HW 7

1] Prove the following identity:

$$Pr(\alpha_1, \ldots, \alpha_n \mid \beta) = Pr(\alpha_1 \mid \alpha_2, \ldots, \alpha_n, \beta) Pr(\alpha_2 \mid \alpha_3, \ldots, \alpha_n, \beta) \ldots Pr(\alpha_n \mid \beta)$$

Looking at RHS,

$$Pr(\alpha_1 \mid \alpha_2 \ldots \alpha_n, \beta) = Pr(\alpha_1, \alpha_2, \ldots, \alpha_n, \beta) / Pr(\alpha_2, \ldots, \alpha_n, \beta)$$
$$Pr(\alpha_2 \mid \alpha_3 \ldots \alpha_n, \beta) = Pr(\alpha_2, \alpha_3, \ldots, \alpha_n, \beta) / Pr(\alpha_3, \ldots, \alpha_n, \beta)$$

[chain rule]

$\vdots$

$$Pr(\alpha_n \mid \beta) = Pr(\alpha_n, \beta) / Pr(\beta)$$

[based on conditional prob. formula: $Pr(A \mid B) = Pr(A, B) / Pr(B)$]

Using these in the RHS,

$$[Pr(\alpha_1, \alpha_2, \ldots, \alpha_n, \beta) / Pr(\cancel{\alpha_2, \ldots \alpha, \beta})][\cancel{Pr(\alpha_2, \alpha_3, \ldots, \alpha_n, \beta)} / Pr(\cancel{\alpha_3, \ldots, \alpha_n, \beta})] \ldots [\cancel{Pr(\alpha_n, \beta)} / Pr(\beta)]$$

* Every denominator gets cancelled by next numerator in this prod.

So, we are finally left with –

$$\text{RHS} = Pr(\alpha_1, \alpha_2, \ldots, \alpha_n, \beta) / Pr(\beta)$$

which in conditional probability terms, is same as

$$\underline{P(\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_n \mid \beta)} = \underline{\text{LHS}}$$

$\therefore$ Hence proved

2] A well is being drilled on a farm. Based on what has happened to similar farms, we judge the probability of oil being present to be 0.5, the probability of natural gas being present to be 0.2, and the probability of neither being present to be 0.3. If oil is present, a geological test will give a positive result with probability 0.9; if natural gas is present, it will give a positive result with probability 0.3; and if neither are present, the test will be positive with probability 0.1. Suppose the test comes back positive. What's the probability that oil is present?

O – Oil
NG – Natural Gas
N – Neither

$P(O) = 0.5$, $P(NG) = 0.2$, $P(N) = 0.3$

$P(+ve \mid O) = 0.9$
$P(+ve \mid NG) = 0.3$
$P(+ve \mid N) = 0.1$

$P(O \mid +ve) = ?$

Using Bayes Theorem,

we know

$$Pr(O|+ve) = \frac{Pr(+ve|O) \times Pr(O)}{Pr(+ve)}$$

← unknown

$$Pr(+ve) = Pr(+ve|O) * Pr(O) + Pr(+ve|NG) * Pr(NG) + Pr(+ve|N) * Pr(N)$$

— case Analysis

$$Pr(+ve) = 0.9 \times 0.5 + 0.3 \times 0.2 + 0.1 \times 0.3$$
$$= 0.45 + 0.06 + 0.03$$
$$= \underline{0.54}$$

$$\therefore Pr(O|+ve) = \frac{0.9 \times 0.5}{0.54} \approx 0.8333$$

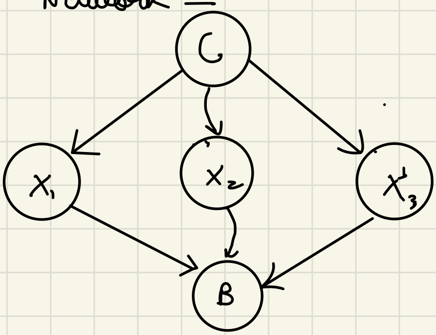
$\therefore$ $Pr(O|+ve)$ required in the question is $\underline{83.33\%}$

3] We have a bag of three biased coins a, b, and c with probabilities of coming up heads of 20%, 40%, and 80%, respectively. One coin is drawn randomly from the bag (with equal likelihood of drawing each of the three coins), and then the coin is flipped three times to generate the outcomes X_1, X_2, and X_3. A bell will ring "on" if all coins flips come out the same. Draw the Bayesian network corresponding to this setup and define the necessary CPTs (Conditional Probability Tables).

$P(H|a) = 0.2$
$P(H|b) = 0.4$
$P(H|c) = 0.8$

Variable : C – coin $\in \{a, b, c\}$
X_1, X_2, X_3 – outcome $\in \{H, T\}$
B – Bell $\in \{On, Off\}$

Network —



CPTs —

C	Pr
a	1/3
b	1/3
c	1/3

C	X	Pr
a	H	0.2
b	H	0.4
c	H	0.8
a	T	0.8
b	T	0.6
c	T	0.2

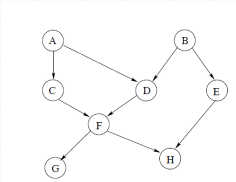
X_1	X_2	X_3	B	Pr
H	H	H	On Off	1 0
H	H	T	On Off	0 1
H	T	H	On Off	0 1
H	T	T	On Off	0 1
T	H	H	On Off	0 1
T	H	T	On Off	0 1
T	T	H	On Off	0 1
T	T	T	On Off	1 0

On only when $X_1 = X_2 = X_3$ otherwise off.

4]

Consider the DAG in Figure 1:

1. List the Markovian assumptions asserted by the DAG.
2. True or false? Why?
   - d_seperated(A, F, E)
   - d_seperated(G, B, E)
   - d_seperated(AB, CDE, GH)
3. Express $\Pr(a, b, c, d, e, f, g, h)$ in factored form using the chain rule for Bayesian networks.
4. Compute $\Pr(A = 1, B = 1)$ and $\Pr(E = 0, A = 0)$. Justify your answers.



a) Assumptions (Markov) -

$I(A, \phi, BE)$ $\quad$ $I(E, B, ACDFG)$

$I(B, \phi, AC)$ $\quad$ $I(F, CD, ABE)$

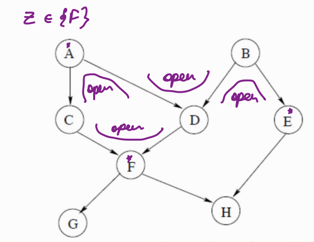
$I(C, A, BDE)$ $\quad$ $I(G, F, ABCDEH)$

$I(D, AB, CE)$ $\quad$ $I(H, EF, ABCDG)$

[I(Node v, Parent(v), Non-descendants(v))]

b) d-sep(A, F, E)

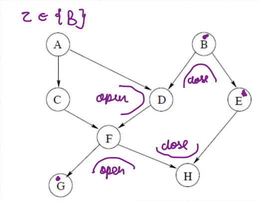
False as F does not block path b/w A & E. There is a path via D.



Alternate: As we can create a series of open paths from A to E, the answer is false

d-sep(G, B, E)

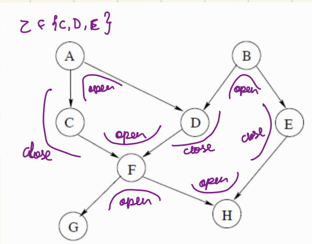
True as there is no active trail from G to E which isn't blocked by B. Removing outgoing edges from B, recursively removing leaves ie not G, B, E gives disconnection b/w G & E.



Alternate: Since every path from G to E involves a closed route, therefore answer is true.

d-sep (AB, CDE, GH)

True as there is no path possible b/w $\{A, B\}$ & $\{G, H\}$ which isn't blocked by least one of nodes from $\{C, D, E\}$
Using the same logic as (b) gives this result.



Alternate: Since every path from $X = \{A, B\}$ to $Y = \{G, H\}$ involves a closed route due to $Z = \{C, D, E\}$, therefore the answer is True.

c) Using chain rule for Bayesian networks

$$Pr(a, b, c, d, e, f, g, h) = Pr(a) * Pr(b) * Pr(c|a) * Pr(d|a, b) * Pr(e|b) * Pr(f|c, d) * Pr(g|f) * Pr(h|e, f)$$

This covers every node in graph and considers child-parent pairs

d) 1. Since A & B are independent statements with no parent each

$$Pr(A=1, B=1) = Pr(A=1) * Pr(B=1) = 0.2 \times 0.7 = \boxed{0.14}$$

2. Since E is dependent of B but independent of A, we compute

$$Pr(A=0, E=0) = Pr(A=0) * [Pr(B=0) * Pr(E=0|B=0) + Pr(B=1) * Pr(E=0|B=1)]$$

$$= 0.8 * [0.3 \times 0.1 + 0.7 \times 0.9] = 0.8 \times 0.66 = \boxed{0.528}$$

5]

Consider the joint probability distribution in Table 2 and the propositional sentence $\alpha : A \Rightarrow B$.

1. List the models of α.
2. Compute the probability $Pr(\alpha)$.
3. Compute the conditional probability distribution $Pr(A, B \mid \alpha)$ in factored form as in Table 2.
4. Compute the probability $Pr(A \Rightarrow \neg B \mid \alpha)$.

	A	B	$Pr(A, B)$
ω_0	T	T	0.3
ω_1	T	F	0.2
ω_2	F	T	0.1
ω_3	F	F	0.4

1.] Since $\alpha : A \Rightarrow B$

The worlds satisfying α are only $\underline{w_0, w_2, w_3}$

2.] $P_r(\alpha) = P_r(w0) + P_r(w2) + P_3(w_3)$

$= 0.3 + 0.1 + 0.4 = \underline{0.8}$

3] $P_r(A, B \mid \alpha)$

That means given α is true so we will consider w_0, w_2, w_3

$\therefore P_r(A, B \mid \alpha) =$ ① $P_r(A = T, B = T \mid \alpha) \longrightarrow P_r(w_0 \mid \alpha) = P_r(w_0) / P_r(\alpha) = 0.375$

② $P_r(A = F, B = T \mid \alpha) \longrightarrow P_r(w_2 \mid \alpha) = P_r(w_2) / P_r(\alpha) = 0.125$

③ $P_r(A = F, B = F \mid \alpha) \longrightarrow P_r(w_3 \mid \alpha) = P_r(w_3) / P_r(\alpha) = 0.5$

$\longrightarrow$ For $P_r(A, B \mid \alpha)$ we have to consider $A = T$, $B = T$ given α

$\therefore P_r(A, B \mid \alpha) = P_r(w_0 \mid \alpha) = \dfrac{P_r(w_0)}{P_r(\alpha)} = \underline{0.375}$

4] $P_r(A \Rightarrow \neg B \mid \alpha)$

Since α is True lets consider $A \Rightarrow \neg B$,
$A \Rightarrow \neg B$ is only true in worlds w_1, w_2, w_3
& since α is satisfied in w_0, w_2, w_3

$\therefore P_r(A \Rightarrow \neg B \mid \alpha) = \dfrac{P_r(w_2)}{P_r(\alpha)} + \dfrac{P_r(w_3)}{P_r(\alpha)} = \dfrac{0.1}{0.8} + \dfrac{0.4}{0.8}$

(w_2, w_3 only true)

$= \underline{0.625}$