Deterministic PG Boltzmann Solver I: Collisional Term Discretization

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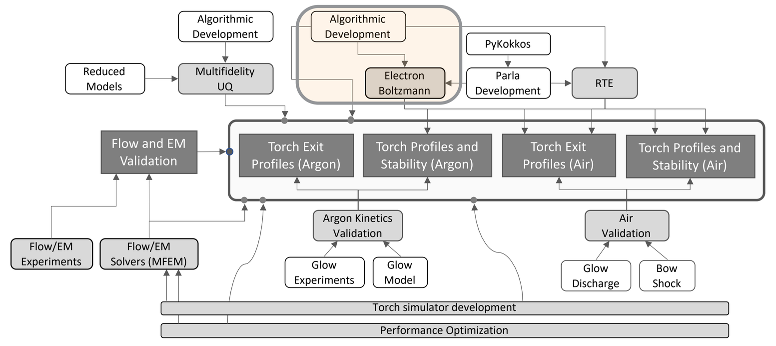


Objectives

- Electron density function $f = f(\boldsymbol{x}, \boldsymbol{v}, t)$ defines transport and kinetic properties of plasma
- Need to couple plasma model with electron kinetics (planned for Y3)
- **Y2 Goal:** Extend standalone electron Boltzmann solver from Y1 to support spatially inhomogeneous case and additional inelastic collisions
- **Poster focus:** discretization of collisional term

Approach: Petrov-Galerkin

- Expansion in terms of spherical harmonics:

$$f(\boldsymbol{v}) = \sum_{k,l,m} h_{k,l,m}(t)\Phi_{kl}(v)\underbrace{Y_{lm}(v_\theta, v_\phi)}_{\text{sph. harm.}}$$

- Choice of radial representation:

$$\Phi_{kl}(v) = \begin{cases} v^l B_k(v) & \text{(B-splines)} \\ v^l M(v) P_{kl}(v) & \text{(Maxwell poly.)} \end{cases}$$

$$M(v) = n_e \left(\frac{m}{2\pi kT}\right)^{\frac{3}{2}} e^{-\frac{mv^2}{2kT}}$$

- Weak formulation:

$$\int_{R^3} C\phi(\boldsymbol{v}_e) d\boldsymbol{v}_e = n_0 \int_{R^3}\int_{S^2} \sigma_a(v,\omega) v f_e(\boldsymbol{v}_e) \times \left(\phi(\boldsymbol{v}_e^{\text{post}}(\boldsymbol{v}_e, \boldsymbol{\omega})) - \phi(\boldsymbol{v}_e)\right) d\boldsymbol{\omega} d\boldsymbol{v}_e$$

- Resulting system of ODEs:

$$\sum_{k,l,m} M_{p,q,s}^{k,l,m} \partial_t h_{k,l,m}(t) = \sum_{k,l,m} L_{p,q,s}^{k,l,m} h_{k,l,m}(t)$$

where

$$L_{k,l,m}^{p,q,s} = n_0 \int_{v_r}\int_{S^2}\int_{S^2} v^2 M(v_r) P_k\left(\frac{v_r}{v_{\text{th}}}\right) Y^{lm}(v_\theta, v_\phi) v_r \sigma(|v_r|, \chi) \times \left(P_p\left(\frac{v_r'}{v_{\text{th}}}\right) Y_{qs}(v_\theta', v_\phi') - P_p\left(\frac{v_r}{v_{\text{th}}}\right) Y_{qs}(v_\theta, v_\phi)\right) d\omega d\omega_v dv$$

Introduction

- **Boltzmann equation**

$$\partial_t f + \boldsymbol{v} \cdot \nabla_{\boldsymbol{x}} f - \boldsymbol{E} \cdot \nabla_{\boldsymbol{v}} f = \overbrace{\boxed{\sum_a C_a(f)}}^{\text{focus}}$$

for example, $a =$ elastic collisions:

$$C_a(f) = n_0 \int_{S^2} v \underbrace{\sigma_a(v,\omega)}_{\text{scat. cross sec.}} (f(v') - f(v))\, d\omega$$

- **Main challenge**: 6+1 dimensions
- **Overall approach**: FEM in $\boldsymbol{x}$, spectral in $\boldsymbol{v}$
- **Collisional Term**: Choice of basis?

Tensorized Implementation

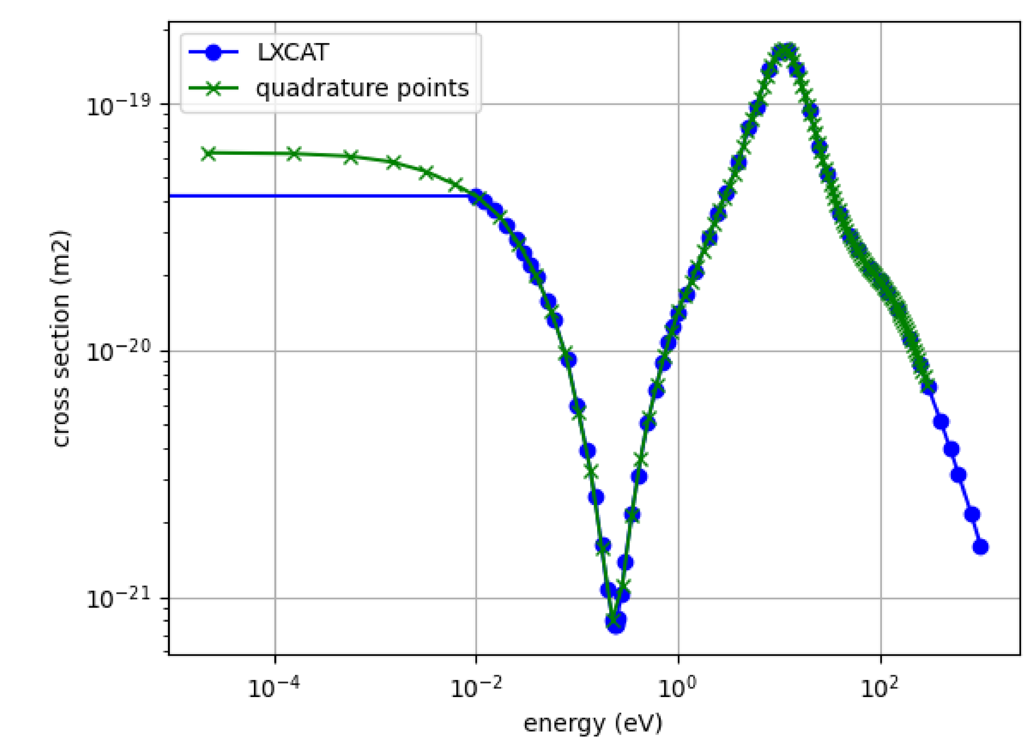
- Tensorized formulation

$$L_{k,l,m}^{p,q,s} = P_k^{r\theta\phi} Y_{lm}^{r\theta\phi} \times \ldots \times W_\phi W_\theta W_r$$
$$\uparrow$$
$$M^{r\theta\phi\chi\gamma} \times \ldots \times W_\chi W_\gamma$$
$$\uparrow$$
$$P_p^{r\theta\phi\chi\gamma}(v_r') Y_{qs}^{r\theta\phi\chi\gamma}(v_\theta', v_\phi') - P_p^{r\theta\phi\chi\gamma}(v_r) Y_{qs}^{r\theta\phi\chi\gamma}(v_\theta, v_\phi)$$

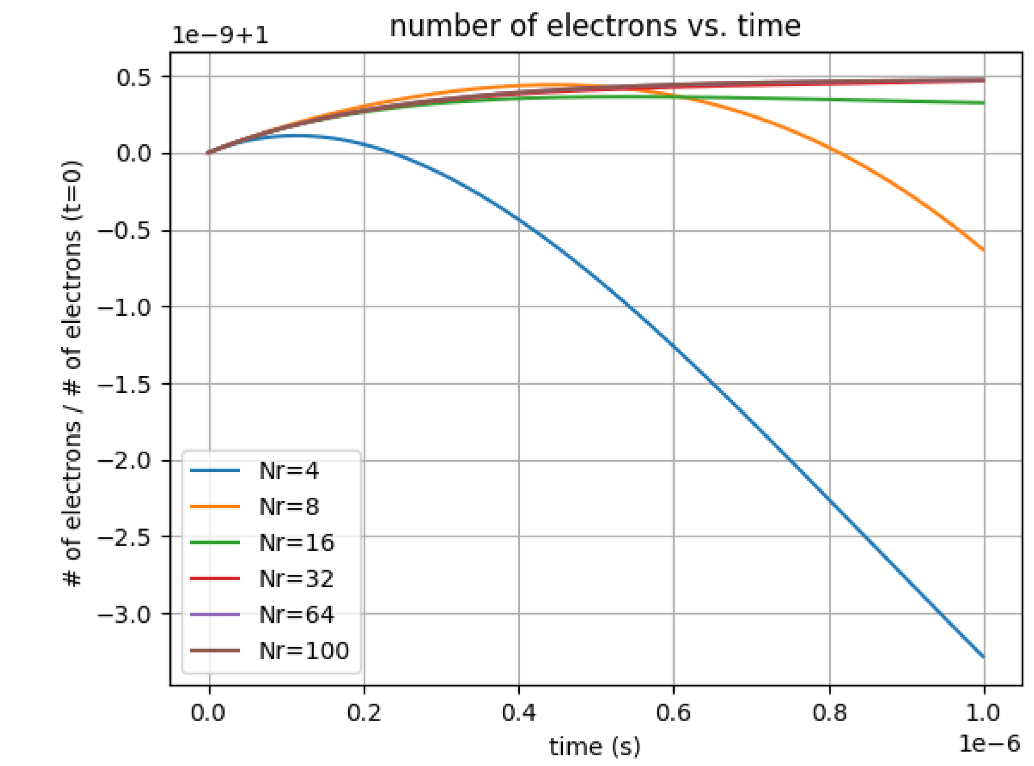
- Tensors:
  - $S_{r'\theta'\phi'}^{r\theta\phi\chi\gamma}$: Scattering velocity tensor, for each $v = (r, \theta, \phi)$ and scattering solid angle (χ, γ) computes (r', θ', ϕ') scattered or newly created particle velocity
  - $P_p^{r\theta\phi\chi\gamma}$: radial polynomial evaluated at differed velocity for given incident particle $(r, \theta, \phi, \chi, \gamma)$
  - $Y_{qs}^{r\theta\phi\chi\gamma}$: qs spherical harmonic mode evaluated differed particle direction for a given incident particle $(r, \theta, \phi, \chi, \gamma)$
  - $M^{r\theta\phi\chi\gamma}$: Maxwellian times v_r evaluated for the differed particle for a given incident particle $(r, \theta, \phi, \chi, \gamma)$
  - $\sigma^{r\theta\phi\chi\gamma}$: differential cross section broadcasted on scattering cross section angles.
  - $P_k^{r\theta\phi}$: radial polynomials evaluated at radial quadrature points.
  - $Y_{lm}^{r\theta\phi}$: spherical harmonics evaluated angular quadrature points.

Results

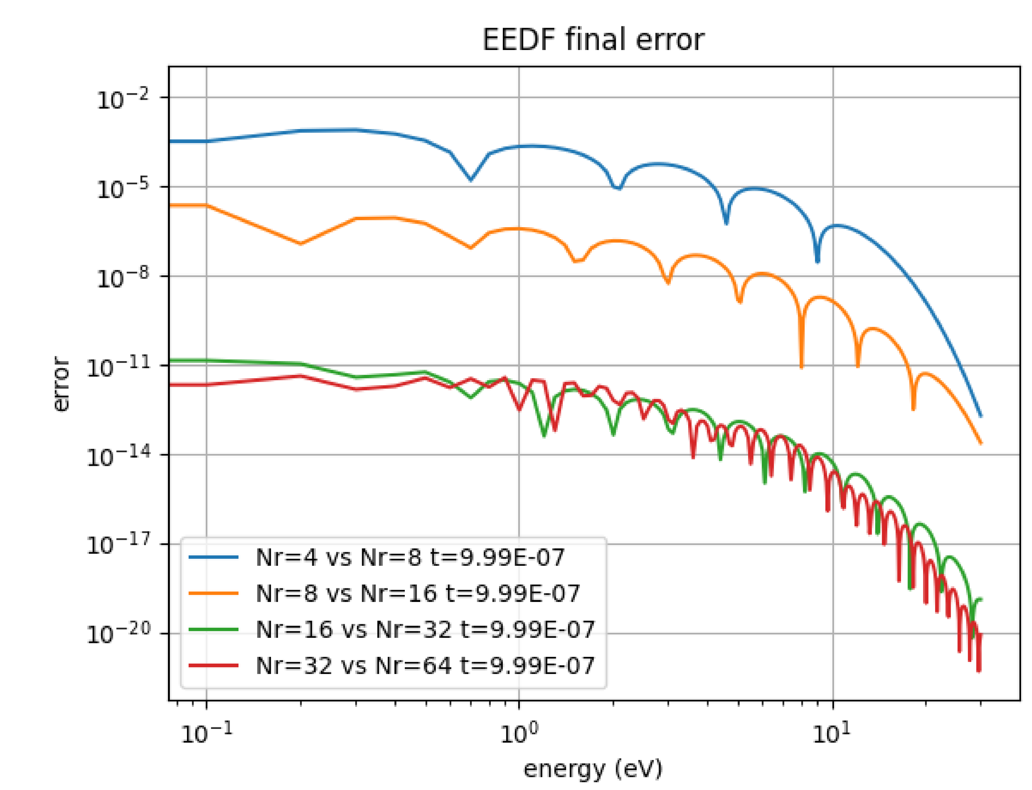
- Cross section graph



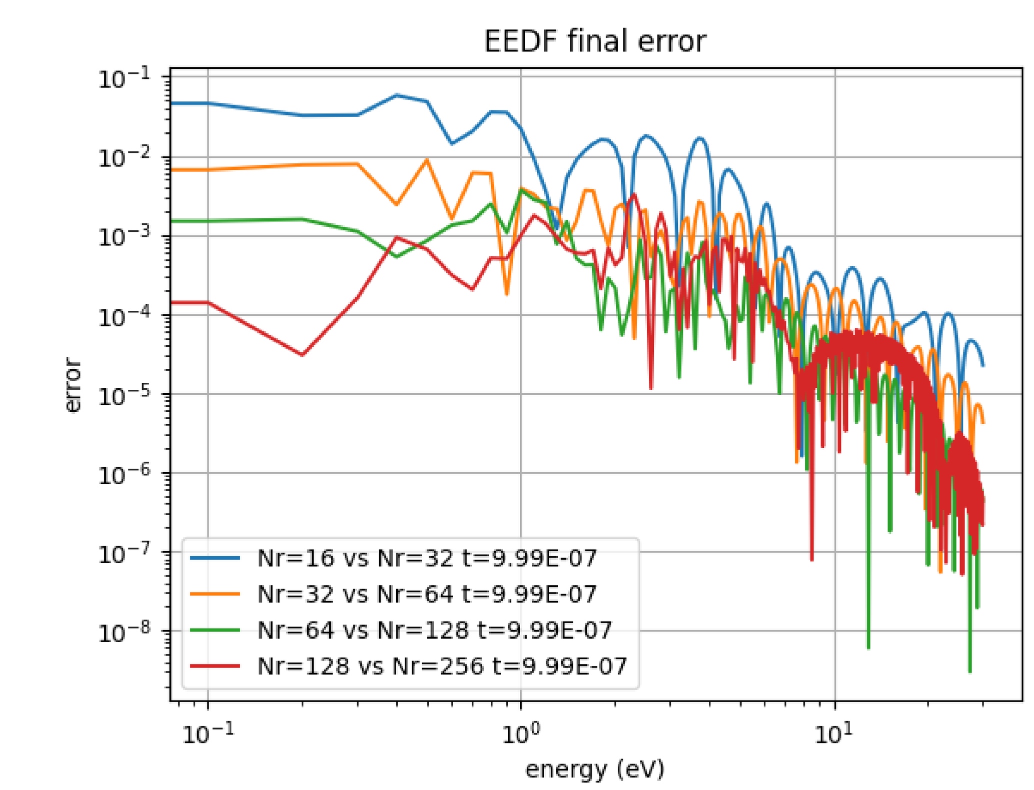
- Mass conservation (?)



- Convergence of Maxwell polynomials



- Convergence of B-splines



Conclusion

- Analytical cross sections are needed to preserve convergence
- Maxwell polynomials are expected to converge faster (?)
- B-splines represent tails better (?)
- Tensorized assembly accelerates execution by several order of magnitudes

Future Research

- Add recombination reactions
- Couple with electric field forcing term
- Extend to spatially inhomogeneous case