II. Epps problem 5.1.15 | p273

(15) $0, -\frac{1}{2}, \frac{2}{3}, -\frac{3}{4}, \frac{4}{5}, -\frac{5}{6}, \frac{6}{7}$

$$a_0 = \frac{(-1)^0 \cdot 1}{0+1} = 0$$

$$a_1 = \frac{(-1)^1 \cdot 1}{1+1} = -\frac{1}{2}$$

$$a_2 = \frac{(-1)^2 \cdot 2}{2+1} = \frac{2}{3}$$

$$a_3 = \frac{(-1)^3 \cdot 3}{3+1} = -\frac{3}{4}$$

$$a_n = \frac{(-1)^n \cdot n}{n+1}$$

where n is an integer and $n \geq 0$

2. Epps problem 5.1.57 | p274

$$\sum_{i=1}^{n-1} \frac{i}{(n-i)^2}$$

when

$j = i - 1$

$i = j + 1$

$i = 1, \; j = 1 - 1 = 0$

$i = n-1 \; j = n-1-1 = n-2$

So,

$$\sum_{j=0}^{n-2} \frac{j+1}{(n-(j+1))^2}$$

$$= \sum_{j=0}^{n-2} \frac{j+1}{(n-1-j)^2}$$

3. Epps problem 5.2.4 (p288)

$$\sum_{i=1}^{n-1} i(i+1) = \frac{n(n-1)(n+1)}{3}$$

a Ans

$$P(2) = \sum_{i=1}^{2-1} i(i+1)$$

$$P(2) = 1(1+1) = 2$$

So,

$$P(n) = \frac{n(n-1)(n+1)}{3}$$

$$= \frac{2(2-1)(2+1)}{3}$$

$$= 2$$

Hence,

$P(2) = 2$ which is true

(b) $P(k) = \sum_{i=1}^{k-1} i(i+1) = \frac{k(k-1)(k+1)}{3}$

(c) $P(k+1)$

So,

$$P(k+1) = \sum_{i=1}^{(k+1)-1} i(i+1)$$

$$= \sum_{i=1}^{k} i(i+1)$$

$$P(k+1) = \frac{(k+1)(k+1-1)(k+1+1)}{3}$$

$$= \frac{(k+1)k(k+2)}{3}$$

$$\therefore P(k+1) = \frac{k(k+1)(k+2)}{3}$$

(d) mathematical induction,

Base case:-

$P(2) = 2$ true

Hypothesis:-

$K \geq 2$

$$P(K) = \sum_{i=1}^{K-1} i(i+1) = \frac{K(K-1)(K+1)}{3}$$

Induction

$n = K+1$

$$P(K+1) = \sum_{i=1}^{K} i(i+1) = \frac{K(K+1)(K+2)}{3}$$

inductive step:-

$$P(k+1) = \sum_{i=1}^{k} i(i+1)$$

Split:-

$$P(k+1) = \sum_{i=1}^{k-1} i(i+1) + k(k+1)$$

$$\sum_{i=1}^{k-1} i(i+1) = \frac{k(k-1)(k+1)}{3}$$

$$P(k+1) = \frac{k(k-1)(k+1)}{3} + k(k+1)$$

$$P(k+1) = k(k+1)\left(\frac{k-1}{3} + 1\right)$$

$$\frac{k-1}{3} + 1 = \frac{k-1+3}{3} = \frac{k+2}{3}$$

$$P(k+1) = k(k+1) \times \frac{k+2}{3}$$

$$P(k+1) = \frac{(k+1)(k+2)}{3}$$

proof by induction

4. Epps problem 5.2.12 | p 287

(12) $\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \dots + \frac{1}{n(n+1)} = \frac{n}{n+1}$

for every integer $n \geq 1$

Base :-

$n = 1$

Left hand side $\frac{1}{1\cdot 2} = \frac{1}{2}$

Righthand side $= \frac{1}{1+1} = \frac{1}{2}$

Hypothesis

$n = k$

$\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \dots + \frac{1}{k(k+1)} = \frac{k}{k+1}$

Inductive step:-

$n = k+1$

$$\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \dots + \frac{1}{k(k+1)} + \frac{1}{(k+1)(k+2)}$$

$$= \frac{k+1}{k+2}$$

Using hypothesis

$$\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \dots \frac{1}{k(k+1)} = \frac{k}{k+1}$$

Left-handside becomes

$$\frac{k}{k+1} + \frac{1}{(k+1)(k+2)}$$

we can,

$$\frac{k(k+2)}{(k+1)(k+2)} + \frac{1}{(k+1)(k+2)}$$

$$= \frac{k(k+2)+1}{(k+1)(k+2)}$$

$$= \frac{k^2+2k+1}{(k+1)(k+2)}$$

$$= \frac{(k+1)^2}{(k+1)(k+2)}$$

$$= \frac{k+1}{k+2}$$

proof:-

Q.5. Epps problems 5.2.27/p288

$5^3+5^4+5^5+\dots+5^k$ where k is any integer $k \geq 3$

$a = 5^3 = 125$

$r = 5$

$$S_n = a \times \frac{r^n - 1}{r - 1}$$

$$= 125 \times \frac{5^{k-2} - 1}{5 - 1}$$

The

$$5^3 + 5^4 + 5^5 \dots 5^k = 125 \times \frac{5^{k-2} - 1}{4}$$

6. Epps problem 5.3.17 | p298

(17) $1+3n \le 4^n$ for $n \ge 0$

Base

$n=0$

$1+3\times 0 \le 4^0$

$1 \le 1$

Hypothesis: $n=k$

$1+3k \le 4^k$ — (i)

Inductive:-

$1+3(k+1) \le 4^{k+1}$

$1+3k+3 \le 4^{k+1}$

from hypothesis — (i)

$4^k+3 \le 4^{k+1}$

$4^k+3 \le 4^k \cdot 4$

proof

Q7. Epp5 problem 5.3.23 | p298

Q: $n^3 > 2n+1$ for $n \geq 2$

Base

$n=2$

$2^3 > 2 \times 2 + 1$

$8 > 5$ (true)

hypothesis

$n=k$

$k^3 > 2k+1$

Inductive,

$n=k+1$

$(k+1)^3 > 2(k+1)+1$

Left side,

$(k+1)^3$

$= k^3 + 3k^2 + 3k + 1$

Right side

$2(k+1)+1$

$= 2k+2+1$

$= 2k+3$

$k^3+3k^2+3k+1 > 2k+3$

$k^3+3k^2+3k+1-(2k+3)>0$

$k^3+3k^2+k-2>0$

$k=2$

$2^3+3(2)^2+2-2$

$=20$

Since

$20>0$

prove:

$n^3>2n+1$

for

$n\geq 2$

(b) $n! > n^2$ for $n \geq 4$

Base $n = 4$

$4! > 4^2$

$24 > 16$ (true)

Hypothesis

$n = k$

$k! > k^2$ — (i)

Inductive:

$(k+1)! > (k+1)^2$

$(k+1)\, k! > k^2 + 2k + 1$

from (i)

$(k+1) k^2 > k^2 + 2k + 1$

$k^3 + k^2 > k^2 + 2k + 1$

$k^3 > 2k + 1$

$k = 4$

$64 > 9$

proof:-

$n! > n^2$

for $n \geq 4$

8. Epps problem 5.4.7 | p311

$g_1 = 3, \; g_2 = 5$

$g_k = 3g_{k-1} - 2g_{k-2}$

for $k \geq 3$

$g_n = 2^n + 1$

for $n \geq 1$

Base:-

$n = 1$

$g_1 = 3$ & $2^1 + 2$

$= 3$

$n = 2$

$g_2 = 5$

$2^2 + 1$

$= 5$

Hypothesis,

$g_k = 2^k + 1$

$1 \leq k \leq 4$

strong induction:-

$$g_{n+1} = 2^{n+1} + 1$$

$$g_{n+1} = 3g_n - 2g_{n-1}$$

So, Using hypothesis

$$g_n = 2^n + 1$$
$$g_{n-1} = 2^{n-1} + 1$$

$$g_{n+1} = 3g_n - 2g_{n-1}$$
$$= 3(2^n + 1) - 2(2^{n-1} + 1)$$

$$g_{n+1} = 3 \cdot 2^n - 2 \cdot 2^{n-1} + 1$$
$$= 2^{n+1} + 1$$

proof

9. Epps problem 5.4.20 / p312

Given,

$b_1 = 0$, $b_2 = 3$

$b_k = 5 \cdot b_{k/2} + 6$ for $k \geq 3$

Prove,

b_n is divisible by 3, $n \geq 1$

Base:- $n = 1$ $\quad b_1 = 0$

$n = 2$ $\quad b_2 = 3$

Hypothesis:-

$2 \leq i \leq k$

Strong Inductive

$b_{k+1} = 5 \cdot b_{\lfloor (k+1)/2 \rfloor} + 6$

So,

$b_{k+1} = 5 \cdot 3m + 6$

$b_{k+1} = 15m + 6$

$b_{k+1} = 3(5m + 2)$

Since,

$b_{k+1} = 3 \times \text{integer}$

Divisible by 3 for $n \geq 1$

10. Epps problem 5.6.8 | p337

$v_k = v_{k-1} - v_{k-2} + 1$ for $k \geq 3$

$v_1 = 1, v_2 = 3$

$v_3 = v_2 + v_1 + 1$

$= 5$

$v_4 = v_3 + v_2 + 1 = 9$

$v_5 = v_4 + v_3 + 1 = 15$

$v_6 = v_5 + v_4 + 1 = 25$

first four terms:-

$v_1 = 1$

$v_2 = 3$

$v_3 = 5$

$v_4 = 9$