Using Adaptive Sparse Grids to Solve High-Dimensional Dynamic Models

Simon Scheidegger
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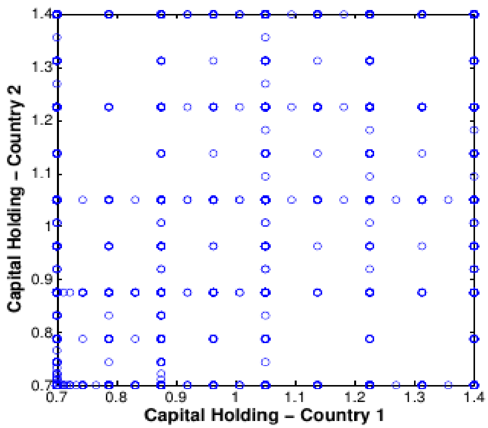
July 16th, 2019
Open Source Economics Laboratory – BFI/UChicago
(joint work with J. Brumm, Econometrica)



Who I am?

- Assistant Prof, University of Lausanne, Department of Finance (https://sites.google.com/site/simonscheidegger)

- Ph.D., theoretical Physics, University of Basel (2010)
- Dissertation title:
  “Gravitational waves from 3D MHD core-collapse supernova simulations with neutrino transport”

Research interests:
Research focus on developing computational methods for high-dimensional dynamic stochastic economic models and applying them to optimal tax policy, monetary policy and option pricing.

Teaching:
Machine Learning, Programming, numerical analysis, & computational methods, software engineering, financial economics.

Roadmap – fast forward:

Day 1, Tuesday – July 16th

1. (Adaptive) Sparse Grids (8.00 – 9.00)

2. Hands-on Session (I) (9.30 – 10.30):
   →log onto MIDWAY cluster (→ more details in parallel programming lecture).
   →install & play with open source sparse grid codes (→ analytical functions).

3. Hands-on Session (II)
   →Dynamic Programming & Time Iteration with Sparse Grids (10.45 – 11.45).

4. An exercise sheet related to the day (11.45 – 12.00).
   →Analytical examples.
   →Introduction to a stochastic growth model, solved with DP and sparse grids.

Roadmap – fast forward (2):

Day 2, Thursday – July 18th

1. Introduction to parallel and high-performance computing (8.00-9.30).

2. An ultra-short introduction to C++ (10.00-10.30 – hands on).

3. Basics on code optimization & OpenMP session I (10.45-11.45 – hands on).

4. Introduction to Projects (11.45-12.00 – hands on).

5. Exercise sheet related to the day's topics (12-00-..).

Roadmap – fast forward (3):

Day 3, Tuesday – July 23th

1. OpenMP session II (8.00-9.00 – hands on).
2. MPI session I (9.30-10.30 – hands on).
3. MPI session II (10.45-11.45 – hands on).
4. Exercise sheet related to the day's topics (11.50-12.00).

Day 4, Thursday – July 25th

1. Hybrid parallelism – OpenMP & MPI (8.00-9.00 – hands on).
2. Hybridize some of the projects together (9.15-10.00 – hands on).
3. Advanced topics (10.15-11.00).
4. Start to present results from the projects (11.10 – 11.50).
5. Exercise sheet related to the day's topic (11.50-12.00 – hands on).

What are these lectures about?

What are my goals for the next 4 days?

- You know how to deal with high-dimensional state spaces.
- You understand the basic concepts of parallel computing.
- You understand the basics of the available HPC hardware.
- Know which parallel programming paradigms are available.
- Be aware which paradigm and which hardware fits your problem.
- Gain hands-on expertise with exercises.

Lecture Slides & Codes on Git

I will post the slides and codes for this session on sparse grids as well as for the parallel programming session here:

https://github.com/sischei/OSE2019.git

KEEP
CALM
AND
LETS GET
STARTED

Today – (Adaptive) Sparse Grids

I. Motivation – “the curse of dimensionality”

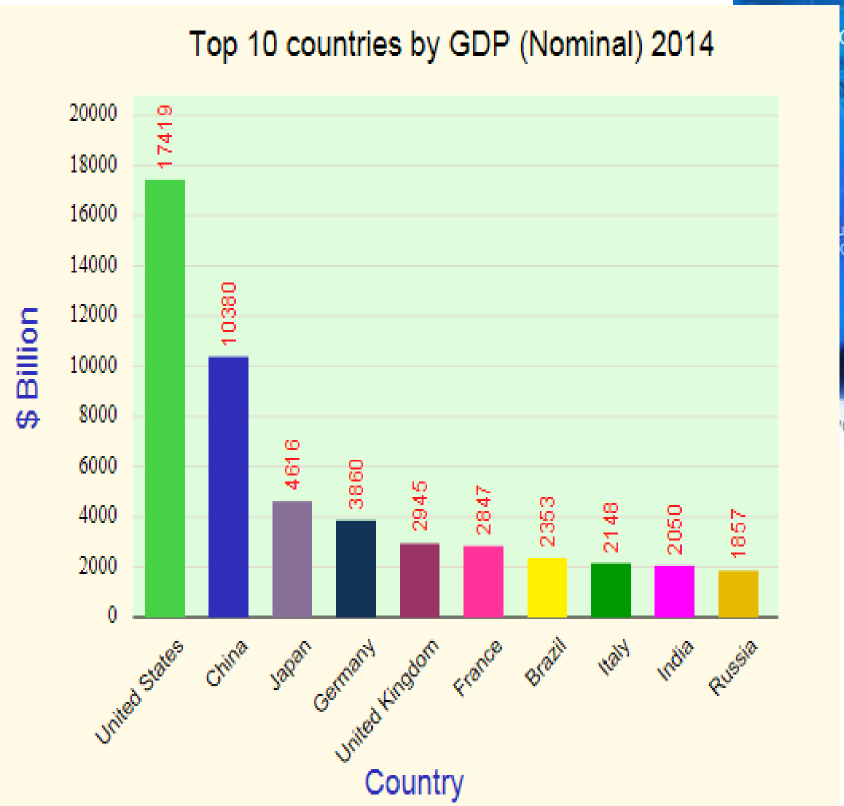
II. From Full (Cartesian) Grids to Sparse Grids

III. Adaptive Sparse Grids

IV. How to integrate ASGs in dynamic economic model

Example – Heterogeneity in IRBC models

- Model trade imbalance
- FX rates
- ...



- How many regions does a minimal model have?
- Are policy functions smooth? (borrowing constraints)

→ **Model heterogeneous & high-dimensional**

Example – Heterogeneity in OLG* models

*Overlapping generation models

To model e.g. social security:

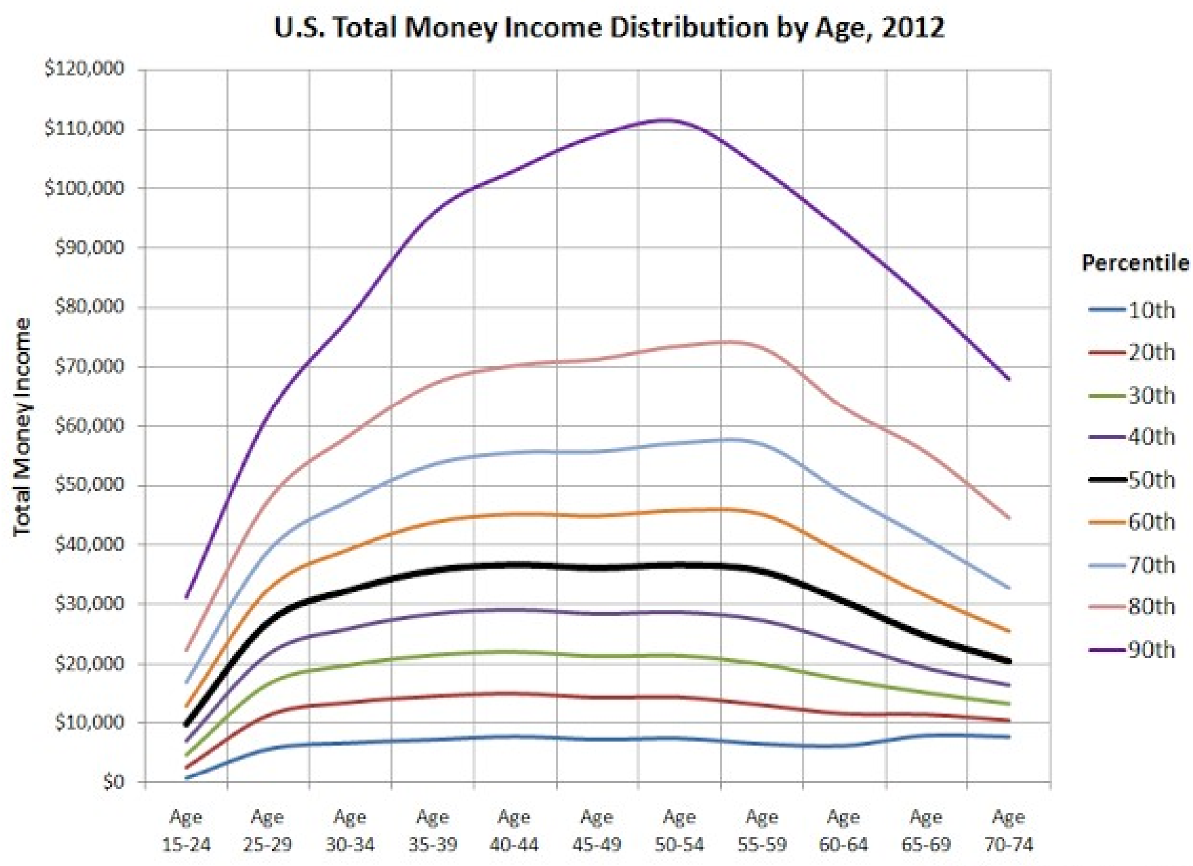
- How many age groups?

- borrowing constraints?

- aggregate shocks?

- …

→ **Model: heterogeneous & high-dimensional**



Financial markets: non-Gaussian returns

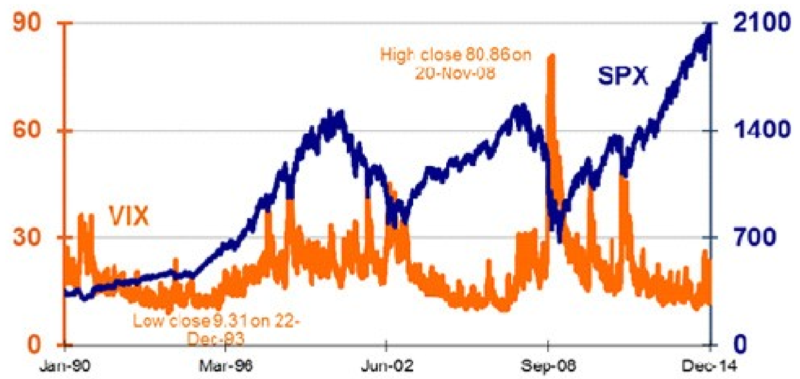
- Derivative contracts giving a right to buy or sell an underlying security.
    - *European* if exercise at expiration only.
    - ***American*** if exercise any time until expiration.

- American options are extremely challenging:
  → Dynamic optimization problem.

- Basic models do not describe dynamics accurately (e.g., Hull (2011)).

- Financial returns are often not Gaussian.

- Realistic models are hard to deal with, as they need many factors.
  → Curse of dimensionality.



Dynamic Programming/Value Function Iteration

e.g. Stokey, Lucas & Prescott (1989), Judd (1998), ...

Dynamic programming seeks a time-invariant policy function ***p*** mapping a state $\mathbf{x}_t$ into the control $\mathbf{u}_t$ such that for all $t \in \mathbb{N}$ $u_t = p(x_t)$

The solution is approached in the limit as $j \longrightarrow \infty$ by iterations on:

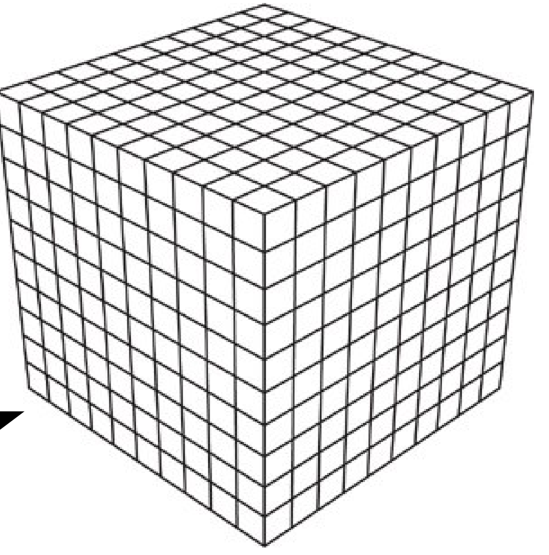
$$\underline{V_{j+1}(x)} = \max_u \{r(x,u) + \beta \underline{V_j(\tilde{x})}\}$$

s.t.

$$\tilde{x} = g(x,u)$$

x: grid point, describes your system.
State-space potentially **high-dimensional**.

`old solution':
high-dimensional function on which **we interpolate**.



→ $\mathbf{N^d}$ points in ordinary discretization schemes.

→ Use-case for (adaptive) sparse grids.

How many is dimensions is high dimensions?

?

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Number of parameters (the dimension)	Number of model runs (at 10 points per dimension)	Time for parameter study (at 1 second per run)
1	10	10 sec
2	100	~ 1.6 min
3	1,000	~ 16 min
4	10,000	~ 2.7 hours
5	100,000	~ 1.1 days
6	1,000,000	~ 1.6 weeks
...	...	...
20	1e20	3 trillion years (240x age of the universe)

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Dimension reduction
Exploit symmetries,...

Deal with #Points
Adaptive Sparse Grids

High-performance computing
Reduces time to solution, but not the problem size

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Adaptive Sparse Grids

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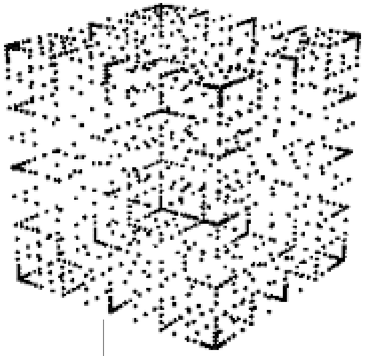
Computational modelling

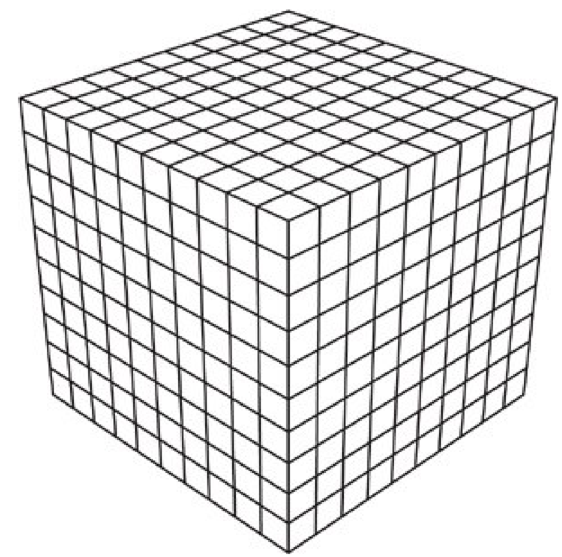
$$
\begin{aligned}
&\lambda_t \cdot \left[1 + \phi \cdot g^j_{t+1}\right] - \mu^j_t \\
&\quad - \beta \mathbb{E}_t \left\{ \lambda_{t+1} \left[a^j_{t+1} A \zeta (k^j_{t+1})^{\zeta - 1} + 1 - \delta + \frac{\phi}{2} g^j_{t+2} \left(g^j_{t+2} + 2 \right) \right] - (1-\delta)\mu^j_{t+1} \right\} = 0, \\
&\qquad 0 \leq \mu^j_t \perp \left(k^j_{t+1} - k^j_t (1 - \delta) \right) \geq 0.
\end{aligned}
$$



Abstract Problem Formulation

i) Dynamic models: heterogeneous & high-dimensional

ii) Want to solve dynamic stochastic models with high-dimensional state spaces



→ Have to approximate and interpolate high-dimensional functions

Problem: curse of dimensionality
→ N^d points in ordinary discretization schemes

iii) **Want to overcome curse of dimensionality**

iv) **Want locality & adaptivity of interpolation scheme**

v) **Speed-up*** → access hybrid HPC systems (MPI, OpenMP, TBB, GPU)

***see the lectures on parallel programming**

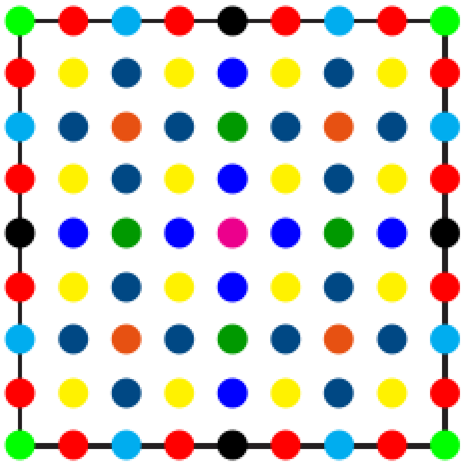
Models where high-dim. state spaces show up

e.g. Stokey, Lucas & Prescott (1989), Ljundquist & Sargent (2004), Krüger & Kübler (2004), Judd et. al. (2013), Brumm & Scheidegger (2017),...

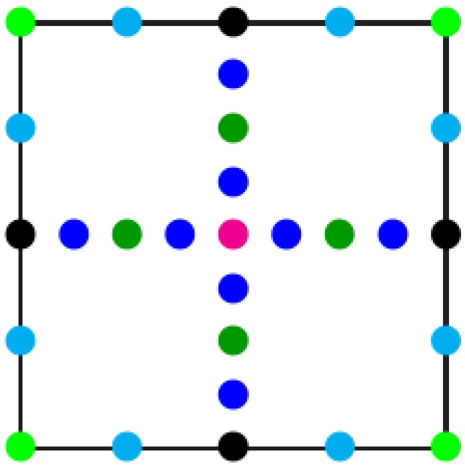
- International Real Business Cycle (IRBC) Models:
  Exchange Rates, Global Trade Imbalances
- Dynamic Stochastic General Equilibrium (DSGE) Models:
  Monetary Policy, Business Cycle Fluctuations
- Overlapping Generations (OLG) Models:
  Demographic Change, Social Security
- Mathematical Finance:
  Option pricing,...

II. From Full Grids to Sparse Grids

(see, e.g. Zenger (1991), Bungartz & Griebel (2004), Garcke (2012), Pflüger (2010),...)



Cartesian Grid



Sparse Grid

Interpolation on a Full Grid

-Consider a **1-dimensional function** $f : \Omega \to \mathbb{R}$ **on [0,1]**

-In numerical simulations:
f might be expensive to evaluate! (solve PDEs/system of non-linear Eqs.)
But: need to be able to evaluate *f* at arbitrary points using a numerical code

-Construct an interpolant ***u*** of ***f*** $f(\vec{x}) \approx u(\vec{x}) := \sum_i \alpha_i \varphi_i(\vec{x})$

-With suitable basis functions: $\varphi_i(\vec{x})$
and coefficients: α_i

- For simplicity: focus on case where $f|_{\partial\Omega} = 0$

Basis Functions

-Hierarchical basis based on **hat functions**

$$\phi(x) = \begin{cases} 1 - |x| & \text{if } x \in [-1, 1] \\ 0 & \text{else} \end{cases}$$

-Used to generate a **family of basis functions** $\phi_{l,i}$ having support $[x_{l,i} - h_l, x_{l,i} + h_l]$ by **dilation** and **translation**

$$\phi_{l,i}(x) := \phi\left(\frac{x - i \cdot h_l}{h_l}\right)$$

Hierarchical Increment Spaces

Hierarchical increment spaces:

$$W_l := \operatorname{span}\{\phi_{l,i} : i \in I_l\}$$

with the **index set**

$$I_l = \{i \in \mathbb{N}, 1 \leq i \leq 2^l - 1, i \text{ odd}\}$$

The corresponding function space:

$$V_l = \bigoplus_{k \leq l} W_k$$

The **1d-interpolant:**

$$f(x) \approx u(x) = \sum_{k=1}^{l} \sum_{i \in I_k} \alpha_{k,i} \phi_{k,i}(x)$$

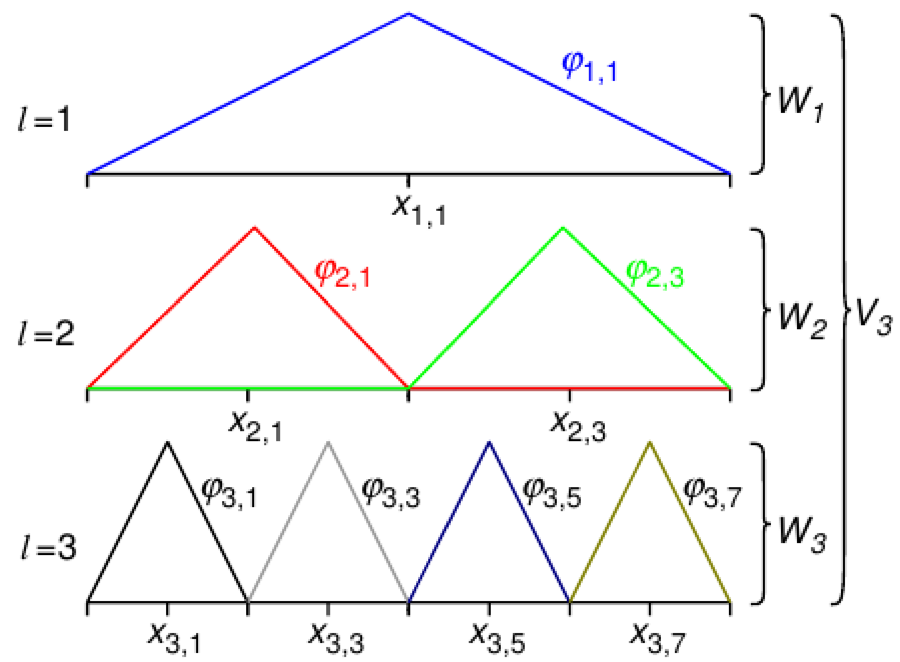


Fig.: 1-d basis functions $\phi_{l,i}$ and the corresponding **grid points** up level ***l* = 3** in the hierarchical basis.

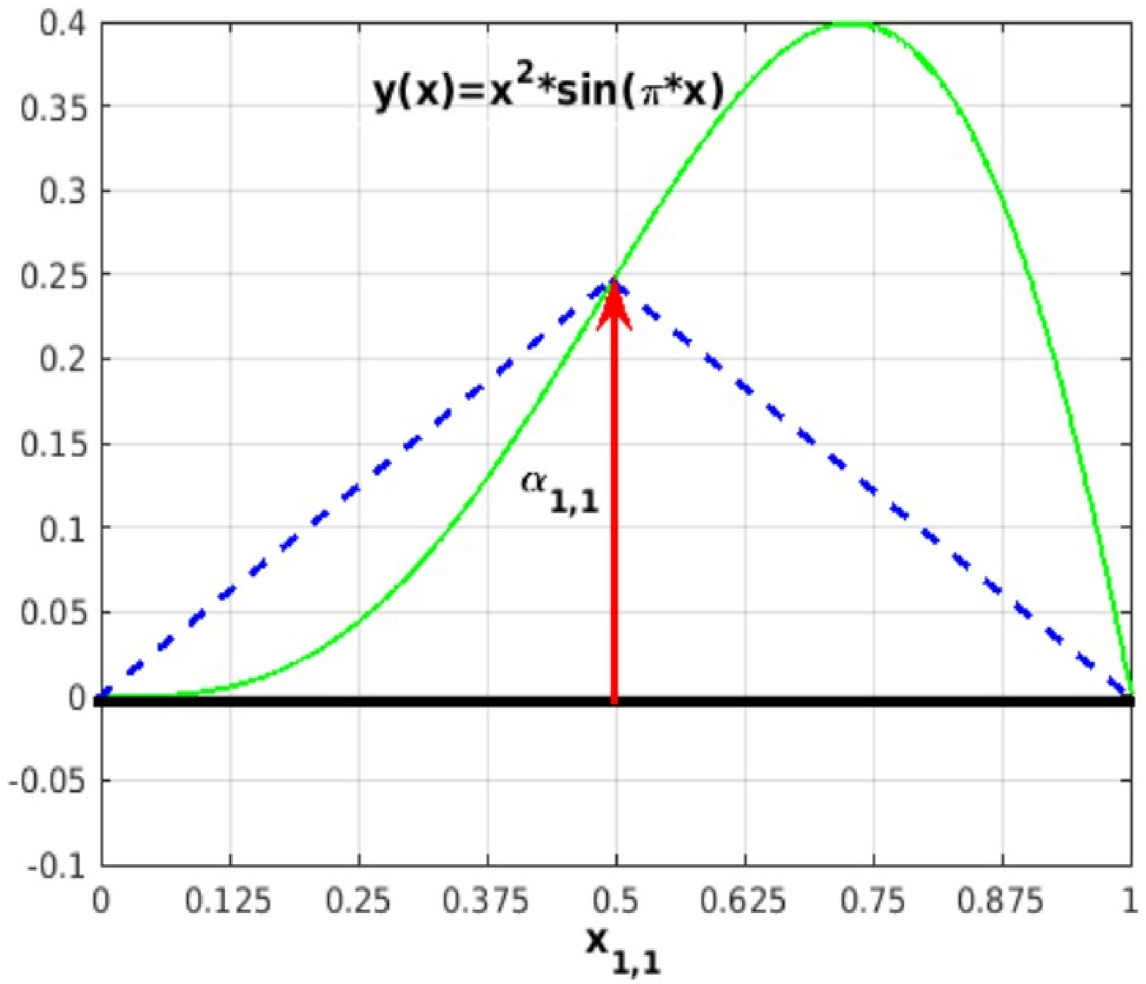
Note: supports of all basis functions of W_k mutually disjoint!

Piecewise Linear Interpolation: Level l

Coefficients:
hierarchical surpluses

They correct the interpolant of level *l*-1 at $\vec{x}_{l,i}$ to the actual value of $f\left(\vec{x}_{l,i}\right)$

Nested structure:
Evaluate function only at points that are unique to the new level.

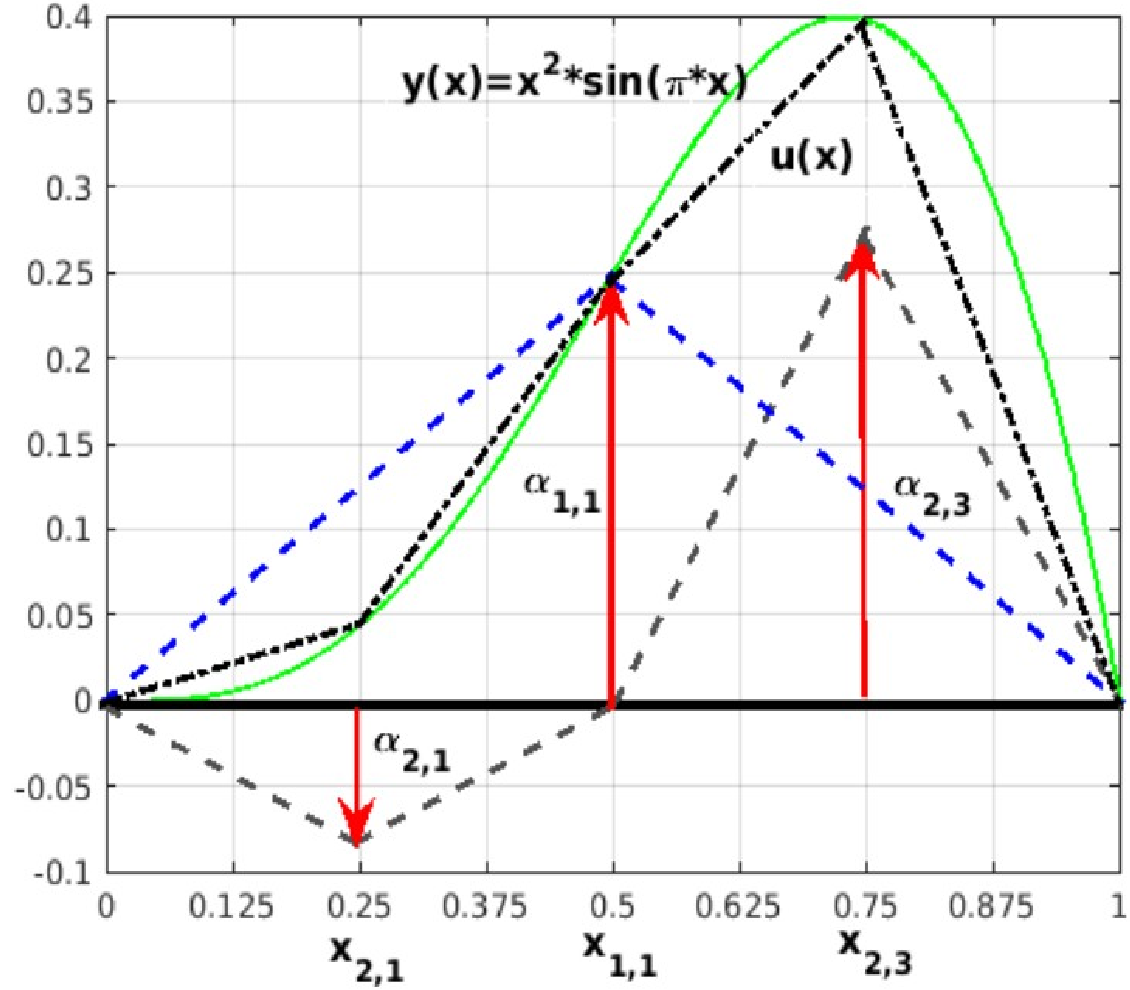


Piecewise Linear Interpolation: Level II

Coefficients:
hierarchical surpluses

They correct the interpolant of level *l*-1 at $\vec{x}_{l,i}$ to the actual value of $f(\vec{x}_{l,i})$

Nested structure:
Evaluate function only at points that are unique to the new level.

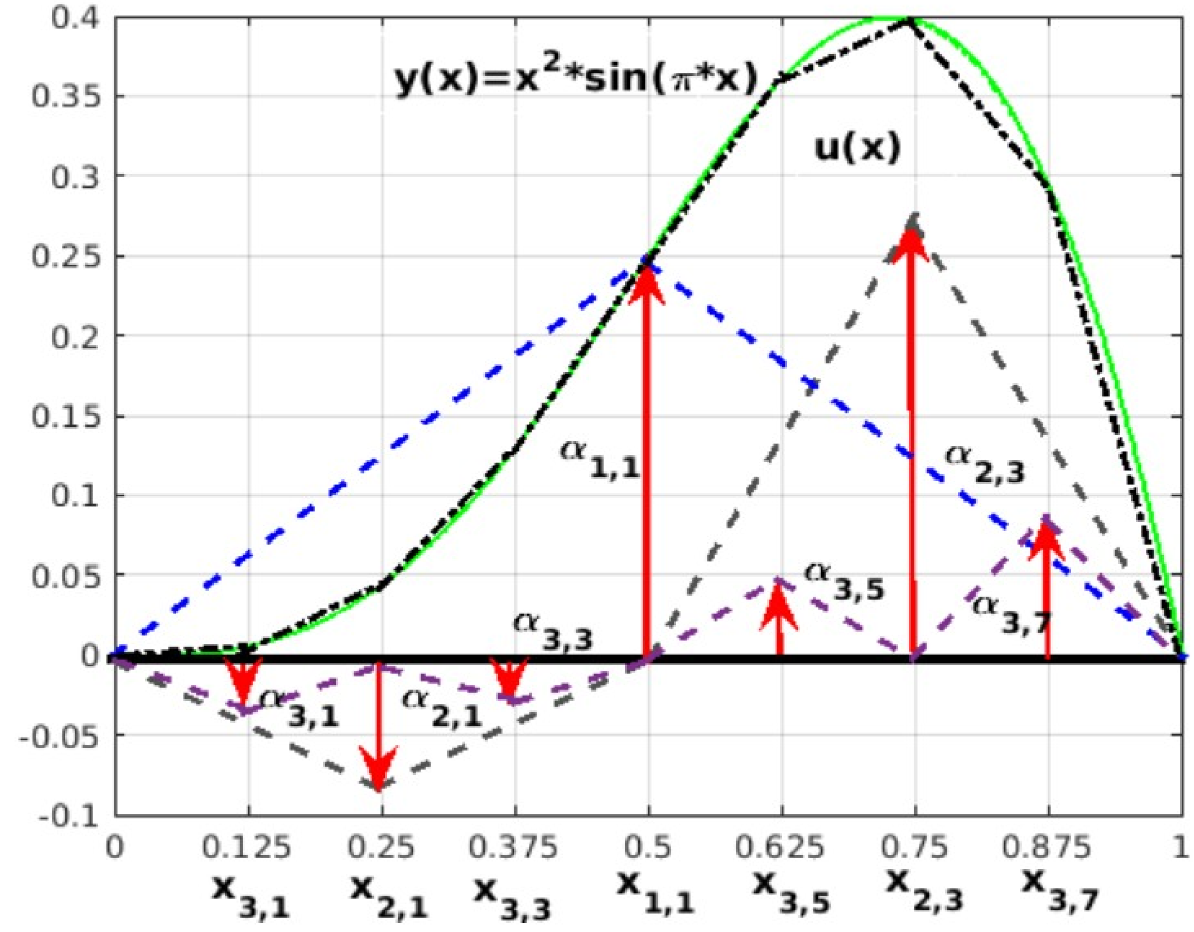


Piecewise Linear Interpolation: Level III

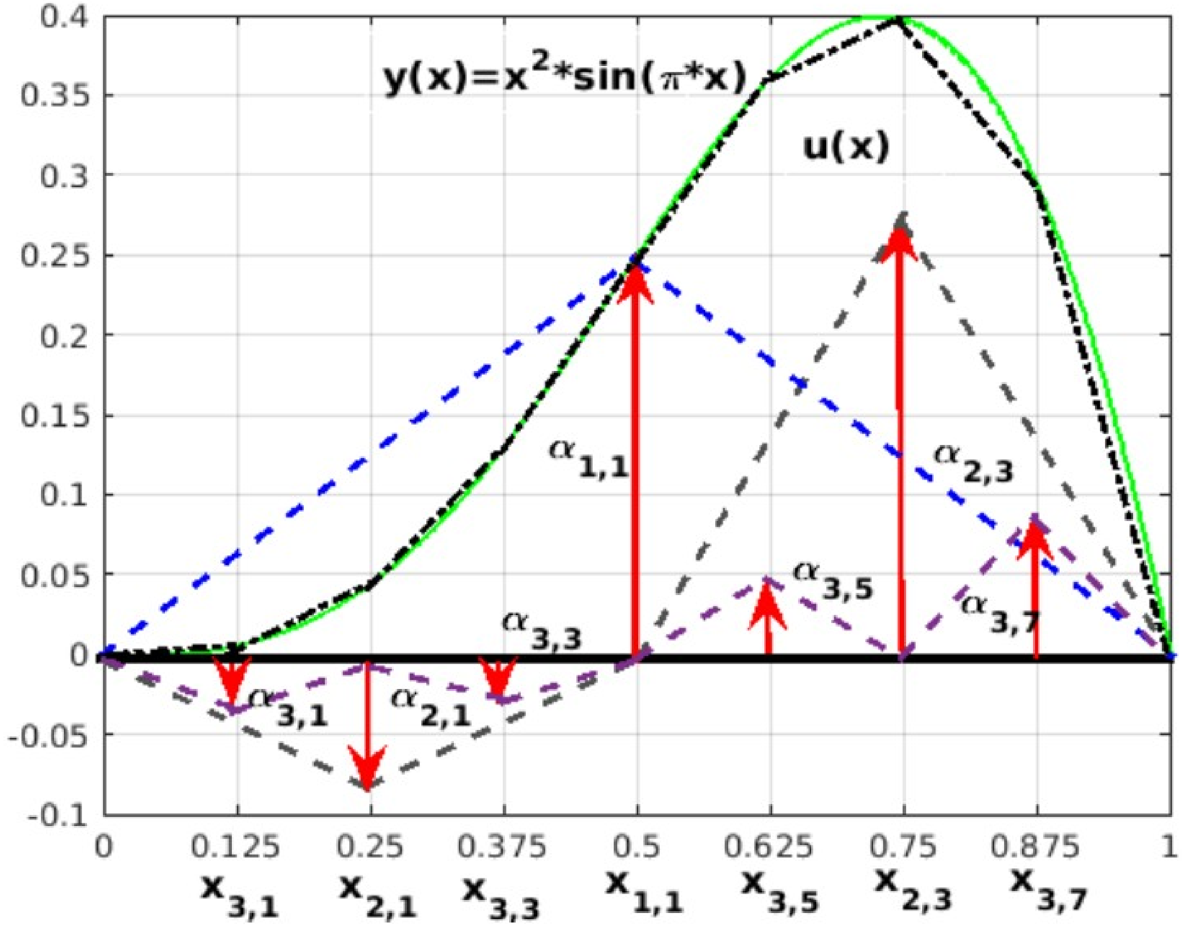
Coefficients:
hierarchical surpluses

They correct the interpolant of level *l*-1 at $\vec{x}_{l,i}$ to the actual value of $f\left(\vec{x}_{l,i}\right)$

Nested structure:
Evaluate function only at points that are unique to the new level.



MOVIE



y(x)=x²*sin(π*x)
u(x)
α1,1
α2,3
α3,5
α3,7
α3,3
α3,1
α2,1
0.4
0.35
0.3
0.25
0.2
0.15
0.1
0.05
0
-0.05
-0.1
0
0.125
0.25
0.375
0.5
0.625
0.75
0.875
1
x3,1
x2,1
x3,3
x1,1
x3,5
x2,3
x3,7

Non-zero Boundary Conditions

Want to be able to handle non-zero boundaries:

$$f|_{\partial\Omega} \neq 0$$

If we add naively points at boundaries, **3^d** support nodes will be added.

Numerically cheapest way:
Modify basis functions and interpolate towards boundary.
Various choices possible!

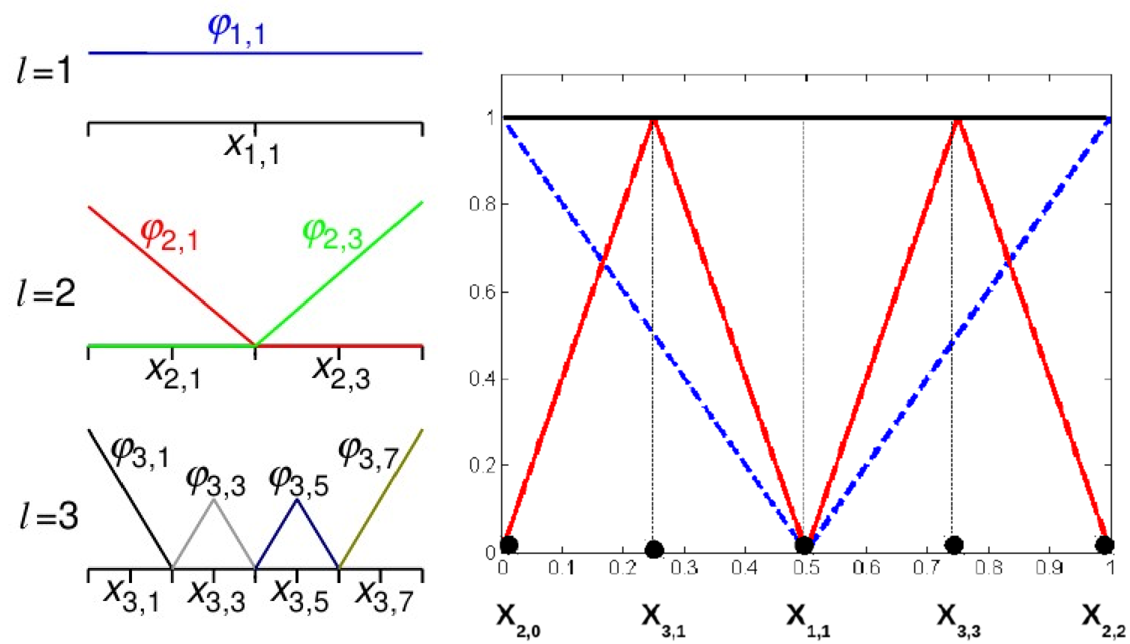


Fig.: Example of modified 1d-basis functions According to Pflüger (2010), which are extrapolating towards the boundary (**left**). They are constant on level 1 and **"folded-up"** if adjacent to the boundary on all other levels. **Right**: "Modified" hat basis.

Examples for basis functions

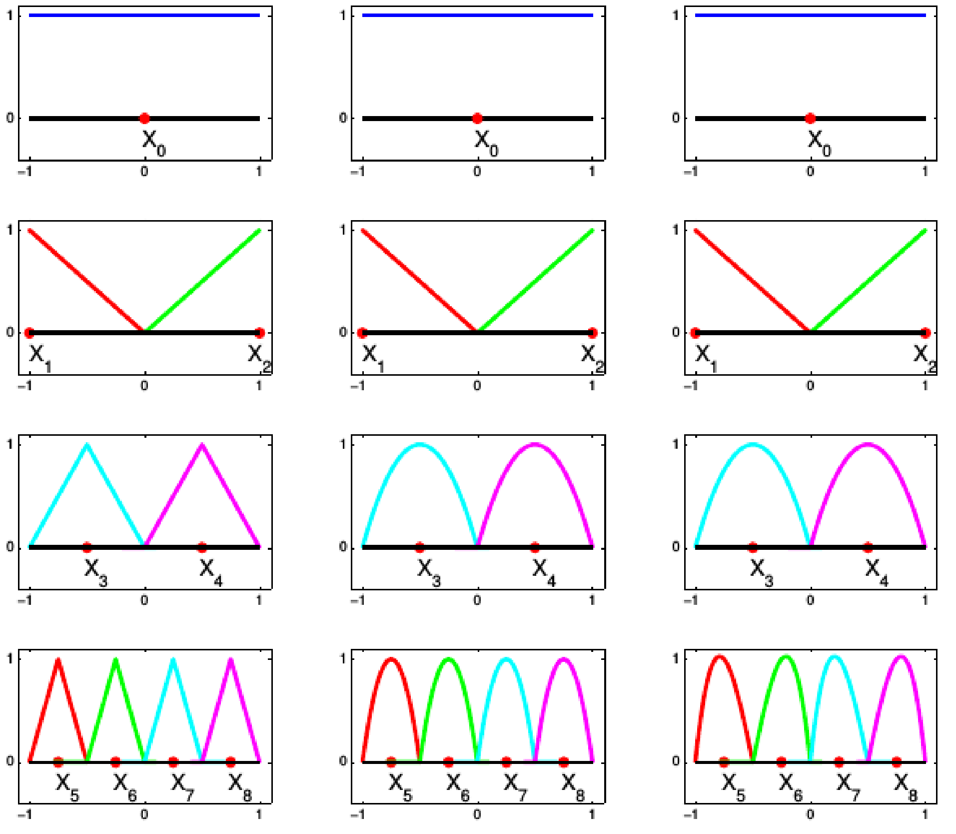


Figure 1: Local polynomial points (*rule_localp*) and functions, left to right: linear, quadratic, and cubic functions.

Fig. taken from Tasmanian User Manual

Some definitions & notation

(see, e.g. Zenger (1991), Bungartz & Griebel (2004), Garcke (2012), Pflüger (2010),...)

- We will focus on the domain **$\Omega = [0,1]^d$**

  d: dimensionality; **other domains: rescale**

- introduce **multi-indices**:

  grid refinement level: $\vec{l} = (l_1, ..., l_d) \in \mathbb{N}^d$

  spatial position: $\vec{i} = (i_1, ..., i_d) \in \mathbb{N}^d$

- Discrete, (Cartesian) full grid $\Omega_{\vec{l}}$ on Ω

- Grid $\Omega_{\vec{l}}$ consists of points: $\boxed{\vec{x}_{\vec{l},\vec{i}} := (x_{l_1,i_1}, ..., x_{l_d,i_d})}$

  Where $x_{l_t,i_t} := i_t \cdot h_{l_t} = i_t \cdot 2^{-l_t}$ and $i_t \in \{0, 1, ..., 2^{l_t}\}$

Multi-Dimensional Interpolant

Extension to multi-d by a **tensor-product construction**:

Multi-d basis: $\phi_{\vec{l},\vec{i}}(\vec{x}) := \prod_{t=1}^{d} \phi_{l_t,i_t}(x_t)$

Index set: $I_{\vec{l}} := \{\vec{i} : 1 \leq i_t \leq 2^{l_t} - 1, i_t \text{ odd}, 1 \leq t \leq d\}$

Hierarchical increments: $W_{\vec{l}} := \text{span}\{\phi_{\vec{l},\vec{i}} : \vec{i} \in I_{\vec{l}}\}$

Multi-d interpolant:

$$\longrightarrow f(\vec{x}) \approx u(\vec{x}) = \sum_{|l|_\infty \leq n} \sum_{\vec{i} \in I_{\vec{l}}} \alpha_{\vec{l},\vec{i}} \cdot \phi_{\vec{l},\vec{i}}(\vec{x})$$

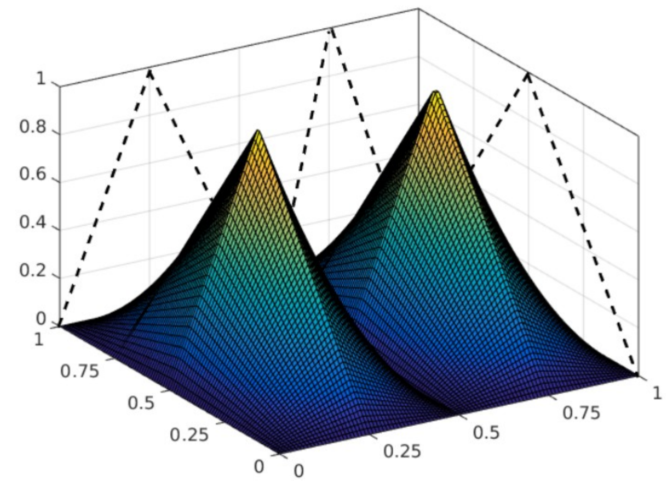


Fig.: Basis functions of the **subspace $W_{2,1}$**

Why reality bites...

Interpolant consists of ***O(2^nd)*** grid points

For **sufficiently smooth *f*** and its interpolant ***u***, we obtain an asymptotic error decay of $\|f(\vec{x}) - u(\vec{x})\|_{L_2} \in \mathcal{O}\left(h_n^2\right)$

But at the cost of $\mathcal{O}\left(h_n^{-d}\right) = \mathcal{O}\left(2^{nd}\right)$

function evaluations → **"curse of dimensionality"**

Hard to handle more than 4 dimensions numerically

→ e.g. d=10, n = 4, 15 points/d, **5.8 x 10^{11}** grid points

`Breaking' the curse of dimensionality I

Question: **"can we construct discrete approximation spaces that are better in the sense that the same number of invested grid points leads to a higher order of accuracy?"** **YES √**

(see, e.g. Bungartz & Griebel (2004))

→ If **second mixed derivatives are bounded**, then the hierarchical **surpluses decay rapidly** with increasing approximation level.

$$|\alpha_{\vec{l},\vec{i}}| = \mathcal{O}\left(2^{-2|\vec{l}|_1}\right)$$

`Breaking' the curse of dimensionality II

(see, e.g. Bungartz & Griebel (2004))

Strategy of constructing sparse grid: leave out those subspaces from full grid that only contribute little to the overall interpolant.

Optimization w.r.t. number of degrees of freedom (grid points) and the approximation accuracy leads to the sparse grid space of level ***n***.

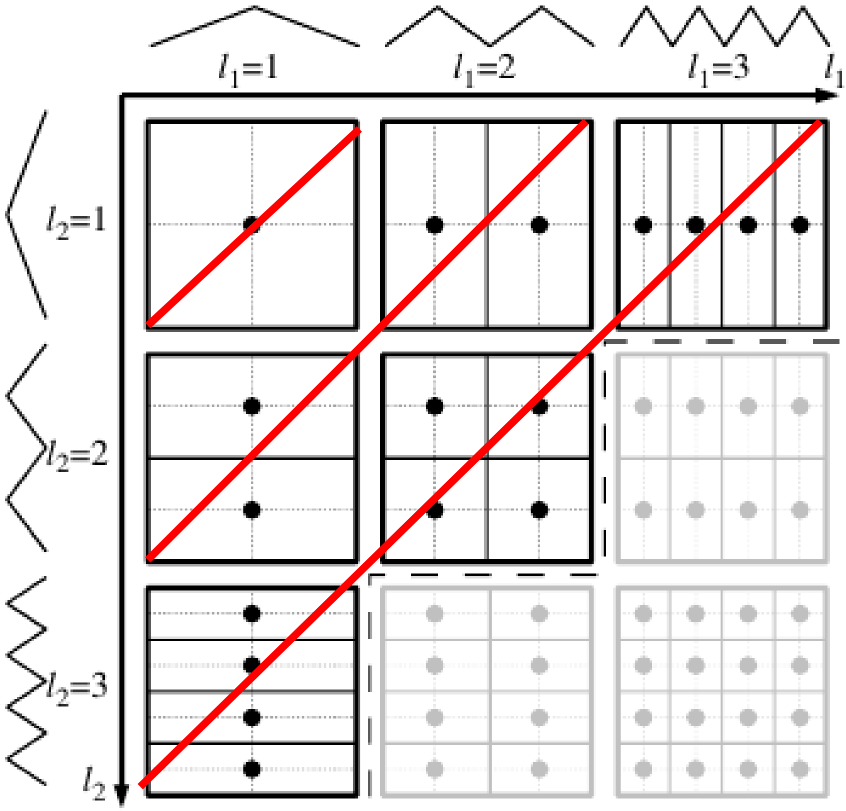
$$V_{0,n}^{S} := \bigoplus_{|\vec{l}|_1 \leq n+d-1} W_{\vec{l}}$$

Interpolant: $$f_{0,n}^{S}(\vec{x}) \approx u(\vec{x}) = \sum_{|l|_1 \leq n+d-1} \sum_{\vec{i} \in I_{\vec{l}}} \alpha_{\vec{l},\vec{i}} \cdot \phi_{\vec{l},\vec{i}}(\vec{x})$$

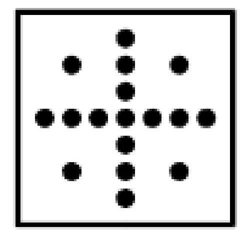
grid points: $\underline{\mathcal{O}\left(h_n^{-1} \cdot \left(\log(h_n^{-1})\right)^{d-1}\right) = \mathcal{O}\left(2^n \cdot n^{d-1}\right)}$ << $\mathcal{O}\left(h_n^{-d}\right) = \mathcal{O}\left(2^{nd}\right)$

Accuracy of the interpolant: $\underline{\mathcal{O}\left(h_n^2 \cdot \log(h_n^{-1})^{d-1}\right)}$ vs. $\mathcal{O}\left(h_n^2\right)$

Sparse grid construction in 2D



$$V_{0,n}^S := \bigoplus_{|\vec{l}|_1 \le n+d-1} W_{\vec{l}}$$



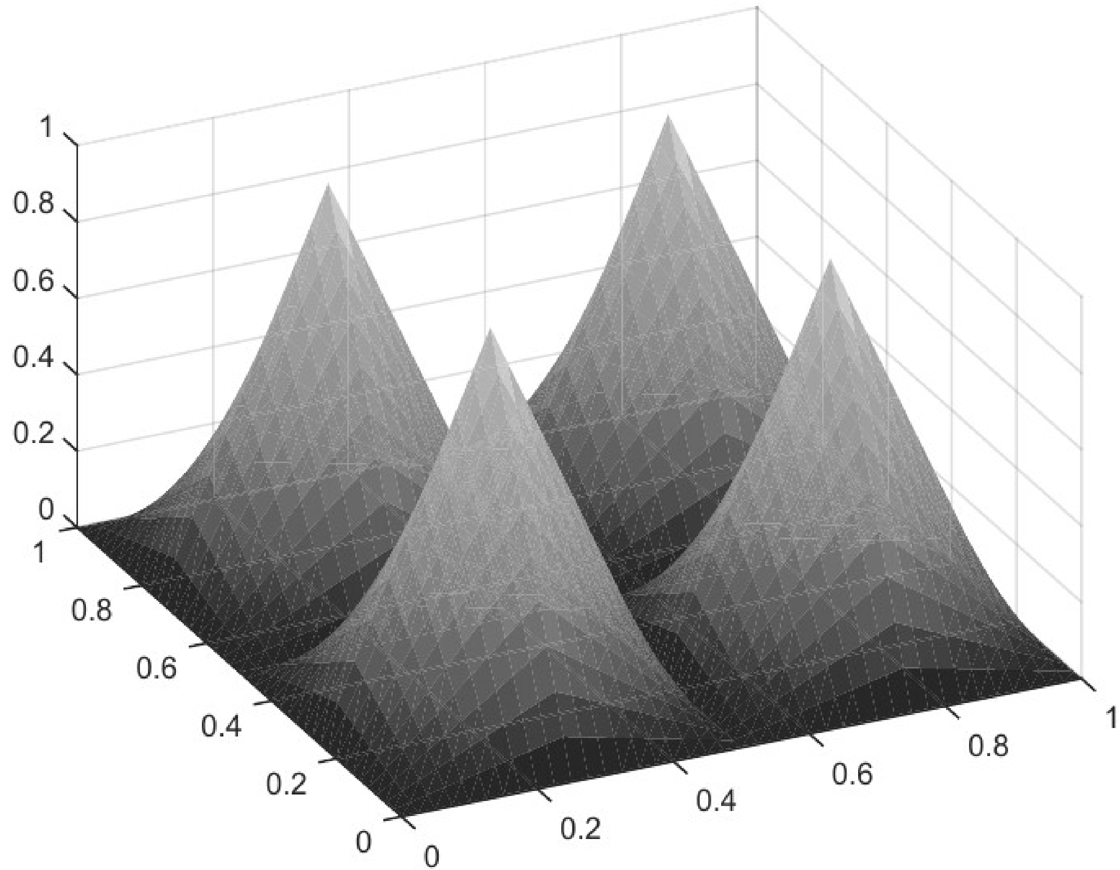
V_3

Sparse grid: 17 pt.
Full grid : 49 pt.

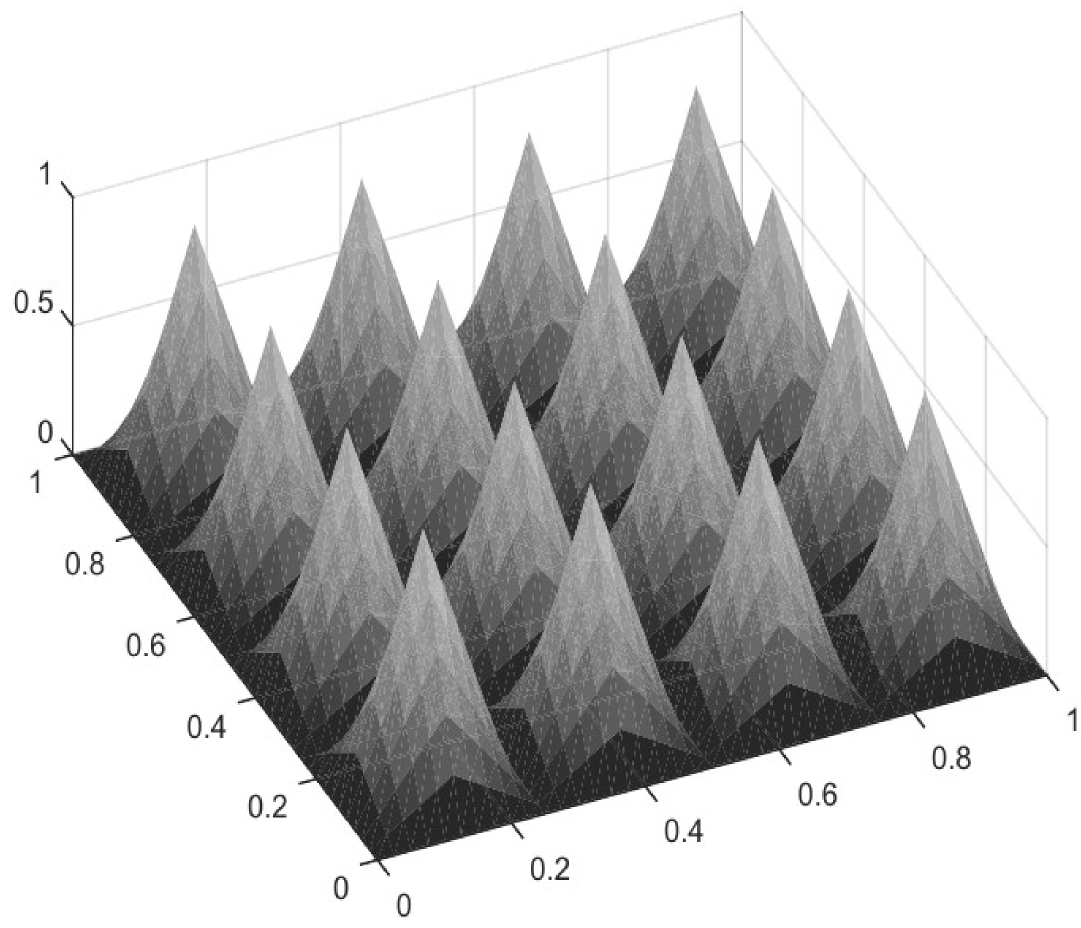
Fig.: Two-dimensional subspaces **W**$_l$ up to *l*=3 (h_3 = 1/8) in each dimension.

The optimal a priori selection of subspaces in shown in black **(left)** and the Corresponding sparse grid of level n = 3 **(right)**.
For the full grid, the gray subspaces have to be used as well.

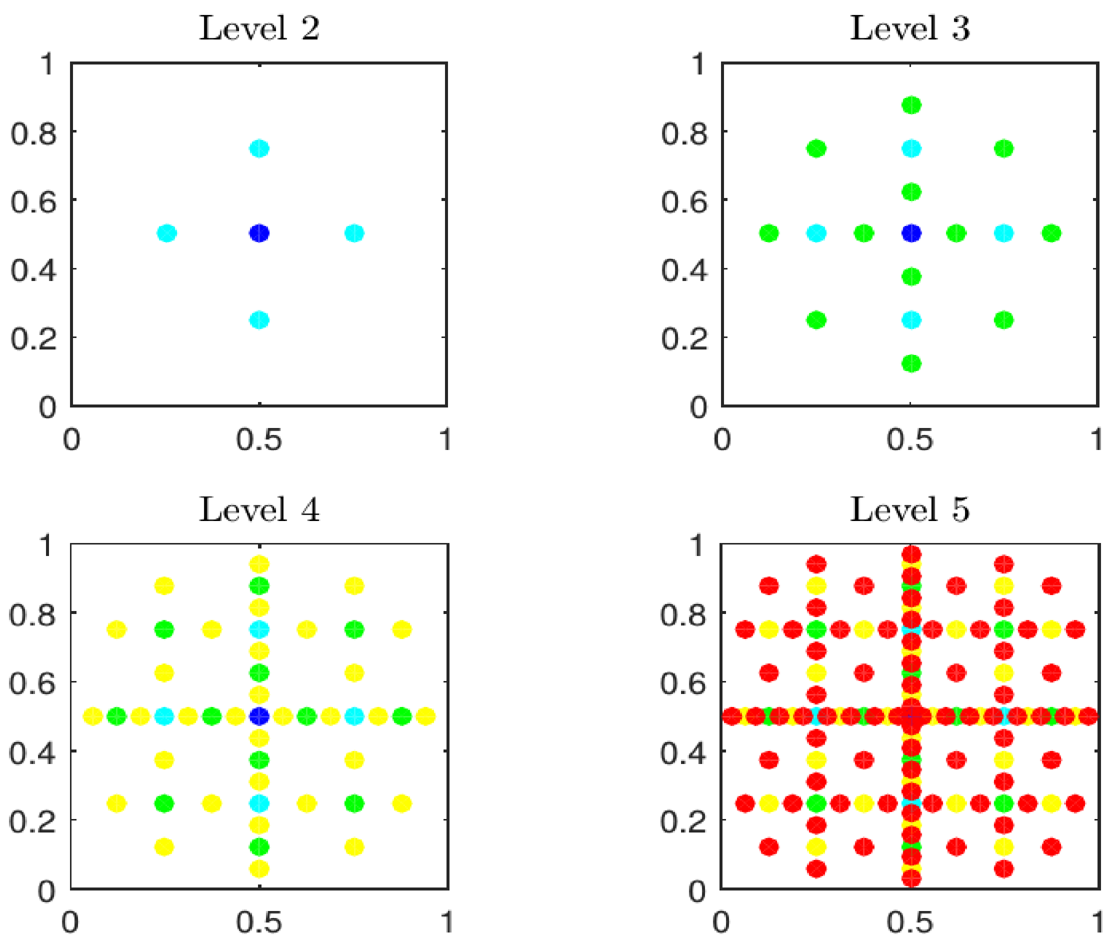
Basis Functions of $W_{2,2}$ — Included in V_3



Basis Functions of $W_{3,3}$ — not Included in V_3

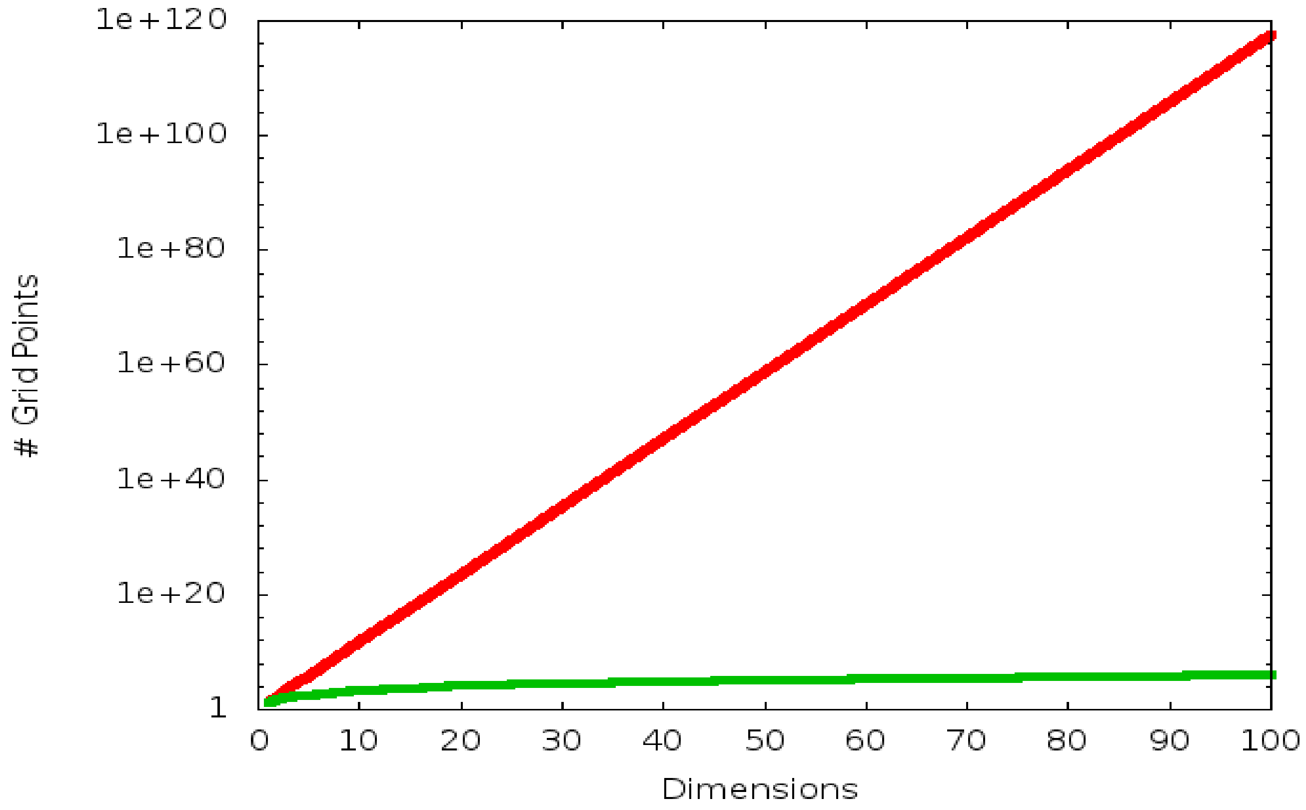


Sparse Grid of Increasing level



Grid Points

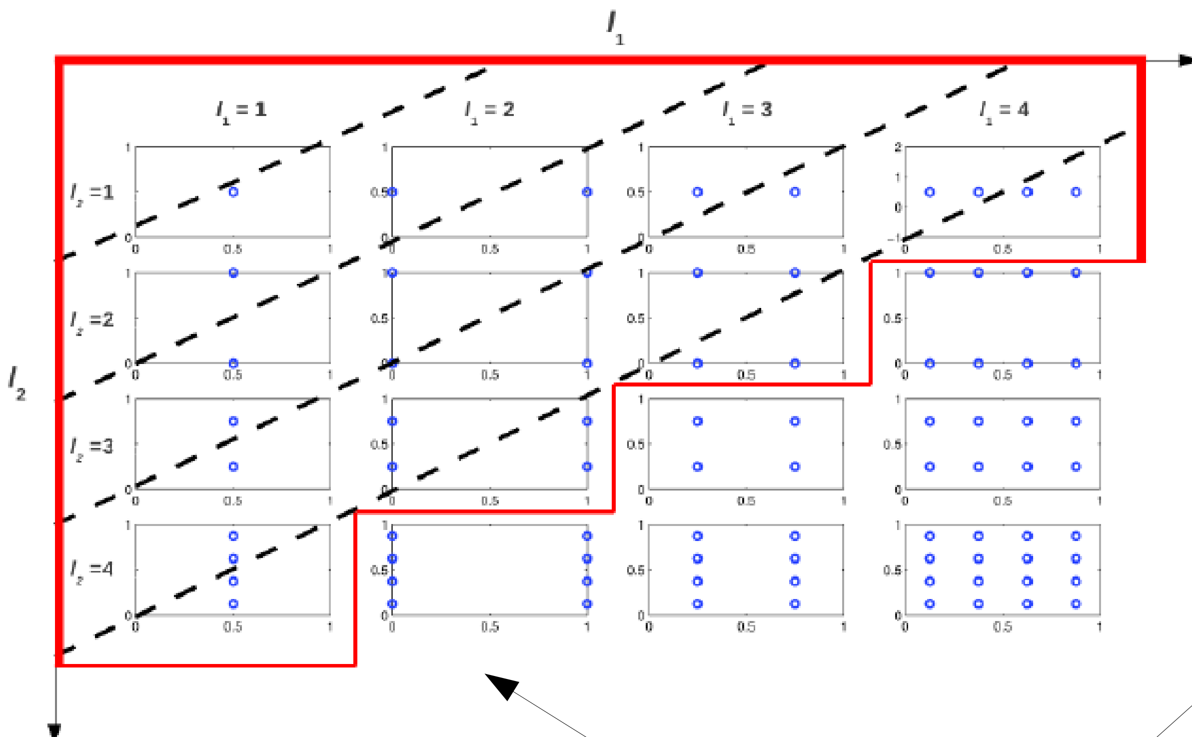
d	$\lvert V_n \rvert$	$\lvert V_{0,n}^S \rvert$
1	15	15
2	225	49
3	3375	111
4	50'625	209
5	759'375	351
10	$5.77 \cdot 10^{11}$	2'001
15	$4.37 \cdot 10^{17}$	5'951
20	$3.33 \cdot 10^{23}$	13'201
30	$1.92 \cdot 10^{35}$	41'601
40	$1.11 \cdot 10^{47}$	95'201
50	$6.38 \cdot 10^{58}$	182'001
100	>Googol	1'394'001

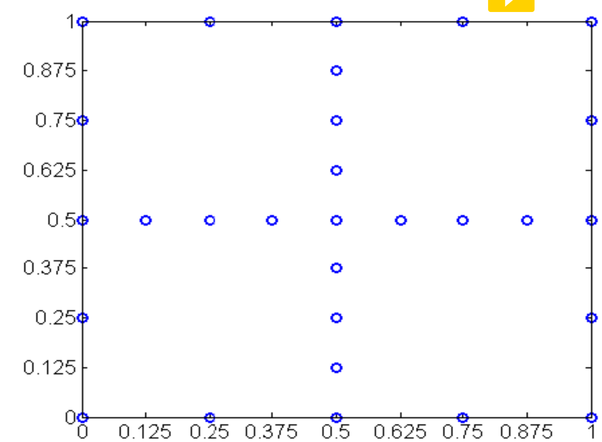


Tab.: Number of grid points for several types of sparse grids of level n = 4.
Middle: Full grid; **right:** **classical sparse grid** with **no points at the boundaries.**
Fig.: Number of grid points growing with dimension (full grid vs. sparse grid).

Sparse Grid with non-zero boundaries

(see, e.g. Bungartz & Griebel (2004))





Sparse grid: 29 pt.
Full grid : 81 pt.

$$V_n^{S,CC} := \bigoplus_{|\vec{l}|_1 \le n+d-1} W_{\vec{l}}.$$

Hierarchical Integration

High-dimensional integration easy with sparse grids, e.g. compute expectations
Let's assume uniform probability density:

$$\mathbb{E}\left[u(\vec{x})\right] = \sum_{|l|_1 \leq n+d-1} \sum_{\vec{i} \in I_{\vec{l}}} \alpha_{\vec{l},\vec{i}} \int_{\Omega} \phi_{\vec{l},\vec{i}}(\vec{x}) d\vec{x}$$

The one-dimensional integral can now be computed analytically (Ma & Zabras (2008))

$$\int_0^1 \phi_{l,i}(x)\, dx = \begin{cases} 1, & \text{if } l = 1 \\ \frac{1}{4} & \text{if } l = 2 \\ 2^{1-l} & \text{else} \end{cases}$$

Note that this result is independent of the location of the interpolant to dilation
And translation properties of the hierarchical basis functions.
→ **Multi-d integrals are therefore again products of 1-d integrals.**

We denote $\int_{\Omega} \phi_{l,i}(\vec{x})\, d\vec{x} = J_{\vec{l},\vec{i}}$

$$\longrightarrow \mathbb{E}\left[u(\vec{x})\right] = \sum_{|l|_1 \leq n+d-1} \sum_{\vec{i} \in I_{\vec{l}}} \alpha_{\vec{l},\vec{i}} \cdot J_{\vec{l},\vec{i}}$$

Where are Sparse Grids used?

For a review, see, e.g. Bungartz & Griebel (2004)

Sparse grid methods date back to Smolyak(1963)

BUT: Smolyak used global polynomials!

So far, methods applied to:

-High-dimensional integration

e.g. Gerstner & Griebel (1998), Bungartz et al. (2003),...

-Interpolation

e.g. Barthelmann et al. (2000), Klimke & Wohlmut (2005),...

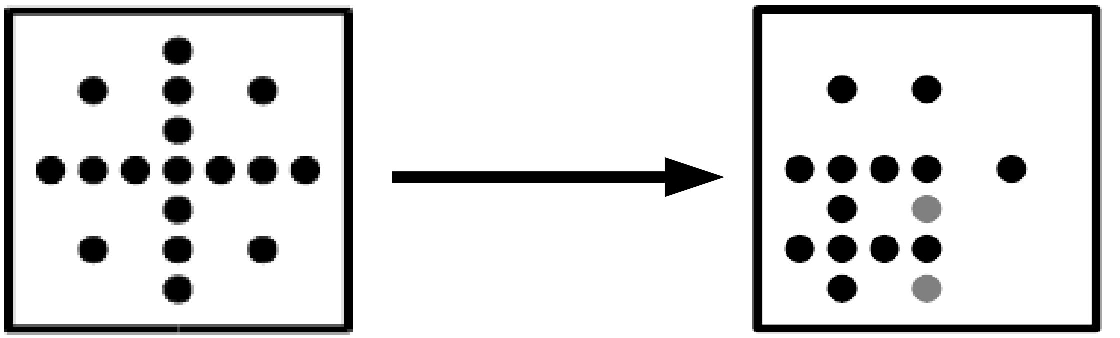
-Solution of PDEs

e.g. Zenger (1991), Griebel (1998),...

More fields of application: regressions, data mining, likelihood estimations, option pricing, data compression, dynamic economic models...

e.g. Kubler & Kruger (2004), Winschel & Kraetzig (2010), Judd et al. (2013) → Smolyak; global basis functions.

III. Adaptive Sparse Grids



Sketch of adaptive refinement

See, e.g. Ma & Zabaras (2008), Pflüger (2010), Bungartz (2003),..

-Surpluses should quickly decay to zero

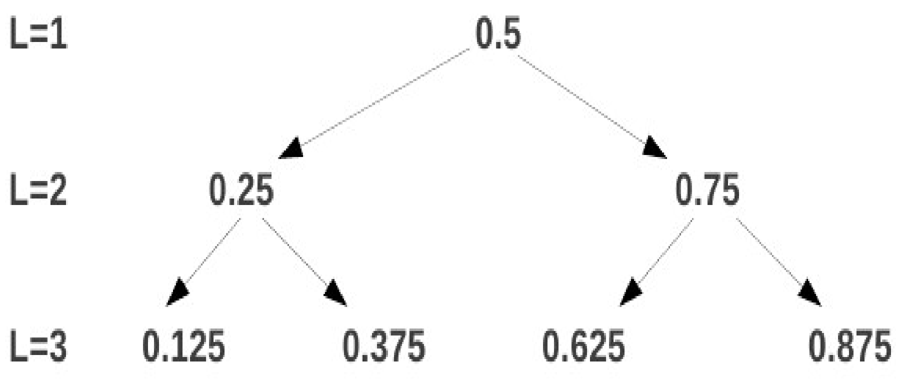
-**Use hierarchical surplus as error indicator**.

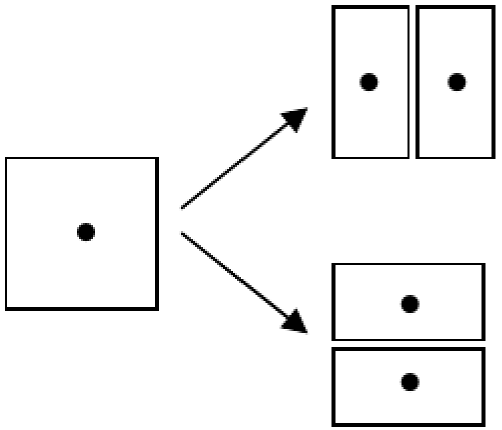
-**Automatically detect "discontinuity regions"** and adaptively refine the points in this region.

-Each grid point has ***2d*** neighbours

-**Add neighbour points, i.e. locally refine interpolation level from *l* to *l+1***

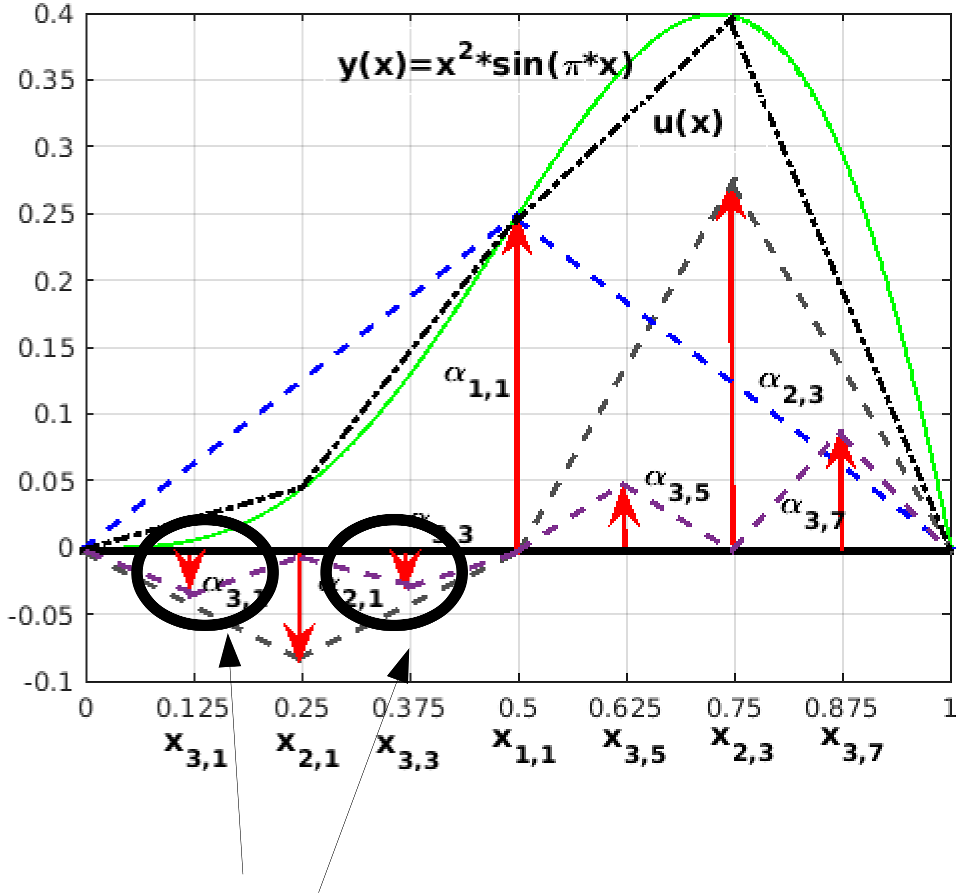
-Criterion: **e.g.** $|\alpha_{\vec{l},\vec{i}}| \geq \epsilon$





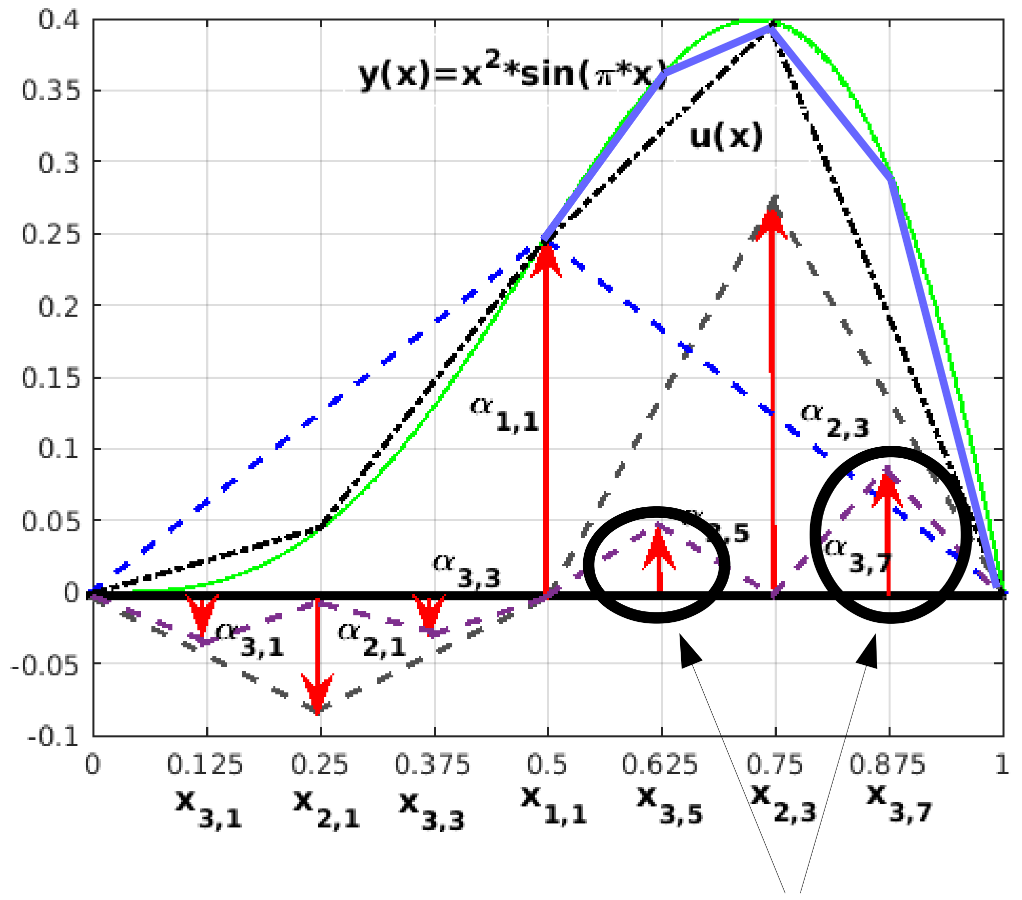
top panel: tree-like structure of sparse grid.
lower panel: locally refined sparse grid in 2D.

Example I



Small – below threshold

Example II



Add points – above threshold

Test in 1d

(See Genz (1984) for test functions)

Test function:

$$f(x) = \frac{1}{|0.5 - x^4| + 0.01}$$

Error both for full grid and adapt. sparse grid of *O(10^-2)*.

Error measure:

→ 1000 random points from [0,1]

$$e = \max_{i=1,\ldots,1000} |f(\vec{x}_i) - u(\vec{x}_i)|$$

Full grid: **1023** points
Adaptive sparse grid: **109** points.

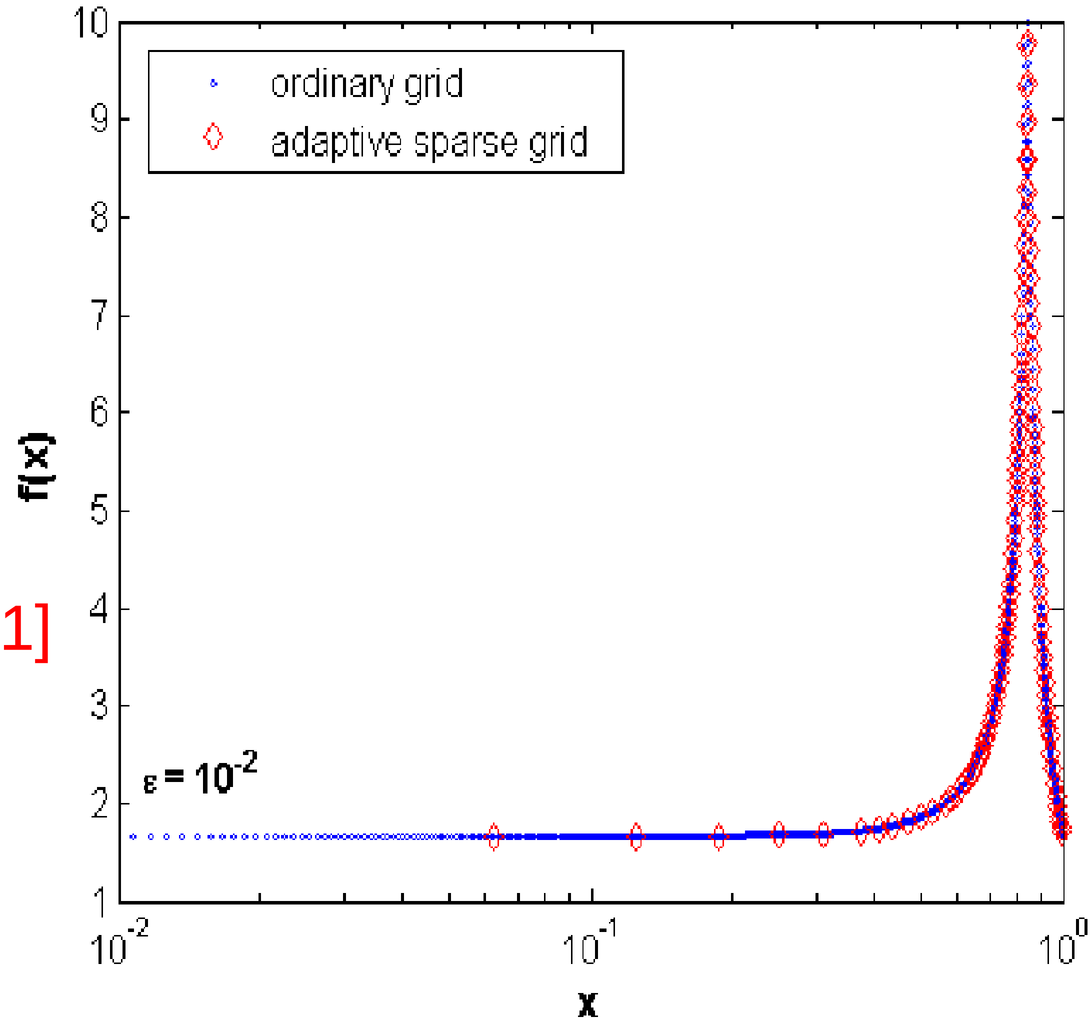


Fig.: Blue: Full grid;red: adaptive sparse grid.

Test in 2d

Test function: $$\frac{1}{|0.5 - x^4 - y^4| + 0.1}$$

Error: *$O(10^{-2})$*

Full grid:
→ *$O(10^9)$* points

Sparse grid:
→ **311,297 points**

Adaptive sparse grid:
→ **4,411 points**

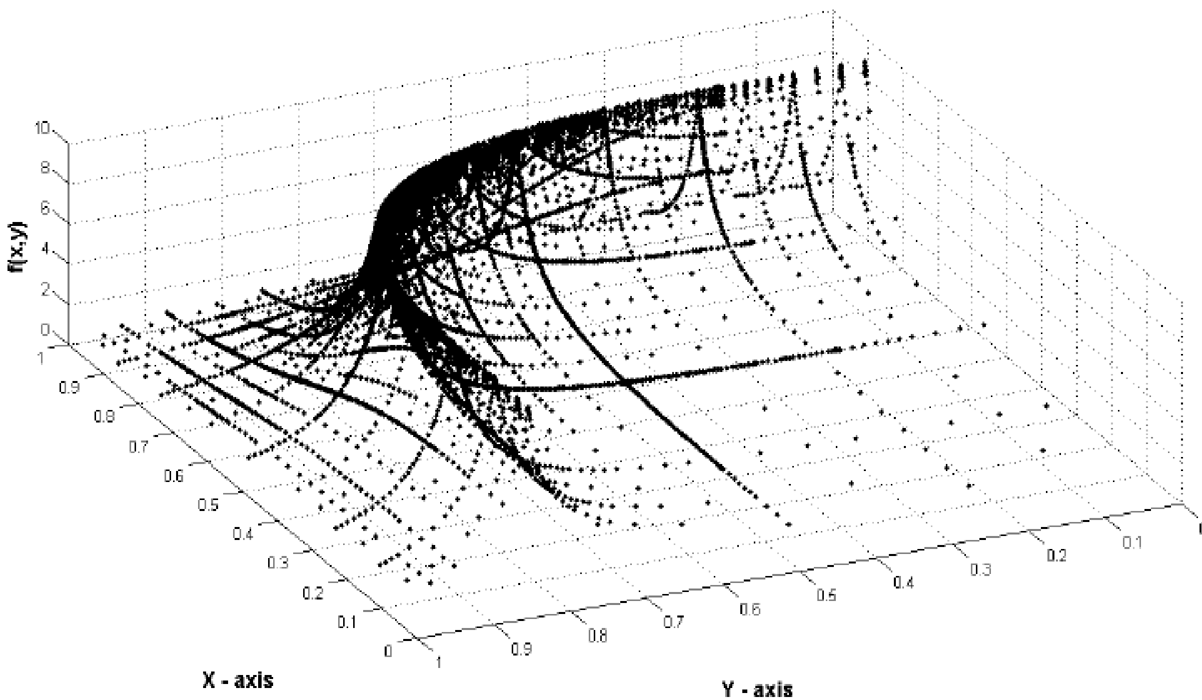


Fig.: 2d test function and its corresponding grid points after 15 refinement steps.

Movie

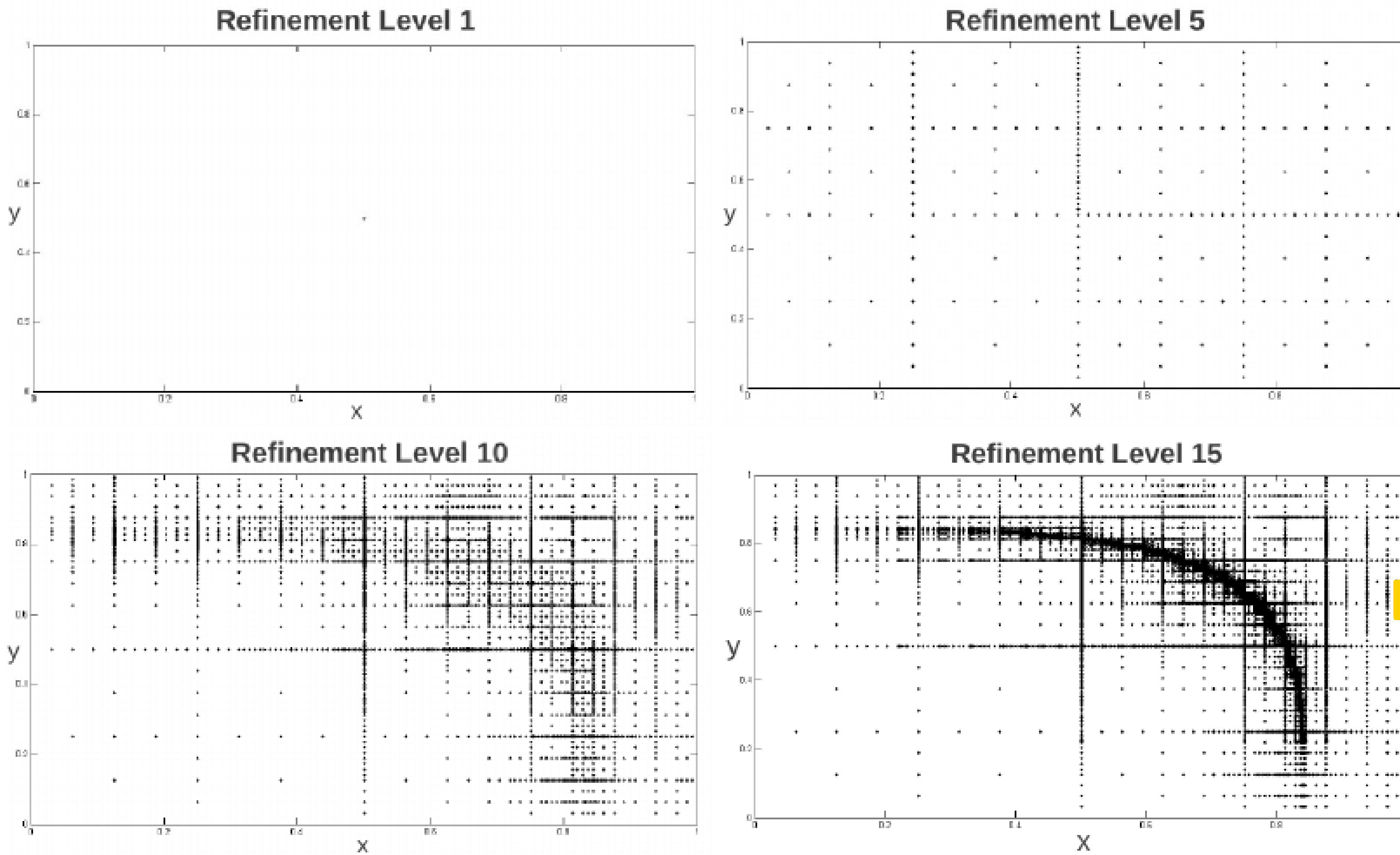


Fig.: Evolution of the adaptive sparse grid with a **threshold for refinement of 10^{-2}** . The refinement levels displayed are L = 1, 5, 10, 15.

Convergence

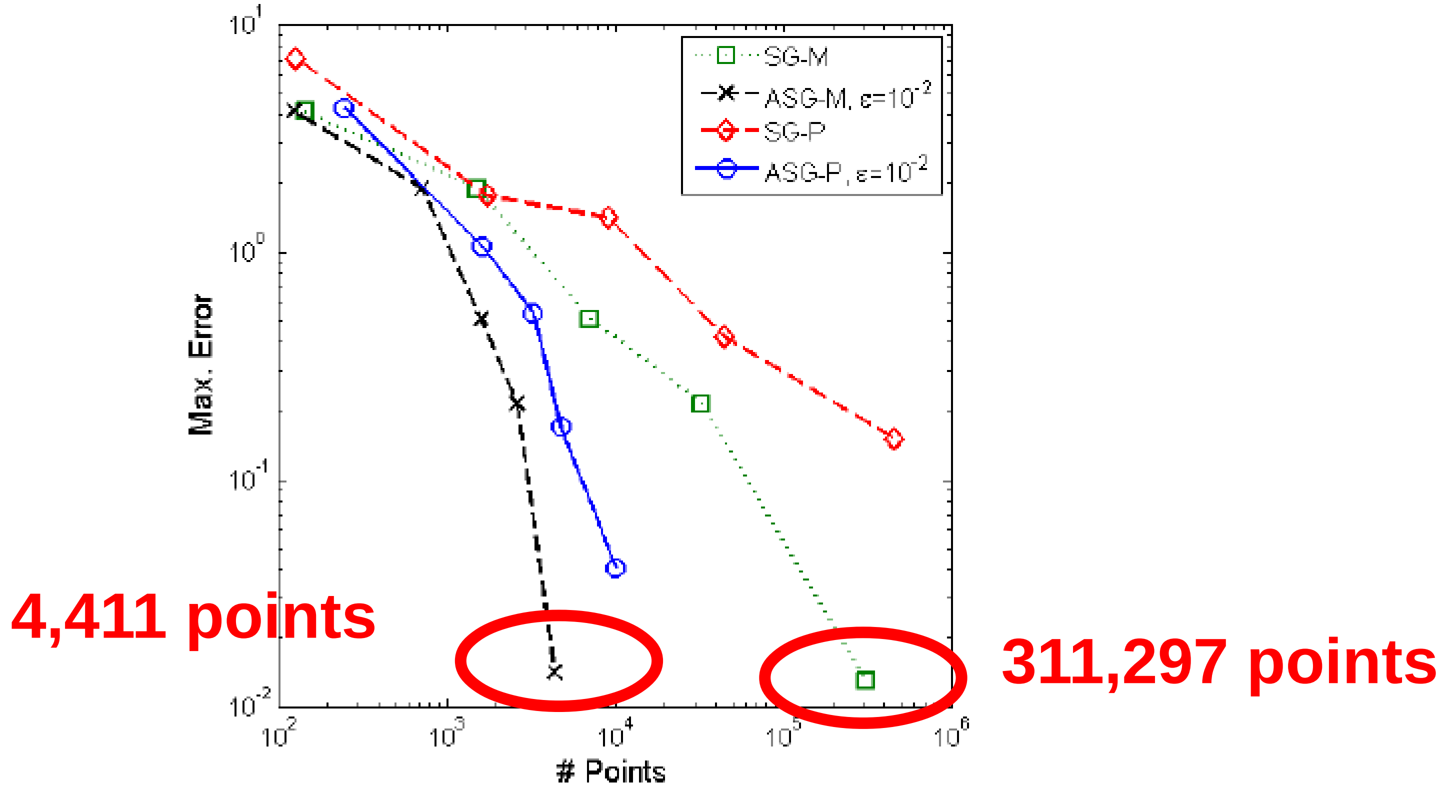
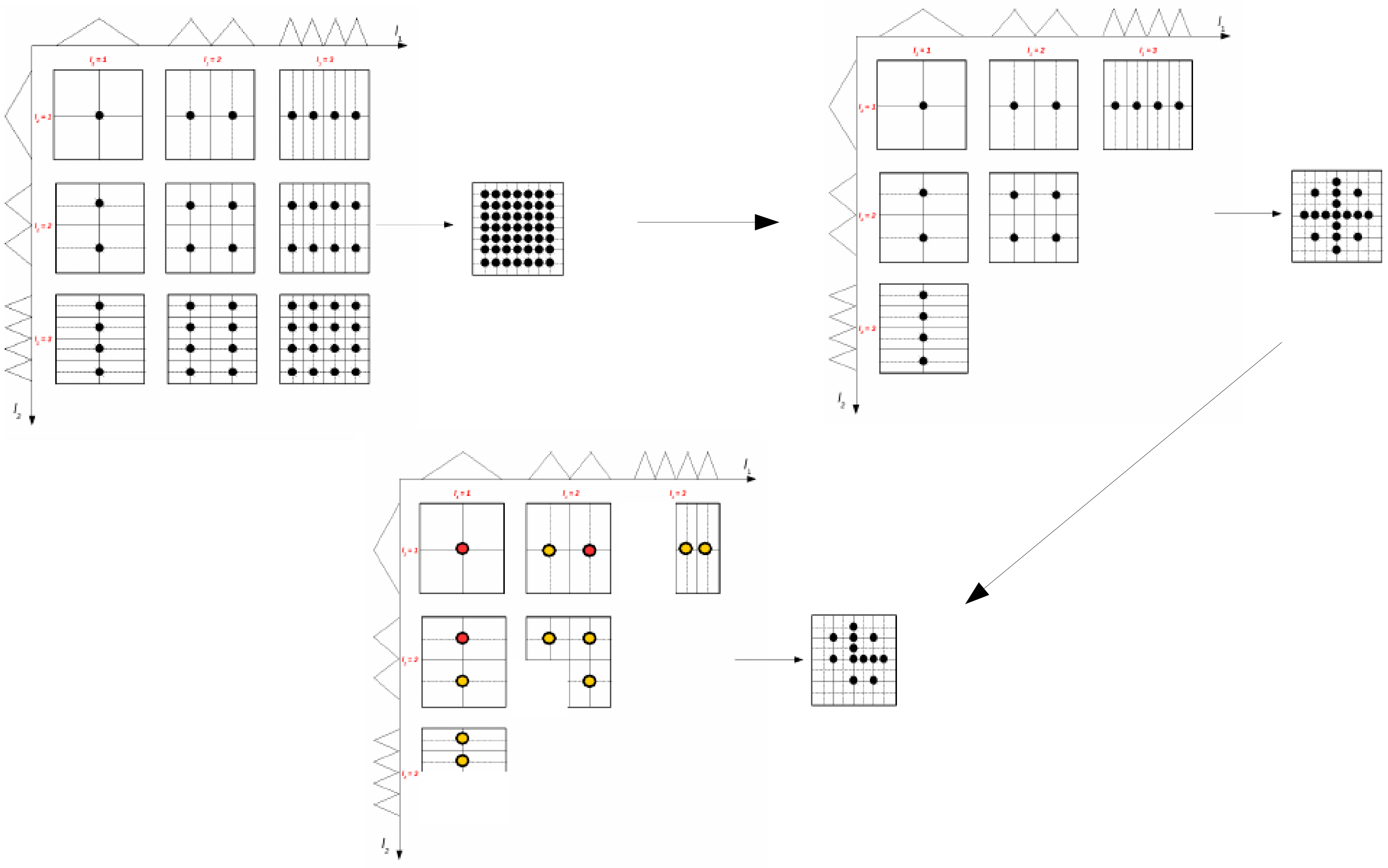


Fig.: Comparison of the interpolation error for **conventional and adaptive sparse grid interpolation** (two different adaptive sparse grid choices).

From Cartesian to adaptive sparse grids



IRBC with Adjustment Costs and Irreversible Investment

- Use test case from JEDC project on computational methods
(Den Haan et al. 2011, Juillard and Villemot 2011, Kollmann et al. 2011)

- *N* countries facing productivity shocks and capital adjustment costs

⇒ countries differ in productivity, ***a*** (stochastic and exogen.), and capital stock, ***k*** (endogen.)
⇒ dimension of the state space is ***2N***
⇒ recursive equilibrium is characterized by
policy $p: R_+^{2N} \Rightarrow R_+^{N+1}$
mapping state into ***N*** capital choices and ***1*** Lagrange multiplier (as markets are complete)

- Extension: Investment in each country is irreversible:

⇒ recursive equilibrium is characterized by policy
$p: R_+^{2N} \Rightarrow R_+^{2N+1}$
mapping state into ***N*** capital choices, ***1*** Lagrange multiplier,
and ***N*** Karush-Kuhn-Tucker multipliers

Time Iteration Algorithm

Algorithm 1 Overview of the crucial steps of the time-iteration algorithm.

Time-Iteration Algorithm:

1. Make an initial guess p_{init} for next period's policy function. Set $p_{next} = p_{init}$. Choose an approximation accuracy $\bar{\eta}$.
2. Make one time-iteration step:
   (a) Choose a maximal refinement level L_{max} and a fixed grid level $L_0 \leq L_{max}$. Set $l = 1$, set $G \subset X$ to be the level 1 grid on X, and set $G_{old} = \emptyset$.
   (b) For each $g \in G \setminus G_{old}$ compute the optimal policies $p(g)$ by solving the system of equilibrium conditions

   $$0 = \mathbb{E}\left\{ f\left(g, x_{t+1}, p(g), p_{next}(x_{t+1})\right) | g, p(g) \right\},$$
   $$x_{t+1} \sim F(\cdot | g, p(g)),$$

   given next period's policy p_{next}. Construct the hierarchical surpluses of level l.
   (c) Generate G_{new} from G by adding for each $g \in G_l \setminus G_{old}$ its $2d$ neighbouring points, if either $l < L_0$ or

   $$\|p(g) - \tilde{p}(g)\|_\infty > \varepsilon,$$

   where the policy $\tilde{p}(g)$ is given by interpolating between $\{p(g)\}_{g \in G_{old}}$. Note that if G_{old} is of level 2, then each point does not have $2d$ but only d neighbouring points (cf. Fig. 3.4).
   (d) If $G_{new} = G$ or $l = L_{max}$, then set $G = G_{new}$ and go to (e), else set $G = G_{new}$, $l = l + 1$ and go to (b).
   (e) Define the policy function p as the sparse grid interpolation of $\{p(g)\}_{g \in G}$.
   (f) Calculate (an approximation for) the error, e.g.

   $$\eta = \|p - p_{next}\|_\infty.$$

   If $\eta > \bar{\eta}$, set $p_{next} = p$ and go to (a), else go to step 3.
3. The (approximate) equilibrium policy function is given by p.

Time Iteration Algorithm Details

N Euler equations

$$g_{t+1}^j := k_{t+1}^j / k_t^j - 1, \ g_{t+2}^j := k_{t+2}^j / k_{t+1}^j - 1,$$

$$\forall j : \lambda_t \cdot \left[1 + \phi \cdot g_{t+1}^j\right] - \beta \cdot \mathbb{E}_t \left\{ \lambda_{t+1} \cdot \left[a_{t+1}^j \cdot A \cdot \alpha \cdot (k_{t+1}^j)^{\alpha-1} + (1-\delta) + \frac{\phi}{2} \cdot g_{t+2}^j \cdot \left(g_{t+2}^j + 2\right) \right] \right\} = 0,$$

$$\sum_{j=1}^{N} \left(a_t^j \cdot A \cdot (k_t^j)^\alpha + k_t^j \cdot \left((1-\delta) - \frac{\phi}{2} \cdot (g_{t+1}^j)^2 \right) - k_{t+1}^j - \left(\frac{\lambda_t}{\tau_j} \right)^{-\gamma^j} \right) = 0.$$

1 aggregate resource constraint

$$(a_t^1, \ldots, a_t^N, k_t^1, \ldots, k_t^N) \longrightarrow \textbf{Current grid points}$$

solve for the unknown policy variables

$$(k_{t+1}^1, \ldots, k_{t+1}^N, \lambda_t) \longrightarrow \textbf{Solution of system of Eqs. At the current iteration}$$

given the known policy functions from last iteration

$$\left(k_{t+2}^1(a_{t+1}, k_{t+1}), \ldots, k_{t+2}^N(a_{t+1}, k_{t+1}), \lambda_{t+1}(a_{t+1}, k_{t+1})\right) \longrightarrow \textbf{Interpolant}$$

evaluated at next periods state

$$(a_t^1, \ldots, a_t^N, k_t^1, \ldots, k_t^N), \text{where } a_{t+1}^j = (a_t^j)^\rho \cdot e^{\sigma(e_t + e_t^j)}.$$

Results for Smooth IRBC Model

Non-adaptive sparse grid of fixed level produces stable accuracy when dimension is increased massively:

Dimension	Level	Points	Max. Error	Avg. Error
4	3	41	-2.95	-3.18
12	3	313	-2.81	-3.27
20	3	841	-2.93	-3.30
50	3	5,101	-2.64	-3.33
100	3	20,201	-2.79	-3.33
4	4	137	-3.04	-3.65
12	4	2,649	-3.04	-3.83
20	4	11,561	-3.00	-3.73

All errors are given in log 10 -scale.

Results for Non-Smooth IRBC Model: Finer Grids

Non-adaptive sparse grid has problems with non-smooth model:

Dimension	Level	Points	Max. Error	Avg. Error
4	5	401	-2.11	-2.93
4	7	2,929	-2.32	-3.12
4	9	18,945	-2.45	-3.32

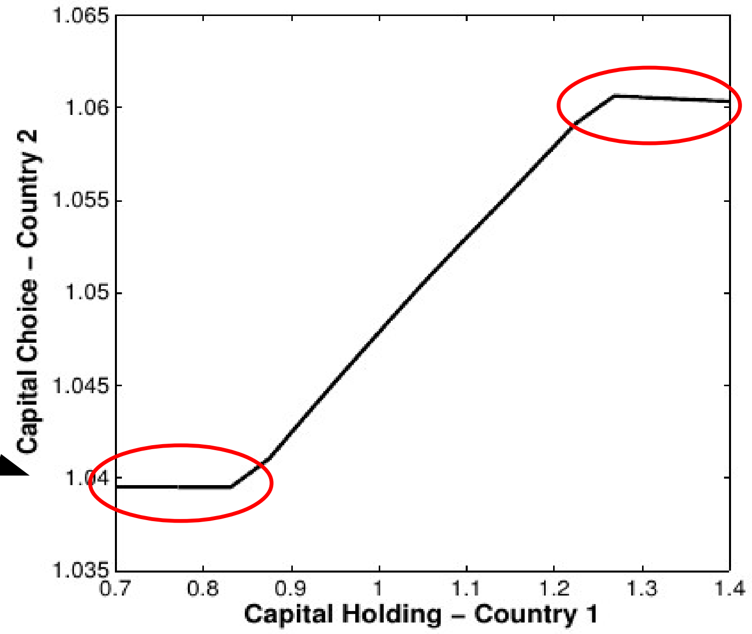
Adaptive sparse grid can overcome problems with non-smooth model
→ they provide higher accuracy with less points:

ϵ	Max. Level Reached	Points	Max. Error	Avg. Error
0.01	7 (2,929)	245	-2.23	-2.88
0.005	9 (1,945)	559	-2.42	-2.98
0.0025	13 (643,073)	2,346	-2.68	-3.32
0.001	14 (3,502,081)	14,226	-2.91	-3.73

All errors are given in log 10 -scale.

IRBC with irreversible investment

Capital in country 1 low:
→ investment opportunities better than in country 2
→ irrev. constraint binding for country 2.

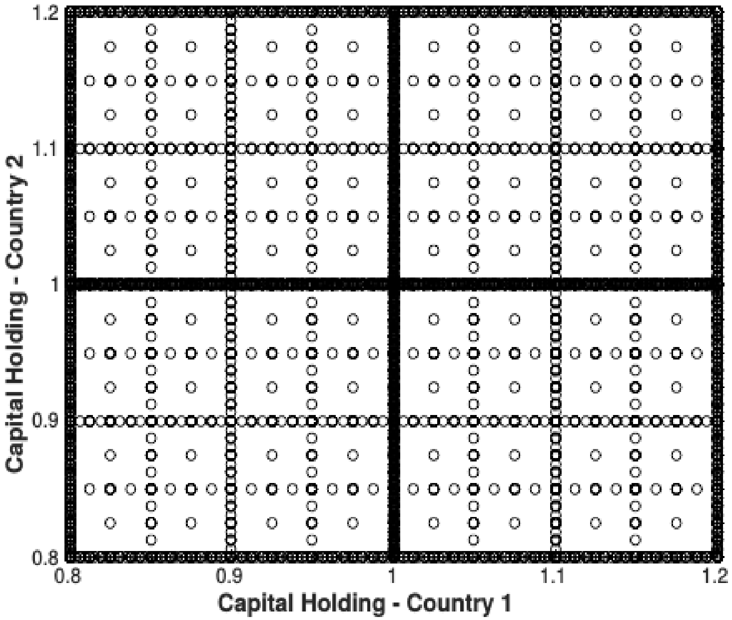


Capital in country 1 high:
→ irrev. constraint binding for country 1.
→ this limits transfer of resources to country 2.

Fig.: Capital choice of country 2 as a function of capital holding of country 1.
All other state variables of this model are kept fixed at steady state (2N = 4d).
The 4-d policy function was interpolated on an adaptive sparse grid (ε = 0.0033).

Note: **kink is (2N -1) - dimensional hypersurface in 2N - dim state space.**

IRBC with binding constraints



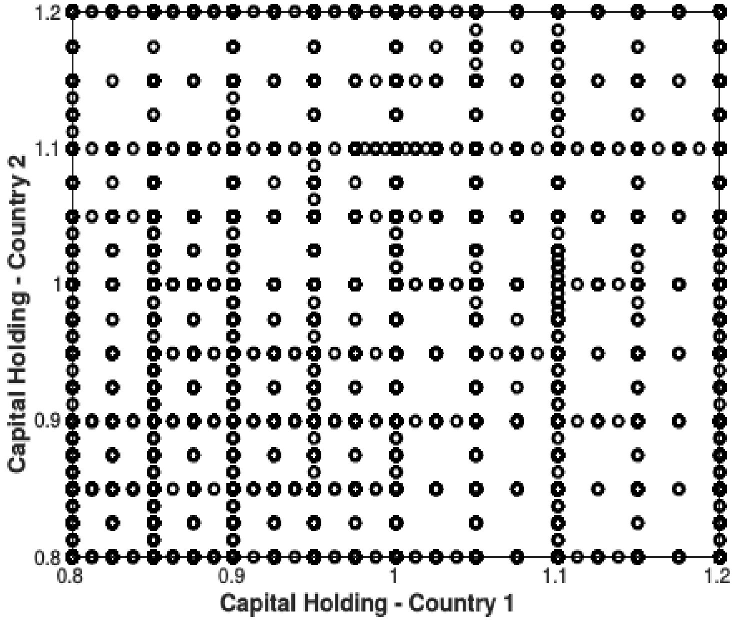


Fig.: 2-d projections of two different grids.
Left: \`classical' sparse grid of level 9 (18,945 points),
Right: adaptive grid with refinement threshold ε = 0.001 (14,226 points).
The x-axis shows capital holding of country 1, the y-axis shows capital holding of country 2, while the productivities of the two countries are kept fixed at their unconditional means.

Models with binding constrains: massive speedup due to adaptivity

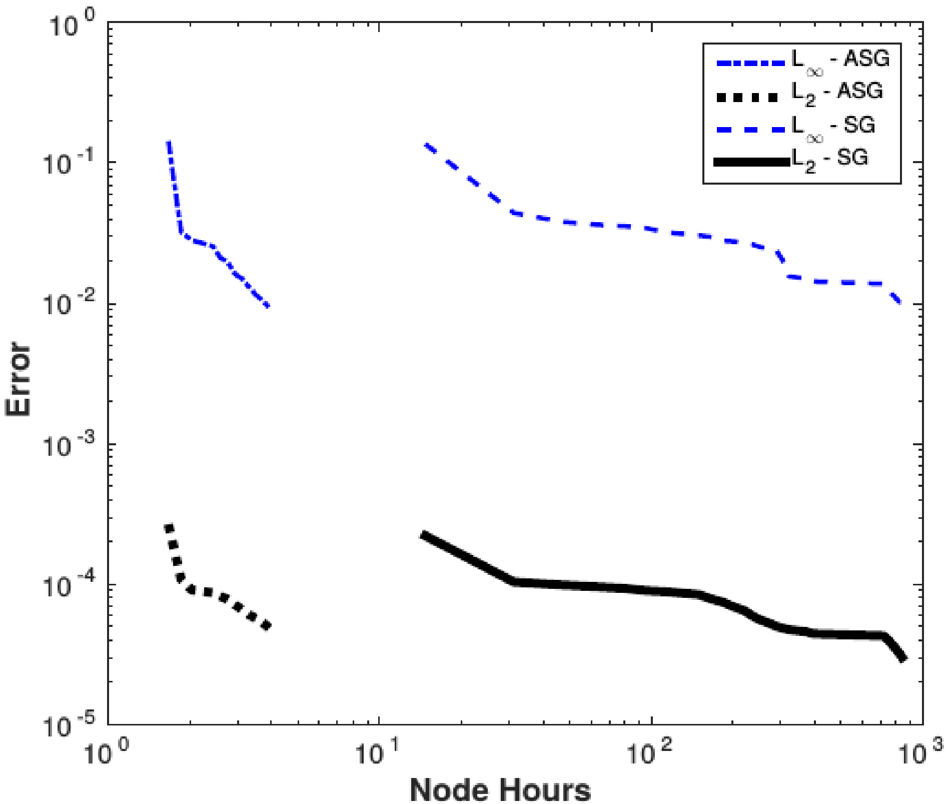
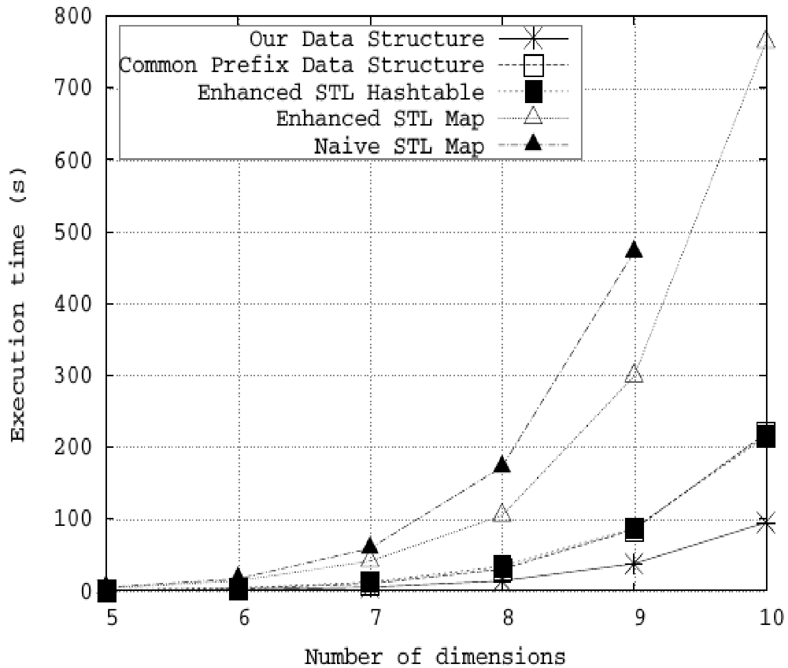


Fig.: 8d model with binding constraints.
model run with/without adaptive sparse grids.
Relative error among two consecutive time-steps.
10k points drawn from uniform distribution.

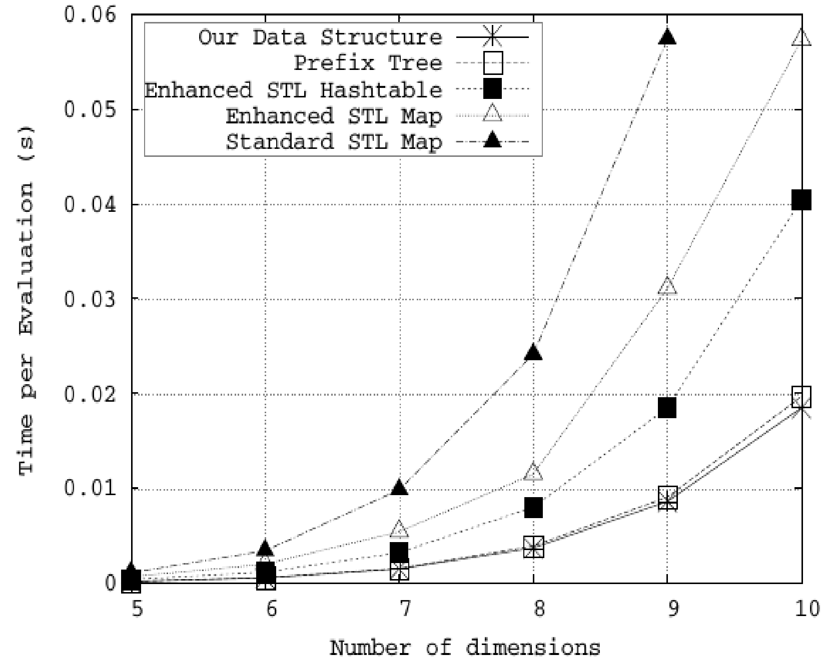
Dimension	Points	Max. Error	Avg. Error	$\lvert V_8^{S,CC} \rvert$
4	245	-2.22	-2.88	18,945
6	684	-2.26	-2.73	127,105
8	931	-2.02	-2.66	609,025
10	2,790	-1.97	-2.54	2,148,960
12	4,239	-1.81	-2.48	7,451,394
16	8,569	-1.94	-2.36	52,789,761
20	9,098	-1.96	-2.35	$\gg 10^8$

Tab.: Comparison of a sparse and
adapt. sparse grid of comparable accuracy.

Limitation of sparse grids: Execution times in higher dimension



(a) Runtime for sequential hierarchization.



(b) Runtime for sequential evaluation.

going to higher dimensions gets polynomially harder → we need parallel programming

Figs. From Murarasu et al. (2011)

Limitations of sparse grids (II)

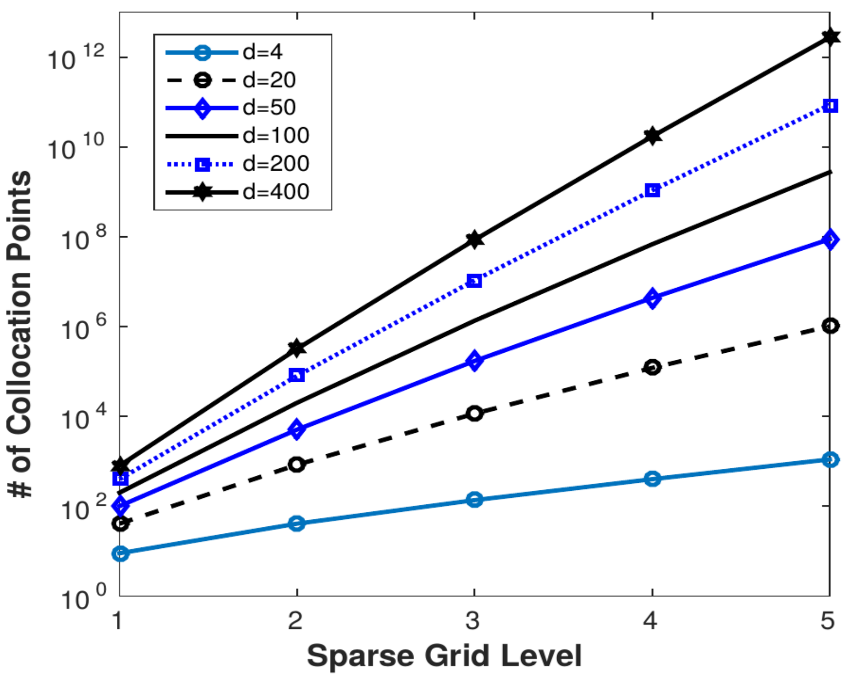


Fig.: classical sparse grids of varying dimension and increasing refinement level

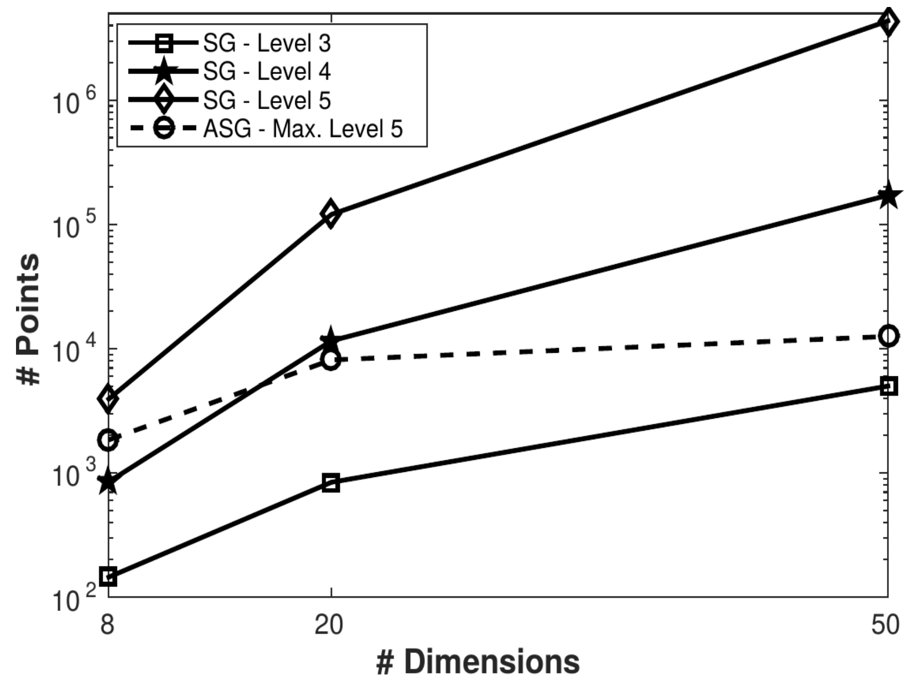


Fig.: IRBC model, solved both with classical sparse grids of varying dimension and increasing refinement level.

Major issue: a complex problem may require a high resolution in order to obtain a “reasonable” solution, i.e., a high sparse grid refinement level. For high-dimensional problems, the amount of points added to the sparse grid grow fast with the increasing level (still slower than exponential) but still make problems quickly intractable **(left panel)**. ASGs can alleviate this issue to some extend **(right panel)**.

How to build a star killer for economic problems?