- Advection equation

$$\partial_t f - \boldsymbol{E} \cdot \nabla_{\boldsymbol{v}} f = 0$$

- In spherical coordinates

$$\nabla_{\boldsymbol{v}} = \hat{\boldsymbol{v}}_r \frac{\partial}{\partial v} + \hat{\boldsymbol{v}}_\theta \frac{1}{v} \frac{\partial}{\partial v_\theta} + \hat{\boldsymbol{v}}_\varphi \frac{1}{v \sin(v_\theta)} \frac{\partial}{\partial v_\varphi}$$

$$\boldsymbol{E} = E \hat{\boldsymbol{z}} = E \left(\cos(v_\theta) \hat{\boldsymbol{v}}_r - \sin(v_\theta) \hat{\boldsymbol{v}}_\theta \right)$$

$$\boldsymbol{E} \cdot \nabla_{\boldsymbol{v}} f = E \left(\cos(v_\theta) \frac{\partial f}{\partial v} - \sin(v_\theta) \frac{1}{v} \frac{\partial f}{\partial v_\theta} \right)$$

- Expansion in terms of Maxwell polynomials and spherical harmonics

$$f = A \exp\left(- \left(\frac{v}{v_{\mathrm{th}}} \right)^2 \right) \sum_{klm} h_{klm} P_k \left(\tfrac{v}{v_{\mathrm{th}}} \right) Y_{lm}(v_\theta, v_\varphi)$$

- Projection:

$$\int_{\mathbb{R}^3} \left(\partial_t f - \boldsymbol{E} \cdot \nabla_{\boldsymbol{v}} f = 0 \right) P_p \left(\tfrac{v}{v_{\mathrm{th}}} \right) Y_{qs}(v_\theta, v_\varphi) \, \mathrm{d}\boldsymbol{v} \, \mathrm{d}v_\omega$$

- Resulting system of ODEs:

$$\partial_t h_{pqs}(t) = \sum_{k,l,m} \epsilon_{pqsklm} h_{klm}(t)$$

- Discretized operator

$$\epsilon_{pqsklm} = \left(d_{pk} - \frac{1}{2} l_{pk}\right) \Psi_{lmqs} + \frac{1}{2} \sum_{j=0}^{N_r} \left(d_{jk} - \frac{1}{2} l_{jk}\right) \left(d_{pj} - \frac{1}{2} l_{pj} + d_{jp}\right) \Phi_{lmqs}$$

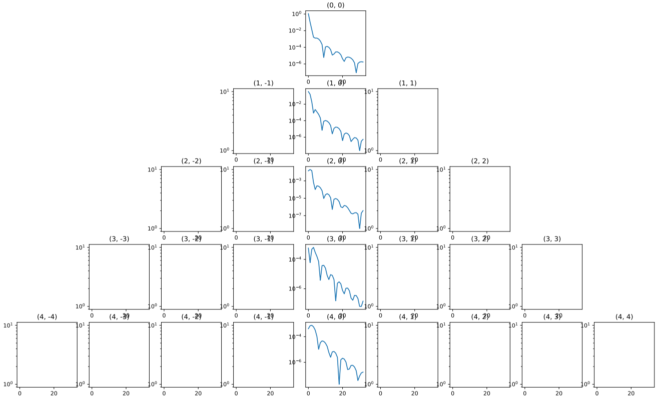
$$d_{jk} : \quad P_k'(x) = \sum_{j=0}^{N_r} d_{jk} P_j(x)$$

$$l_{jk} : \quad x P_k(x) = \sum_{j=0}^{N_r} l_{jk} P_k(x)$$

$$\Psi_{lmqs} = \int \cos(v_\theta) Y_{lm}(v_\theta, v_\varphi) Y_{qs}(v_\theta, v_\varphi) \, \mathrm{d}v_\omega$$

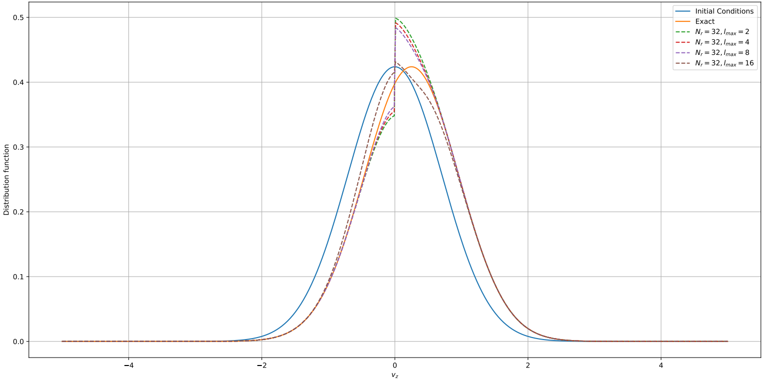
$$\Phi_{lmqs} = \int \sin(v_\theta) Y_{qs}(v_\theta, v_\varphi) \frac{\partial}{\partial v_\theta} Y_{lm}(v_\theta, v_\varphi) \, \mathrm{d}v_\omega$$

Sanity check: solution stays axisymmetric



Convergence test: discontinuities at $\boldsymbol{v} = 0$

- $\boldsymbol{E} = \text{const} \quad \rightarrow \quad f(t, \boldsymbol{v}) = f(0, \boldsymbol{v} + \boldsymbol{E}t)$
- $N_r = 32$ is fixed, $l_{\max}$ is increasing



Convergence test: discontinuities at $\boldsymbol{v} = 0$

- $\boldsymbol{E} = \text{const} \rightarrow f(t, \boldsymbol{v}) = f(0, \boldsymbol{v} + \boldsymbol{E}t)$
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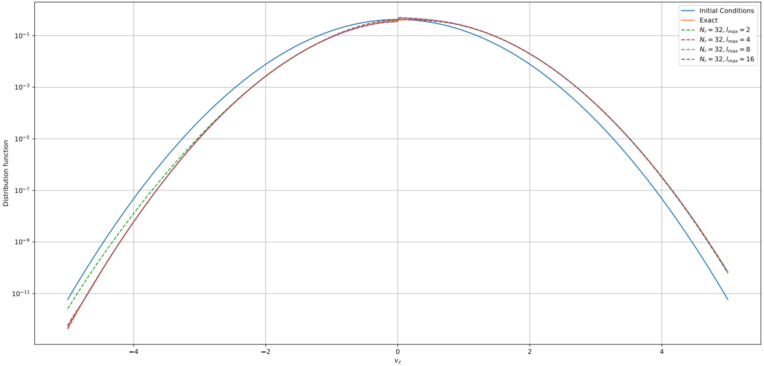
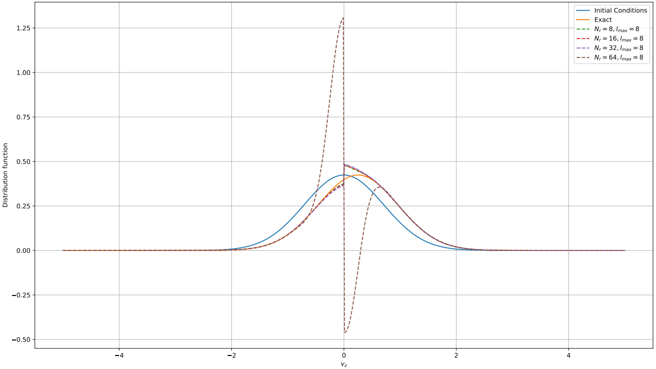


Convergence test: discontinuities at $\boldsymbol{v} = 0$

- $\boldsymbol{E} = \text{const} \quad \rightarrow \quad f(t, \boldsymbol{v}) = f(0, \boldsymbol{v} + \boldsymbol{E}t)$
- N_r is increasing, $l_{\max} = 8$ is fixed



- Kinda makes sense: generally singular at $\boldsymbol{v} = 0$

$$f = A \exp\left(- \left(\frac{v}{v_{\mathrm{th}}} \right)^2 \right) \sum_{lm} \left(\sum_k h_{klm} P_k\left(\frac{v}{v_{\mathrm{th}}} \right) \right) Y_{lm}(v_\theta, v_\varphi)$$

- Continuous if

$$\left(\sum_k h_{klm} P_k(0) \right) = 0$$

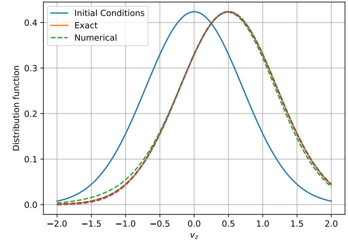
- Possible fix (Following Gamba&Rjasanow 2018):

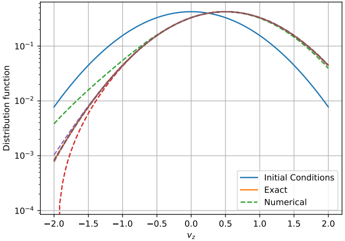
$$f = A \exp\left(- \left(\frac{v}{v_{\mathrm{th}}} \right)^2 \right) \sum_{lm} \left(\sum_k h_{klm} \left(\frac{v}{v_{\mathrm{th}}} \right)^l P_{kl} \left(\frac{v}{v_{\mathrm{th}}} \right) \right) Y_{lm}(v_\theta, v_\varphi)$$

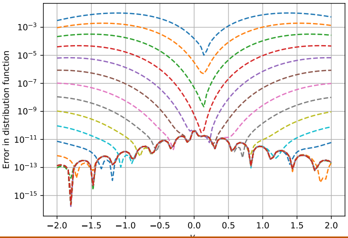
- where (associated Maxwell polynomials?)

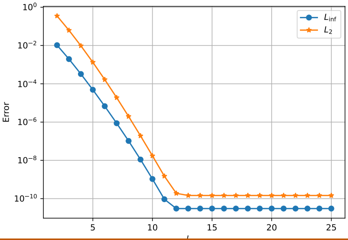
$$\int_0^\infty P_{kl}(x) P_{jl}(x) x^{2l+2} e^{-x^2} \, \mathrm{d}x = \delta_{k,j}$$

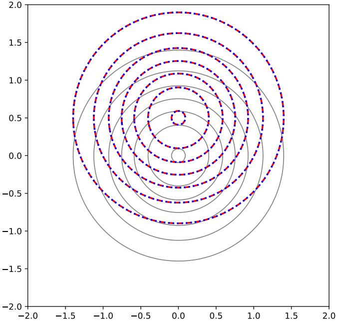
Convergence test ($N_r = 16$ is fixed, $l_{\max}$ is increasing)



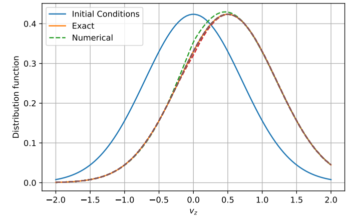


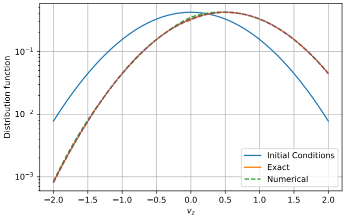


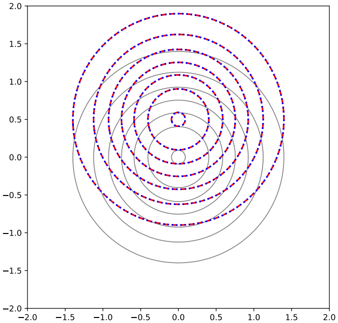


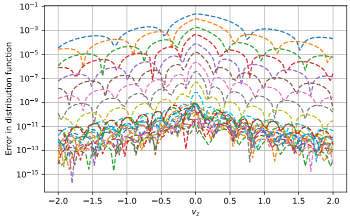


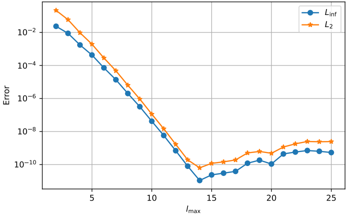
Convergence test (N_r is increasing, $l_{\max} = 16$ is fixed)



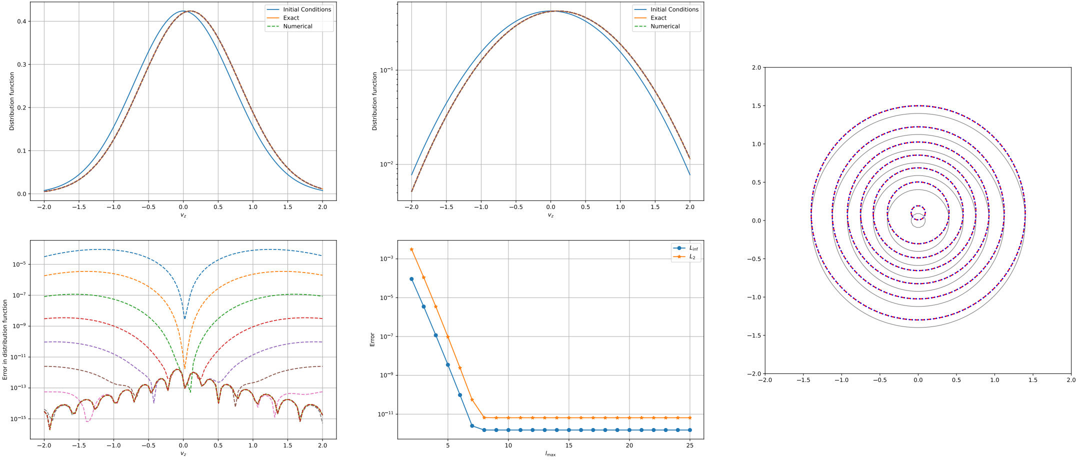




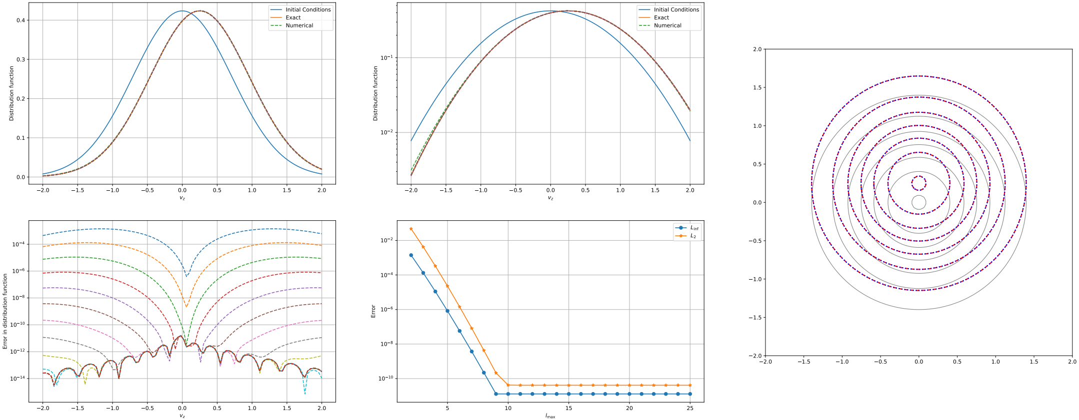




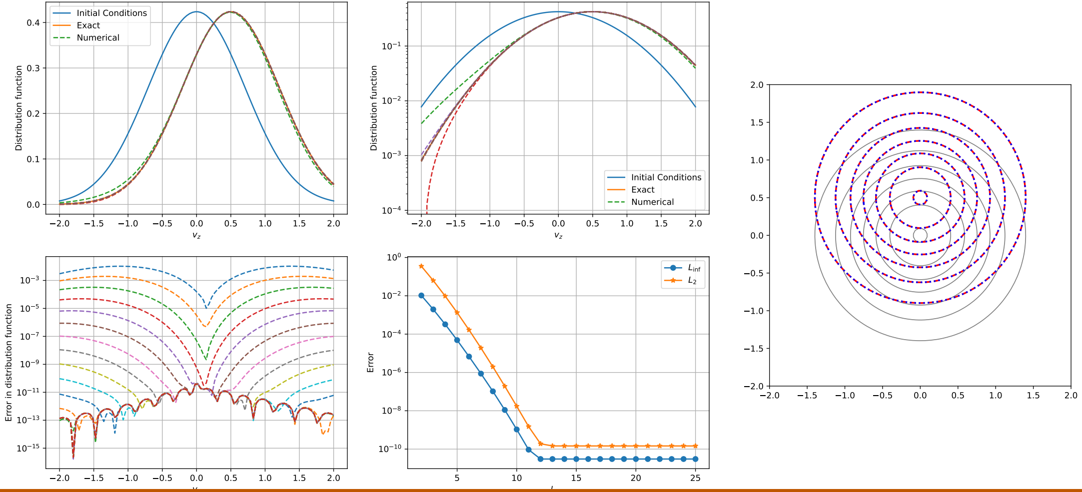
Convergence test ($N_r = 16$ is fixed, $l_{\max}$ is increasing)



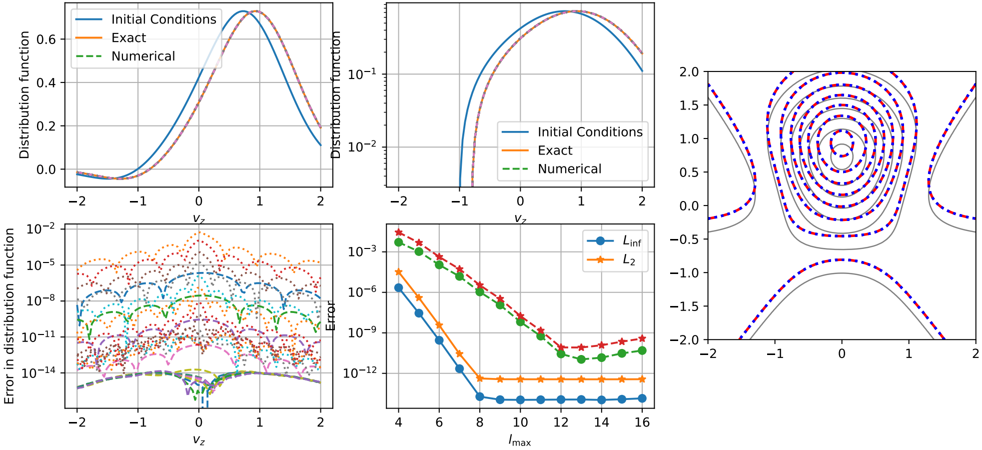
Convergence test ($N_r = 16$ is fixed, $l_{\max}$ is increasing)



Convergence test ($N_r = 16$ is fixed, $l_{\max}$ is increasing)



Convergence test: Maxwell vs Laguerre (N_r is increasing, $l_{\max} = 16$ is fixed)



Dealing with advection term

Eulerian

$$\partial_t f - \boldsymbol{E} \cdot \nabla f = C[f]$$

- Might need more spherical harmonics

Lagrangian + operator splitting

$$\begin{aligned} \partial_t f - \boldsymbol{E} \cdot \nabla f &= 0 \\ \partial_t f &= C[f] \end{aligned}$$

- Have to reassemble collision operator
- Operator splitting error