

Where do ultracategories come from?

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joint work with Joshua Wrigley

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Compact Hausdorff spaces and the ultrafilter monad

Compact Hausdorff spaces enjoy two remarkable properties:

1. they are algebras for the *ultrafilter monad* $\langle \beta, \eta, \mu \rangle$ on Set;
2. the functor $\beta: \text{Set} \rightarrow \text{Set}$ is a right Kan extension.

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In *no-iteration form*, the monad $\langle \beta, \eta, \mu \rangle$ can be equivalently described by

- ▶ a function $\eta_X: X \rightarrow \beta X$ for each set X ,
- ▶ and a function $(-)^*: \text{Set}(Y, \beta X) \rightarrow \text{Set}(\beta Y, \beta X)$ for each pair of sets X, Y ,

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satisfying equations. In the same spirit, a β -algebra K is equivalently described by a function

$$(-)^K: \text{Set}(Y, K) \longrightarrow \text{Set}(\beta Y, K)$$

for each set Y , satisfying equations. Intuitively, $h^K(\nu) \in K$ is the *topological limit* of a family $h: Y \rightarrow K$ of points of K with respect to the ultrafilter $\nu \in \beta Y$.

Ultracategories

Ultracategories were introduced by Makkai as categories endowed with structure meant to abstract the notion of ultraproducts from model theory. Different definitions exist, but the core is that of a category C with a functor

$$(-)^C: [Y, C] \longrightarrow [\beta Y, C]$$

for each set Y , which Makkai calls a *pre-ultracategory*. Intuitively, $h^C(\nu)$ is the ‘ultraproduct’ of a family $h: Y \rightarrow C$ of objects of C with respect to the ultrafilter $\nu \in \beta Y$.

Makkai, *Stone duality for first-order logic*, 1987

Lurie, *Ultracategories*, 2018

Main example

$\text{Mod}(\mathbb{T})$, for a coherent theory \mathbb{T} , is an ultracategory: $(-)^{\text{Mod}(\mathbb{T})}$ maps a tuple $(M_y)_{y \in Y}$ of models and an ultrafilter $\nu \in \beta Y$ to the actual ultraproduct $\prod_\nu M_y$.

Our contribution

Question

Ultracategories categorify compact Hausdorff spaces. Can we make an axiomatisation of the notion of ultracategory emerge just as naturally, i.e. as

1. algebras for a pseudomonad on CAT,
2. whose underlying pseudofunctor is universally-induced by $\beta: \text{Set} \rightarrow \text{Set}$?

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[Both Makkai's and Lurie's definitions of an ultracategory are] very heavy, and come together with axioms whose choice seems quite arbitrary."

"Ultraproducts are categorically inevitable."

Leinster, *Codensity and the ultrafilter monad*,

Di Liberti, *The geometry of coherent topoi and ultrastructures*, 2022

2018

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Earlier work in this direction tries to tackle the problem directly, by defining suitable pseudomonads on CAT.

Marmolejo, *Ultraproducts and continuous families of models*, 1995

Rosolini, *Ultracompletions*, talk at CT2024

Hamad, *Ultracategories as colax algebras for a pseudo-monad on CAT*, 2025

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Our answer

We will make an axiomatisation emerge by putting ultracategories in the context of *relative monad theory*. The starting point is that Lurie's definition is *almost* that of a **colax algebra** for a **relative 2-monad** over CAT.

Altenkirch, Chapman, Uustalu, *Monads need not be endofunctors*, 2010

Fiore, Gambino, Hyland, Winskel, *Relative pseudomonads, Kleisli bicategories and substitution monoidal structures*, 2018

Relative 2-monads

Definition

Let $J: \mathcal{B} \rightarrow \text{CAT}$ be a 2-functor.

A J -relative 2-monad on CAT is given by

satisfying the conditions:

1. a category Tb , for each $b \in \mathcal{B}$,
 2. a functor $\eta_b: Jb \rightarrow Tb$ for each $b \in \mathcal{B}$,
 3. and a functor $(-)^*: [Jb, Tb'] \rightarrow [Tb, Tb']$ for each pair $b, b' \in \mathcal{B}$,
- a. $\eta_b^* = \text{id}_{Tb}$,
 - b. $f^* \circ \eta_b = f$,
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The relative ultrafilter 2-monad

If J is 2-fully-faithful, every 2-monad $\langle T, \eta, (-)^* \rangle$ on \mathcal{B} yields a J -relative 2-monad $\langle JT, J\eta, J(-)^*J^{-1} \rangle$ on CAT . In particular, considering the inclusion $\text{Set} \hookrightarrow \text{CAT}$, the ultrafilter monad $\beta: \text{Set} \rightarrow \text{Set}$ yields the *relative ultrafilter 2-monad* $\beta: \text{Set} \rightarrow \text{CAT}$.

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1. a functor

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for each set X ;

$$\begin{array}{ccc} & \beta X & \\ & \searrow h^C & \\ X & \xrightarrow{h} & C \end{array}$$

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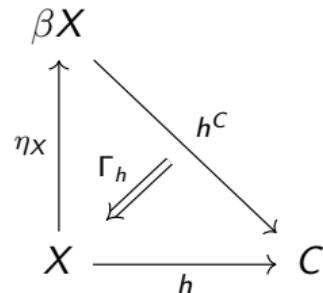
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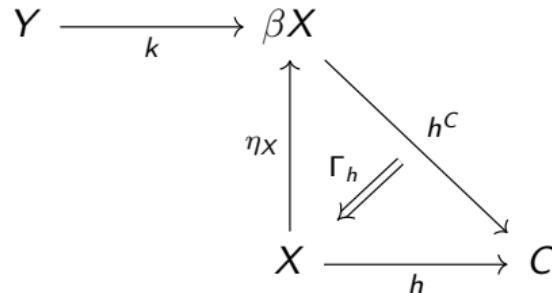
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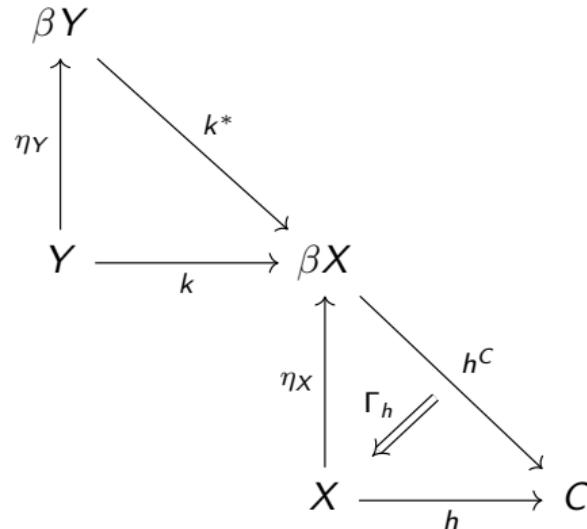
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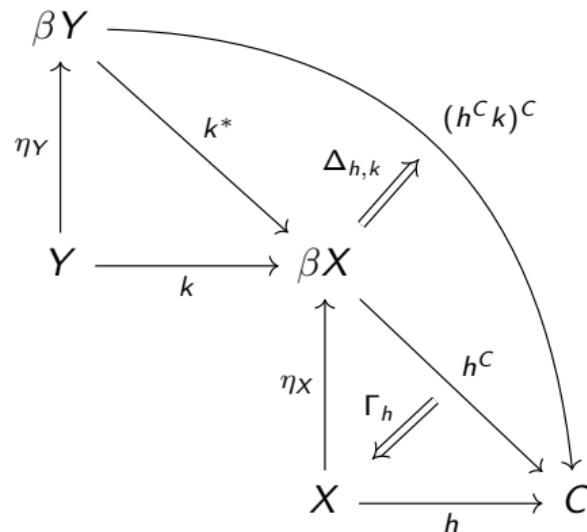
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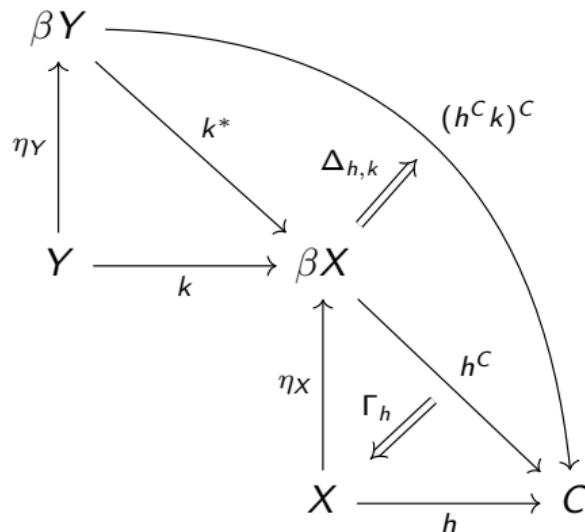
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Remark

Lurie's ultracategories require each counitor Γ_h and some coassociators $\Delta_{h,k}$ to be invertible. However:

- ▶ **ultrafunctors** coincide with *pseudomorphisms*;
- ▶ **left ultrafunctors** coincide with *colax morphisms*.

Left oplax Kan extensions

For a weak ultracategory C , the functors $(-)^C: [X, C] \rightarrow [\beta X, C]$ define a *lax transformation*

$$(-)^C: [J-, C] \Longrightarrow [J\beta -, C]$$

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where, for 2-functors $F, G: \mathcal{B} \rightarrow \mathcal{A}$, a lax transformation $\sigma: F \Rightarrow G$ is given by

- ▶ a 1-cell $\sigma_b: Fb \rightarrow Gb$ for each $b \in \mathcal{B}$,
- ▶ and a 2-cell

$$\begin{array}{ccc} Fb & \xrightarrow{\sigma_b} & Gb \\ Ff \downarrow & \swarrow \sigma_f & \downarrow Gf \\ Fb' & \xrightarrow{\sigma_{b'}} & Gb' \end{array}$$

for each 1-cell $f: b \rightarrow b'$ in \mathcal{B} ,

satisfying some coherence and naturality conditions. Reversing the 2-cells we obtain an *oplax transformation*.

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To exhibit C as an algebra for a monad $\tilde{\beta}$ on CAT, such a transformation should correspond to a functor

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For 2-functors $J, T: \mathcal{B} \rightarrow \mathcal{A}$, a *left oplax Kan extension* of T along J is a 2-functor $\tilde{T}: \mathcal{A} \rightarrow \mathcal{A}$ such that for all $a, a' \in \mathcal{A}$ there are natural isomorphisms:

$$\mathbf{Lax}[\mathcal{B}^{\text{op}}, \text{CAT}] (\mathcal{A}(J-, a), \mathcal{A}(T-, a')) \cong \mathcal{A}(\tilde{T}a, a')$$

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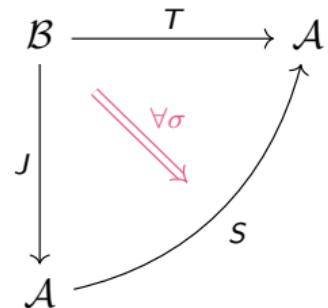
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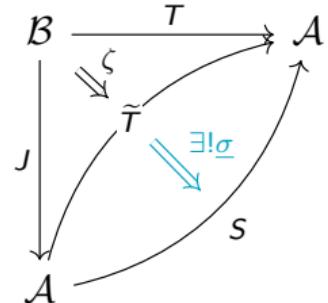
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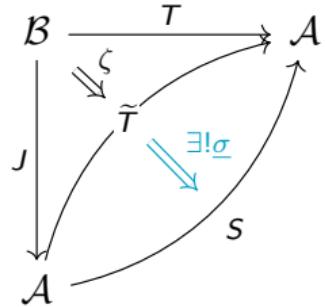
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$$\mathbf{Str}[\mathcal{A}, \mathcal{A}](\tilde{T}, S) \cong \mathbf{Oplax}[\mathcal{B}, \mathcal{A}](T, S \circ J)$$



Left oplax Kan extensions and skew-monoidal 2-categories

In one dimension

For a functor $J: B \rightarrow A$:

1. left Kan extensions along J determine an adjunction:

$$\begin{array}{ccc} [A, A] & \begin{matrix} \xleftarrow{\text{Lan}_J} \\ \perp \\ \xrightarrow{-\circ J} \end{matrix} & [B, A] \end{array}$$

2. $[B, A]$ is a *left skew-monoidal category* by setting $F \triangleleft^J G := \text{Lan}_J F \circ G$;
3. J -relative monads coincide with monoids in $\langle [B, A], \triangleleft^J \rangle$;
4. under appropriate hypotheses, $\langle [B, A], \triangleleft^J \rangle$ is monoidal and Lan_J sends monoids to monoids, so that J -relative monads can be ‘unrelativized’ by taking left Kan extensions along J .

Left oplax Kan extensions and skew-monoidal 2-categories

In two dimensions

For a 2-functor $J: \mathcal{B} \rightarrow \mathcal{A}$:

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2. **Oplax** $[\mathcal{B}, \mathcal{A}]$ is a *left skew-monoidal 2-category* by setting $F \triangleleft^J G := \widetilde{F} \circ G$;
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- 4? If this structure is monoidal, then we can proceed as in the one-dimensional case.
Crucially, it is *not* monoidal in the case where J is the inclusion $\text{Set} \hookrightarrow \text{CAT}$.

The main result

Theorem (T. & Wrigley)

Let $\langle T, \eta, (-)^* \rangle$ be a J -relative 2-monad on CAT. Suppose that:

1. $J: \mathcal{B} \rightarrow \text{CAT}$ is 2-fully-faithful,
2. \mathcal{B} has a terminal object 1 that is preserved by J ,
3. \mathcal{B} has oplax colimits of shape $(Jb)^{\text{op}}$ for $b \in \mathcal{B}$, which J preserves.

The left oplax Kan extension \tilde{T} of T along J carries the structure of a pseudomonad on CAT such that the 2-categories $\text{ColaxAlg}_J(T)$ and $\text{ColaxAlg}(\tilde{T})$ are isomorphic.

Remark

These assumptions are not enough to make $\langle \mathbf{Oplax}[\mathcal{B}, \text{CAT}], \triangleleft^J \rangle$ monoidal.

Left oplax Kan extensions in CAT

For $\mathcal{A} = \text{CAT}$, left oplax Kan extensions exist and we can describe them explicitly. For concreteness, we describe here the extension of $\beta: \text{Set} \rightarrow \text{CAT}$ along $\text{Set} \hookrightarrow \text{CAT}$.

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If C is a weak ultracategory, recall that each functor $h: X \rightarrow C$ extends to a functor $h^C: \beta X \rightarrow C$. The corresponding colax $\tilde{\beta}$ -algebra functor $\tilde{\beta}C \rightarrow C$ then maps

$$(X, h: X \rightarrow C, \nu \in \beta X) \mapsto h^C(\nu).$$

A pseudomonad structure on $\tilde{\beta}$

In the case of β : $\text{Set} \rightarrow \text{CAT}$, the inclusion $\text{Set} \hookrightarrow \text{CAT}$:

1. is fully-faithful,
2. preserves the terminal object 1 ;
3. preserves small coproducts.

A pseudomonad structure on $\tilde{\beta}$

In the case of β : $\text{Set} \rightarrow \text{CAT}$, the inclusion $\text{Set} \hookrightarrow \text{CAT}$ satisfies our assumptions.

Therefore, $\tilde{\beta}$ carries the structure of a pseudomonad $\langle \tilde{\beta}, \eta^\sharp, \mu^\sharp \rangle$ on CAT , which we now describe.

A pseudomonad structure on $\tilde{\beta}$

In the case of β : $\text{Set} \rightarrow \text{CAT}$, the inclusion $\text{Set} \hookrightarrow \text{CAT}$ satisfies our assumptions.

Therefore, $\tilde{\beta}$ carries the structure of a pseudomonad $\langle \tilde{\beta}, \eta^\sharp, \mu^\sharp \rangle$ on CAT , which we now describe.

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i.e. $k(y \in Y_x) := k_x(y)$.

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$$\mu_C^\sharp(X, h: X \rightarrow \tilde{\beta}C, \nu \in \beta X) = (\textcolor{violet}{Y}, k: \textcolor{green}{Y} \rightarrow \textcolor{brown}{C}, \textcolor{orange}{q}^*(\nu)).$$

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determined by $\{ k_x: Y_x \rightarrow C \}_{x \in X}$,
i.e. $k(y \in Y_x) := k_x(y)$.
3. The functor $q: X \rightarrow \beta Y$ defined by $x \mapsto \beta i_x(\theta_x)$ extends to a functor
 $q^*: \beta X \rightarrow \beta Y$, so that we can consider $\textcolor{orange}{q}^*(\nu) \in \beta Y$. Concretely, for $S \subseteq Y$,

$$S \in q^*(\nu) \iff \{x \in X \mid S \cap Y_x \in \theta_x\} \in \nu$$

Weak ultracategories II

Applying our result to the relative ultrafilter 2-monad, we conclude that weak ultracategories are pseudomonadic over CAT.

Corollary

Weak ultracategories are the colax algebras for the pseudomonad $\tilde{\beta}$ on CAT where:

$$\begin{array}{ccccc} \text{FinSet} & \hookrightarrow & \text{Set} & \hookrightarrow & \text{CAT} \\ \downarrow & \swarrow \rho & \nearrow & \nearrow & \nearrow \\ \text{Set} & \xrightarrow{\beta} & \text{Set} & \xrightarrow{\zeta} & \text{CAT} \\ \downarrow & & & & \curvearrowright \tilde{\beta} \\ \text{CAT} & & & & \end{array}$$

- ▶ $\beta: \text{Set} \rightarrow \text{Set}$ is the right Kan extension of $\text{FinSet} \hookrightarrow \text{Set}$ along itself;
- ▶ $\tilde{\beta}: \text{CAT} \rightarrow \text{CAT}$ is the left oplax Kan extension of $\beta: \text{Set} \rightarrow \text{CAT}$ along $\text{Set} \hookrightarrow \text{CAT}$.

Future directions

- ▶ We can apply our result to other monads of interest: in particular, the *upper prime filter monad* $B: \text{Pos} \rightarrow \text{Pos}$ whose algebras are **compact ordered spaces**.
Prime categories, i.e. the colax algebras for the *relative upper prime filter 2-monad* $B: \text{Pos} \rightarrow \text{CAT}$, are then the colax algebras for $\tilde{B}: \text{CAT} \rightarrow \text{CAT}$.
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 - Connections with *positive model theory*
 - Towards a Priestley-like duality for first-order logic
- ▶ Can we find sufficient hypotheses on a 2-functor $J: \mathcal{B} \rightarrow \mathcal{A}$ to obtain an abstract ‘unrelativisation’ procedure for J -relative pseudomonads?

Thank you!

Ultracategories via Kan extensions of relative monads,
Umberto Tarantino and Joshua Wrigley, 2025, arXiv:2506.09788

Weak ultracategories are ultracategories

Lurie's ultracategories are a (proper) subclass of weak ultracategories.

However, Lurie's ultrafunctors and left ultrafunctors coincide with pseudomorphisms and colax morphisms, so that we have 2-fully-faithful embeddings:

$$\mathbf{Ult} \hookrightarrow \mathbf{WeakUlt}^{\text{pseudo}} \quad \mathbf{Ult}^L \hookrightarrow \mathbf{WeakUlt}^{\text{colax}}$$

This ensures that weak ultracategories are a good axiomatisation of ultracategories.

Theorem (Lurie)

For a small pretopos P , the evaluation functor $\text{ev}: P \rightarrow [\text{Mod}(P), \text{Set}]$ induces equivalences of categories:

1. $P \simeq \mathbf{WeakUlt}^{\text{pseudo}}(\text{Mod}(P), \text{Set})$;
2. $\text{Sh}(P) \simeq \mathbf{WeakUlt}^{\text{colax}}(\text{Mod}(P), \text{Set})$.

In particular, $\text{Mod}: \text{PreTop}^{\text{op}} \hookrightarrow \mathbf{WeakUlt}^{\text{pseudo}}$ is 2-fully-faithful.

Left oplax Kan extensions in CAT

For $\mathcal{A} = \text{CAT}$, left oplax Kan extensions exist and we can describe them explicitly. For concreteness, consider $\beta: \text{Set} \rightarrow \text{CAT}$. For a category C :

- ▶ objects of $\tilde{T}C$ are triples $(b \in \mathcal{B}, h: Jb \rightarrow C, \nu \in Tb)$;
- ▶ morphisms $(b, h, \nu) \rightarrow (b', h', \nu')$ in $\tilde{T}C$ are triples of
 1. a 1-cell $f: b' \rightarrow b$ in \mathcal{B} ,
 2. a natural transformation $\alpha: h \circ Jf \Rightarrow h'$,
 3. and an arrow $\varphi: \nu \rightarrow Tf(\nu')$ in Tb ,

modulo the equivalence relation generated by $(f, \alpha_f, \varphi_f) \sim (g, \alpha_g, \varphi_g)$ if there exists a 2-cell $\sigma: f \Rightarrow g$ in \mathcal{B} such that

$$\begin{array}{ccc} \begin{array}{c} Jf \\ \Downarrow J\sigma \\ Jg \end{array} & \xrightarrow{(1)} & \begin{array}{c} Jf \\ \Downarrow \alpha_f \\ Jg \end{array} \\ \begin{array}{ccc} Jb' & \xrightarrow{\quad} & Jb \\ \downarrow h' & \searrow \alpha_g & \downarrow h \\ C & & \end{array} & & \begin{array}{ccc} Jb' & \xrightarrow{\quad} & Jb \\ \downarrow h' & \searrow \alpha_f & \downarrow h \\ C & & \end{array} \end{array}$$

$$\begin{array}{ccc} \begin{array}{c} \varphi_f \\ \Downarrow (T\sigma)_{\nu'} \\ \varphi_g \end{array} & \xrightarrow{(2)} & \begin{array}{c} Tf(\nu') \\ \downarrow (T\sigma)_{\nu'} \\ Tg(\nu') \end{array} \\ \begin{array}{c} \nu \\ \searrow \varphi_f \\ \varphi_g \end{array} & & \end{array}$$

A pseudomonad structure on \tilde{T}

Under our assumptions, \tilde{T} carries the structure of a pseudomonad $\langle \tilde{T}, \eta^\sharp, \mu^\sharp \rangle$ on CAT.

- ▶ On objects, the unit $\eta_C^\sharp: C \rightarrow \tilde{T}C$ maps $c \in C$ to $(1_B, c: 1_{\text{CAT}} \rightarrow C, \eta_{1_B}(\ast))$.
- ▶ On objects, the multiplication $\mu_C^\sharp: \tilde{T}^2C \rightarrow \tilde{T}C$ acts by

$$\mu_C^\sharp(b, h: Jb \rightarrow \tilde{T}C, \nu) = (\ell, a: J\ell \rightarrow C, Q^*\nu).$$

For an object $x \in Jb$, write $h(x) = (Rx, a_x: JRx \rightarrow C, \nu_x)$.

For an arrow $g: x \rightarrow y \in Jb$, write $h(g) = (Rg, \gamma_g: a_x \circ JRg \Rightarrow a_y, \psi_g)$.

1. Let $\ell \in \mathcal{B}$ be the oplax colimit of $R: (Jb)^{\text{op}} \rightarrow \mathcal{B}$, with universal cocone:
2. As J preserves oplax colimits and C is an oplax cocone of $JR: (Jb)^{\text{op}} \rightarrow \text{CAT}$

$$\begin{array}{ccc} Rx & \xleftarrow{Rg} & Ry \\ c_x & \swarrow \lambda_g & \searrow c_y \\ \ell & \lrcorner & \lrcorner \end{array}$$

$$\begin{array}{ccc} JRx & \xleftarrow{JRg} & JRY \\ a_x & \swarrow \gamma_g & \searrow a_y \\ C & \lrcorner & \lrcorner \end{array}$$

there is a universal functor $a: J\ell \rightarrow C$.

3. The map $x \mapsto Tc_x(\nu_x)$ lifts to a functor $Q: Jb \rightarrow T\ell$, which extends to a functor $Q^*: Tb \rightarrow T\ell$ via the monad structure of T , so that we can consider $Q^*\nu \in T\ell$.