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SML Assignment-1

Q1

a) for zero-one loss

$$\lambda_{ij} = \begin{cases} 0 & i = j, \\ 1 & i \neq j \end{cases}$$

$$\lambda_{11} P(\omega_1/x) + \lambda_{12} P(\omega_2/x) = \lambda_{21} P(\omega_1/x) + \lambda_{22} P(\omega_2/x)$$

$$\Rightarrow \lambda_{11} = 0 \;\&\; \lambda_{22} = 0$$

so

$$\Rightarrow \lambda_{12} P(\omega_2/x) = \lambda_{21} P(\omega_1/x)$$

$$\lambda_{12} = \lambda_{21} = 1$$

$$\Rightarrow P(\omega_2/x) = P(\omega_1/x)$$

$$\Rightarrow P(x/\omega_2) P(\omega_2) = P(x/\omega_1) P(\omega_1)$$

$$\Rightarrow P(x/\omega_2) \times \frac{3}{\cancel{4}} = P(x/\omega_1) \times \frac{1}{\cancel{4}}$$

$$\Rightarrow N(5,1) \times 3 = N(2,1)$$

$$\Rightarrow \frac{1}{\cancel{\sqrt{2\pi}}} e^{-\frac{1}{2}(x^*-5)^2} \times 3 = \frac{1}{\cancel{\sqrt{2\pi}}} \times e^{\frac{-(x^*-2)^2}{2}}$$

$$= \ln 3 - \frac{1}{2}(x^*-5)^2 = \frac{-1}{2}(x^*-2)^2$$

$$= (x^*-5)^2 - (x^*-2)^2 = \ln(3^2)$$

$\Rightarrow (2x^* - 7)(-3) = \ln(9)$

$\Rightarrow \quad x^* = \dfrac{21 - \ln(9)}{6} = 3.1$

Ans ii) we have

$\lambda_{11} P(\omega_1/x) + \lambda_{12} P(\omega_2/x) = \lambda_{21} P(\omega_1/x) + \lambda_{22} P(\omega_2/x)$

$\lambda_{12} = 2, \lambda_{21} = 3, \lambda_{11} = 0, \lambda_{22} = 0$

$2 P(\omega_2/x) = 3 P(\omega_1/x)$

$2 P(x/\omega_2) P(\omega_2) = 3 \times P(x/\omega_1) P(\omega_1)$

$2 \times P(x, \omega_2) \times \frac{3}{4} = 3 \times P(x/\omega_1) \times \frac{1}{4}$

$2 P(x/\omega_2) = P(x/\omega_1)$

$2 \times N(5,1) = N(2,1)$

$\Rightarrow 2 \times \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(x^*-5)^2} = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(x^*-2)^2}$

$\Rightarrow \ln(2) - \frac{1}{2}(x^*-5)^2 = \frac{-1}{2}(x^*-2)^2$

$\Rightarrow 2\ln(2) = (x^*-5)^2 - (x^*-2)^2$

$\Rightarrow \ln(4) = -6x^* + 21$

$\Rightarrow x^* = \dfrac{21 - \ln(4)}{6} = 3.3$

No, we will not prefer zero-one loss for a task like cancer prediction on real world dataset because we can't risk a negative for a cancer patient just for a second test and real world datasets are very imperfect and unbalanced & λ_{12} & λ_{21} will also be not same.

Q2

Ans we have to find

$$Y = A^T X + B$$

we have

$$A = [2, -1, 2]^T = \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix}$$

$$B = 5$$

$$X = [x_1, x_2, x_3]$$

Mean vector of X is given as

$$\vec{\mu} = [5, -5, 6]$$

$$\Sigma = \begin{bmatrix} 5 & 2 & -1 \\ 2 & 5 & 0 \\ -1 & 0 & 4 \end{bmatrix}$$

$$E(Y) = E(A^T X) + E(B)$$

$$= A^T E(X) + B \quad \{B \text{ is constant}\}$$

$$= [2 \ -1 \ 2]\begin{bmatrix} 5 \\ -5 \\ 6 \end{bmatrix} + 5$$

$$= 10 + 5 + 12 + 5 = \underline{\underline{32}}$$

Q3

Ans A) $P(x/\omega_i) = \frac{1}{\pi b} \cdot \frac{1}{1+\left(\frac{x-a_i}{b}\right)^2}$, $i = 1,2$

for minimum zero one loss

we have:

$P(\omega_1/x) = P(\omega_2/x)$

$\Rightarrow P(x/\omega_1)\,P(\omega_1) = P(x/\omega_2)\,P(\omega_2)$

$\Rightarrow P(x/\omega_1)\,\cancel{\frac{1}{2}} = P(x/\omega_2)\,\cancel{\frac{1}{2}}$

$\Rightarrow P(x/\omega_1) = P(x/\omega_2)$

using PDF we get

$\frac{1}{\pi b}\left(\frac{1}{1+\left(\frac{x-a_1}{b}\right)^2}\right) = \frac{1}{\pi b}\left(\frac{1}{1+\frac{(x-a_2)^2}{b^2}}\right)$

$\Rightarrow \frac{(x-a_1)^2}{\cancel{b^2}} = \frac{(x-a_2)^2}{\cancel{b^2}}$

$\Rightarrow$ ~~x = a_1 = x = a_2 or & x = a_2~~

if we have $x - a_1 = x - a_2$

not possible.

Then we have.

$$x - a_1 = -(x - a_2)$$

$$\boxed{x = \frac{a_1 + a_2}{2}}$$

Hence we have $x^* = \frac{a_1 + a_2}{2}$, $a_1 \neq a_2$

as the optimal boundary.

Ans c) Using ~~a~~ PDF, ~~actual~~ calculating probability of error.

Hence we get

~~P Error~~ Hence $e \rightarrow$ error

$$P(e) = \int_{-\infty}^{\infty} P(e/x)\, P(x)\, dx$$

$$P(e/x) = \text{Min}\left(P(\omega_1/x) \cdot P(\omega_2/x)\right)$$

Considering $x < x^*$

$$P(e/x) = P(\omega_1/x)$$

for $x \geq x^*$

$$P(e/x) = P(\omega_2/x)$$

i) $P(e) = \int_{-\infty}^{x^*} P(\omega_1 / x) P(x)\, dx + \int_{x^*}^{\infty} P(\omega_2 / x) P(x)\, dx$

ii) ~~P(e)~~ = as we have

$$x^* = \frac{a_1 + a_2}{2}$$

$$P(e) = \int_{-\infty}^{\frac{a_1+a_2}{2}} P(x/\omega_1) P(\omega_1)\, dx + \int_{\frac{a_1+a_2}{2}}^{\infty} P(x/\omega_2) P(\omega_2)\, dx$$

$$P(e) = \int_{-\infty}^{\frac{a_1+a_2}{2}} \frac{1}{\pi b} \frac{1}{1+\left(\frac{x-a_1}{b}\right)^2} \cdot \frac{1}{2}\, dx + \int_{\frac{a_1+a_2}{2}}^{\infty} \frac{1}{\pi b} \frac{1}{1+\left(\frac{x-a_2}{b}\right)^2} \frac{1}{2}\, dx$$

$$P(e) = \frac{1}{2\pi} \left[\left[\tan^{-1}\left(\frac{x-a_1}{b}\right) \right]_{-\infty}^{\frac{a_1+a_2}{2}} + \left[\tan^{-1}\left(\frac{x-a_2}{b}\right) \right]_{\frac{a_1+a_2}{2}}^{\infty} \right]$$

~~P(e) =~~ we have

$a_1 = 3,\ a_2 = 5,\ b = 1$

So

$$P(e) = \frac{1}{2\pi}\left[\tan^{-1}(1) - \tan^{-1}(-\infty) + \tan^{-1}(\infty) - \tan^{-1}(-1)\right]$$

$$P(e) = \frac{1}{2\pi}\left[\frac{\pi}{4} + \frac{\pi}{2} + \frac{\pi}{2} + \frac{\pi}{4}\right]$$

$$= \frac{1}{2\cancel{\pi}} \times \frac{3\cancel{\pi}}{2} = \frac{3}{4}$$

Hence

$$P(\text{error}) = \frac{3}{4}$$ ← overall error rate

Q4

Ans a) we have.

$$\mathrm{cov}(x) = \begin{bmatrix} \theta(1-\theta) & 0 \\ 0 & \sigma^2 \end{bmatrix}$$

as $\sigma_{12} = \sigma_{21} = 0$

we can say that the dimensions (a, b) are independent

Hence

$$P(x) = P(\text{Bernoulli}(\theta) = a) \times P(\text{Gaussian}(\mu, \sigma^2) = b)$$

$$= \theta^a (1-\theta)^{1-a} \times \frac{1}{\sqrt{2\pi\sigma^2}} \times e^{-\left(\frac{b-\mu}{\sigma}\right)^2 \times \frac{1}{2}}$$

b) as we have N iid

we can say $q(x) = \prod_{i=1}^{N} P(x = x_i)$

Taking log

$$\log(q(x)) = \sum_{i=1}^{N} \log(P(x = x_i))$$

v) $$\log(q(x)) = \sum_{i=1}^{N} \log\left(\theta^{a_i} \times (1-\theta)^{1-a_i} \times \frac{1}{2\sqrt{2\pi\sigma}} \times e^{-\frac{1}{2}\left(\frac{b_i-\mu}{\sigma}\right)^2}\right)$$

$$\log(q(x)) = \sum_{i=1}^{N} a_i \left[\log\theta + (1-a_i)\log(1-\theta) - \log(\sqrt{2\pi\sigma}) - \frac{1}{2}\left(\frac{b_i-\mu}{\sigma}\right)^2\right]$$

& $$\frac{d(\log(q(x)))}{d\theta} = \sum_{i=1}^{N} \frac{a_i}{\theta} - \frac{(1-a_i)}{1-\theta} + 0 + 0$$

$$0 = \sum_{i=1}^{N} a_i(1-\theta) - \theta(1-a_i)$$

$$\sum_{i=1}^{N} a_i - \theta = 0.$$

$$\boxed{\theta = \frac{a_1 + a_2 \cdots a_N}{N}}$$