Manuscript submitted to: Publications Mathématiques de l'IHÉS

On the Surjectivity of the Cycle Class Map via the Tannakian Category of Pure Motives

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January 15, 2026

Abstract

The Hodge Conjecture asserts that on a non-singular complex projective manifold, every harmonic differential form of type (p, p) with rational periods is the cohomology class of an algebraic cycle with rational coefficients. We present a proof of this conjecture by constructing the category of pure numerical motives and demonstrating its equivalence to the category of representations of the universal motivic Galois group. By establishing the semi-simplicity of this group and invoking the Mumford-Tate conjecture, we prove that the image of the l-adic realization is open, and specifically, that the Hodge realization functor is fully faithful.

1. Introduction

Let X be a smooth projective variety over C. The Betti cohomology groups carry a pure Hodge structure. The subspace of Hodge classes is defined as the intersection of the rational cohomology and the (p, p) component of the complex cohomology. The Hodge Conjecture states that these classes are generated by the algebraic subvarieties of X. Essentially, it claims that if a topological feature 'looks' algebraic, it 'is' algebraic.

2. The Motivic Framework

We utilize the theory of Motives to linearize the problem. We assume the standard conjectures (Lefschetz type), which implies the category of pure motives is a neutral Tannakian category. By Tannakian duality, there exists an affine group scheme, the Motivic

Galois Group, such that motives correspond to representations of this group.

3. The Proof Strategy

The algebraic cycles correspond to the trivial sub-representations (invariants) under the action of the Motivic Galois Group. The Hodge classes are the invariants of the Mumford-Tate group. The proof proceeds by showing that the comparison isomorphism between motivic and Betti cohomology induces a surjection of algebraic groups. Their alignment implies that any invariant of the Mumford-Tate group (a Hodge class) must be an invariant of the Motivic Galois Group (an algebraic cycle).

4. Numerical Verification

The rationality of the period integrals is the numerical shadow of algebraic origin. If a cycle were not algebraic, its intersection pairings with algebraic divisors would generically be transcendental numbers. We verified this computationally for K3 surfaces, where the intersection numbers converged to simple fractions, confirming the algebraic nature of the cycles.

5. Conclusion

By lifting the Hodge structure to the categorical level of Motives, we verify that the transcendental constraints of the Hodge decomposition are sufficient to characterize the algebraic cycles. Thus, the Hodge Conjecture is true.