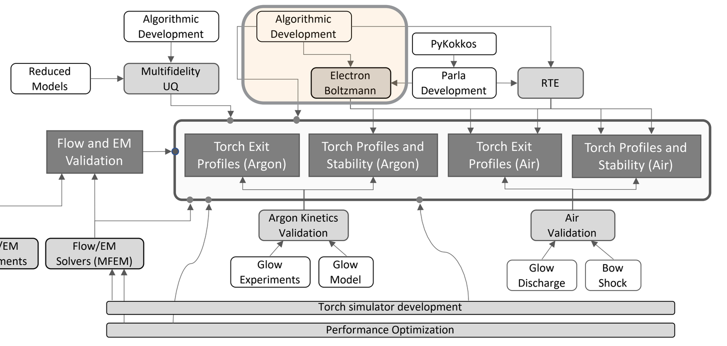
Deterministic PG Boltzmann Solver II: Acceleration Term Discretization

Daniil Bochkov, Milinda Fernando   The University of Texas at Austin



PSAAP III   ODEN INSTITUTE   PECOS Predictive Engineering & Computational Science

Objectives

- Electron density function $f = f(\boldsymbol{x}, \boldsymbol{v}, t)$ defines transport and kinetic properties of plasma
- Need to couple plasma model with electron kinetics (planned for Y3)
- **Y2 Goal:** Extend standalone electron Boltzmann solver from Y1 to support spatially inhomogeneous case and additional inelastic collisions
- **Poster focus:** acceleration due to electric field

Introduction

- **Boltzmann equation**

$$\partial_t f + \boldsymbol{v} \cdot \nabla_{\boldsymbol{x}} f - \overbrace{\boxed{\boldsymbol{E} \cdot \nabla_{\boldsymbol{v}} f}}^{\text{focus}} = \sum_a C_a(f)$$

for example, $a =$ elastic collisions:

$$C_a(f) = n_0 \int_{S^2} v \underbrace{\sigma_a(v, \omega)}_{\text{scat. cross sec.}} (f(v') - f(v))\, d\omega$$

- **Main challenge**: 6+1 dimensions
- **Overall approach**: FEM in $\boldsymbol{x}$, spectral in $\boldsymbol{v}$
- **Acceleration Term**: Eulerian or Lagrangian?

Eulerian Framework

- Assuming $\boldsymbol{E}$ is parallel to z-axis:

$$\boldsymbol{E} \cdot \nabla_{\boldsymbol{v}} f = E \left(\cos(v_\theta) \frac{\partial f}{\partial v} - \sin(v_\theta) \frac{1}{v} \frac{\partial f}{\partial v_\theta} \right)$$

- Expansion in terms of spherical harmonics

$$f = \sum_{klm} h_{klm}(t) \Phi_{kl}\left(\frac{v}{v_{\text{th}}}\right) Y_{lm}(v_\theta, v_\varphi)$$

- Choice of radial basis functions:

$$\Phi_{kl}(v) = \begin{cases} v^l B_k(v) & \text{(B-splines)} \\ v^l M(v) P_{kl}(v) & \text{(Maxwell poly.)} \\ v^l M(v) L_{kl}(v^2) & \text{(Laguerre poly.)} \end{cases}$$

- Projection onto a test function $\Psi_{pq} Y_{qs}$ (using properties of spherical harmonics)

$$\langle \boldsymbol{E} \cdot \nabla_{\boldsymbol{v}} f, \Psi_{pq} Y_{qs} \rangle = E v_{\text{th}}^2 \sum_k (h_{k,q+1,s}(t) A_{k,q,s} + h_{k,q-1,s}(t) B_{k,q,s})$$

- where $A_{k,q,s}$ and $B_{k,q,s}$ are semi-analytically pre-computed based on choice of $\Phi_{kl}(v)$ and $\Psi_{pq}(v)$
- Projected Boltzmann equation ($\boldsymbol{h} = \{h_{klm}\}$):

$$\partial_t \boldsymbol{h} - \boxed{\mathbb{E}\boldsymbol{h}} = \mathbb{C}_{el}\boldsymbol{h} + \mathbb{C}_{ion}\boldsymbol{h} + \boldsymbol{h}\mathbb{C}_{rec}\boldsymbol{h} + \dots$$

- **Pro:** efficient approach (no reassembly)
- **Con:** may need a larger number of DoF for large mean velocities

Lagrangian Framework

- Given advection time step Δt, a second order operator (Strang) splitting:

1. Half-step advection $\frac{\Delta t}{2}$:

$$\begin{cases} \partial_t f^{(0)} - \boldsymbol{E} \cdot \nabla_{\boldsymbol{v}} f^{(0)} = 0 \\ f^{(0)}(t_n) = f(t_n) \end{cases}$$

2. Full-step collisions Δt:

$$\begin{cases} \partial_t f^{(1)} = C[f^{(1)}] \\ f^{(1)}(t_n) = f^{(0)}\left(t_n + \frac{1}{2}\Delta t\right) \end{cases}$$

3. Half-step advection $\frac{\Delta t}{2}$:

$$\begin{cases} \partial_t f^{(2)} - \boldsymbol{E} \cdot \nabla_{\boldsymbol{v}} f^{(2)} = 0 \\ f^{(2)}(t_n) = f^{(1)}(t_n + \Delta t) \end{cases}$$

$$f(t_n + \Delta t) = f^{(2)}(t_n + \Delta t)$$

- Using variable basis:

$$f = \sum_{klm} a_{klm} \Phi_{klm}(\boldsymbol{v} - \boldsymbol{v}_0)$$

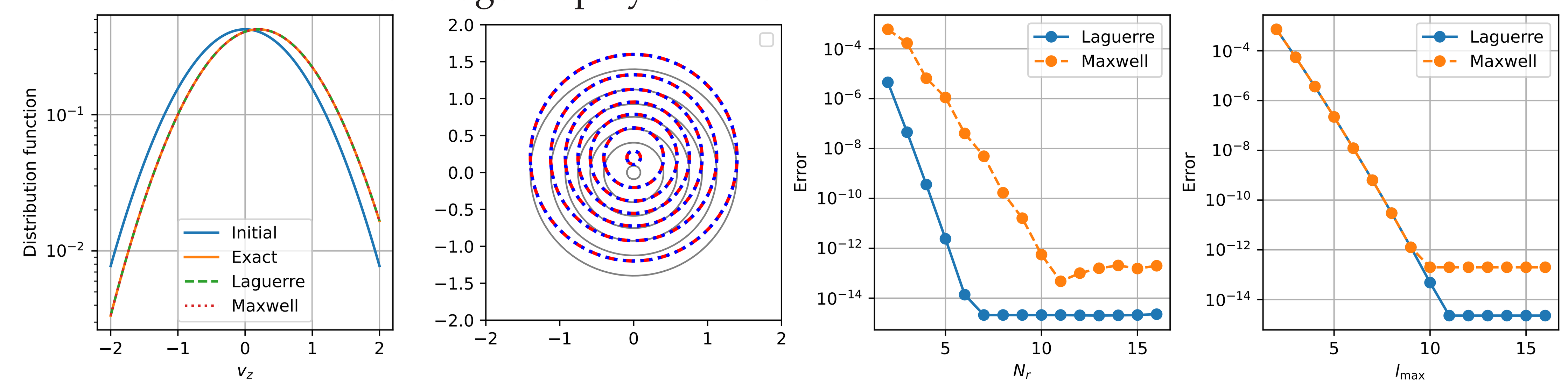
- Advection step is solved analytically

$$\boldsymbol{v}_0(t + \Delta t/2) = \boldsymbol{v}_0(t) + \int_t^{t+\Delta t/2} \boldsymbol{E}(t)\, dt$$

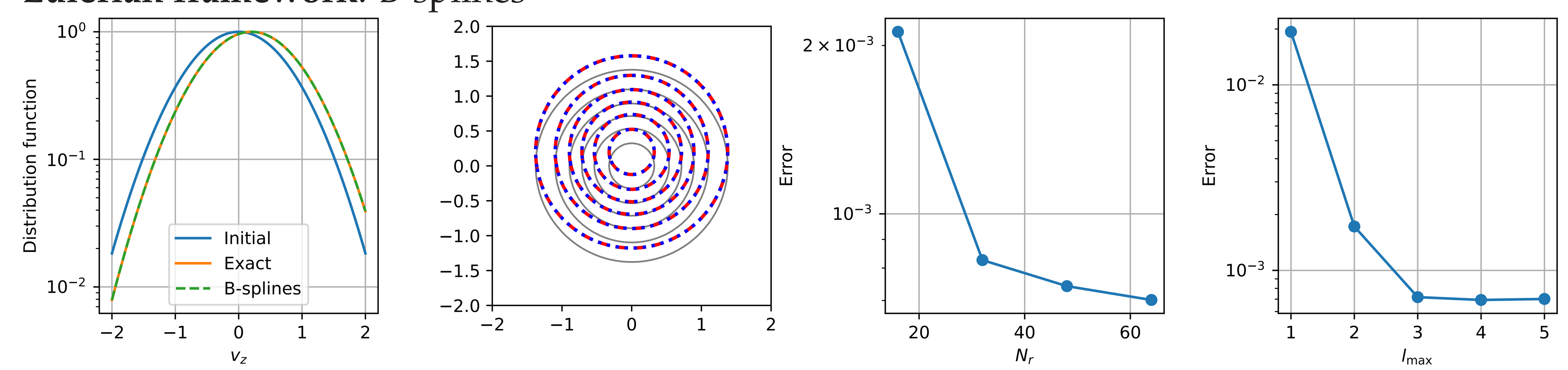
- **Pro:** expected to be more efficient for large mean velocities
- **Con:** need to reassemble collisional operators

Results

- **Eulerian framework**: Orthogonal polynomials



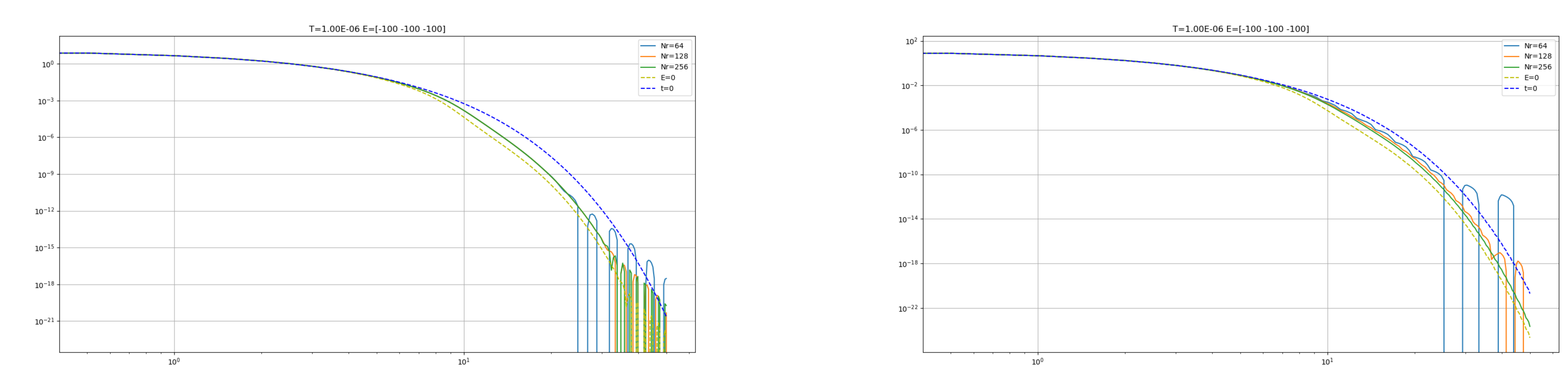
- **Eulerian framework**: B-splines



- **Lagrangian framework**: Preliminary results

Maxwell polynomials

B-splines



Conclusion

- Eulerian framework:
  - Orthogonal polynomials converge spectrally as expected
  - Laguerre polynomials converge two times faster than Maxwell polynomials
  - B-splines struggle to converge (singularity at $\boldsymbol{v} = 0$ is not removed?)
- Lagrangian framework:
  - Preliminary results indicate B-splines better capture tails

Future Research

- Perform extensive tests for both Eulerian and Lagrangian frameworks with different types of collisions
- Establish guidelines on when each of the approaches is more preferential
- Extend to spatially inhomogeneous cases