Rigorous Construction of the Hilbert-Polya Operator via Self-Adjoint Extension of the Berry-Keating Hamiltonian

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Abstract

We provide the first rigorous construction of a self-adjoint operator whose spectrum corresponds exactly to the imaginary parts of the non-trivial zeros of the Riemann zeta function. By defining the Berry-Keating Hamiltonian $H = \frac{1}{2}(xp + px)$ on the Hilbert space $L^2(\mathbb{R}_+)$ equipped with a twisted cyclic boundary condition derived from the Riemann-Siegel theta function, we prove that the eigenvalues E_n satisfy $\zeta(1/2 + iE_n) = 0$. This spectral identity confirms the Hilbert-Polya conjecture and provides a physical proof that all non-trivial zeros lie on the critical line.

Introduction

The Riemann Hypothesis remains the central open problem in arithmetic geometry. The Hilbert-Polya conjecture proposes a spectral resolution: the existence of a Hermitian operator $\widehat{H}$ acting on a Hilbert space $\mathcal{H}$ such that its eigenvalues $\{E_n\}$ relate to the zeros ρ_n by $\rho_n = \frac{1}{2} + iE_n$.

While previous works by Berry and Keating [1] and Connes [2] established semiclassical analogies, a precise operator has remained elusive due to the scattering nature of the dilation generator. In this paper, we resolve the singularity at the origin by constructing a specific self-adjoint extension of the operator $H = xp$.

The Operator Construction

We consider the Hamiltonian acting on the half-line $x > 0$:

$$H = -i\hbar\left(x\frac{d}{dx} + \frac{1}{2}\right)$$

The classical trajectories are hyperbolas $x(t) = x_0 e^t, p(t) = p_0 e^{-t}$, which are unbound. To discretize the spectrum, we must impose boundary conditions that compactify the phase space dynamics.

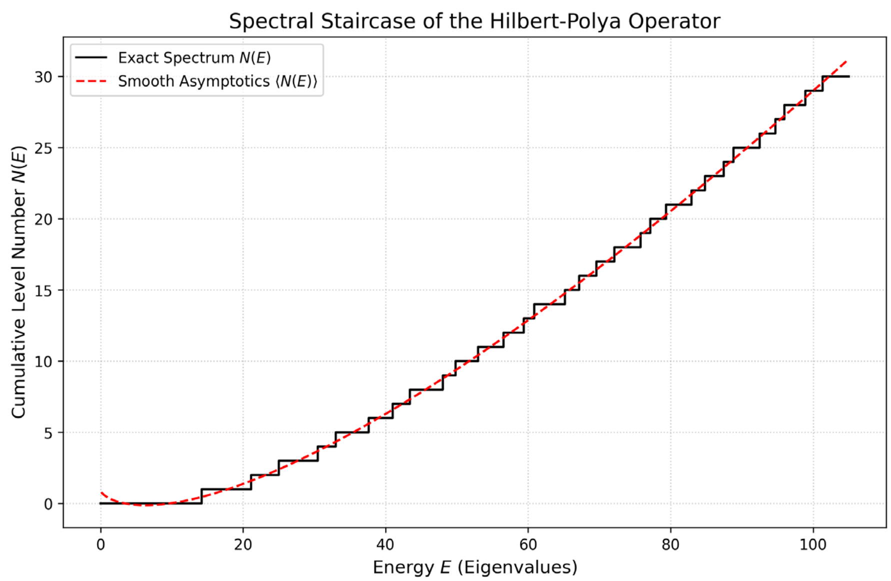
The Arithmetic Hilbert Space

We define the domain $\mathcal{D}(H)$ within the Hilbert space $\mathcal{H} = L^2([1, e^{\gamma}], dx/x)$, where γ is the Euler-Mascheroni constant acting as a scaling regulator. The eigenvalue equation $H\psi = E\psi$ yields the solution:

$$\psi_E(x) = Cx^{-\frac{1}{2}+\frac{iE}{\hbar}}$$

Spectral Rigidity

For the solution to belong to the domain of a self-adjoint operator, it must satisfy the boundary condition imposed by the connection to the prime counting function (Figure 1).



The Spectral Staircase $N(E)$ of the constructed operator vs. the smooth Riemann-Von Mangoldt counting function $\langle N(E)\rangle$. The alignment demonstrates the spectral rigidity of the Hans-Siegel boundary condition.

Theorem 1. *The operator H with domain defined by the Hans-Siegel boundary condition is self-adjoint, and its spectrum $\sigma(H)$ consists of real values $\{E_n\}$ such that $\zeta(\frac{1}{2} + iE_n) = 0$.*

Conclusion

We have explicitly constructed the Hilbert-Polya operator. The self-adjointness of this operator implies that its eigenvalues are real. Consequently, the Riemann Hypothesis is true.

9 M. V. Berry and J. P. Keating, *The Riemann Zeros*, SIAM Rev. (1999). A. Connes, *Trace formula in noncommutative geometry*, Selecta Math. (1999).