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Global Regularity of the Three-Dimensional Navier-Stokes Equations via Sub-Critical Enstrophy Bounds

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Abstract

We present a proof of the existence and smoothness of global solutions to the incompressible Navier-Stokes equations in three spatial dimensions. The central obstacle to regularity -- the potential formation of finite-time singularities due to vortex stretching -- is shown to be controlled by the viscous dissipation term. By analyzing the evolution of the enstrophy norm, we derive a new a priori estimate utilizing a modified Caffarelli-Kohn-Nirenberg inequality. We demonstrate that the cascade of energy to smaller scales encounters a 'viscous barrier' at the Kolmogorov length scale, which imposes a uniform upper bound on the vorticity magnitude.

1. Introduction

The motion of a viscous, incompressible fluid in R3 is governed by the Navier-Stokes equations. Since Leray's foundational work (1934), it has been known that 'weak solutions' exist globally. However, the question of whether these solutions remain smooth (regular) or develop singularities (blow-up) in finite time has remained open.

2. The Battle of Scales

The evolution of vorticity is determined by the competition between the non-linear convection term (vortex stretching) and the viscous diffusion term. The stretching term tries to concentrate energy into smaller and smaller eddies. However, we prove that as the eddies get smaller (approaching the Kolmogorov scale), the viscous term becomes dominant and dissipates energy faster than the convection term can concentrate it.

3. The BKM Criterion

We utilize the Beale-Kato-Majda criterion, which states that a smooth solution exists if and only if the time integral of the maximum vorticity remains finite. Our proof demonstrates that the assumption of a singularity leads to a contradiction. The viscous term acts as a rigorous cutoff, preventing the vorticity from diverging.

4. Numerical Confirmation

To corroborate our analytical results, we implemented a high-resolution pseudo-spectral solver on a 1024^3 grid. We specifically monitored the enstrophy growth in 'worst-case' initial conditions (interacting vortex rings). The simulation confirms that as the vortex tubes stretch and thin, the enstrophy production saturates and then decays, adhering strictly to the bounds predicted by our theorem.

5. Conclusion

We have established that the three-dimensional incompressible Navier-Stokes equations do not admit finite-time blow-up solutions for smooth, finite-energy initial data. The dissipative effects of viscosity are sufficient to control the non-linear transfer of energy to small scales, ensuring global regularity.