

Toposes with enough points as categories of étale spaces

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Compact Hausdorff spaces and convergence

Theorem (Manes)

CompHaus $\cong \text{Alg}(\beta)$, where $\beta: \text{Set} \rightarrow \text{Set}$ is the ultrafilter monad.

This means that, for a compact Hausdorff space X , every function $f: I \rightarrow X$ extends to a function $f^*: \beta I \rightarrow X$ which we can think of as computing the *limit* of f with respect to each $\nu \in \beta I$. Concretely:

$$\begin{array}{ccc} \beta I & & \\ \eta_I \uparrow & \searrow f^* & \\ I & \xrightarrow{f} & X \end{array} \quad f^*(\nu) = x \iff \forall U \subseteq X \text{ open, if } x \in U \text{ then } f^{-1}(U) \in \nu$$

In particular, the algebra map $\text{id}_X^*: \beta X \rightarrow X$ specifies, for each ultrafilter ν on X , the unique point of X all of whose open neighborhoods lie in ν .

Topological spaces and generalized convergence

For an arbitrary topological space X , these limits may not exist nor be unique, so that the previous definition of id_X^* determines a relation $\beta X \rightarrowtail X$.

Theorem (Barr)

The ultrafilter monad β extends to a monad $\underline{\beta}: \mathbf{Rel} \rightarrow \mathbf{Rel}$, and $\mathbf{Top} \cong \text{LaxAlg}(\underline{\beta})$.

This means that a topology on a set X can be equivalently specified by a relation $\xi: \beta X \rightarrowtail X$ such that:

$$\begin{array}{ccc} X & \xlongequal{\quad} & X \\ & \searrow \eta_X \quad \nearrow \cap! & \\ & \beta X & \end{array} \qquad \begin{array}{ccc} \beta^2 X & \xrightarrow{\beta\xi} & \beta X \\ \mu_X \downarrow & \curvearrowright & \downarrow \xi \\ \beta X & \xrightarrow{\xi} & X \end{array}$$

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Notation

For $f: I \rightarrow X$ and $\nu \in \beta I$, we write $x \rightsquigarrow \lim_{i \rightarrow \nu} f(i)$ in case $\xi(x, \beta f(\nu))$ holds.

Now: one dimension higher!

Ultracategories and convergence of ultrafamilies

Going one dimension higher, the role of β is played by the *ultracompletion pseudomonad* $\beta: \mathbf{CAT} \rightarrow \mathbf{CAT}$. For a category C , the category βC has:

- ▶ as objects, triples (I, y, ν) of a set I , a functor $y: I \rightarrow C$, and an ultrafilter $\nu \in \beta I$;
- ▶ as morphisms $(I, y, \nu) \rightarrow (I', y', \nu')$, pairs of a function $h: I' \rightarrow I$ such that $\beta h(\nu') = \nu$ and a family of arrows $(\alpha_i: y_{h(i)} \rightarrow y'_i)_{i \in I'}$ in C , both considered up to ν' -equivalence.

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Intuitively, an *ultracategory* is a category C endowed with a functor $\Phi: \beta C \rightarrow C$, assigning a unique *limit* in C to each *ultrafamily* (I, y, ν) in C . Formally, we define:

$$\mathbf{UltCat} := \text{PsAlg}(\beta)$$

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Ultracategories categorify compact Hausdorff spaces

CompHaus $\hookrightarrow \mathbf{UltCat}$ as those algebras whose carrier category is small and discrete.

Ultracategories and coherent toposes

Ultracategories were originally introduced by Makkai to prove a **reconstruction theorem** for (coherent) first-order logic.

Theorem (Makkai; Lurie)

Let \mathbb{T} be a coherent theory. Then, $\text{Mod}(\mathbb{T})$ is an ultracategory by setting the limit of an ultrafamily (I, M_-, ν) of models to be their ultraproduct $\prod_{i \rightarrow \nu} M_i$, and $\mathbf{UltCat}(\text{Mod}(\mathbb{T}), \mathbf{Set})$ is the classifying topos of \mathbb{T} .

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Identifying coherent theories with coherent toposes, and restricting to the subcategory \mathbf{UltCat}_* of ultracategories C such that $\mathbf{UltCat}(C, \mathbf{Set})$ is a topos, we have:

$$\begin{array}{ccc} & \mathbf{UltCat}(-, \mathbf{Set}) & \\ \mathbf{Topos}_{coh} & \xleftarrow{\perp} & \mathbf{UltCat}_* \\ & \curvearrowright \text{pt}(-) & \end{array}$$

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$$\mathbf{Set}^I \xrightarrow{\prod_{i \rightarrow \nu}(-)} \mathbf{Set}$$

are coherent, but not necessarily geometric functors.

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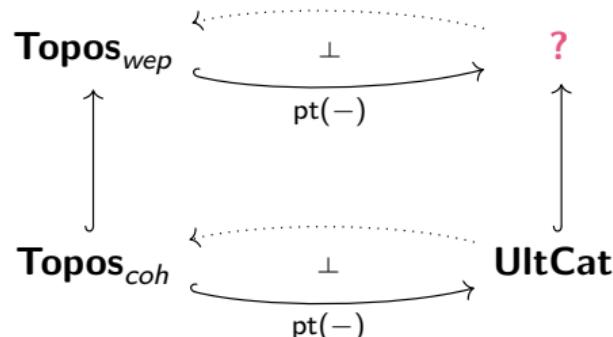
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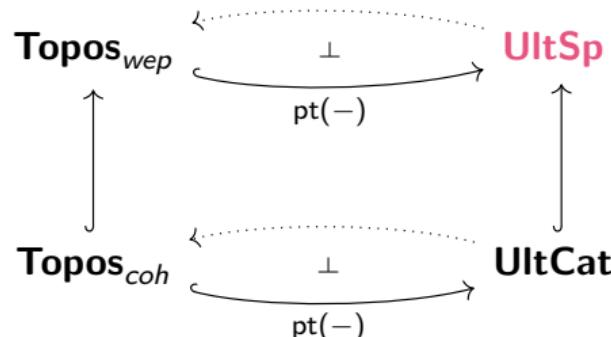


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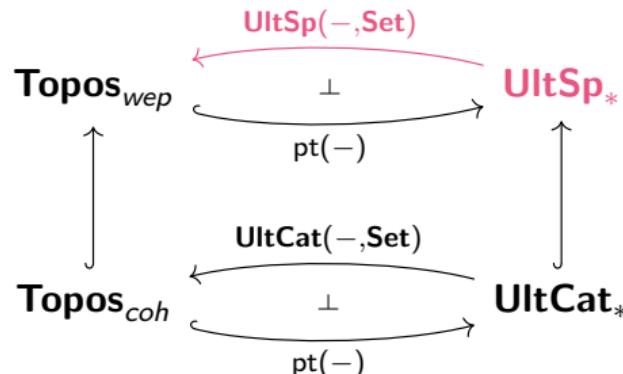


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- ▶ for every $x \in X$, an *identity* ultra-arrow $\text{id}_x: x \rightsquigarrow \lim_{* \rightarrow 1} x$;
- ▶ for every ultra-arrow $r: x \rightsquigarrow \lim_{i \rightarrow \mu} y_i$ and every ultrafamily of ultra-arrows $(s_i: y_i \rightsquigarrow \lim_{j \rightarrow \nu_i} z_{i,j})_{i \rightarrow \mu}$, a *composite* ultra-arrow $(s_i)_{i \rightarrow \mu} \cdot r: x \rightsquigarrow \lim_{(i,j) \rightarrow \sum_{i \rightarrow \mu} \nu_i} z_{i,j}$,

satisfying some equational axioms.

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Similarly, we can extend the notion of continuity to this **Set**-valued convergence relation, which now becomes *structure* rather than *property*.

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Definition

A **continuous map** of ultraconvergence spaces is a functor $f: X \rightarrow X'$ together with a family of functions

$$\Xi(x, (I, y, \nu)) \longrightarrow \Xi'(f(x), (I, fy, \nu))$$

$$r: x \rightsquigarrow \lim_{i \rightarrow \nu} y_i \longmapsto f(r): f(x) \rightsquigarrow \lim_{i \rightarrow \nu} f(y_i)$$

also satisfying some equational axioms.

With appropriate 2-cells, ultraconvergence spaces define a 2-category **UltSp**.

Examples

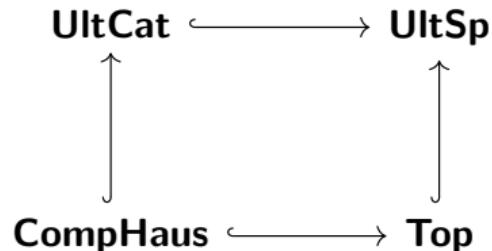
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The main theorem

As promised, the notion of ultraconvergence space allows us to obtain a reconstruction theorem for geometric logic: in topos-theoretical terms, it reads as follows.

Theorem (Saadia; Hamad; van Gool, Marquès, T.)

If \mathcal{E} is a topos with enough points, then $\mathcal{E} \simeq \mathbf{UltSp}(\mathrm{pt}(\mathcal{E}), \mathbf{Set})$.

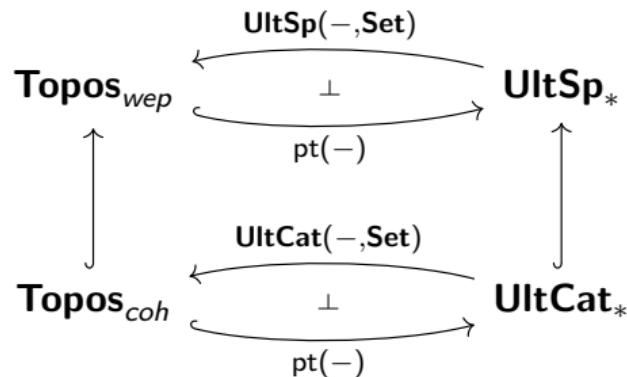
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In other words, restricting to the subcategory \mathbf{UltSp}_* of ultraconvergence spaces X such that $\mathbf{UltSp}(X, \mathbf{Set})$ is a topos, we have what we wanted:



Étale spaces

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Étale maps over B form a category $\text{Et}(B)$, equivalent to **UltSp**(B , **Set**).

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- ▶ For every object $\varphi \in \mathcal{E}$, we can define an étale space $\pi_\varphi: [\![\varphi]\!] \longrightarrow X$ where:
 - ▶ the fiber of π_φ at $x \in X$ is given by $x(\varphi)$;
 - ▶ an ultra-arrow $(x, \nu) \rightsquigarrow \lim_{i \rightarrow \nu} (y_i, w_i)$ in $[\![\varphi]\!]$ is given by an ultra-arrow $r: x \rightsquigarrow \lim_{i \rightarrow \nu} y_i$ in X such that $r_\varphi(\nu) = (w_i)_{i \rightarrow \nu}$.

This assignment defines the *evaluation functor* $[\![-\!]\!]: \mathcal{E} \longrightarrow \text{Et}(X)$.

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*Let \mathbb{T} be a geometric theory which is complete with respect to its **Set**-models. Then, $\text{Mod}(\mathbb{T})$ is an ultraconvergence space by setting ultra-arrows $M \rightsquigarrow \lim_{i \rightarrow \nu} N_i$ to be structure morphisms $M \rightarrow \prod_{i \rightarrow \nu} N_i$, and $\text{Et}(\text{Mod}(\mathbb{T}))$ is the classifying topos of \mathbb{T} .*

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The localic/propositional case

In particular, if a localic topos \mathcal{E} has enough points, i.e. $\mathcal{E} \simeq \text{Sh}(\mathcal{O}(X))$ for some topological space X , then $\mathcal{E} \simeq \text{Et}(X)$.

Proof sketch

Our proof is substantially different from both Saadia's and Hamad's, who use Butz-Moerdijk's representation theorem for toposes with enough points. Instead, we proceed similarly to Makkai's original work, in two main steps.

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1. $\llbracket - \rrbracket: \mathcal{E} \longrightarrow \text{Et}(X)$ is **full on subobjects**: every subobject of $\pi_\varphi: \llbracket \varphi \rrbracket \longrightarrow X$ in $\text{Et}(X)$ is the restriction of π_φ to $\llbracket \psi \rrbracket \subseteq \llbracket \varphi \rrbracket$ for some subobject $\psi \rightarrowtail \varphi$ in \mathcal{E} .

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2. $\llbracket - \rrbracket: \mathcal{E} \longrightarrow \text{Et}(X)$ is **covering**: every étale space $p: Y \longrightarrow X$ is covered by an epimorphism $\alpha: \pi_\varphi \twoheadrightarrow p$ in $\text{Et}(X)$ for some object $\varphi \in \mathcal{E}$.

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1. $\llbracket - \rrbracket: \mathcal{E} \longrightarrow \text{Et}(X)$ is **full on subobjects**: every subobject of $\pi_\varphi: \llbracket \varphi \rrbracket \longrightarrow X$ in $\text{Et}(X)$ is the restriction of π_φ to $\llbracket \psi \rrbracket \subseteq \llbracket \varphi \rrbracket$ for some subobject $\psi \rightarrowtail \varphi$ in \mathcal{E} .
2. $\llbracket - \rrbracket: \mathcal{E} \longrightarrow \text{Et}(X)$ is **covering**: every étale space $p: Y \longrightarrow X$ is covered by an epimorphism $\alpha: \pi_\varphi \twoheadrightarrow p$ in $\text{Et}(X)$ for some object $\varphi \in \mathcal{E}$.

Two points of view

Concretely, (1) entails fully-faithfulness, while (2) entails essential surjectivity of $\llbracket - \rrbracket$.

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Two points of view

Concretely, (1) entails fully-faithfulness, while (2) entails essential surjectivity of $\llbracket - \rrbracket$. However, we can also interpret (1) as stating that $\llbracket - \rrbracket$ defines a hyperconnected geometric morphism, and (2) as stating that it defines a localic geometric morphism.

Ongoing work: ultraconvergence spaces as lax algebras

As it turns out, the inspiration from Barr's theorem can be pushed even further: in joint work in progress with Quentin Aristote, we have the following.

Theorem

*The ultracompletion pseudomonad β extends to a pseudomonad $\underline{\beta}$: **PROF** \longrightarrow **PROF**, and **UltSp** \cong discLaxAlg($\underline{\beta}$).*

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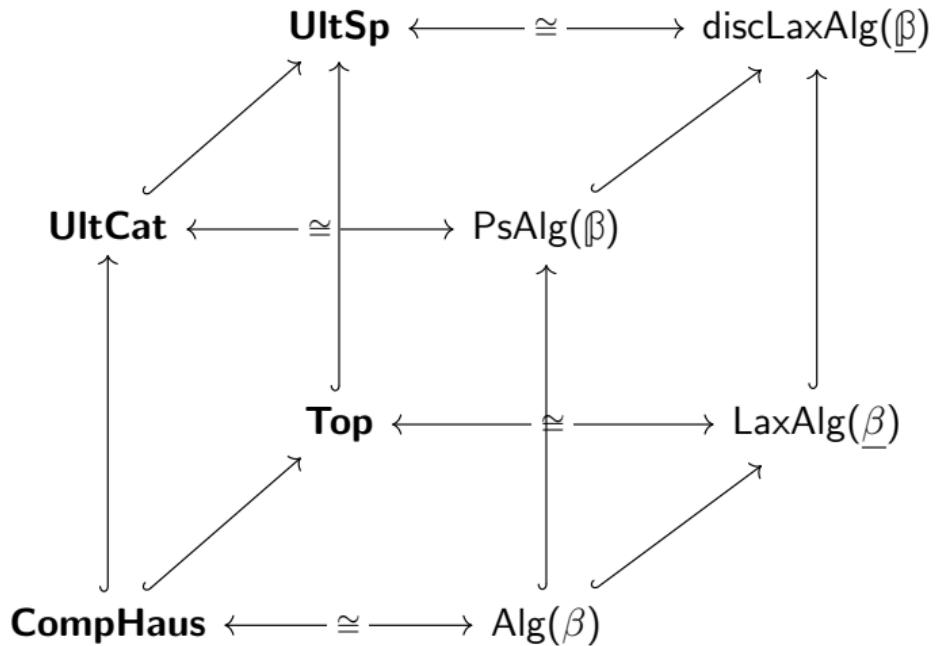
Theorem

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This means that an ultraconvergence structure on a discrete category X can be equivalently specified by a profunctor $\Xi: \beta X \nrightarrow X$ and two transformations

$$\begin{array}{ccc} X & \xlongequal{\quad} & X \\ \nwarrow \eta_X & \downarrow u & \nearrow \Xi \\ \beta X & & \end{array} \qquad \begin{array}{ccc} \beta^2 X & \xrightarrow{\beta \Xi} & \beta X \\ \downarrow \mu_X & \swarrow m & \downarrow \Xi \\ \beta X & \xrightarrow{\Xi} & X \end{array}$$

satisfying appropriate axioms.



Future work

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- ▶ Towards step (2) of our proof, we prove a kind of **Beth definability theorem** for geometric logic. What does this perspective entail?
- ▶ Can we describe the equivalences induced by the two adjunctions?

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Thank you!