The Blueprint For Formalizing Geometric Algebra in Lean

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Introduction

The goal of this document is to provide a detailed account of the formalization of Geometric Algebra (GA) a.k.a. Clifford Algebra [Hestenes and Sobczyk(1984)] in the Lean 4 theorem prover and programming language [Moura and Ullrich(2021), de Moura et al.(2015), Ullrich(2023)] and using its Mathematical Library Mathlib [The mathlib Community(2020)].

The web version of this blueprint is available here.

1 Preliminaries

This section introduces the algebraic environment of Clifford Algebra, covering vector spaces, groups, algebras, representations, modules, multilinear algebras, quadratic forms, filtrations and graded algebras.

The material in this section should be familiar to the reader, but it is worth reading through it to become familiar with the notation and terminology that is used, as well as their counterparts in Lean, which usually require some additional treatment, both mathematically and technically (probably applicable to other formal proof verification systems).

No details will be given as these are given in standard textbooks such as [Mac Lane and Birkhoff(1999)], or see the references in corresponding section.

1.1 Basics

In this section, we follow [Jadczyk(2019)], with supplements from [Garling(2011), Chen(2016)], and modifications to match the counterparts in Lean's Mathlib.

Definition 1.1.1 (Group)**.** A **group** is a pair $(G, *)$, where G is a set, $*$ is a binary operation, satisfying:

1. $(g * h) * j = g * (h * j)$ for all $g, h, j \in G$ (**associativity**)

2. there exists e in G such that $e * g = g * e = g$ for all $g \in G$

3. for each $g \in G$ there exists $g^{-1} \in G$ such that $g * g^{-1} = g^{-1} * g = e$

> **Remark 1.1.2 —** It then follows that e, the **identity element**, is unique, and that for each $g \in G$ the **inverse** g^{-1} is unique.
>
> A group G is abelian, or **commutative**, if $g * h = h * g$ for all $g, h \in G$.

Definition 1.1.4 (Monoid)**.** A **monoid** is a pair $(R, *)$, satisfying:

1. $(a * b) * c = a * (b * c)$ for all $a, b, c \in R$ (**associativity**),

2. There is an element $1 \in R$ such that $1 * a = a * 1 = a$ for all a in R (that is 1 is the multiplicative identity (**neutral element**).

Definition 1.1.5 (Ring)**.** A **ring** is a triple $(R, +, *)$, satisfying:

1. The elements of R form a **commutative group** under $+$;

2. The elements of R form a **monoid** under $*$;

3. If a, b, c are elements of R, we have

$$a * (b + c) = a * b + a * c,$$

$$(a + b) * c = a * c + b * c$$

(left and right **distributivity** over $+$).

> **Remark 1.1.6** — In applications to Clifford algebras R will be always assumed to be **commutative**.

Definition 1.1.7 (Division ring)**.** A ring containing at least two elements, in which every nonzero element a has a multiplicative inverse a^{-1} is called a **division ring** (sometimes also called a "skew field").

Definition 1.1.8 (Module)**.** Let R be a commutative ring. A **module** over R (in short R-module) is a set M such that

1. M has a structure of an additive group,

2. For every $a, b \in R$, $x, y \in M$, an operation $\bullet : R \to M \to M$ called scalar multiplication is defined, and we have

   i $a \bullet (x + y) = a \bullet x + b \bullet y$,

   ii $(a + b) \bullet x = a \bullet x + b \bullet x$,

   iii $a * (b \bullet x) = (a * b) \bullet x$,

   iv $1_R \bullet x = x$.

Definition 1.1.10 (Vector space). If R is a **division ring**, then a module M over R is called a **vector space**.

Definition 1.1.13 (Algebra). An **algebra** A over R is a module over R with a multiplication which makes A a ring and satisfying

$$\alpha(xy) = (\alpha x)y = x(\alpha y), (x, y \in A, \alpha \in R).$$

Definition 1.1.15 (Ring homomorphism). Let $(\alpha, +_\alpha, *_\alpha)$, and $(\beta, +_\beta, *_\beta)$ be rings.

A **ring homomorphism**, from α to β is a function $f : \alpha \to_{+*} \beta$ such that

(i) $f(x +_\alpha y) = f(x) +_\beta f(y)$ for each $x, y \in \alpha$.

(ii) $f(x *_\alpha y) = f(x) *_\beta f(y)$ for each $x, y \in \alpha$.

(iii) $f(1_\alpha) = 1_\beta$.

Definition 1.1.16 (FreeAlgebra). TODO

Definition 1.1.17 (LinearMap). TODO

Definition 1.1.18 (RingQuot). TODO

Definition 1.1.19 (TensorAlgebra relation). TODO

Definition 1.1.20 (Tensor algebra). Let M be a module over R. An algebra T is called a **tensor algebra** over M (or "of M ") if it satisfies the following universal property

2. T is an algebra containing M as a **submodule**, and it is **generated by M**,

3. Every linear mapping λ of M into an algebra A over R, can be extended to a **homomorphism** θ of T into A.

> **Remark 1.1.21 —** The properties above are equivalent to the following:
>
> 2. T is the free (associative, unital) R-algebra generated by M.
>
> 3. additional relations making the inclusion of M into an R-linear map
>
> As ideals haven't been formalized for the non-commutative case, Mathlib uses RingQuot which is the quotient of a non-commutative ring by an arbitrary relation.

2 Foundations

2.1 Clifford algebras - definition

Throughout this section:

Let M be a module over a commutative ring R, equipped with a quadratic form $Q : M \to R$.
Let $\iota : M \to_{l[R]} T(M)$ be the canonical R-linear map for the tensor algebra $T(M)$.
Let $\iota_a : R \to_{+*} T(M)$ be the canonical map from R to $T(M)$, as a ring homomorphism.

Definition 2.1.1 (Clifford relation). $\forall m \in M, \iota(m)^2 \sim \iota_a(Q(m))$

We say that ι **is Clifford** if this relation holds.

Definition 2.1.2 (Clifford algebra). A **Clifford algebra** over M, denoted $\mathcal{C}\ell(M)$, is the quotient of the **tensor algebra** $T(M)$ by **Clifford relation** 2.1.1.

> **Remark 2.1.3 —** In literatures, M is often written V, and the quotient is taken by the two-sided ideal I_Q generated from the set $\{v \otimes v - Q(v) \mid v \in V\}$.
>
> As of writing, Mathlib does not have direct support for two-sided ideals, but it does support the equivalent operation of taking the quotient by a suitable closure of a relation like $v \otimes v \sim Q(v)$.
>
> Hence the definition above.

> **Example 2.1.4** (Clifford algebra over a vector space)
>
> Let V be a vector space $\mathbb{R}^n$ over $\mathbb{R}$, equipped with a quadratic form Q.
>
> Since $\mathbb{R}$ is a commutative ring and V is a module, definition 2.1.2 of Clifford algebra applies.

References

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