Q1

A

Let

$g_1(x)$ & $g_2(x)$ be discriminant function for ~~both~~ both the classes.

Class 1

$$p = \begin{bmatrix} p_1 \\ \vdots \\ p_d \end{bmatrix}$$

Class 2

$$q = \begin{bmatrix} q_1 \\ \vdots \\ q_d \end{bmatrix}$$

$$g_1(x) = \ln(P(\omega_1 | x))$$

$$= \ln\left(\prod_{i=1}^{d} p_i^{x_i} (1-p_i)^{(1-x_i)}\right) + \ln(p(\omega_1))$$

$$\cancel{= \sum_{i=1}^{d} \ln(p_i^{x_i}) + \sum_{i=1}^{d} \ln((1-p_i)^{(1-x_i)})}$$

$$= \sum_{i=1}^{d} x_i \ln(p_i) + \sum_{i=1}^{d} (1-x_i)\ln(1-p_i) + \ln(p(\omega_1))$$

$$\Rightarrow \cancel{\ln} \sum_{i=1}^{d} \cancel{\ln} \left[x_i \ln\left(\frac{p_i}{1-p_i}\right) + \ln(1-p_i) \right]$$

$$+ \ln(P(\omega_1))$$

$$g_2(x) = \sum_{i=1}^{d} \left(x_i \ln\left(\frac{q_i}{1-q_i}\right) + \ln(1-q_i) \right)$$

$$+ \ln(P(\omega_2))$$

$$\boxed{g(x) = g_1(x) - g_2(x)}$$

$$\cancel{= \sum_{i=1}^{d} \left[\ln\left(\frac{p_i(1-q_i)}{q_i(1-p_i)}\right) x_i \right] +}$$

$$g(x) = \sum_{i=1}^{d} w_i x_i + w_0$$

$$w_i = \ln\left(\frac{p_i(1-q_i)}{q_i(1-p_i)}\right)$$

$$w_0 = \ln\left(\frac{1-p_i}{1-q_i}\right) + \ln\left(\frac{P(\omega_1)}{P(\omega_2)}\right)$$

So $g(x) > 0 \rightarrow$ class 1

$g(x) < 0 \rightarrow$ class 2

2019213
Utkarsh Dubey

SML Assignment-2

Q2

A a) ~~An~~ Hypothetical prior $\rightarrow \theta^T e^{-\theta}$

We have.

$$\theta_{MAP} = \arg\max p(\theta/D) = p(D/\theta)p(\theta)$$

taking log

$$F(\theta) = \ln(p(D/\theta)) + \ln(p(\theta))$$

$$= \ln\left(\prod_{i=1}^{n} p(x_i/\theta)\right) + \ln(p(\theta))$$

$$= \ln\left(\prod_{i=1}^{n}\prod_{j=1}^{d} \theta_j^{x_{ij}} \times (1-\theta_{ij})^{(1-x_{ij})}\right) + \ln(p(\theta))$$

$$= \sum_{i=1}^{n}\sum_{j=1}^{d} x_{ij}\ln(\theta_j) + \sum_{i=1}^{n}\sum_{j=1}^{d}(1-x_{ij})\ln(1-\theta_j)$$
$$+ \ln(p(\theta))$$

for θ_{MAP} we have. $\frac{\partial(F(\theta))}{\partial\theta} = 0$

$$\frac{\partial(F(\theta))}{\delta\theta} = \sum_{i=1}^{n}\left(\frac{x_{ij}}{\theta_j} - \frac{(1-x_{ij})}{1-\theta_j}\right) = 0$$
$$+ \frac{\partial(\ln(P(\theta)))}{\delta\theta}$$

$$= \sum_{i=1}^{n} \frac{x_{ij} - \theta_j}{\theta_j - \theta_j^2} + \frac{\delta(\ln(P(\theta)))}{\delta \theta_j} = 0$$

We have, $\theta^T e^{-\theta}$

Hence $P(\theta) = (\theta_1 \cdot \theta_2 \cdots \theta_d)(e^{-\theta_1} e^{-\theta_2} \cdots e^{-\theta_d})$

$$\ln(P(\theta)) = \sum_{i=1}^{d} (\ln(\theta_i) - \theta_i)$$

$$\frac{\ln(P(\theta))}{\delta \theta} = \frac{1}{\theta_j} - 1$$

$$\frac{\delta(f(\theta))}{\delta \theta_j} = \sum_{i=1}^{n} \left(\frac{x_{ij} - \theta_j}{(\theta_j - \theta_j^2)} \right) + \frac{1}{\theta_j} - 1 = 0$$

$$\sum_{i=1}^{n} (x_{ij}) - n\theta_j + 1 - \theta_j - \theta_j + \theta^2 = 1$$

$$\theta_j^2 - (n+2)\theta_j + 1 + \sum_{i=1}^{n} x_{ij} = 0$$

$$\theta_{jMAP} = \frac{n + 2 \pm \sqrt{n^2 + 4 + 4n - 4 - 4\sum_{i=1}^{n} x_{ij}}}{2}$$

$$\boxed{\theta_{jMAP} = \frac{n + 2 \pm \sqrt{n^2 + 4n - 4\sum_{i=1}^{n} x_{ij}}}{2}}$$

b) We have

$$X = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix}, d = 2, N = 4$$

Putting values in $\theta_1 MAP$ ~~calc~~ calculated in part (a)

$$\theta_{1MAP} = \frac{4+2 \pm \sqrt{16+16-4\sum_{i=1}^{n} x_{i1}}}{2}$$

$$= \frac{6 \pm \sqrt{32-4\times 3}}{2}, \text{ we have } \sum_{i=1}^{n} x_{i1} = 3$$

$$= \frac{6 \pm \sqrt{20}}{2} = 3 \pm \sqrt{5}$$

$$\theta_{2MAP} = \frac{4+2 \pm \sqrt{16+16-4\sum_{i=1}^{n} x_{i2}}}{2}$$

$$\Rightarrow \sum_{i=1}^{n} x_{i2} = 1$$

$$= \frac{6 \pm \sqrt{32-4}}{2}$$

$$\theta_{2MAP} = 3 \pm \sqrt{7}$$

as probability can only be $[0,1]$

So possible values are $3-\sqrt{5}$ and $3-\sqrt{7}$

Q3

A $X = \begin{bmatrix} 1 & 3 \\ 4 & 6 \end{bmatrix}$, $\mu \begin{bmatrix} 2 \\ 5 \end{bmatrix}$

$X_c = X - \mu = \begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix}$

$Cov(X_c) = \frac{1}{N} X_c \times X_c^T = \frac{1}{2}\begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} -1 & -1 \\ 1 & 1 \end{bmatrix}$

$= \frac{1}{2}\begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$

$|Cov(X_c) - \lambda I| = 0$

$\begin{vmatrix} 1-\lambda & 1 \\ 1 & 1-\lambda \end{vmatrix} = 0$

$(1-\lambda)^2 - 1 = 0$

$\cancel{1} + \lambda^2 - 2\lambda \cancel{-1} = 0$

$\lambda^2 - 2\lambda = 0$

$\lambda(\lambda - 2) = 0$, $\lambda = 0$ & $\lambda = 2$

$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \lambda \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

for $\lambda = 0$

$$\mu_1 + \mu_2 = 0$$
$$\mu_1 = -\mu_2$$

Eigen Vector $\begin{bmatrix} \mu_1 \\ -\mu_1 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{bmatrix}$

for $\lambda = 2$

$$\mu_1 + \mu_2 = 2\mu_1$$
$$\mu_1 + \mu_2 = 2\mu_2$$

$$\mu_1 = \mu_2$$

Eigen Vector $= \begin{bmatrix} \mu_1 \\ \mu_1 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$

$U = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$ $\quad x_c = \begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix}$

$$Y = U^T X_c$$

$$= \begin{bmatrix} \frac{1}{\sqrt{2}} & -1/\sqrt{2} \\ \frac{1}{\sqrt{2}} & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 \\ -\sqrt{2} & \sqrt{2} \end{bmatrix}$$

b) $UY + \text{mean}(x) = UU^T x_c + \cancel{m} \mu_{(x)}$

$$UU^T x_c = UY = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 0 & 0 \\ -\sqrt{2} & \sqrt{2} \end{bmatrix}$$

$$= \begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix}$$

$$UY + \mu = \begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix} + \begin{bmatrix} 2 & 2 \\ 5 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 3 \\ 4 & 6 \end{bmatrix} = X$$

Hence $MSE(X, X) = 0$

c) ~~Ye~~ Yes, code matches with the calculations.