

- Advection equation

$$\partial_t f - \mathbf{E} \cdot \nabla_{\mathbf{v}} f = 0$$

- In spherical coordinates

$$\nabla_{\mathbf{v}} = \hat{\mathbf{v}}_r \frac{\partial}{\partial v} + \hat{\mathbf{v}}_\theta \frac{1}{v} \frac{\partial}{\partial v_\theta} + \hat{\mathbf{v}}_\varphi \frac{1}{v \sin(v_\theta)} \frac{\partial}{\partial v_\varphi}$$

$$\mathbf{E} = E \hat{\mathbf{z}} = E (\cos(v_\theta) \hat{\mathbf{v}}_r - \sin(v_\theta) \hat{\mathbf{v}}_\theta)$$

$$\mathbf{E} \cdot \nabla_{\mathbf{v}} f = E \left( \cos(v_\theta) \frac{\partial f}{\partial v} - \sin(v_\theta) \frac{1}{v} \frac{\partial f}{\partial v_\theta} \right)$$

- Expansion in terms of Maxwell polynomials and spherical harmonics

$$f = A \exp\left(-\left(\frac{v}{v_{th}}\right)^2\right) \sum_{klm} h_{klm} P_k\left(\frac{v}{v_{th}}\right) Y_{lm}(v_\theta, v_\varphi)$$

- Projection:

$$\int_{\mathbb{R}^3} \left( \partial_t f - \mathbf{E} \cdot \nabla_{\mathbf{v}} f = 0 \right) P_p\left(\frac{v}{v_{th}}\right) Y_{qs}(v_\theta, v_\varphi) d\mathbf{v} dv_\omega$$

- Resulting system of ODEs:

$$\partial_t h_{pqs}(t) = \sum_{k,l,m} \epsilon_{pqsklm} h_{klm}(t)$$

- Discretized operator

$$\epsilon_{pqsklm} = \left(d_{pk} - \frac{1}{2}l_{pk}\right) \Psi_{lmqs} + \frac{1}{2} \sum_{j=0}^{N_r} \left(d_{jk} - \frac{1}{2}l_{jk}\right) \left(d_{pj} - \frac{1}{2}l_{pj} + d_{jp}\right) \Phi_{lmqs}$$

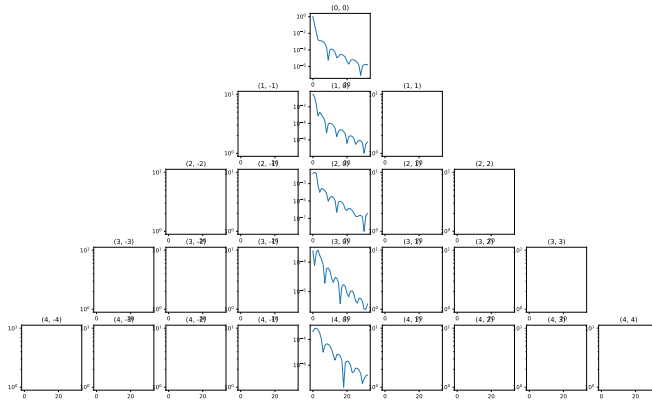
$$d_{jk} : \quad P'_k(x) = \sum_{j=0}^{N_r} d_{jk} P_j(x)$$

$$l_{jk} : \quad x P_k(x) = \sum_{j=0}^{N_r} l_{jk} P_k(x)$$

$$\Psi_{lmqs} = \int \cos(v_\theta) Y_{lm}(v_\theta, v_\varphi) Y_{qs}(v_\theta, v_\varphi) dv_\omega$$

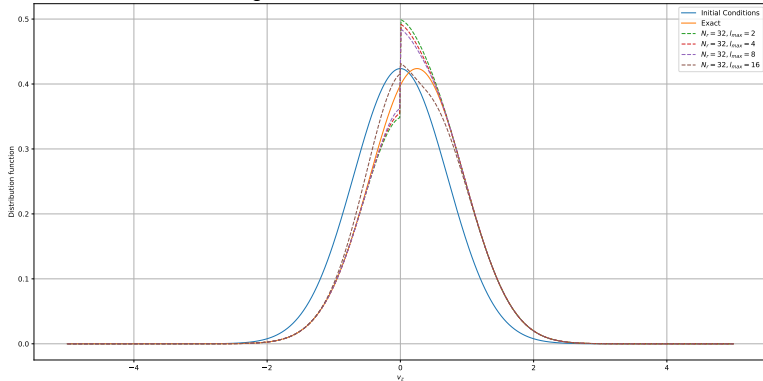
$$\Phi_{lmqs} = \int \sin(v_\theta) Y_{qs}(v_\theta, v_\varphi) \frac{\partial}{\partial v_\theta} Y_{lm}(v_\theta, v_\varphi) dv_\omega$$

# Sanity check: solution stays axisymmetric



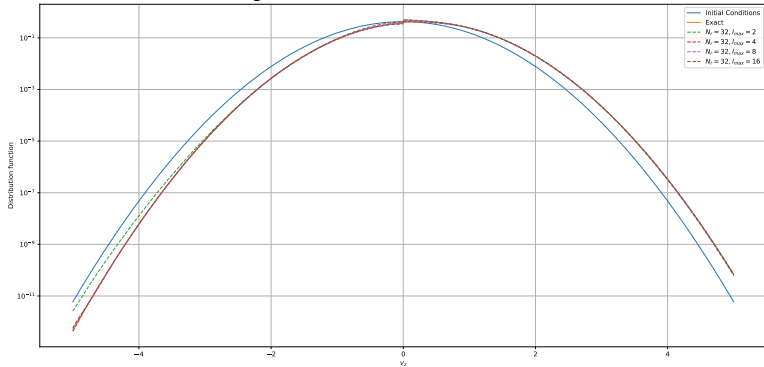
# Convergence test: discontinuities at $v = 0$

- $\mathbf{E} = \text{const} \rightarrow f(t, \mathbf{v}) = f(0, \mathbf{v} + \mathbf{E}t)$
- $N_r = 32$  is fixed,  $l_{\max}$  is increasing



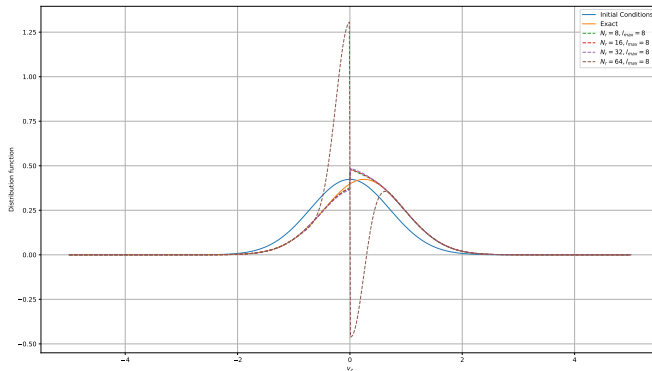
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# Convergence test: discontinuities at $v = 0$

- $E = \text{const} \rightarrow f(t, v) = f(0, v + Et)$
- $N_r$  is increasing,  $l_{\max} = 8$  is fixed



- Kinda makes sense: generally singular at  $\mathbf{v} = 0$

$$f = A \exp\left(-\left(\frac{v}{v_{\text{th}}}\right)^2\right) \sum_{lm} \left(\sum_k h_{klm} P_k\left(\frac{v}{v_{\text{th}}}\right)\right) Y_{lm}(v_\theta, v_\varphi)$$

- Continuous if

$$\left(\sum_k h_{klm} P_k(0)\right) = 0$$



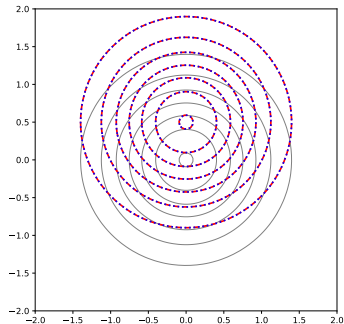
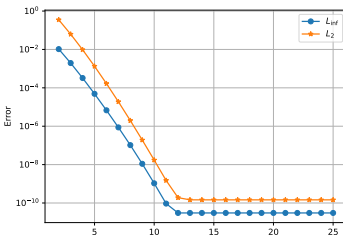
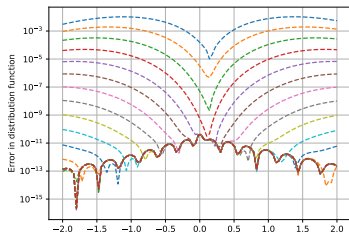
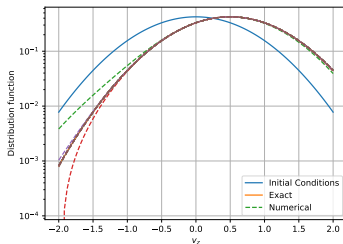
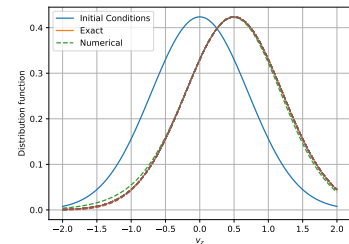
- Possible fix (Following Gamba&Rjasanow 2018):

$$f = A \exp\left(-\left(\frac{v}{v_{\text{th}}}\right)^2\right) \sum_{lm} \left( \sum_k h_{klm} \left(\frac{v}{v_{\text{th}}}\right)^l P_{kl}\left(\frac{v}{v_{\text{th}}}\right) \right) Y_{lm}(v_\theta, v_\varphi)$$

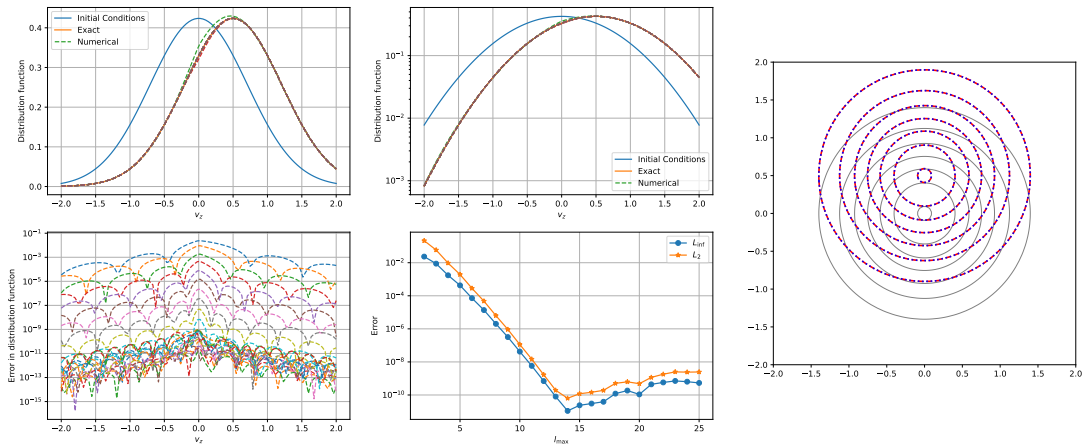
- where (associated Maxwell polynomials?)

$$\int_0^\infty P_{kl}(x) P_{jl}(x) x^{2l+2} e^{-x^2} dx = \delta_{k,j}$$

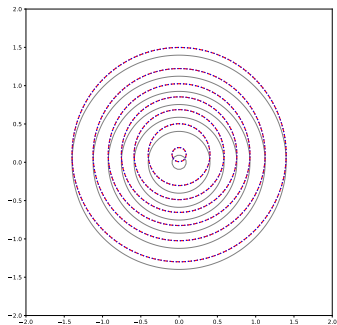
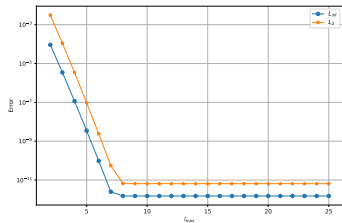
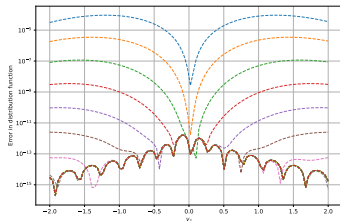
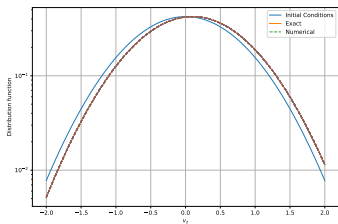
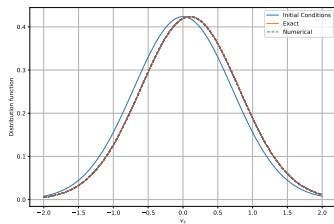
# Convergence test ( $N_r = 16$ is fixed, $l_{\max}$ is increasing)



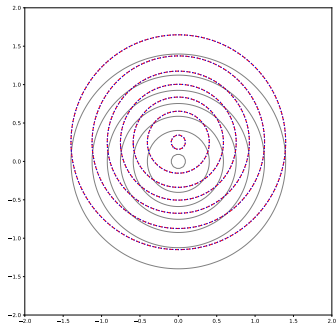
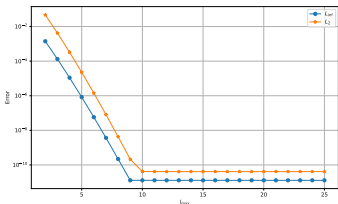
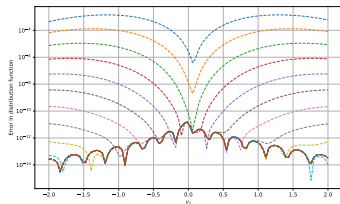
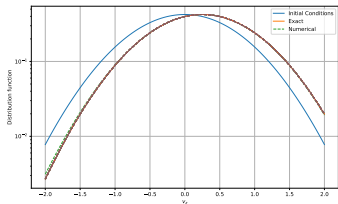
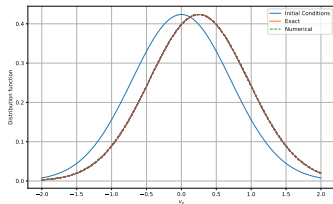
## Convergence test ( $N_r$ is increasing, $l_{\max} = 16$ is fixed)



## Convergence test ( $N_r = 16$ is fixed, $l_{\max}$ is increasing)



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