KREATRYX

Appraise Education, Reprise Innovations

K Notes

SIGNALS AND SYSTEMS

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Manual for K-Notes

Why K-Notes?

Towards the end of preparation, a student has lost the time to revise all the chapters from his / her class notes / standard text books. This is the reason why K-Notes is specifically intended for Quick Revision and should not be considered as comprehensive study material.

What are K-Notes?

A 40 page or less notebook for each subject which contains all concepts covered in GATE Curriculum in a concise manner to aid a student in final stages of his/her preparation. It is highly useful for both the students as well as working professionals who are preparing for GATE as it comes handy while traveling long distances.

When do I start using K-Notes?

It is highly recommended to use K-Notes in the last 2 months before GATE Exam (November end onwards).

How do I use K-Notes?

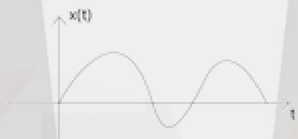
Once you finish the entire K-Notes for a particular subject, you should practice the respective Subject Test / Mixed Question Bag containing questions from all the Chapters to make best use of it.

BASIC CONCEPTS

- In continuous time signals independent variable is continuous and thus these signals are defined for a continuum of values of independent variable.
- Discrete time signals are only defined at discrete times and consequently for these signals the independent variable takes discrete set of values.

Representation of continuous time signals

- We use symbol 't' to denote independent variable for continuous time signal.
- These signals can be represented by a wave form as shown below



- If possible, these can also be represented by a mathematical function like

$$x(t) = \sin t$$

Representation of discrete time signal

- We use symbol 'n' to denote independent variable for discrete time signal.
- These signals can be represented as a series of numbers like

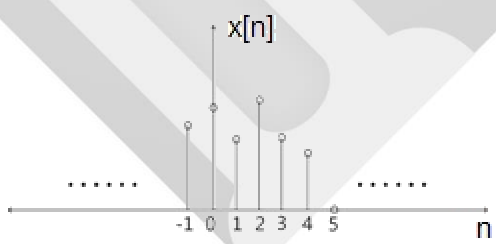
$$x[n] = [5, 4, 5, \underset{\uparrow}{7}, 9, 2 \ldots\ldots]$$

Arrow indicates reference point or x [0]

- If possible, we can represent the same by a function like

$$x[n] = \sin\left(\frac{n\pi}{4}\right)$$

- Also these signals can be represented by a wave form as shown below



Energy & Power Signals

Interval $(-\infty, \infty)$

Energy of continuous time signal

$$E_\infty = \lim_{T \to \infty} \int_{-T}^{T} |x(t)|^2 \, dt = \int_{-\infty}^{\infty} |x(t)|^2 \, dt$$

Energy of discrete time signal

$$E_\infty = \lim_{T \to \infty} \sum_{n=-N}^{N} |x[n]|^2 = \sum_{n=-\infty}^{\infty} |x[n]|^2$$

Power of continuous time signals

$$P_\infty = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} |x(t)|^2 \, dt$$

Power of discrete time signals

$$P_\infty = \lim_{N \to \infty} \frac{1}{2N+1} \sum_{n=-N}^{N} |x[n]|^2$$

- Signals having non-zero (finite) power and infinite energy are called as Power Signals.
  ex. $x(t) = \sin t$
- Signals having finite (non-zero) energy and zero power are called as Energy Signals.
  ex. $x[n] = [1, 2, 3, 4]$
- The bounded signal radiate finite energy and periodic signal radiate finite average power.

Even & Odd signals

- A signal is said to be "even" if it satisfies the condition
  $x(t) = x(-t)$ or $x[n] = x[-n]$
- A signal is said to be "odd" if it satisfies the condition
  $x(t) = -x(-t)$ or $x[n] = -x[-n]$
- Any signal (even those which are neither odd nor even) can be broken into odd & even parts

Odd Part

$$x_0(t) = \frac{x(t) - x(-t)}{2} \;;\quad x_0[n] = \frac{x[n] - x[-n]}{2}$$

Even Part

$$x_e(t) = \frac{x(t) + x(-t)}{2} \;;\quad x_e[n] = \frac{x[n] + x[-n]}{2}$$

Periodic and Aperiodic Signals

- A signal is said to be periodic with period "T" or "N" if

  $x(t + T) = x(t)$

  $x[n + N] = x[n]$

- Otherwise, the signals are said to be aperiodic.

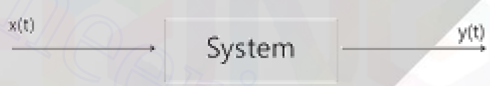
Classification of systems

(i) Linear & Non-Linear Systems

For Linearity

if $x_1(t) \longrightarrow y_1(t)$

$x_2(t) \longrightarrow y_2(t)$



then, this condition must be true

$$\alpha_1 x_1(t) + \alpha_2 x_2(t) \longrightarrow \alpha_1 y_1(t) + \alpha_2 y_2(t)$$

Example : $y(t) = t\,x(t)$ is linear

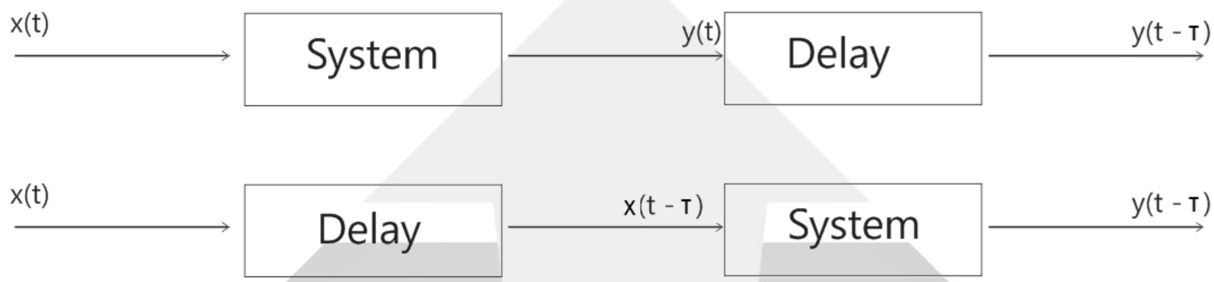
$y[n] = 2x[n] + 3$ is non-linear

(ii) Time Invariant & Time-variant Systems

For system to be time-invariant the following condition must hold true

$x(t - \tau) \longrightarrow y(t - \tau)$

It means that following two realizations must be equivalent



- The simplest way to verify this is to check the coefficient of 't' inside x(t)

  eg. y(t) = tx(t) is time invariant

  but y(t) = tx(2t) is time variant as coefficient of 't' in side x(t) is not '1'

- Otherwise, you need to verify the system equivalence shown above.

(iii) Causal & Non-causal Systems

- The output should depend only on present & past values of input.

$$h(t) = 0 \quad \forall\ t < 0$$

- For discrete time system

$$h[n] = 0 \ \forall \ n < 0$$

(iv) Stable & Unstable Systems

- Every Bounded input should produce a bounded output.

$$DT : \sum_{K=-\infty}^{\infty} |h[k]| < \infty \quad ; \quad CT : \int_{-\infty}^{\infty} |h(\tau)| d\tau < \infty$$

(v) LTI systems with or without memory

- The output at any time should depend only on value of input at the same time.
- For discrete time system

$$h[n] = 0 \ \forall \ n \neq 0$$

$$\Rightarrow h[n] = k\delta\ [n]$$

- For continuous time system

  h(t) = 0 ∀ t ≠ 0

  ⇒ h [t] = k δ [t]

(vi) Invertible Systems

The system is invertible if there exists $h_1(t)$ such that

Thus $h(t) * h_1(t) = \delta(t)$

$$x[n] \longrightarrow \boxed{h[n]} \longrightarrow \boxed{h_1[n]} \longrightarrow x[n]$$

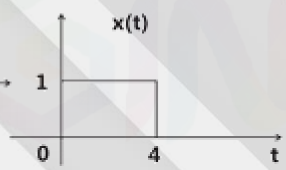
For discrete time, $h[n] * h_1[n] = \delta[n]$

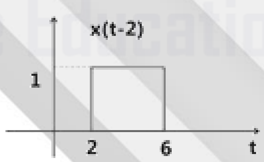
Shifting and Scaling operations

- **Shifting**

  Delay

  if x(t) →

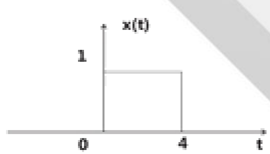


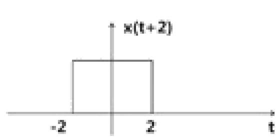


⇒ shift the waveform right by the amount of delay

Advance

if x(t) →



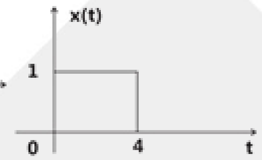


⇒ shift the waveform left by the amount of advance

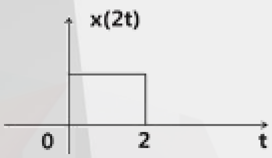
- **Scaling**

Compression

if x(t) →

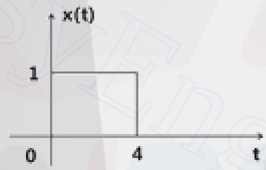


Replace upper & lower limit by original limit divided by compression factor

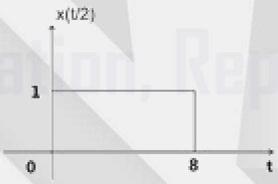


Expansion

if x(t) →

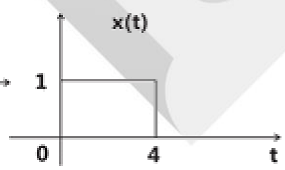


Replace upper & lower limit by original limit multiplied by expansion factor.

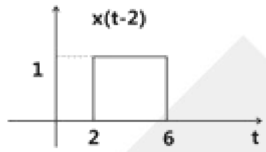


Note : If both scaling and shifting are given in the question .
Ex. x(3t-2)

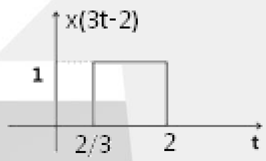
x(t) →



1. shift the waveform right by the amount of delay



2. Replace upper & lower limit by original limit divided by compression factor



- This method is applicable for both continuous and discrete time signal.

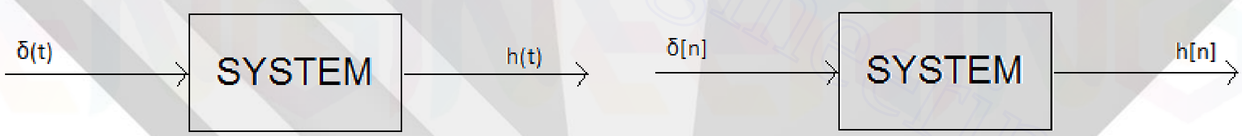
LTI system (Linear Time Invariant Systems)

- Any continuous time or discrete time system can be represented in terms of impulses.

$$x(t)=\int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)d\tau$$

$$x[n]=\sum_{k=-\infty}^{\infty} x(k)\ \delta(n-k)$$

- LTI systems are characterized on the basis of Impulse Response h(t) or h[n]



The response of a system with impulse as an input is called as impulse response.

- Due to time invariance property of LTI system

$$\text{if } \delta[n]\longrightarrow h[n]$$

$$\delta[n-k]\longrightarrow h[n-k]$$

$$\text{since } x[n]=\sum_{K=-\infty}^{\infty} x[k]\ \delta[n-k]$$

$$\Rightarrow y[n]=\sum_{k=-\infty}^{\infty} x[k]h[n-k]=x[n]*h[n] \ = \text{convolution sum}$$

for continuous time domain

$$y(t)=\sum_{k=-\infty}^{\infty} x(\tau)h(t-\tau)=x(t)*h(t) \ = \text{convolution integral}$$

- The condition for causality of system then becomes

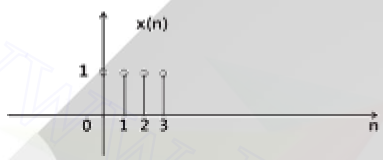
  $h[n] = 0 \ \forall \ n < 0 \ ; \ h(t) = 0 \ \forall \ t < 0$

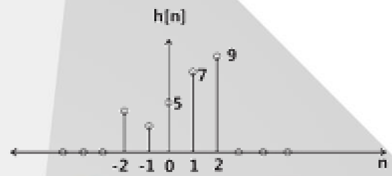
Calculating convolution sum

- Suppose $x[n] = u[n]$

  $h[n] = [1, 2, \underset{\uparrow}{5}, 7, 9]$

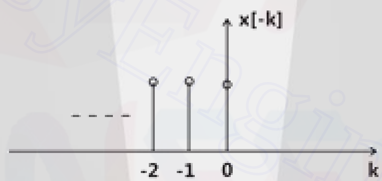
- Draw plots of both $x[n]$ & $h[n]$





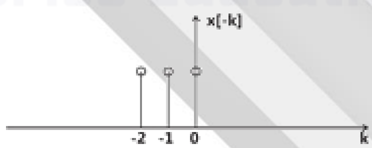
- Flip either $x[n]$ or $h[n]$ about y-axis

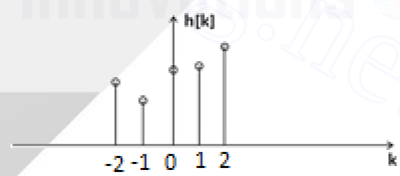
  Here, we flip $x[n]$



For calculating $y[n]$, shift $x[-k]$ to right by amount 'n'

For $y[0]$

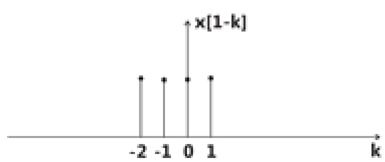


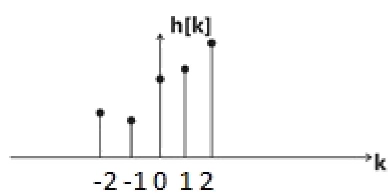


The only overlapping between the two is at $k = 0, -1, -2$

$$y[0] = x[0]\,h[0] + x[1]\,h[-1] + x[2]\,h[-2]$$
$$= 1 \times 5 + 1 \times 2 + 1 \times 1$$
$$= 8$$

For $y[1]$





$$y[1] = x[0]\,h[1] + x[1]\,h[0] + x[2]\,h[-1] + x[3]\,h[-2]$$
$$= 1 \times 7 + 1 \times 5 + 1 \times 1 \times 1 \times 2 = 15$$

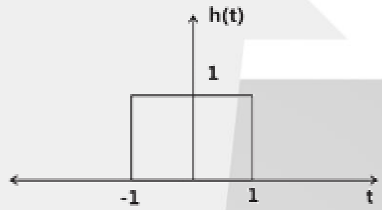
Similarly, we can calculate all values of y[n]

$$y[n] = [2, 3, 8, 15, 24, 24........]$$

Calculating Convolution Integral

Assume x (t) = u (t)

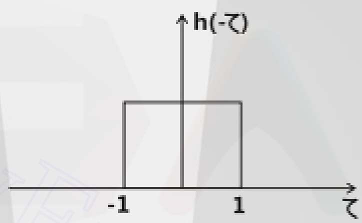
h (t) =



- Step 1

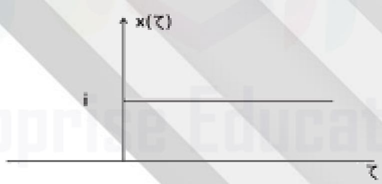
  Flip either x(t) or h(t)

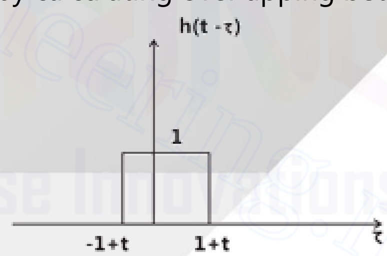
  Here, we flip h(t)



- Step 2

  Shift $h(\tau)$ by amount "t" to the right to calculate y(t) by calculating overlapping between $h(t-\tau)\,\&\,x(\tau)$



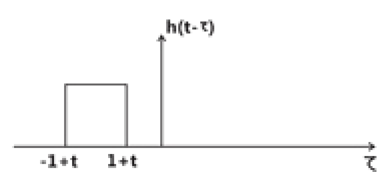


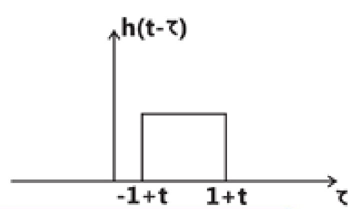
- Overlapping area

$$= \int_0^{(1+t)} 1.1\,d\tau = (1+t)$$

if $t < -1$

so, overlapping area = 0

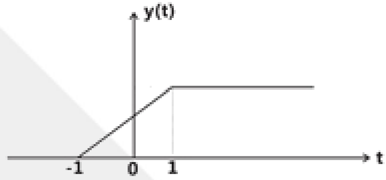




if t > 1

overlapping area = 2

y (t) is shown in adjoining figure:



Properties of Convolution Sum

1) Commutative Property

$x[n] * h[n] = h[n] * x[n]$

2) Distributive Property

$y_1[n] = x[n] * h_1[n]$

$y_2[n] = x[n] * h_2[n]$

$y[n] = y_1[n] + y_2[n] = x[n] * h_1[n] + x_2[n]*h_2[n]$

$= x[n] * \{ h_1[n] + h_2[n] \}$

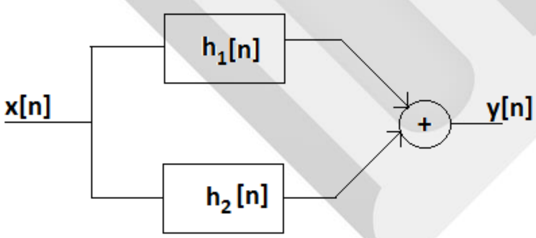
3) Associative Property

$\{x[n] * h_1[n] \}* h_2[n] = \{ x[n]* h_2[n] \} * h_1 [n]$

Same properties will apply for continuous time domain for convolution integral.

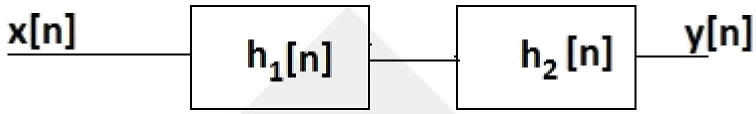
Parallel & Cascade structure of LTI systems

Parallel:



$y[n] = x[n] * [h_1 [n] + h_2 [n]]$

Cascade:



$$y[n] = x[n] * ([h_1[n] + h_2[n]])$$

Frequency Response

The frequency response of any LTI system is given by its Fourier Transform.

$$\text{DT: } H\left(e^{jw}\right) = \sum_{n=-\infty}^{\infty} h[n]e^{-jwn}$$

$$\text{CT: } H(jw) = \int_{-\infty}^{\infty} h(t)e^{-jwt}dt$$

Group delay & Phase delay

Assuming transfer function of system is H(s)

input is $x(t) = e^{jwt}$

Output: $H(jw)e^{jwt} = \left[|H(jw)|e^{j\phi(w)}\right]e^{jwt}$

$$= |H(jw)|e^{j(wt+\phi(w))}$$

$$\phi(w) = \text{Arg}\{H(jw)\}$$

Group Delay, $\tau_g(w) = -\dfrac{d\phi(w)}{dw}$

Phase Delay, $\tau_\phi(w) = -\dfrac{\phi(w)}{w}$

Continuous – Time Fourier series

Fourier states that any periodic signal can be represented by a set of complex exponential signals provided that it satisfies Drichlet Conditions.

Drichlet conditions

(i) Over any period x(t) is absolutely integrable

i.e., $\int_0^T |x(t)|dt < \infty$

(ii) In a finite time interval, x(t) has a finite number of maxima & minima

(iii) It should have finite number of discontinuities in the given interval

Note : for distortion less transmission of the of a signal with some finite frequency content through a continuous time LTI system , the frequency response of the system must satisfy these two conditions.

1. The magnitude response $|H(j\omega)|$ must be constant for all frequencies of interest ; that is, we must have

$$|H(j\omega)| = C$$

For some constant C

2. For the same frequencies of interest, the phase response $\arg\{H(j\omega)\}$ must be linear in frequency, with slope $-t_o$ and intercept zero ; that is, we must have

$$\arg\{H(j\omega)\} = -\omega t_o$$

Fourier series as generally expressed in 2 forms.

- Trigonometric
- Exponential

Trigonometric Fourier Series

Analysis equations

$$a_0 = \frac{1}{T}\int_0^T x(t)\,dt$$

$$a_k = \frac{2}{T}\int_0^T x(t)\cos k\omega_0 t\,dt \qquad \text{where } \omega_0 = 2\pi/T$$

$$b_k = \frac{2}{T}\int_0^T x(t)\sin k\omega_0 t\,dt$$

Synthesis equations

$$x(t) = a_0 + \sum_{\substack{k=-\infty \\ k\neq 0}}^{\infty} a_k \cos k\,\omega_0 t + \sum_{\substack{k=-\infty \\ k\neq 0}}^{\infty} b_k \sin k\,\omega_0 t$$

Exponential Fourier Series

Analysis equations

$$C_k = \frac{1}{T}\int_0^T x(t)e^{-jk\omega_0 t}dt$$

Synthesis equations

$$x(t) = \sum_{k=-\infty}^{\infty} C_K e^{jk\omega_0 t} \quad \text{where } \omega_0 = 2\pi/T$$

Relation between T.F.S. and E.F.S.

- $c_o = a_0$
- $C_n = \dfrac{a_n - jb_n}{2}$
- $C_{-n} = \dfrac{a_n + jb_n}{2}$

Important facts about Trigonometric Fourier series

(i) Any odd signal contains only sine terms in Fourier series.

(ii) Any even signal contains only cosine terms in Fourier series.

(iii) For half–wave symmetric signal

$$x\left(t + T/2\right) = -x(t)$$

Only odd harmonics are present

i.e., k = 1, 3, 5.......

Properties of complex exponential Fourier Series

(i) Linearity

If $x(t) \xrightarrow{F.S.} a_k$

$y(t) \xrightarrow{F.S.} b_k$

then Ax (t) + By (t) $\xrightarrow{F.S.} A a_k + B b_k$

(ii) Time-shifting

if $x(t) \xrightarrow{F.S.} a_k$

$x(t - t_0) \xrightarrow{F.S.} e^{-jk\omega_0 t_0} a_k$ where $\omega_0 = 2\pi/T$

(iii) Time-Reversal

if $x(t) \xrightarrow{F.S.} a_k$

$x(-t) \xrightarrow{F.S.} a_{-k}$

For odd signal	For even signal
$x(t) = -x(-t)$	$x(t) = x(-t)$
$\Rightarrow a_k = -a_{-k}$	$\Rightarrow a_k = a_{-k}$

(iv) Time – Scaling

$$\text{if } x(t) \xrightarrow{\text{F.S.}} a_k$$

$$x(\alpha t) \xrightarrow{\text{F.S.}} a_k$$

but ω_0 is replaced by $(\omega_0\alpha)$, though Fourier series coefficients remain same.

(v) Multiplication

$$\text{if } x(t) \xrightarrow{\text{F.S.}} a_k$$

$$y(t) \xrightarrow{\text{F.S.}} b_k$$

$$z(t) = x(t)y(t) \xrightarrow{\text{F.S.}} c_k$$

$$C_k = \sum_{P=-\infty}^{\infty} b_p a_{k-p} = \text{convolution sum}$$

(vi) Parseval's Relation

Energy in time domain = Energy frequency Domain

$$\frac{1}{T}\int_{<T>} \left|x(t)^2\right| dt = \sum_{k=-\infty}^{\infty} |a_k|^2$$

$$\text{where } x(t) \xrightarrow{\text{F.S.}} a_k$$

Discrete –Time Fourier series

For a discrete-time signal, with period 'N' the following equations are used for Fourier series.

Analysis equations

$$C_k = \sum_{<N>} x[n] e^{-j\frac{2\pi}{N}Kn}$$

$$\Omega_0 = 2\pi/N$$

$$C_k = \sum_{<N>} x[n] e^{-j\Omega_0 Kn}$$

Synthesis equations

$$x[n] = \sum_{<N>} C_K e^{j\Omega_0 K n}$$

The properties of Fourier series coefficients are same as continuous time Fourier series with one additional property.

$$C_{K+N} = C_K$$

That is, Fourier series coefficients are periodic

IMPORTANT DUALITY

A signal discrete in one domain is periodic in other domain & vice versa.
Example: For continuous Time Fourier Series, x (t) is periodic in time domain & hence Fourier Series exists where coefficients exist for frequency integral multiple of "ω_0" & hence is discrete.

Fourier Transform

Fourier series exists only for periodic signals, Fourier series converges to Fourier Transform which is continuous as compared to Fourier series which is discrete.

Continuous Time Fourier Transform

Analysis equation

$$X(jw) = \int_{-\infty}^{\infty} x(t) e^{-jwt} dt$$

Synthesis equation

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} x(jw) e^{jwt} dw$$

Properties of Continuous Time Fourier Transform

Signal	Fourier Transform
$x(t)$	$X(jw)$
$y(t)$	$Y(jw)$
$Ax(t)+By(t)$	$AX(jw)+BY(jw)$
$x(t-t_0)$	$e^{-j\omega t_0}X(w)$
$x^*(t)$	$X^*(-w)$
$x(-t)$	$X(-w)$
$x(at)$	$\frac{1}{\lvert a \rvert}X\left(\frac{jw}{a}\right)$
$x(t)*y(t)$	$X(jw)Y(jw)$
$\frac{d}{dt}x(t)$	$jwX(jw)$
$x(t)y(t)$	$\frac{1}{2\pi}X(w)*Y(w)$
$\int_{-\infty}^{t} x(\tau)d\tau$	$\frac{1}{jw}X(jw)+\pi X(0)\delta(w)$
$tx(t)$	$j\frac{d}{dw}X(jw)$
$Ev\{x(t)\}$	$Re\{X(jw)\}$
$Od\{x(t)\}$	$jIm\{X(jw)\}$
$X(t)$	$2\pi x(-w)$
$e^{j\omega_0 t}x(t)$	$X(w-w_0)$

Parseval's Relation

$$\int_{-\infty}^{\infty} \lvert x(t)\rvert^2 = \frac{1}{2\pi}\int_{-\infty}^{\infty} \lvert x(w)\rvert^2 \, dw$$

Some common Fourier Transform Pairs

Signal	Fourier Transform
$\sum_{K=-\infty}^{\infty} a_k e^{jkw_0 t}$	$2\pi \sum_{k=-\infty}^{\infty} a_k (\omega - k\omega_0)$
$e^{jkw_0 t}$	$2\pi\delta(\omega-\omega_0)$
$\cos w_0 t$	$\pi[\delta(\omega+\omega_0)+\delta(\omega-\omega_0)]$
$\sin w_0 t$	$\pi/j[\delta(\omega-\omega_0)-\delta(\omega+\omega_0)]$
1	$2\pi\delta(\omega)$
$\sum_{n=-\infty}^{\infty} \delta(t-nT)$	$\frac{2\pi}{T}\sum_{K=-\infty}^{\infty}\delta\left(\omega-\frac{2\pi k}{T}\right)$
$x(t)=\begin{cases}1, & \lvert t\rvert < T_1\\ 0, & \lvert t\rvert > T_1\end{cases}$	$\frac{2\sin\omega T_1}{\omega}$
(sin wt)/πt	$x(\omega)=\begin{cases}1, & \lvert\omega\rvert < w\\ 0, & \lvert\omega\rvert > w\end{cases}$
$\delta(t)$	1
u(t)	$\frac{1}{j\omega}+\pi\delta(\omega)$
$\delta(t-t_0)$	$e^{-j\omega t_0}$
$e^{-at}u(t), \mathrm{Re}(a)>0$	$\frac{1}{a+j\omega}$

Discrete Time Fourier Transform

Analysis equation

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$$

Synthesis Equation

$$x[n] = \frac{1}{2\pi}\int_{<2\pi>} X(e^{j\omega})e^{j\omega n}d\omega$$

Properties of Discrete Time Fourier Transform

Signal	Fourier Transform
$x[n]$ $y[n]$	$\left.\begin{matrix} X(\Omega) \\ Y(\Omega) \end{matrix}\right\}$ periodic with period 2π
$ax[n] + by[n]$	$aX(\Omega) + bY(\Omega)$
$x[n-n_0]$	$e^{-j\Omega n_0} X[\Omega]$
$x^*[n]$	$X^*(-\Omega)$
$e^{j\Omega_0 n} x[n]$	$X(\Omega-\Omega_0)$
$x[-n]$	$X(-\Omega)$
$x_k[n] = \begin{cases} x[n \mid k], & \text{if } n \text{ is multiple of } k \\ 0, & \text{is } n \text{ is not multiple of } k \end{cases}$	$X(k\Omega)$
$x[n] * y[n]$	$X(\Omega)Y(\Omega)$
$n\,x[n]$	$j\dfrac{dx(\Omega)}{d\Omega}$
$Ev\{x[n]\}$	$Re\{X(\Omega)\}$
$Od\{x[n]\}$	$j\,Im\{X(\Omega)\}$

Parseval's Relation

$$\sum_{n=-\infty}^{\infty} \left|x[n]\right|^2 = \frac{1}{2\pi} \int_{<2\pi>} \left|X(\Omega)\right|^2 d\Omega$$

Some common Fourier Transform Pairs

Signal	Fourier Transform
$\sum_{K=\langle N\rangle} a_k e^{jk\left(\frac{2\pi}{N}\right)n}$	$2\pi \sum_{k=-\infty}^{\infty} a_k \; \delta\left[\Omega - \frac{2\pi k}{N}\right]$
$e^{j\Omega_0 n}$	$2\pi \sum_{\ell=-\infty}^{\infty} \delta\left(\Omega - \Omega_0 - 2\pi\ell\right)$
$\cos\Omega_0 n$	$\pi \sum_{\ell=-\infty}^{\infty} \left[\delta\left(\Omega - \Omega_0 - 2\pi\ell\right) + \delta\left(\Omega + \Omega_0 - 2\pi\ell\right)\right]$
$\sin\Omega_0 n$	$\frac{\pi}{j} \sum_{\ell=-\infty}^{\infty} \left[\delta\left(\Omega - \Omega_0 - 2\pi\ell\right) - \delta\left(\Omega + \Omega_0 - 2\pi\ell\right)\right]$
$x[n] = 1$	$2\pi \sum_{\ell=-\infty}^{\infty} \delta\left(\Omega - 2\pi\ell\right)$
$x[n] = \begin{cases} 1, & \lvert n \rvert \le N_1 \\ 0, & \lvert n \rvert > N_1, \lvert n \rvert \le \frac{N}{2} \end{cases}$ and $x[n+N] = x[n]$	$2\pi \sum_{k=-\infty}^{\infty} a_k \; \delta\left(\Omega - \frac{2\pi k}{N}\right)$
$\sum_{k=-\infty}^{\infty} \delta[n - kN]$	$\frac{2\pi}{N} \sum_{k=-\infty}^{\infty} \delta\left(\Omega - \frac{2\pi k}{N}\right)$
$x[n] = \begin{cases} 1, & \lvert n \rvert \le N_1 \\ 0, & \lvert n \rvert > N_1 \end{cases}$	$\frac{\sin\left[\Omega\left(N_1 + \frac{1}{2}\right)\right]}{\sin\left(\frac{\Omega}{2}\right)}$
$\frac{\sin Wn}{\pi n} = \frac{W}{\pi} \operatorname{sinc}\left(\frac{Wn}{\pi}\right)$	$x[\Omega] = \begin{cases} 1, & 0 \le \lvert \Omega \rvert \le W \\ 0, & W < \lvert \Omega \rvert \le \pi \end{cases}$
$\delta[n - n_0]$	$e^{-j\Omega n_0}$

Laplace Transform

- Laplace Transform is more general than Fourier Transform but can only be computed in Region of Convergence (ROC), so it cannot be computed $\not\forall$ s

$$\text{ROC} = \begin{cases} S = \sigma + jw; \text{ such that} \\ \int \left|x(t)e^{-\sigma t}\right| dt < \infty \end{cases}$$

Laplace transform becomes Fourier transform for $\sigma = 0$, if it lies in ROC.

- **Analysis Equations**

for bilateral Laplace Transform

$$H(s) = \int_{-\infty}^{\infty} h(t)e^{-st}dt$$

for unilateral Laplace Transform

$$H(s) = \int_{0}^{\infty} h(t)e^{-st}dt$$

- **Synthesis Equation**

$$x(t) = \frac{1}{2\pi j}\int_{\sigma-j\infty}^{\sigma+j\infty} x(s)e^{st}ds$$

Properties of ROC

(i) ROC consists of a collection of lines parallel to jw–axis in s–plane.

$$\text{such that } \int_{-\infty}^{\infty} \left|x(t)e^{-\sigma t}\right| dt < \infty$$

(ii) If X (s) is rational, then ROC does not contain any poles.

(iii) If x(t) is of finite duration & absolutely integrable, then ROC is entire s-plane.

(iv) If x(t) is right sided signal (i.e., it is zero before some time) and if Re(s) = σ_0 is in the ROC, then all values of s for which Re(s) > σ_0 are also in ROC.

(v) If x(t) is left sided, (i.e., if it is zero after some time), and if Re (s) = σ_0 is in ROC, then all values of s for which Re(s) < σ_0 are also in ROC.

(vi) If x(t) is two–sided signal and if the line Re (S) = σ_0 is in ROC, then the ROC consists of a strip in s–plane include the line Re (S) = σ_0

(vii) If X(s) is rational, and
x(t) is right sided signal, then ROC is right of right most pole.
x(t) is left sided signal, then ROC is left of left most pole.

Properties of Laplace Transform

Signal	Transform	ROC
x(t)	X(s)	R
$x_1(t)$	$X_1(s)$	R_1
$x_2(t)$	$X_2(s)$	R_2
$ax_1(t) + bx_2(t)$	$aX_1(s) + bX_2(s)$	At least $R_1 \cap R_2$
$x(t-t_0)$	$e^{-st_0}X(s)$	R
$e^{s_0t}x(t)$	$X(s-s_0)$	Shifted version R [i.e., s is in ROC if $(s-s_0)$ is in R]
x (at)	$\frac{1}{\lvert a \rvert}X\left(s/a\right)$	Scaled ROC i.e., s is ROC if $\left(s/a\right)$ is in R
$x_1(t) * x_2(t)$	$X_1(s)\ X_2(s)$	At least $R_1 \cap R_2$
$\frac{d}{dt}x(t)$	$sX(s)$	At least R
tx(t)	$\frac{-d}{ds}x(s)$	R
$\int_{-\infty}^{t} x(\tau)d\tau$	$\frac{1}{s}X(s)$	At least R

Some common Laplace Transform Pairs

Signal	Transform	ROC
$\delta(t)$	1	All s
u(t)	$1/s$	Re {s} > 0
−u(−t)	$1/s$	Re {s} < 0
$\frac{t^{n-1}}{(n-1)!}u(t)$	$1/s^n$	Re {s} > 0
$\frac{-t^{n-1}}{(n-1)!}u(-t)$	$1/s^n$	Re {s} < 0
$e^{-at}u(t)$	$\frac{1}{s+a}$	Re {s} > −a
$-e^{-at}u(-t)$	$\frac{1}{s+a}$	Re {s} < −a
$\frac{t^{n-1}}{(n-1)!}e^{-at}u(t)$	$\frac{1}{(s+a)^n}$	Re {s} < −a
$\frac{-t^{n-1}}{(n-1)!}e^{-at}u(t)$	$\frac{1}{(s+a)^n}$	Re {s} > −a
$\delta(t-T)$	e^{-sT}	All s
$[\cos\omega_0 t]u(t)$	$\frac{s}{s^2+\omega_0^2}$	Re {s} > 0
$[\sin\omega_0 t]u(t)$	$\frac{\omega_0}{s^2+\omega_0^2}$	Re {s} > 0
$[e^{-at}\cos\omega_0 t]u(t)$	$\frac{s+a}{(s+a)^2+\omega_0^2}$	Re {s} > −a
$[e^{-at}\sin\omega_0 t]u(t)$	$\frac{\omega_0}{(s+a)^2+\omega_0^2}$	Re {s} > −a

Initial and Final Value Theorem

$$x(0^+) = \lim_{s \to \infty} sX(s) \to \text{initial value}$$

$$x(\infty) = \lim_{s \to 0} s\,X(s) \to$$ Final value, first stability should be ensured, else final value does not exist.

Analysis of LTI system using Laplace Transform

- **Stability**

  $\int_{-\infty}^{\infty} |h(t)|\,dt < \infty$; ROC of H(s) should include $\sigma = 0$.

- **Causality**

  h(t) = 0, t < 0 i.e., right sided signal
  ROC should be right sided
  ROC should include Right half plane.
  but converse is not true.

Z – Transform

It is generalization of Discrete Time Fourier Transform

Analysis Equation

$$H(z) = \sum_{k=-\infty}^{\infty} h[k] z^{-k}$$

Synthesis Equation

$$h[n] = \frac{1}{2\pi j} \oint H(z) z^{n-1} dz$$

Indicates integration around counter clockwise circular contour centered at origin & with radius r.

ROC for Z-Transform

Z – Transform also exists only inside ROC

$\sum_{n=-\infty}^{\infty} \left| x[n] r^{-n} \right| < \infty$ is the condition for ROC.

Mapping from s-plane from z–plane

- The jw-axis is mapped to unit circle in z–plane.
- Right Half plane is mapped to exterior of unit circle.
- Left Half plane is mapped to interior of unit circle.

Properties of ROC

(i) The ROC x(z) consists of a ring in the z – plane centered about the origin.

(ii) The ROC does not contain any poles.

(iii) If x[n] is of finite duration, then ROC is the entire z – plane except possibility at z = 0 and/or z = ∞

(iv) If x[n] is a right sided sequence and if the circle, | z | = r_0 is in the ROC, then all finite values of z, for which | z | > r_0 will also be in ROC.

(v) If x[n] is a left sided sequence, and the circle | z | = r_0 is in ROC, then all finite value of z, for which 0 < | z | < r_0 will be in ROC.

(vi) If x[n] is two sided sequence and if circle | z | = r_0 is in the ROC. Then ROC will consist of a ring in z-plane which consist of ring | z | = r_0.

(vii) If X (z) is rational and

x[n] is right sided than ROC is outside of outer most pole.

x[n] is left sided then ROC is inside of inner most pole.

(viii) If x[n] is causal, ROC includes z = ∞ provided x[n] = 0, n < 0.

If x [n] is anti – causal, ROC includes z = 0 provided x [n] = 0, n > 0.

(ix) A causal LTI system with rational system function is stable if all poles inside the unit circle that is have magnitude, | z | < 1.

Properties of z–Transform

Signal	Transform	ROC
$x[n]$	$X(z)$	R_x
$x_1[n]$	$X_1(z)$	R_1
$x_2[n]$	$X_2(z)$	R_2
$ax_1[n]+bx_2[n]$	$aX_1(z)+bX_2(z)$	At least $R_1 \cap R_2$
$x[n-n_0]$	$z^{-n_0}X(z)$	R_x with addition or deletion of origin
$e^{j\Omega_0 n}x[n]$	$X\left(e^{-j\Omega_0}z\right)$	R_x
$z_0^n\, x[n]$	$X\left(z/z_0\right)$	z_0R_x
$x[-n]$	$X\left(z^{-1}\right)$	$z^{-1}\left(\text{s.t } z\in R_x\right)$
$w[n]=\begin{cases} x[r], & n=rk \\ 0, & n\neq rk \text{ for some } r\end{cases}$	$X\left(z^k\right)$	$R_x^{1/k}\left(\text{i.e., } z^{1/k}\ \text{s.t } z\in R_x\right)$
$x_1[n]*x_2[n]$	$X_1(z)X_2(z)$	At least $R_1 \cap R_2$
$nx[n]$	$\dfrac{-zdX(z)}{dz}$	R_x except addition or deletion of zero
$\sum_{k=-\infty}^{n} x[k]$	$\dfrac{1}{1-z^{-1}}X(z)$	$R_x\bigcap\left[\lvert z\rvert>1\right]$

Some common Z –Transform pairs

Signal	Transform	ROC
$\delta[n]$	1	All z
$u[n]$	$\dfrac{1}{1-z^{-1}}$	$\lvert z \rvert > 1$
$u[-n-1]$	$\dfrac{1}{1-z^{-1}}$	$\lvert z \rvert < 1$
$\delta[n-m]$	z^{-m}	All z except 0 (if m > 0) or ∞ (if m < 0)
$a^n u[n]$	$\dfrac{1}{1-az^{-1}}$	$\lvert z \rvert > \lvert a \rvert$
$-a^n u[-n-1]$	$\dfrac{1}{1-az^{-1}}$	$\lvert z \rvert < \lvert a \rvert$
$na^n u[n]$	$\dfrac{az^{-1}}{\left(1-az^{-1}\right)^2}$	$\lvert z \rvert > \lvert a \rvert$
$-na^n u[-n-1]$	$\dfrac{az^{-1}}{\left(1-az^{-1}\right)^2}$	$\lvert z \rvert < \lvert a \rvert$

Initial & Final value Theorem

$$x\left[0^+\right] = \lim_{z \to \infty} X(z) \rightarrow \text{ Initial value}$$

$$x\left[\infty\right] = \lim_{z \to 1} \left(1 - \frac{1}{z}\right) X(z) \rightarrow \text{ Final value}$$

In z – transform also, stability must be verified before using final value theorem.

Sampling

Nyquist Sampling Theorem

It states that if sampling frequency is greater than twice the maximum frequency in the signal for the signal to be recovered from its samples.

$$w_S \geq 2w_M$$

Note: For this condition signal spectrum should be centered around y-axis.

Band-pass Sampling Theorem

If the signal spectrum is band-pass which means it has minimum & maximum frequency

f_L = lower frequency ; f_u = upper frequency

$K = \left[\dfrac{f_u}{f_u - f_L}\right]$, where $[\bullet] \rightarrow$ indicates Greatest Integer function

$$w_S \geq \frac{2f_u}{K}$$

- $x_p(t) = x(t)\,p(t)$

$$p(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT)$$

T = sampling interval ; $x_p(t) \rightarrow$ Sampled signal

x(t) = continuous time signal

$$x_p(t) = \sum_{n=-\infty}^{\infty} x(t)\delta(t - nT)$$

$$X_P(w) = \frac{1}{2\pi}\left[X(w) * P(w)\right]$$

$$P(w) = \frac{2\pi}{T}\sum_{k=-\infty}^{\infty} \delta(w - kw_s)$$

$$X_P(w) = \frac{1}{T}\sum_{k=-\infty}^{\infty} X(w - kw_s) \quad ; \quad w_s = \frac{2\pi}{T}$$

- The spectrum of sampled signal is just repetition of actual spectrum at integral multiples of w_s.

If $w_s < 2w_M$, adjacent samples of spectrum overlap, called as aliasing.

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