

Ultracategories via Kan extensions of relative monads

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joint work with Joshua Wrigley

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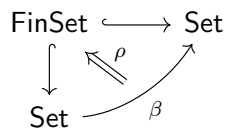
ItaCa Fest

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Compact Hausdorff spaces and the ultrafilter monad

Compact Hausdorff spaces enjoy two remarkable properties:

1. they are algebras for the *ultrafilter monad* $\langle \beta, \eta, \mu \rangle$ on \mathbf{Set} ;
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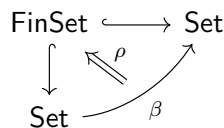
In *no-iteration form*, the monad $\langle \beta, \eta, \mu \rangle$ can be equivalently described by

- ▶ a function $\eta_X: X \rightarrow \beta X$ for each set X ,
 - ▶ and a function $(-)^*: \mathbf{Set}(Y, \beta X) \rightarrow \mathbf{Set}(\beta Y, \beta X)$ for each pair of sets X, Y ,
- satisfying equations.

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satisfying equations. In the same spirit, a β -algebra K is equivalently described by a function

$$(-)^K: \mathbf{Set}(Y, K) \longrightarrow \mathbf{Set}(\beta Y, K)$$

for each set Y , satisfying equations. Intuitively, $h^K(\nu) \in K$ is the *topological limit* of a family $h: Y \rightarrow K$ of points of K with respect to the ultrafilter $\nu \in \beta Y$.

Ultracategories

Ultracategories were introduced by Makkai as categories endowed with structure meant to abstract the notion of **ultraproducts** from model theory. Different definitions exist, but the core is that of a category C with a functor

$$(-)^C: [Y, C] \longrightarrow [\beta Y, C]$$

for each set Y , which Makkai calls a *pre-ultracategory*. Intuitively, $h^C(\nu)$ is the ‘ultraproduct’ of a family $h: Y \rightarrow C$ of objects of C with respect to the ultrafilter $\nu \in \beta Y$.

Makkai, *Stone duality for first-order logic*, 1987

Lurie, *Ultracategories*, 2018

Main example

$\text{Mod}(\mathbb{T})$, for a coherent theory \mathbb{T} , is an ultracategory: $(-)^{\text{Mod}(\mathbb{T})}$ maps a tuple $(M_y)_{y \in Y}$ of models and an ultrafilter $\nu \in \beta Y$ to the actual ultraproduct $\prod_\nu M_y$.

Our contribution

Question

Ultracategories categorify compact Hausdorff spaces. Can we make an axiomatisation of the notion of ultracategory emerge just as naturally, i.e. as

1. algebras for a pseudomonad on \mathbf{CAT} ,
2. whose underlying pseudofunctor is universally-induced by $\beta: \mathbf{Set} \rightarrow \mathbf{Set}$?

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“[Both Makkai’s and Lurie’s definitions of an ultracategory are] very heavy, and come together with axioms whose choice seems quite arbitrary.”

Di Liberti, *The geometry of coherent topoi and ultrastructures*, 2022

“Ultraproducts are categorically inevitable.”

Leinster, *Codensity and the ultrafilter monad*,
2018

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Earlier work in this direction tries to tackle the problem directly, by defining suitable pseudomonads on \mathbf{CAT} .

Marmolejo, *Ultraproducts and continuous families of models*, 1995

Rosolini, *Ultracompletions*, talk at CT2024

Hamad, *Ultracategories as colax algebras for a pseudo-monad on \mathbf{CAT}* , 2025

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Our answer

We will make an axiomatisation emerge by putting ultracategories in the context of *relative monad theory*. The starting point is that Lurie's definition is *almost* that of a **colax algebra** for a **relative 2-monad** over \mathbf{CAT} .

Altenkirch, Chapman, Uustalu, *Monads need not be endofunctors*, 2010

Fiore, Gambino, Hyland, Winskel, *Relative pseudomonads, Kleisli bicategories and substitution monoidal structures*, 2018

Relative 2-monads

Definition

Let $J: \mathcal{B} \rightarrow \mathbf{CAT}$ be a 2-functor.

A *J*-relative 2-monad on \mathbf{CAT} is given by

1. a category Tb , for each $b \in \mathcal{B}$,
2. a functor $\eta_b: Jb \rightarrow Tb$ for each $b \in \mathcal{B}$,
3. and a functor $(-)^*: [Jb, Tb'] \rightarrow [Tb, Tb']$ for each pair $b, b' \in \mathcal{B}$,

satisfying the conditions:

- a. $\eta_b^* = \text{id}_{Tb}$,
- b. $f^* \circ \eta_b = f$,
- c. $(f^* \circ g)^* = f^* \circ g^*$.

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The relative ultrafilter 2-monad

If J is 2-fully-faithful, every 2-monad $\langle T, \eta, (-)^* \rangle$ on \mathcal{B} yields a J -relative 2-monad $\langle JT, J\eta, J(-)^* J^{-1} \rangle$ on \mathbf{CAT} . In particular, considering the inclusion $\mathbf{Set} \hookrightarrow \mathbf{CAT}$, the ultrafilter monad $\beta: \mathbf{Set} \rightarrow \mathbf{Set}$ yields the *relative ultrafilter 2-monad* $\beta: \mathbf{Set} \rightarrow \mathbf{CAT}$.

Weak ultracategories I

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A *weak ultracategory* is a colax algebra for the relative ultrafilter 2-monad.

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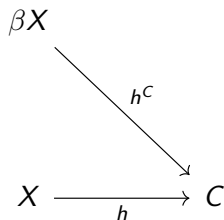
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Explicitly, this means a category C equipped with:

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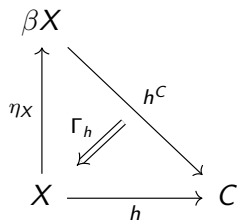
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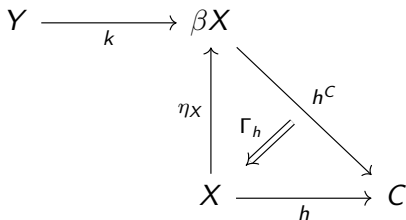
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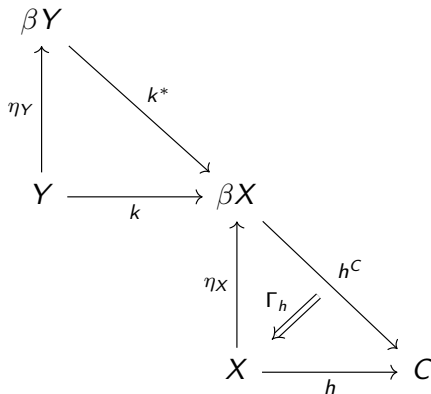
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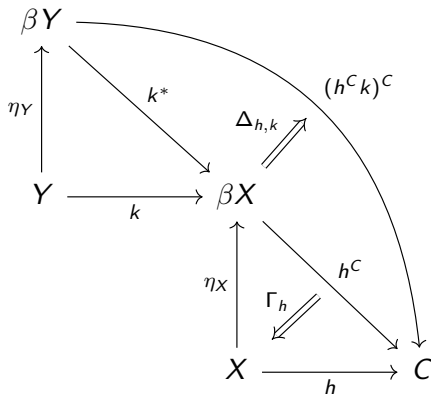
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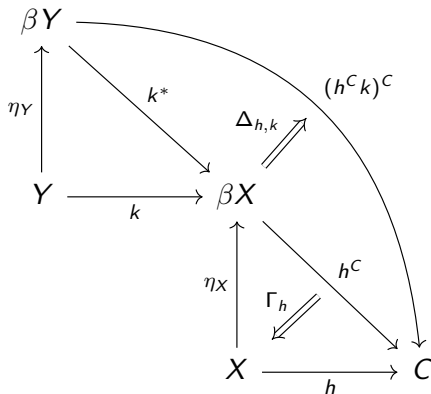
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Remark

Lurie's ultracategories require each counitor Γ_h and some coassociators $\Delta_{h,k}$ to be invertible. However:

- ▶ **ultrafunctors** coincide with *pseudomorphisms*;
- ▶ **left ultrafunctors** coincide with *colax morphisms*.

Left oplax Kan extensions

For a weak ultracategory C , the functors $(-)^C: [X, C] \rightarrow [\beta X, C]$ define a *lax transformation*

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where, for 2-functors $F, G: \mathcal{B} \rightarrow \mathcal{A}$, a lax transformation $\sigma: F \Rightarrow G$ is given by

- ▶ a 1-cell $\sigma_b: Fb \rightarrow Gb$ for each $b \in \mathcal{B}$,
- ▶ and a 2-cell

$$\begin{array}{ccc} Fb & \xrightarrow{\sigma_b} & Gb \\ Ff \downarrow & \swarrow \sigma_f & \downarrow Gf \\ Fb' & \xrightarrow{\sigma_{b'}} & Gb' \end{array}$$

for each 1-cell $f: b \rightarrow b'$ in \mathcal{B} ,

satisfying some coherence and naturality conditions. Reversing the 2-cells we obtain an *oplax* transformation.

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To exhibit C as an algebra for a monad $\tilde{\beta}$ on \mathbf{CAT} , such a transformation should correspond to a functor

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$$\mathbf{Lax}[\mathcal{B}^{\mathrm{op}}, \mathbf{CAT}] (\mathcal{A}(J-, a), \mathcal{A}(T-, a')) \cong \mathcal{A}(\tilde{T}a, a')$$

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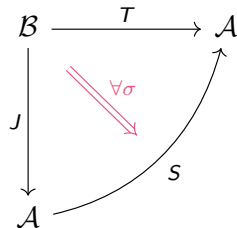
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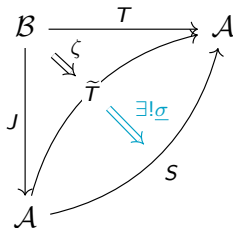
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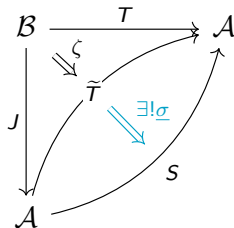
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$$\mathbf{Str}[\mathcal{A}, \mathcal{A}] (\tilde{T}, S) \cong \mathbf{Oplax}[\mathcal{B}, \mathcal{A}] (T, S \circ J)$$



Left oplax Kan extensions and skew-monoidal 2-categories

In one dimension

For a functor $J: B \rightarrow A$:

1. left Kan extensions along J determine an adjunction:

$$\begin{array}{ccc} & \text{Lan}_J & \\ \curvearrowleft & & \curvearrowright \\ [A, A] & \perp & [B, A] \\ \curvearrowright & & \curvearrowleft \\ & - \circ J & \end{array}$$

2. $[B, A]$ is a *left skew-monoidal category* by setting $F \triangleleft^J G := \text{Lan}_J F \circ G$;
3. J -relative monads coincide with monoids in $\langle [B, A], \triangleleft^J \rangle$;
4. under appropriate hypotheses, $\langle [B, A], \triangleleft^J \rangle$ is monoidal and Lan_J sends monoids to monoids, so that J -relative monads can be ‘unrelativized’ by taking left Kan extensions along J .

Left oplax Kan extensions and skew-monoidal 2-categories

In two dimensions

For a 2-functor $J: \mathcal{B} \rightarrow \mathcal{A}$:

1. left oplax Kan extensions along J determine a 2-adjunction:

$$\begin{array}{ccc} & \widetilde{(-)} & \\ \text{Str}[\mathcal{A}, \mathcal{A}] & \xleftarrow{\quad} & \mathbf{Oplax}[\mathcal{B}, \mathcal{A}] \\ & \perp & \\ & \xrightarrow{- \circ J} & \end{array}$$

2. $\mathbf{Oplax}[\mathcal{B}, \mathcal{A}]$ is a *left skew-monoidal 2-category* by setting $F \triangleleft^J G := \widetilde{F} \circ G$;
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- 4? If this structure is monoidal, then we can proceed as in the one-dimensional case.
Crucially, it is *not* monoidal in the case where J is the inclusion $\mathbf{Set} \hookrightarrow \mathbf{CAT}$.

The main result

Theorem (T. & Wrigley)

Let $\langle T, \eta, (-)^* \rangle$ be a J -relative 2-monad on \mathbf{CAT} . Suppose that:

1. $J: \mathcal{B} \rightarrow \mathbf{CAT}$ is 2-fully-faithful,
2. \mathcal{B} has a terminal object 1 that is preserved by J ,
3. \mathcal{B} has oplax colimits of shape $(Jb)^{\mathrm{op}}$ for $b \in \mathcal{B}$, which J preserves.

The left oplax Kan extension \tilde{T} of T along J carries the structure of a pseudomonad on \mathbf{CAT} such that the 2-categories $\mathrm{ColaxAlg}_J(T)$ and $\mathrm{ColaxAlg}(\tilde{T})$ are isomorphic.

Remark

These assumptions are not enough to make $\langle \mathbf{Oplax}[\mathcal{B}, \mathbf{CAT}], \triangleleft^J \rangle$ monoidal.

T. and Wrigley, *Ultracategories via Kan extensions of relative monads*, 2025

Left oplax Kan extensions in CAT

For $\mathcal{A} = \mathbf{CAT}$, left oplax Kan extensions exist and we can describe them explicitly. For concreteness, we describe here the extension of $\beta: \mathbf{Set} \rightarrow \mathbf{CAT}$ along $\mathbf{Set} \hookrightarrow \mathbf{CAT}$.

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 2. and a natural transformation $\alpha: h \circ f \Rightarrow h'$.

If C is a weak ultracategory, recall that each functor $h: X \rightarrow C$ extends to a functor $h^C: \beta X \rightarrow C$. The corresponding colax $\tilde{\beta}$ -algebra functor $\tilde{\beta}C \rightarrow C$ then maps

$$(X, h: X \rightarrow C, \nu \in \beta X) \mapsto h^C(\nu).$$

A pseudomonad structure on $\tilde{\beta}$

In the case of $\beta: \mathbf{Set} \rightarrow \mathbf{CAT}$, the inclusion $\mathbf{Set} \hookrightarrow \mathbf{CAT}$:

1. is fully-faithful,
2. preserves the terminal object 1;
3. preserves small coproducts.

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In the case of $\beta: \mathbf{Set} \rightarrow \mathbf{CAT}$, the inclusion $\mathbf{Set} \hookrightarrow \mathbf{CAT}$ satisfies our assumptions.

Therefore, $\tilde{\beta}$ carries the structure of a pseudomonad $\langle \tilde{\beta}, \eta^\sharp, \mu^\sharp \rangle$ on \mathbf{CAT} , which we now describe.

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For an element $x \in X$, write $h(x) = (Y_x, k_x: Y_x \rightarrow C, \theta_x \in \beta Y_x) \in \tilde{\beta}C$.

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$$\mu_C^\sharp(X, h: X \rightarrow \tilde{\beta}C, \nu \in \beta X) = (\textcolor{violet}{Y}, -, -).$$

For an element $x \in X$, write $h(x) = (Y_x, k_x: Y_x \rightarrow C, \theta_x \in \beta Y_x) \in \tilde{\beta}C$.

1. Let $\textcolor{violet}{Y}$ be the coproduct $\coprod_{x \in X} Y_x$,
with inclusions $\{i_x: Y_x \hookrightarrow \textcolor{violet}{Y}\}_{x \in X}$.

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A pseudomonad structure on $\tilde{\beta}$

In the case of $\beta: \text{Set} \rightarrow \text{CAT}$, the inclusion $\text{Set} \hookrightarrow \text{CAT}$ satisfies our assumptions.

Therefore, $\tilde{\beta}$ carries the structure of a pseudomonad $\langle \tilde{\beta}, \eta^\sharp, \mu^\sharp \rangle$ on CAT , which we now describe.

- On objects, the unit $\eta_C^\sharp: C \rightarrow \tilde{\beta}C$ maps $c \in C$ to $(1, c: 1_{\text{CAT}} \rightarrow C, * \in \beta 1)$.
- On objects, the multiplication $\mu_C^\sharp: \tilde{\beta}^2 C \rightarrow \tilde{\beta}C$ acts by

$$\mu_C^\sharp(X, h: X \rightarrow \tilde{\beta}C, \nu \in \beta X) = (\textcolor{violet}{Y}, k: \textcolor{green}{Y} \rightarrow \textcolor{blue}{C}, q^*(\nu)).$$

For an element $x \in X$, write $h(x) = (Y_x, k_x: Y_x \rightarrow C, \theta_x \in \beta Y_x) \in \tilde{\beta}C$.

1. Let $\textcolor{violet}{Y}$ be the coproduct $\coprod_{x \in X} Y_x$,
with inclusions $\{i_x: Y_x \hookrightarrow \textcolor{violet}{Y}\}_{x \in X}$.
2. Let $k: \textcolor{green}{Y} \rightarrow \textcolor{blue}{C}$ be the unique functor
determined by $\{k_x: Y_x \rightarrow C\}_{x \in X}$,
i.e. $k(y \in Y_x) := k_x(y)$.
3. The functor $q: X \rightarrow \beta Y$ defined by $x \mapsto \beta i_x(\theta_x)$ extends to a functor
 $q^*: \beta X \rightarrow \beta Y$, so that we can consider $q^*(\nu) \in \beta Y$. Concretely, for $S \subseteq Y$,

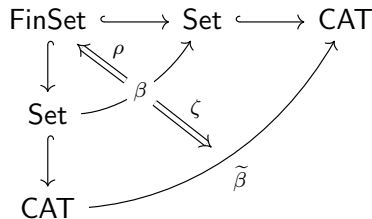
$$S \in q^*(\nu) \iff \{x \in X \mid S \cap Y_x \in \theta_x\} \in \nu$$

Weak ultracategories II

Applying our result to the relative ultrafilter 2-monad, we conclude that weak ultracategories are pseudomonadic over CAT.

Corollary

Weak ultracategories are the colax algebras for the pseudomonad $\tilde{\beta}$ on CAT where:



- ▶ $\beta: \mathbf{Set} \rightarrow \mathbf{Set}$ is the right Kan extension of $\mathbf{FinSet} \hookrightarrow \mathbf{Set}$ along itself;
- ▶ $\tilde{\beta}: \mathbf{CAT} \rightarrow \mathbf{CAT}$ is the left oplax Kan extension of $\beta: \mathbf{Set} \rightarrow \mathbf{CAT}$ along $\mathbf{Set} \hookrightarrow \mathbf{CAT}$.

Future directions

- ▶ We can apply our result to other monads of interest: in particular, the *upper prime filter monad* $B: \mathbf{Pos} \rightarrow \mathbf{Pos}$ whose algebras are **compact ordered spaces**. *Prime categories*, i.e. the colax algebras for the *relative upper prime filter 2-monad* $B: \mathbf{Pos} \rightarrow \mathbf{CAT}$, are then the colax algebras for $\tilde{B}: \mathbf{CAT} \rightarrow \mathbf{CAT}$.
 - Connections with *positive model theory*
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 - Connections with *positive model theory*
 - Towards a Priestley-like duality for first-order logic
- ▶ Can we find sufficient hypotheses on a 2-functor $J: \mathcal{B} \rightarrow \mathcal{A}$ to obtain an abstract ‘unrelativisation’ procedure for J -relative pseudomonads?

Thank you!

Ultracategories via Kan extensions of relative monads,
Umberto Tarantino and Joshua Wrigley, 2025, arXiv:2506.09788

Weak ultracategories are ultracategories

Lurie's ultracategories are a (proper) subclass of weak ultracategories.

However, Lurie's ultrafunctors and left ultrafunctors coincide with pseudomorphisms and colax morphisms, so that we have 2-fully-faithful embeddings:

$$\mathbf{Ult} \hookrightarrow \mathbf{WeakUlt}^{\text{pseudo}} \qquad \mathbf{Ult}^{\text{L}} \hookrightarrow \mathbf{WeakUlt}^{\text{colax}}$$

This ensures that weak ultracategories are a good axiomatisation of ultracategories.

Theorem (Lurie)

For a small pretopos P , the evaluation functor $\text{ev}: P \rightarrow [\text{Mod}(P), \text{Set}]$ induces equivalences of categories:

1. $P \simeq \mathbf{WeakUlt}^{\text{pseudo}}(\text{Mod}(P), \text{Set});$
2. $\text{Sh}(P) \simeq \mathbf{WeakUlt}^{\text{colax}}(\text{Mod}(P), \text{Set}).$

In particular, $\text{Mod}: \text{Pretop}^{\text{op}} \hookrightarrow \mathbf{WeakUlt}^{\text{pseudo}}$ is 2-fully-faithful.

Left oplax Kan extensions in CAT

For $\mathcal{A} = \text{CAT}$, left oplax Kan extensions exist and we can describe them explicitly. For concreteness, consider $\beta: \text{Set} \rightarrow \text{CAT}$. For a category C :

- ▶ objects of $\tilde{T}C$ are triples $(b \in \mathcal{B}, h: Jb \rightarrow C, \nu \in Tb)$;
- ▶ morphisms $(b, h, \nu) \rightarrow (b', h', \nu')$ in $\tilde{T}C$ are triples of
 1. a 1-cell $f: b' \rightarrow b$ in \mathcal{B} ,
 2. a natural transformation $\alpha: h \circ Jf \Rightarrow h'$,
 3. and an arrow $\varphi: \nu \rightarrow Tf(\nu')$ in Tb ,

modulo the equivalence relation generated by $(f, \alpha_f, \varphi_f) \sim (g, \alpha_g, \varphi_g)$ if there exists a 2-cell $\sigma: f \Rightarrow g$ in \mathcal{B} such that

$$\begin{array}{ccc}
 Jb' & \xrightarrow{Jf} & Jb \\
 \downarrow J\sigma & & \downarrow \alpha_f \\
 Jb' & \xrightarrow{Jg} & Jb \\
 \downarrow \alpha_g & & \downarrow \alpha_f \\
 C & & C
 \end{array}
 \quad (1)$$

(Note: The diagram above is a simplified representation of the commutative diagram in the image. The full diagram shows a square with Jb' at the top-left, Jb at the top-right, C at the bottom-left, and C at the bottom-right. Arrows are: $Jb' \xrightarrow{Jf} Jb$ (top), $Jb' \xrightarrow{Jg} Jb$ (bottom), $Jb' \xrightarrow{h'} C$ (left), $Jb \xrightarrow{h} C$ (right), $Jb' \xrightarrow{J\sigma} Jb$ (vertical), $Jb \xrightarrow{\alpha_g} C$ (vertical), $Jb' \xrightarrow{\alpha_g} C$ (vertical). The diagram is labeled (1).)

$$\begin{array}{ccc}
 \nu & \xrightarrow{\varphi_f} & Tf(\nu') \\
 \downarrow \varphi_g & & \downarrow (T\sigma)_{\nu'} \\
 \nu & \xrightarrow{\varphi_g} & Tg(\nu')
 \end{array}
 \quad (2)$$

(Note: The diagram above is a simplified representation of the commutative diagram in the image. The full diagram shows a square with ν at the top-left, $Tf(\nu')$ at the top-right, $Tg(\nu')$ at the bottom-right, and ν at the bottom-left. Arrows are: $\nu \xrightarrow{\varphi_f} Tf(\nu')$ (top), $\nu \xrightarrow{\varphi_g} Tg(\nu')$ (bottom), $Tf(\nu') \xrightarrow{(T\sigma)_{\nu'}} Tg(\nu')$ (right), $\nu \xrightarrow{\varphi_g} Tg(\nu')$ (left). The diagram is labeled (2).)

A pseudomonad structure on \tilde{T}

Under our assumptions, \tilde{T} carries the structure of a pseudomonad $\langle \tilde{T}, \eta^\sharp, \mu^\sharp \rangle$ on CAT.

- ▶ On objects, the unit $\eta_C^\sharp: C \rightarrow \tilde{T}C$ maps $c \in C$ to $(1_B, c: 1_{\text{CAT}} \rightarrow C, \eta_{1_B}(*))$.
- ▶ On objects, the multiplication $\mu_C^\sharp: \tilde{T}^2 C \rightarrow \tilde{T}C$ acts by

$$\mu_C^\sharp(b, h: Jb \rightarrow \tilde{T}C, \nu) = (\ell, a: J\ell \rightarrow C, Q^*\nu).$$

For an object $x \in Jb$, write $h(x) = (Rx, a_x: JRx \rightarrow C, \nu_x)$.

For an arrow $g: x \rightarrow y \in Jb$, write $h(g) = (Rg, \gamma_g: a_x \circ JRg \Rightarrow a_y, \psi_g)$.

1. Let $\ell \in \mathcal{B}$ be the oplax colimit of $R: (Jb)^{\text{op}} \rightarrow \mathcal{B}$, with universal cocone:
2. As J preserves oplax colimits and C is an oplax cocone of $JR: (Jb)^{\text{op}} \rightarrow \text{CAT}$

$$\begin{array}{ccc} Rx & \xleftarrow{Rg} & Ry \\ & \searrow \lambda_g & \swarrow \\ c_x & & c_y \\ & \searrow & \swarrow \\ & \ell & \end{array}$$

$$\begin{array}{ccc} JRx & \xleftarrow{JRg} & J Ry \\ & \searrow \gamma_g & \swarrow \\ a_x & & a_y \\ & \searrow & \swarrow \\ & C, & \end{array}$$

there is a universal functor $a: J\ell \rightarrow C$.

3. The map $x \mapsto T_{c_x}(\nu_x)$ lifts to a functor $Q: Jb \rightarrow T\ell$, which extends to a functor $Q^*: Tb \rightarrow T\ell$ via the monad structure of T , so that we can consider $Q^*\nu \in T\ell$.