tilde{Q}_2, .., ., \tilde{Q}_M)^T$, computed CO $\qquad \mathbf{Q} = (Q_1, Q_2, , ., ., Q_M)^T$ conserved CO

Let ω_j be the integration weights where $j = 1, 2, ..., M$. Define $\mathbf{C} =$'*vector' of moments*, $\mathbf{a} =$ '*vector of conserved quantities*:

$$\mathbf{C}^e_{(m(d)\times M} = \begin{pmatrix} \langle \omega_j \rangle \\ \langle v_j \omega_j \rangle \\ \vdots \\ \langle \times^{m(d)} v_j \, \omega_j \rangle \end{pmatrix} \qquad \text{and} \qquad \mathbf{a}^e_{m(d)\times 1} = \begin{pmatrix} \mathbf{a}_1 \\ \mathbf{a}_2 \\ \vdots \\ \mathbf{a}_m \end{pmatrix}$$

Then, the conservation method can be written as a constrained optimization problem: Find $\mathbf{Q}$ such that

$$(*) \left\{ \min \|\tilde{\mathbf{Q}} - \mathbf{Q}\|_2^2 : \mathbf{C}^e\mathbf{Q} = \mathbf{a}^e; \mathbf{C}^e \in \mathbb{R}^{d+2\times M}, \tilde{\mathbf{Q}} \in \mathbb{R}^M, \mathbf{a}^e \in \mathbb{R}^{d+2} \right\}$$

To solve $(*)$, one can employ the Lagrange multiplier method.

Let $\gamma \in \mathbb{R}^{d+2}$ be the Lagrange multiplier vector. Then the scalar objective function to be optimized is given by

$$\boxed{L(\tilde{\mathbf{Q}}, \gamma) = \sum_{j=1}^{M} |\tilde{Q}_j - Q_j|^2 + \gamma^T(\mathbf{C}^e\mathbf{Q} - \mathbf{a}^e)\,.}$$

Can be solved explicitly for the corrected value and the resulting equation of correction is implemented numerically in the code.

Taking the derivative of $L(\tilde{\mathbf{Q}}, \lambda)$ with respect to $f_j, j = 1, ..., M$, and $\gamma_i, i = 1, ..., m(d)$, i.e., gradients of L, retrieve the constrains by

$$\begin{aligned}
\frac{\partial L}{\partial \tilde{Q}_j} &= 0 \quad j = 1, ..., M\,, \quad \Rightarrow \quad \mathbf{Q} = \tilde{\mathbf{Q}} + \frac{1}{2}(\mathbf{C}^e)^T\gamma \\
\frac{\partial L}{\partial \gamma_1} &= 0; \quad i = 1, ..., d+2\,, \Rightarrow \quad \mathbf{C}^e\mathbf{Q} = \mathbf{a}^e\,,
\end{aligned}$$

and solve for γ,

$$\mathbf{C}^e(\mathbf{C}^e)^T\gamma = 2(a^e - \mathbf{C}^e\tilde{Q})\,.$$

Now $\mathbf{C}^e(\mathbf{C}^e)^T$ is symmetric and positive definite so its inverse exists $\Rightarrow$

$$\boxed{\gamma = 2(\mathbf{C}^e(\mathbf{C}^e)^T)^{-1}(a^e - \mathbf{C}^e\tilde{\mathbf{Q}})\,.}$$

Substituting γ into (3), and since $\mathbf{a}^e = \mathbf{0}$ (*collision invariants,*)

$$\begin{aligned} \mathbf{Q} &= \tilde{\mathbf{Q}} + (\mathbf{C}^e)^T(\mathbf{C}^e(\mathbf{C}^e)^T)^{-1}(a^e - \mathbf{C}^e\tilde{\mathbf{Q}}) \\ &= \left[\mathbb{I}_M - (\mathbf{C}^e)^T(\mathbf{C}^e(\mathbf{C}^e)^T)^{-1}\mathbf{C}^e\right]\tilde{\mathbf{Q}} \\ &= \Lambda_M(\mathbf{C}^e)\tilde{\mathbf{Q}}\,, \end{aligned}$$

Discrete Conservation operator

$\Rightarrow$ Define $\Lambda_M(\mathbf{C}^e) : \mathbb{I}_M - (\mathbf{C}^e)^T(\mathbf{C}^e(\mathbf{C}^e)^T)^{-1}\mathbf{C}^e$ Then this procedure is

$$\text{Conserve}(\tilde{\mathbf{Q}}) = \mathbf{Q} = \Lambda_M(\mathbf{C}^e)\tilde{\mathbf{Q}}\,.$$

Then, for $D_t\mathbf{f}$ any **order time discretization** of $\frac{\partial f}{\partial t} \Rightarrow$

$$D_t\mathbf{f} = \Lambda_M(\mathbf{C}^e)\tilde{\mathbf{Q}}\,;$$ ***'conserve'* algorithm**

This identity summarizes the whole conservation process:

- Required observables are conserved
- The approximate solution to the *elastic homogeneous BE* approaches a stationary state, since

$$\lim_{n\to\infty} \|\Lambda_M(C)\,\tilde{\mathbf{Q}}(f_j^n, f_j^n)\|_\infty = 0$$

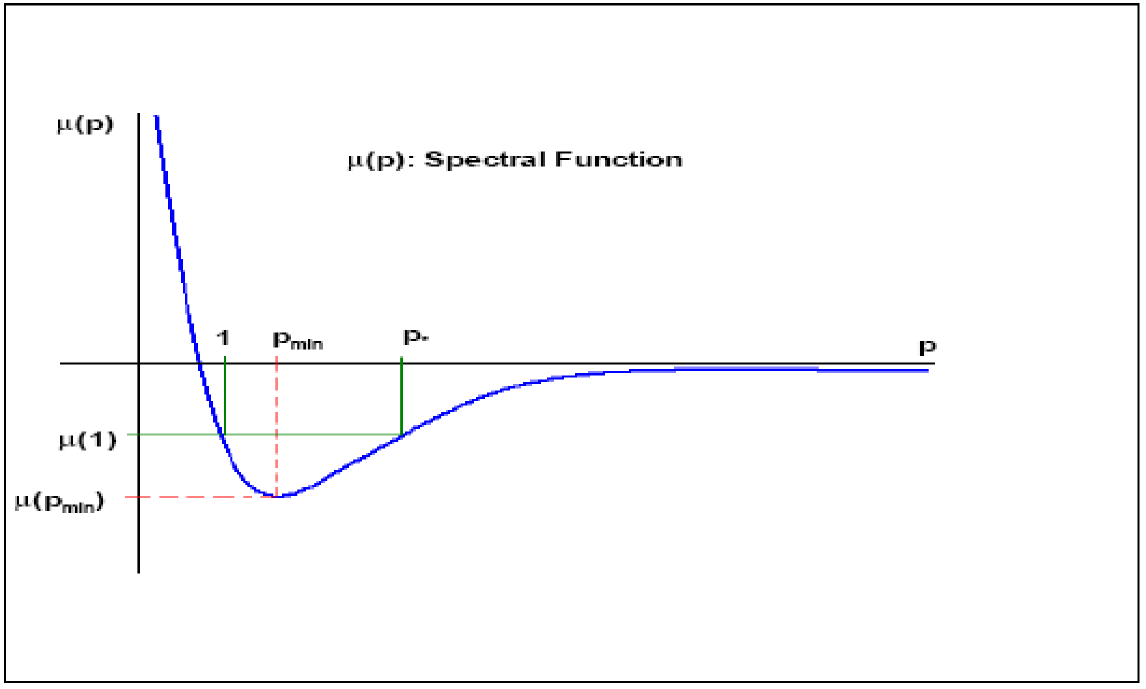
Stabilization property

Self-similar solutions and Power-like Tails

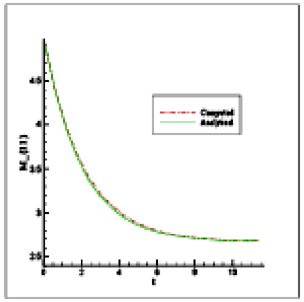
Theorem: *(Bobylev, Cercignani, I.M.G,06)* The self-similar asymptotic function $F_{\mu(p)}(|v|)$ does NOT have finite moments of all orders if the energy dissipates, i.e. $\mu(1) < 0$.

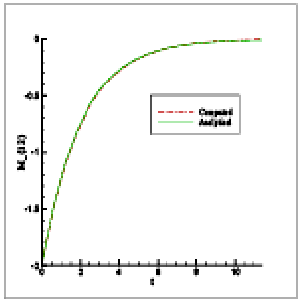
If $0 \le p \le 1$ then, $m_q = \int_{\mathbb{R}^3} F_{\mu(p)}(|v|)|v|^q dv \le \infty; \quad 0 \le q \le p$

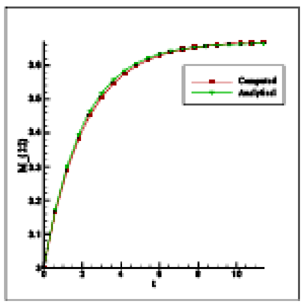
If $p = 1$ (finite initial energy) then, $m_q \le \infty$ only for $0 \le q \le p_*$, where $p_* \ge 1$ is the unique maximal root of the equation $\mu(p_*) = \mu(1)$.

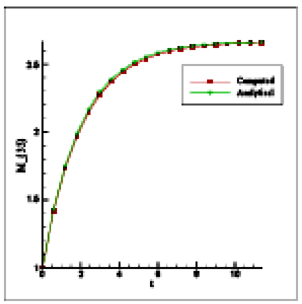


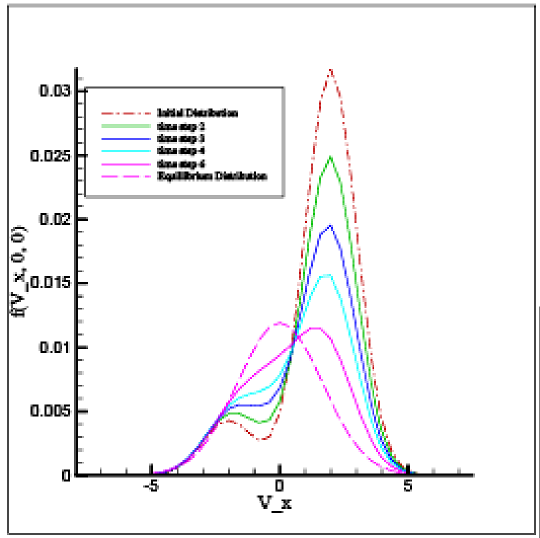
Testing - Maxwell Elastic Collisions

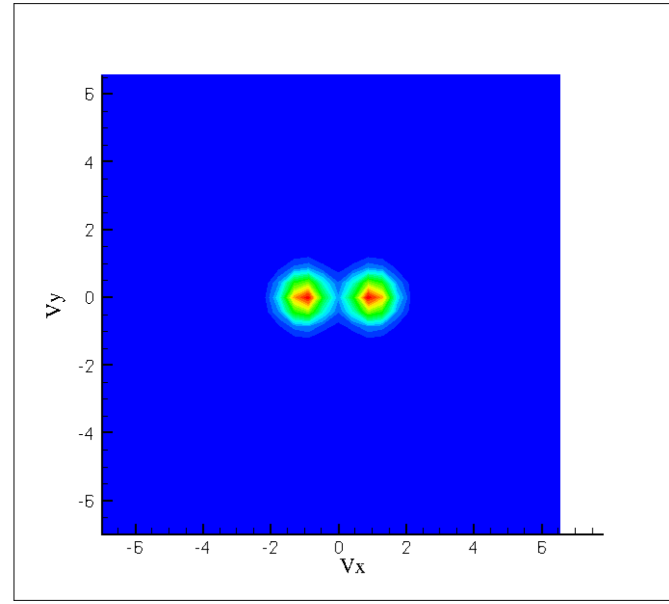












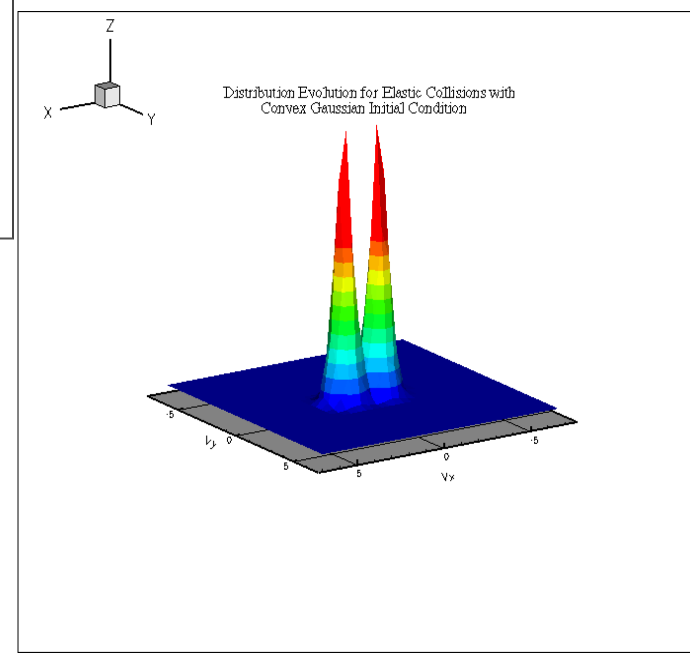
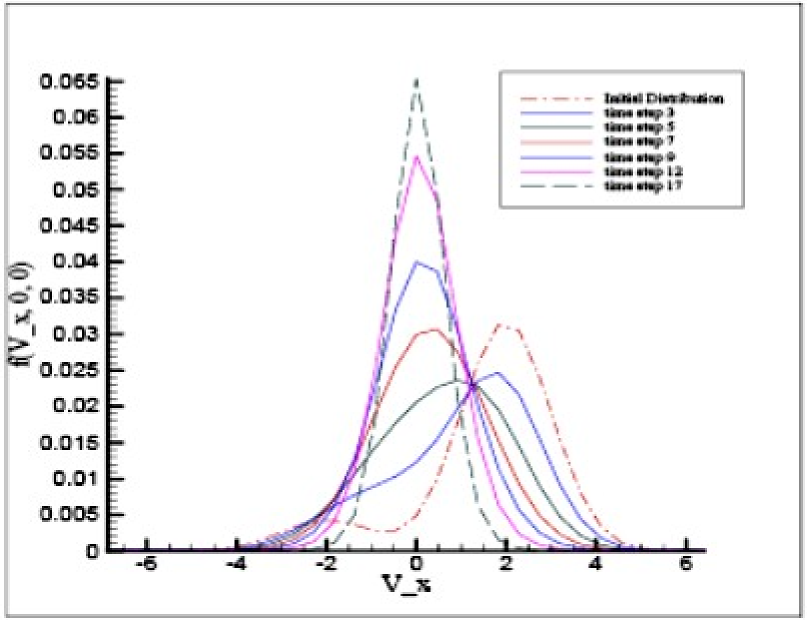


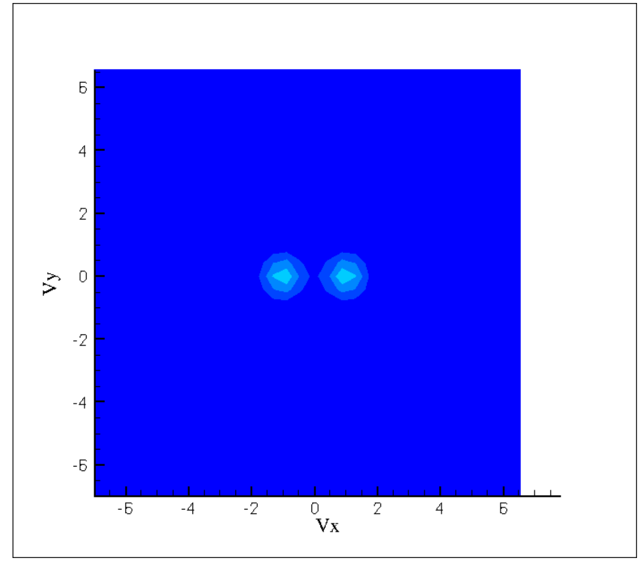
Figure 1: Left four graphs: Momentum Flow -Right graph: **pdf** evolution

t_r = reference time = mft

Δt= 0.25 mft.

Testing - Maxwell Inelastic Collisions





$$K'(t) = \beta(1-\beta)(\frac{|V|^2}{2} - K(t)) \Rightarrow K(t) = K(0)e^{-\beta(1-\beta)t} + \frac{|V|^2}{2}(1 - e^{-\beta(1-\beta)t})$$

where $K(0)$ is Kinetic Energy at time $t = 0$ and V - momentum (constant) of the distribution function.

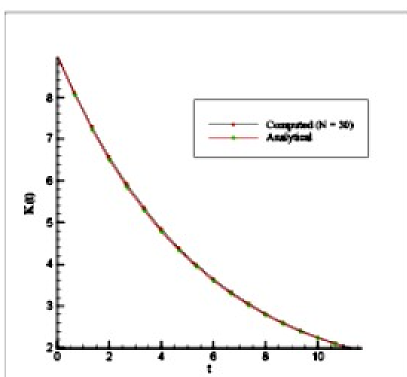
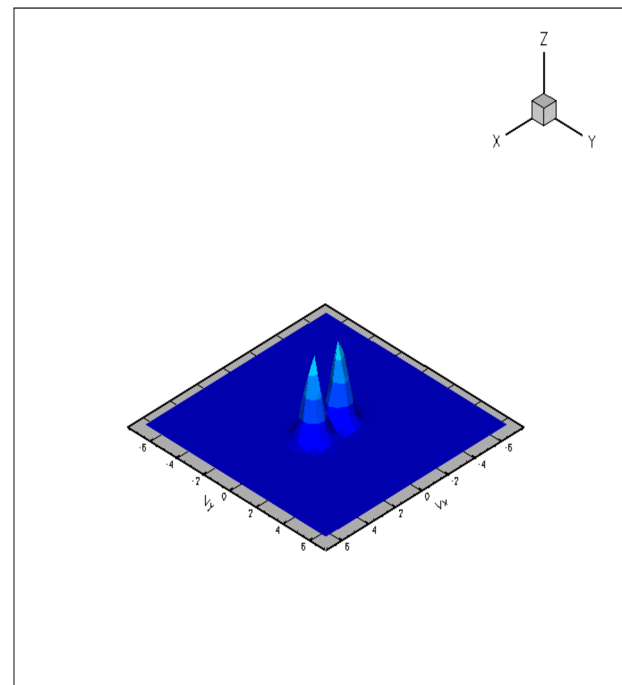


Figure 2: Kinetic Energy for N = 30

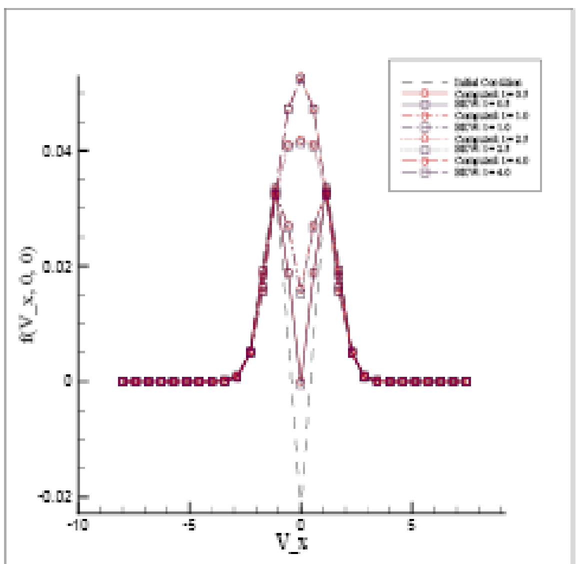


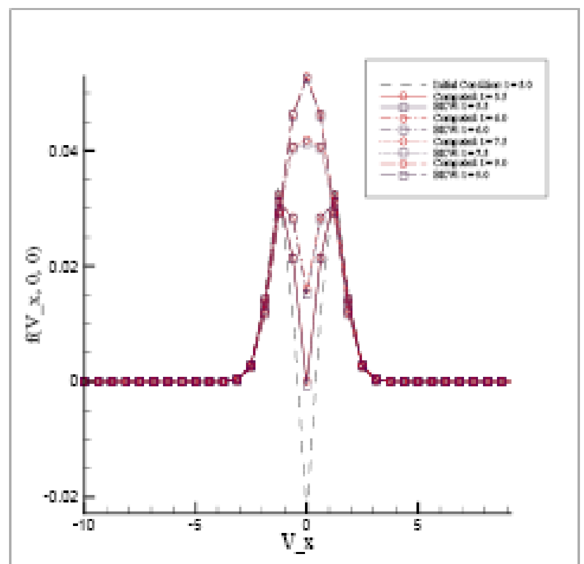
Testing - BKW solutions

A explicit solution was derived by Bobylev-Krook-Wu ’70s (BKW solutions of elastic BTE of Maxwell type) which is given by *convolutions of solutions by self-similar transformations and Maxwellians*. This solutions converges to the Maxwellian equilibrium state.

$$f(v,t) = \frac{e^{-|v|^2/(2K(t)\eta^2)}}{2(2\pi K(t)\eta^2)^{3/2}}\left(\frac{5K(t)-3}{K(t)} + \frac{1-K(t)}{K^2(t)}\frac{|v|^2}{\eta^2}\right)$$

where $K(t) = 1 - e^{-t/6}$ and $\eta = \int |v|^2 f_0(v)dv$ initial distribution temperature.
Therefore for $K \geqslant \frac{3}{5}$ or $t \geqslant t_0 \equiv 6\ln(\frac{5}{2}) \sim 5.498,\ f$ is non-negative.
Setting the initial distribution function to be the BKW solution, the numerical approximation to the BKW solution and the exact solution are plotted for different values of N





Self - Similar Asymptotics

Example: Description of the Weakly Coupled Binary Mixture Problem *(Bobylev, I.M.G. JSP '06)*
Construction of **explicit solutions** to:

$$\frac{\partial f(v,t)}{\partial t} = \int_{w\in\mathbb{R}^3}\int_{\sigma\in S^2} B(|u|,\mu)[f('v,t)f('w,t) - f(v,t)f(w,t)]d\sigma dw$$

$$+ \quad \theta_b \int_{w\in\mathbb{R}^3}\int_{\sigma\in S^2} B(|u|,\mu)[f('v,t)M_T('w) - f(v,t)M_T(w)]d\sigma dw$$

with $M_T(v) = \frac{e^{\frac{-|v|^2}{(2T)}}}{(2\pi T)^{3/2}}$, $B(|u|,\mu) = C_\lambda = \frac{1}{4\pi}$, $\beta = 1.0$, θ_b - depending on the asymptotics and T being the background temperature.

- A system of two different particles with the same mass is considered. One set of particles is assumed to be at equilibrium i.e., with a Maxwellian distribution with temperature $T(t)$.
- Second set of particles is assumed to collide with themselves (first integral) and the background particles(Linear Boltzmann Collision Integral).

The collisions are assumed to be ***locally* elastic** i.e., $|v|^2 + |v_*|^2 = |v'|^2 + |v'_*|^2$ but the above form leads to ***global*** energy dissipation i.e., $\int_{\mathbb{R}^3} |v|^2 f(v,t)dv \neq 0$.

Self - Similar Asymptotics elastic BTE with thermostat

- For self similar asymptotics we study $t \to \infty$ so $\hat{T} \to T$ in $f_T^{ss}(v,t)$ (i.e. the particle distribution temperature approaches the background temperature as expected due to the linear coll. op.)

- Interesting NESS behavior can be observed if $T \to 0$: Set $\hat{T} = s^2 e^{\frac{-2t}{3}}$ so $f_0^{ss}(|v|)$ is explicit.

- Then $f(|v|e^{-t/3}, t) \to_{t\to\infty} e^t f_0^{ss}(|v|)$ where $f_0^{ss}(|v|) = \frac{4}{\pi}\int_0^\infty \frac{e^{-|v|^2/(2s^2)}}{(2\pi s^2)(1+s^2)^2} ds$

- $\boxed{f_0^{ss}(|v|) = O(\frac{1}{|v|^6}) \ \text{ as } \ |v| \to \infty,}$ and
  $\boxed{f_0^{ss}(|v|) = O(\frac{1}{|v|^2}) \ \text{ as } |v| \to 0}$

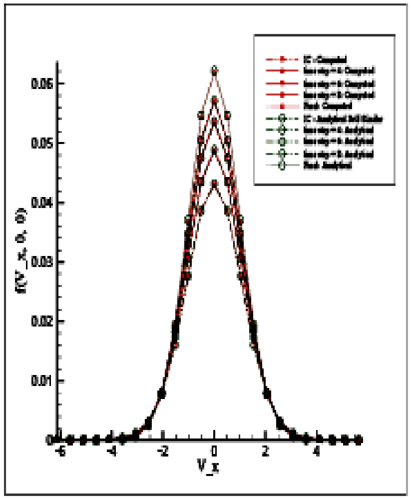
Soft condensed matter phenomena

Remark: **The numerical algorithm is based on the evolution of the continuous spectrum of the solution as in Greengard-Lin'00 spectral calculation of the free space heat kernel, i.e. self-similar of the heat equation in all space.**

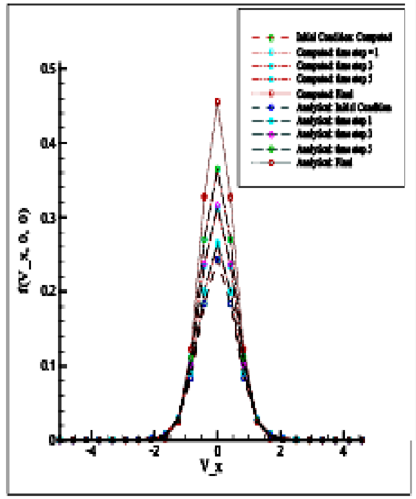
Evolution of the Particle Density $(N = 32, T = 0.25e^{-2t/3})$

Testing - Mixture Problem

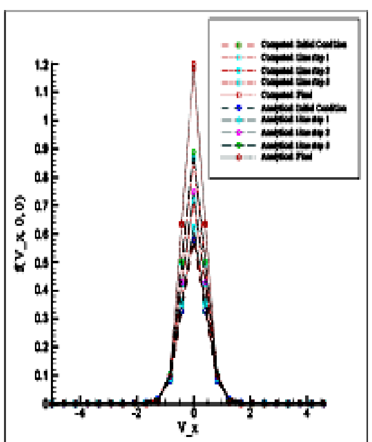
Computed Vs. Analytical Distribution:



$(N = 24, T = 1)$

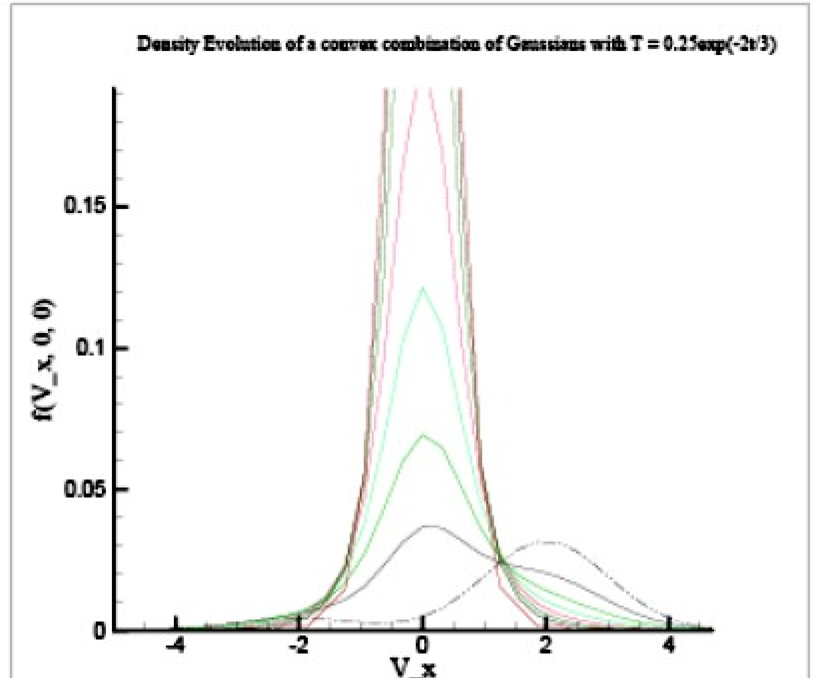


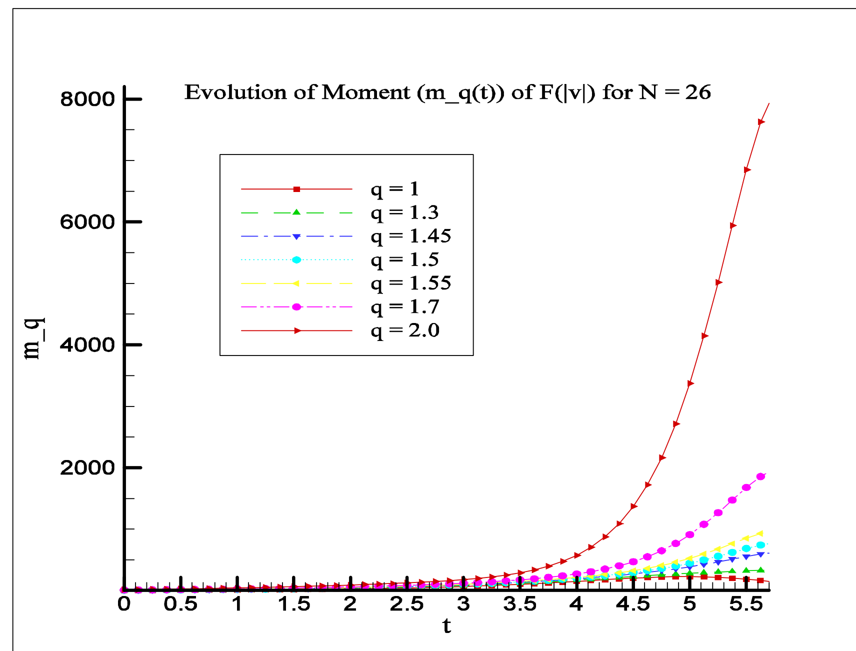
$(N = 24, T = 0.25)$



$(N = 24, T = 0.125)$

Setting $\hat{T} = e^{-\frac{2}{3}t}(\frac{1}{4} + s^2)$





Convergence: spectral accuracy and consistency *with H. Tharkabhushanam*

- Set of trigonometric polynomials
  $\mathbb{P}^N = span\{e^{i\zeta_k \cdot v} | -L_\zeta \le \zeta_k^l < L_\zeta, l = 1,2,3; -N/2 \le k < N/2\}$
- Let $\Pi : L_2(\Omega_v) \to \mathbb{P}^N$ to be the orthogonal projection operator upon $\mathbb{P}^N$ in the $L_2(\Omega_v)$ inner product such that $\boxed{f^\Pi(v) = \sum_k \hat{f}_N(\zeta_k) e^{i\zeta_k \cdot v}}$ with $\sum_{k=-N/2}^{N/2+1} = \sum_k$
- $Q(f^\Pi)$ - Classic collision integral evaluated with the truncated Fourier series of $f(v)$
- $Q^\Pi(f^\Pi) = \Pi Q(f^\Pi)$ - Projection of $Q(f^\Pi) = \sum_k \hat{Q}(\zeta_k) e^{i\zeta_k \dot{v}}$
- $Q_C^\Pi(f^\Pi)$ - Computed conserved form of $Q^\Pi(f^\Pi)$.

Optimization problem:

minimize $A(q_c) = \int_{\Omega_L} |Q^\Pi(f^\Pi) - Q_C^\Pi(f^\Pi)|^2 dv$ subject to $\int_{\Omega_L} Q_C^\Pi(f^\Pi) \begin{pmatrix} 1 \\ v_i \\ |v|^2 \end{pmatrix} dv = 0$

Lemma: *(Conservation Method - An Extended Isoperimetric problem)* ***Isomoment estimates***

$$A^e(q_c) = \|q_u - q_c\|^2_{L^2(\Omega_v)} = 2L^3\gamma_1^2 + \frac{2L^5(\gamma_2^2 + \gamma_3^2 + \gamma_4^2)}{3} + \gamma_5^2 \frac{38}{15} L^7 + \gamma_1\gamma_5 4L^5\,,$$

where γ_j, for $j = 1, \ldots, 5$, are Lagrange multipliers associated with the elastic optimization problem

spectral accuracy

- *Conservation Correction Estimate for collisional operators*: The accuracy of the the conservation scheme is inversely proportional to the size of the velocity domain L and the number of discretizations N as

$$A(q_c) = \|Q_C^{\Pi}(f^{\Pi}, f^{\Pi}) - Q^{\Pi}(f^{\Pi}, f^{\Pi})\|_{L_2(\Omega_L)} \leq C^{\mu,e}\|Q(f,f) - Q^{\Pi}(f^{\Pi}, f^{\Pi})\|_{L_2(\Omega_L)}$$

- **Recall** *Fourier Approximation Estimate*: Let $u \in H_0^{\alpha}(\Omega_L) \cap \mathcal{S}(\Omega_L)$, $u_N = \Pi u = \sum_k \hat{u}_N(\zeta_k) e^{i\zeta_k \cdot v}$ and $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ be a multi index. Then,

$$\|u - u_N\|_{L_2(\Omega_v)} \leq \frac{C}{N^{|\alpha|}} \|u\|_{H_0^{\alpha}}.$$

Shannon Sampling theorem

- The collisional operator satisfies, for f and g in $H_k^{\alpha}(\Omega_L)$ for all k,

$$\|Q(f,g)\|_{H^{\alpha}(\Omega_L)} \leq C_{\lambda}\|f\|_{H_{\lambda}^{\alpha}(\Omega_L)}\|g\|_{H_{\lambda}^{\alpha}(\Omega_L)}$$

Panferov, Villani, I.M.G., CMP'04.

- *Convergence Estimate*: For $f, f^{\Pi} \in H_k^{\alpha}(\Omega_L)$ for all k,

$$\|Q(f,f) - Q_C^{\Pi}(f^{\Pi}, f^{\Pi})\|_{H^{\alpha}(\Omega_L)} \leq C\frac{\bar{C}}{N^{|\alpha|}}\|f\|^2_{H_{\lambda}^{\alpha}(\Omega_L)}.$$

- Solutions of the space homogeneous problem is in the Schwarz class $\mathcal{S}(\mathbb{R}^3)$

Space inhomogeneous simulations

$$\frac{\partial f}{\partial t} + v_1 \frac{\partial f}{\partial x} = Q(f,f)$$

mean free time := the average time between collisions

mean free path := average speed **x** **mft** (average distance traveled between collisions)

→ Set the scaled equation for 1= Kn := **mfp**/geometry of length scale

Spectral-Lagrangian methods in 3D-velocity space and 1D physical space discretization in the **simplest setting**:

Finite difference scheme with splitting into a convective and a collision step:

Define $CFL := \Delta t \frac{v^j}{\Delta x}$ and $t^n = n\Delta t$ $\Rightarrow$ set $f(x^k, v^j, t^n) := f^n_{k,j}$

– Convective Step Space discretization of $O(\Delta x)$: $\frac{\partial f}{\partial t} + v_1 \frac{\partial f}{\partial x} = 0$, $f(x,v,0) = f^n_{k,j}$

$$\tilde{f}_{j,k} = \begin{cases} (1 - CFL) * f^n_k + CFL * f^n_{k-1,j} & \text{if } v_1 > 0 \\ (1 + CFL) * f^n_k - CFL * f^n_{k+1,j} & \text{if } v_1 < 0 \end{cases}$$

– Collision Step Time discretization of $O(\Delta t)$ - first forward Euler (or second order Runge Kutta)

on the "conserve" algorithm for: $\frac{\partial f}{\partial t} = Q(f,f)$, $f(x,v,0) = \tilde{f}_{\cdot,j}$, uniformly in x

$$\tilde{Q}_n = \text{Conserve}(Q(f_n, f_n)), \quad \Rightarrow \quad f_{n+1/2}(x,v) = f_n(x,v) + \frac{dt}{2}\tilde{Q}_n,$$

$$Q_n = \text{Conserve}(Q(f_{n+1/2}, f_{n+1/2})), \quad \Rightarrow \quad \boxed{f_{n+1}(x,v) = f_n(x,v) + dtQ_n.}$$

Spatial mesh size **Δx** = **0.01** mfp   Time step Δt = r **mft ,**

N= Number of Fourier modes in each **j**-direction in

mft= reference time

Resolution of discontinuity 'near the wall' for diffusive boundary conditions:

(K.Aoki, Y. Sone, K. Nijino, H. Sugimoto, 1991)

Sudden heating: *Constant moments initial state with a discontinuous **pdf** at the boundary wall, with wall kinetic temperature increased by **twice** its magnitude*:

Initial state $f_0(x,v) = \frac{1}{(\pi T(x))^{3/2}} e^{-\frac{|v|^2}{T(x)}}$ with $T(0) = T_0$ and $T(x) = 2T_0$ for $x > 0$

Boundary Conditions for sudden heating:

$$f(0,v,t) = \frac{\sigma_w}{(\pi T_w)^{3/2}} e^{-\frac{|v|^2}{T_w}} \quad \text{with} \quad \sigma_w = \left(\frac{8\pi}{T_w}\right)^{3/2} \int_{v_1>0} v_1 f(0,v,t)\,dv$$

with $T_w(0,0) = T_0$ and $T_w(0,t) = 2T_0$ for $t > 0$

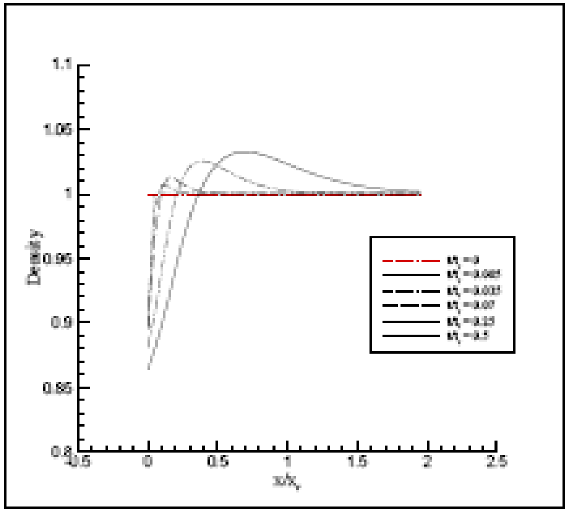
Calculations in the next two pages:

Mean free path ***l_0*** *= 1.*

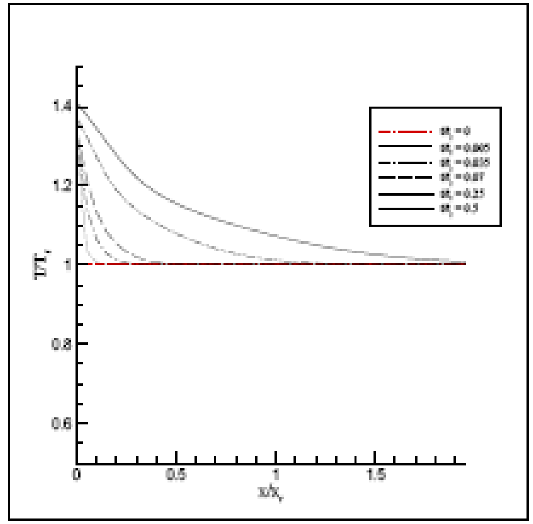
Number of Fourier modes $N = 24^3$,

Spatial mesh size $\Delta x = 0.01$ ***l_0****.*

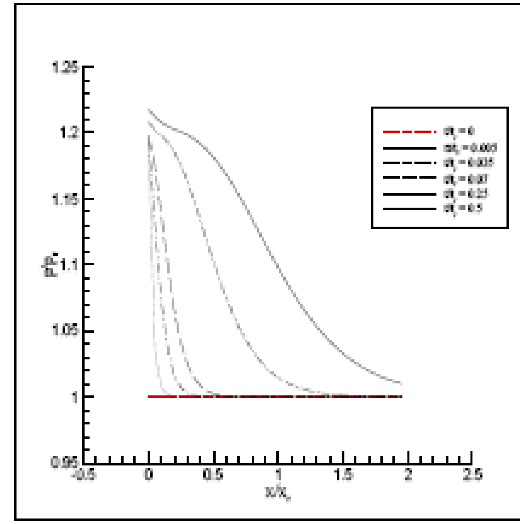
Time step $\Delta t = r$ ***mft***



Density Profile

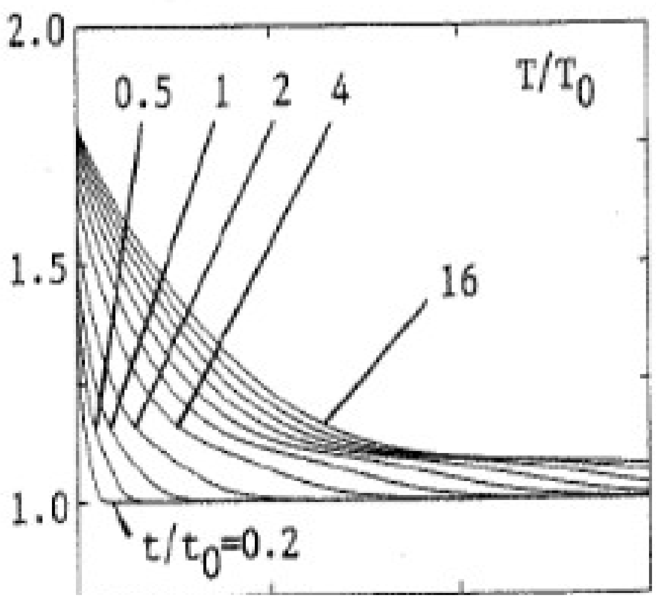


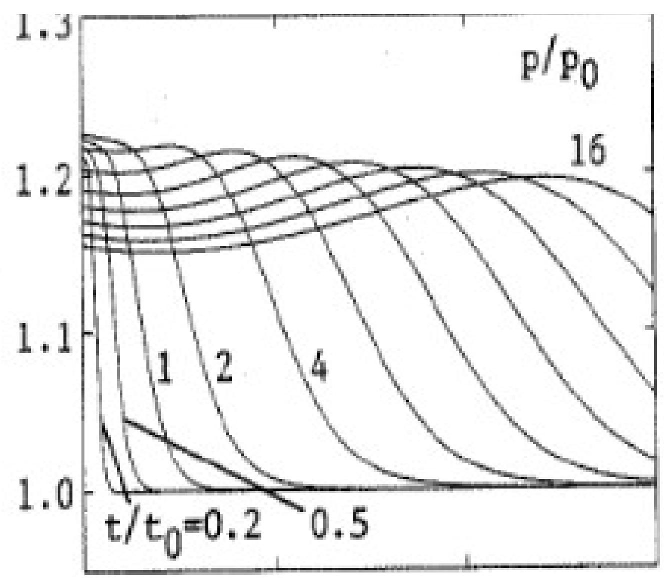
Temperature Profile



Pressure Profile

Formation of a shock wave by an initial sudden change of wall temperature from T_0 to $2T_0$.

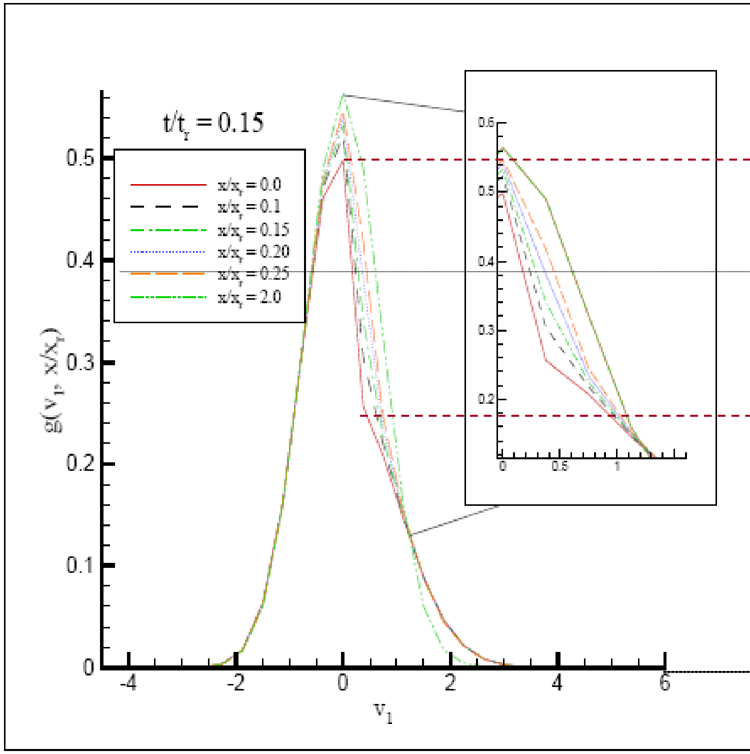




heating problem (BGK eq. with lattice Boltzmann solvers)

. Sone, K. Nijino, H. Sugimoto, 1991

sons with K.Aoki, Y. Sone, K. Nijino, H. Sugimoto, 1991 (Lattice Boltzma



Jump in po

Marginal Distribution at $t = 0.15t_r$ for $N = 16$.

Sudden heating problem

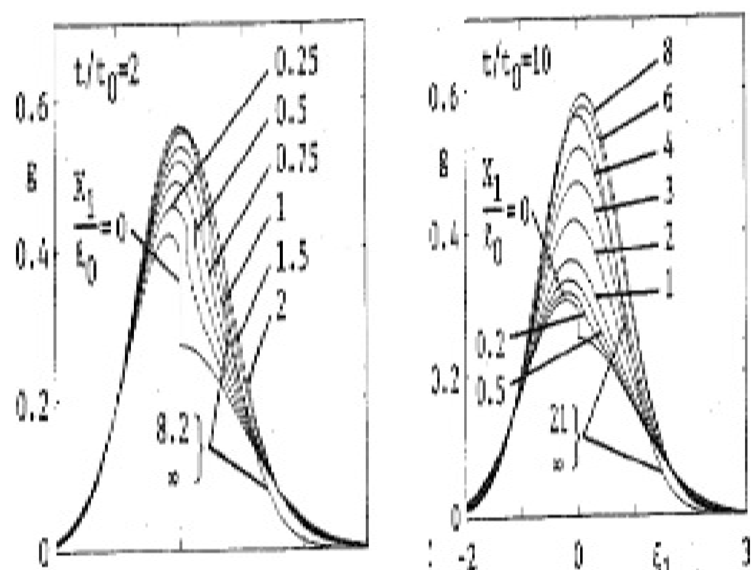
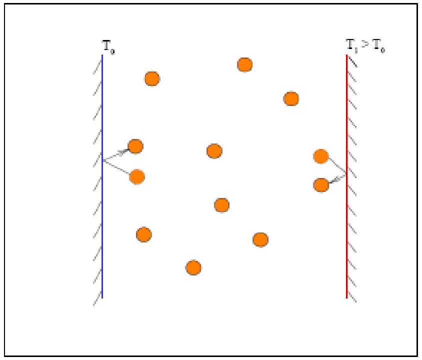


Fig. 5. The reduced velocity distribution function g for $T_1/T_0 = 2$.

Heat transfer problem:

Initial state $f_0(x,v) = \frac{1}{(\pi T(x))^{3/2}} e^{-\frac{|v|^2}{T(x)}}$ with $T(0) = T_0$ and $T(x) = 2T_0$ for $x > 0$

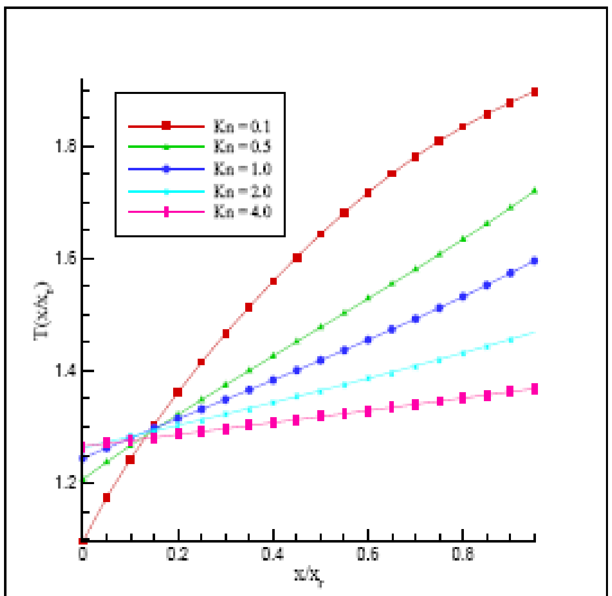


Diffusive boundary conditions

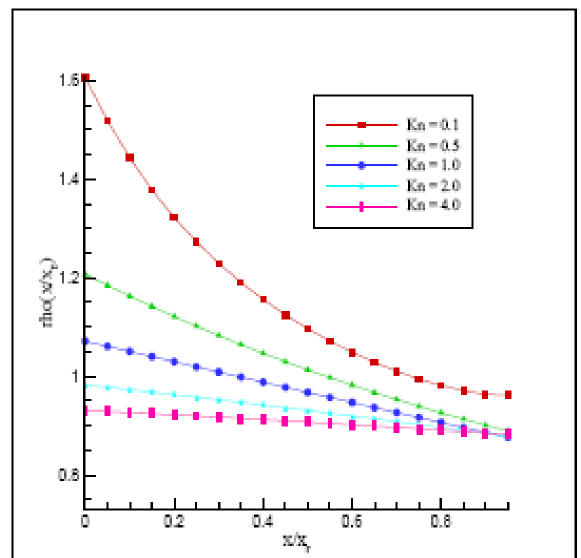
$$f(0,v,t) = \frac{\sigma_w}{(\pi T_w)^{3/2}} e^{-\frac{|v|^2}{T_w}} \qquad \text{with} \quad \sigma_w = (\frac{8\pi}{T_w})^{3/2} \int_{v_1>0} v_1 f(0,v,t) dv$$

Temperature: T_0 given at x_o=0
and $T_1 = 2T_0$ at $x_1 = 1$.

Knudsen Kn = 0.1, 0.5, 1, 2, 4



Stationary Temperature Profile for increasing Knudsen number values.



Stationary Density Profile for increasing Knudsen number values.

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