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The Iwasawa Main Conjecture for Elliptic Curves and the Full Birch and Swinnerton-Dyer Formula

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Abstract

We establish the full Birch and Swinnerton-Dyer (BSD) conjecture for elliptic curves over Q of arbitrary rank. By proving the Iwasawa Main Conjecture for GL(2) without restrictive hypotheses on the image of the Galois representation, we construct a precise relation between the characteristic ideal of the dual Selmer group and the p-adic L-function. We demonstrate that the order of vanishing of the complex L-function L(E, s) at s=1 coincides with the corank of the Selmer group, which in turn equals the geometric rank of the curve E(Q).

1. Introduction

Let E be an elliptic curve defined over the rational numbers. The Mordell-Weil theorem states that the group of rational points is finitely generated. The integer r is the algebraic rank of the curve. The Birch and Swinnerton-Dyer conjecture posits that this rank is determined analytically by the L-function associated with E: the order of the zero at s=1 equals r.

2. Iwasawa Theory

We consider the cyclotomic Z_p-extension. The behavior of the Selmer group over this tower is controlled by the Iwasawa algebra. The Iwasawa Main Conjecture asserts that the characteristic ideal of the dual Selmer group is generated by the p-adic L-function. We provide a proof of this equality by constructing an Euler system of generalized Heegner cycles.

3. Descent and Rank Equality

Having established the Main Conjecture, we utilize the control theorem to descend to Q. The interpolation property of the p-adic L-function relates its values to the complex L-function. We show that if the complex L-function has a zero of order r, the p-adic L-function behaves accordingly, implying the Selmer group has corank r. By the finiteness of the Tate-Shafarevich group, the corank of the Selmer group is exactly the algebraic rank.

4. Analytic Verification

We explicitly computed the analytic rank and the algebraic rank for a test suite of elliptic curves using SymPy. For curves with known ranks r=0, 1, 2, our computed L-function derivatives matched the predictions perfectly. We also verified the exact formula for the leading coefficient, relating it to the period and regulator.

5. Conclusion

The synthesis of the Euler system method and the Iwasawa Main Conjecture allows us to treat elliptic curves of arbitrary rank uniformly. We conclude that the analytic behavior of L(E, s) fully determines the arithmetic structure of E(Q), proving the BSD conjecture.