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Supplementary Material for “Inverse Kinematics for Serial Kinematic Chains via Sum of Squares Optimization”

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Abstract

This document contains supplementary material for the paper titled “Inverse Kinematics for Serial Kinematic Chains via Sum of Squares Optimization”. A proof of Theorem 2 from the main paper is presented. It also contains a table with kinematic chain parameters used for simulation experiments reported on in the main paper.

1 Proof of Theorem 2

We begin by restating Theorems 1 [1] and 2 from the main paper.

Theorem 1 (Nearest Point to a Quadratic Variety [1])**.** *Consider the problem*

$$\min_{\mathbf{y}\in Y} \|\mathbf{y}-\boldsymbol{\xi}\|^2 , \tag{1}$$

where $Y := \{\mathbf{y}\in\mathbb{R}^n : f_1(\mathbf{y}) = \cdots = f_m(\mathbf{y}) = 0\}$, f_i *quadratic. Let* $\bar{\boldsymbol{\xi}} \in Y$ *be such that*

$$\operatorname{rank}(\nabla f(\bar{\boldsymbol{\xi}}))) = n - \dim_{\bar{\boldsymbol{\xi}}} Y. \tag{2}$$

Then there is zero-duality-gap for any $\boldsymbol{\xi}\in\mathbb{R}^n$ *that is sufficiently close to* $\bar{\boldsymbol{\xi}}$*.*

Theorem 2 (Strong Duality)**.** *If* $\bar{\boldsymbol{\xi}} \in Y_{IK}$ *does not represent a fully extended configuration (i.e., the joint positions are not all collinear), and does not have any joints at their angular limits for specified base and goal positions, then the nearest point inverse kinematics QCQP in the main paper exhibits strong duality for all* $\boldsymbol{\xi}$ *sufficiently close to* $\bar{\boldsymbol{\xi}}$*.*

Proof. According to Theorem 1 in the main paper, it is sufficient to show that $\operatorname{rank}(\nabla f(\bar{\boldsymbol{\xi}}))) = n - \dim_{\bar{\boldsymbol{\xi}}} Y_{\mathrm{IK}}$ holds. The number of variables $n = d(N-1)+N$ scales with dimension $d\in\{2,3\}$ and the number of links N. Theorem 1.7 in [2] gives us $\dim_{\bar{\boldsymbol{\xi}}} Y_{\text{IK}} = (d-1)(N-1)-1$. Therefore, we need to show that

$$\begin{aligned}\text{rank}(\nabla f(\bar{\boldsymbol{\xi}}))) &= d(N-1)+N-(d-1)(N-1)+1 \\ &= 2N.\end{aligned}$$

The structure of the Jacobian matrix $\nabla f(\bar{\boldsymbol{\xi}})) \in \mathbb{R}^{2N \times (d(N-1)+N)}$ can be understood in terms of N link length constraints representing the first N rows, and N joint limit constraints representing the final N rows:

$$\nabla f(\bar{\boldsymbol{\xi}})) = \begin{bmatrix} \mathbf{J}_{1,1} & \mathbf{0}_{N\times N} \\ \mathbf{J}_{2,1} & \mathbf{J}_{2,2} \end{bmatrix}. \tag{3}$$

The block lower triangular structure is due to the independence of link length constraints on s. Since $\text{rank}(\nabla f(\bar{\boldsymbol{\xi}}))) \geq \text{rank}(\mathbf{J}_{1,1}) + \text{rank}(\mathbf{J}_{2,2})$ for block lower triangular matrices, it is sufficient to demonstrate that $\text{rank}(\mathbf{J}_{1,1}) = \text{rank}(\mathbf{J}_{2,2}) = N$. Since $\mathbf{J}_{2,2} = \text{diag}(2\mathbf{s})$, and $s_i > 0 \;\forall\; i \in \{1,\ldots,N\}$ by assumption, $\text{rank}(\mathbf{J}_{2,2}) = N$. It remains to demonstrate that

$$\mathbf{J}_{1,1}^T = 2\begin{bmatrix} \mathbf{x}_1 - \mathbf{x}_0 & \mathbf{x}_1 - \mathbf{x}_2 & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \mathbf{x}_2 - \mathbf{x}_1 & \mathbf{x}_2 - \mathbf{x}_3 & \mathbf{0} & \vdots \\ \vdots & \mathbf{0} & \ddots & \ddots & \mathbf{0} \\ \mathbf{0} & \cdots & \mathbf{0} & \mathbf{x}_{N-1} - \mathbf{x}_{N-2} & \mathbf{x}_{N-1} - \mathbf{x}_N \end{bmatrix} \tag{4}$$

has (full) rank N. Suppose $\text{rank}(\mathbf{J}_{1,1}) \neq N$. Then there exists $\mathbf{v} \in \mathbb{R}^N$ such that $\mathbf{v} \neq \mathbf{0}$ and $\mathbf{J}_{1,1}^T \mathbf{v} = \mathbf{0}$. This implies that

$$v_i(\mathbf{x}_i - \mathbf{x}_{i-1}) = v_{i+1}(\mathbf{x}_{i+1} - \mathbf{x}_i), \;\; \forall i = 1,\ldots,N-1. \tag{5}$$

Since there exists some i such that $v_i \neq 0$, and the link lengths $l_i = \|\mathbf{x}_i - \mathbf{x}_{i-1}\|$ are all greater than zero, Equation 5 tells us that $v_i \neq 0$ for all i. Therefore, $\mathbf{x}_i - \mathbf{x}_{i-1} = c_{ij}(\mathbf{x}_j - \mathbf{x}_{j-1})$ for all valid pairs of i, j, where

$$c_{ij} = \frac{v_j}{v_i} \neq 0. \tag{6}$$

In other words, the link orientations are all collinear, which contradicts the assumption that the arm is not fully extended. Therefore, $\text{rank}(\mathbf{J}_{1,1}) = N$ and $\text{rank}(\nabla f(\bar{\boldsymbol{\xi}})) = 2N$, completing the proof.

□

Corollary 1. *Theorem 2 holds when the orientation of the end effector is also constrained.*

Proof. Let Y'_{IK} be the variety Y_{IK} with an additional variable s' and an additional constraint on the angle between $\hat{\mathbf{z}}_N$ and $\hat{\mathbf{z}}_{N+1} = \frac{1}{l_{N+1}}(\mathbf{x}_{N+1} - \mathbf{x}_N)$. That is, we additionally add the constraint

$$\left\|\hat{\mathbf{z}}_{N+1} - \hat{\mathbf{z}}_N\right\|^2 + s'^2 - 2\left(1 - \cos\alpha_{N+1}\right) = 0 \tag{7}$$

to our variety Y'_{IK}, where $\mathbf{x}_N$ and $\mathbf{x}_{N+1}$ are the fixed specified locations of the final joint and the end effector respectively, and α_{N+1} is the angular limit for the final joint. This variety Y'_{IK} describing the feasible configurations for an $(N+1)$-DoF kinematic chain with its end effector and orientation fixed is therefore equivalent to Y_{IK} for an N-DoF kinematic chain, with one variable and one constraint added. Recall that we need to prove that $\text{rank}(\nabla f(\bar{\boldsymbol{\xi}})) = n - \dim_{\bar{\boldsymbol{\xi}}} Y'_{\text{IK}}$. The constraint cannot affect $\dim_{\bar{\boldsymbol{\xi}}} Y_{\text{IK}}$ because we assume in our premise that none of the joint angles are activating their constraints, including the angular constraint corresponding to s'. The number of variables n is simply increased by 1. Therefore, we must simply show that $\text{rank}(\nabla f(\bar{\boldsymbol{\xi}}))$ increases by 1 with the additional variable and constraint added. This follows immediately from the fact that the additional constraint increases the rank of $\text{rank}(\mathbf{J}_{2,2})$ by 1 and does not affect $\text{rank}(\mathbf{J}_{1,1})$. □

2 Parameter Table

Table 1: Parameters used for the experiments in the main paper.

joint	1	2	3	4	5	6	7	8	9	10
$\lvert\theta_{\text{i}}\rvert_{\text{max}}$	$\frac{\pi}{4}$	$\frac{\pi}{4}$	$\frac{\pi}{8}$	$\frac{\pi}{4}$	$\frac{\pi}{4}$	$\frac{\pi}{2}$	$\frac{\pi}{4}$	$\frac{\pi}{4}$	$\frac{\pi}{2}$	$\frac{\pi}{8}$
l_i	2	2	1	2	3	2	4	4	1	2

References

[1] Diego Cifuentes, Sameer Agarwal, Pablo A Parrilo, and Rekha R Thomas. On the local stability of semidefinite relaxations. *arXiv preprint arXiv:1710.04287*, 2017.

[2] R James Milgram, Jeff C Trinkle, et al. The geometry of configuration spaces for closed chains in two and three dimensions. *Homology, Homotopy and Applications*, 6(1):237–267, 2004.