Quantitative methods, Assignment Chapter 9.

Problem 1 (20 points)

Compute the differentials of

1. (5 points) $W^2(t)$

Ito's formula is

$$df(t,X(t)) = f_t(t,X(t))dt + f_x(t,X(t))\,dX(t) + \frac{1}{2} f_{xx}(t,X(t))\,dX(t)\,dX(t) \quad (1)$$

Here $X(t) = W(t)$ $\therefore dX(t) = dW(t)$

$f(t,X(t)) = (W(t))^2 = (X(t))^2$

$f_t(t,X(t)) = 0$

$f_x(t,X(t)) = 2X(t) = 2W(t)$

$f_{xx}(t,X(t)) = 2$

eq^n (1) becomes,

$$dW^2(t) = 0 + 2X(t)\,dX(t) + \frac{1}{2}\cdot 2\,dX(t)\,dX(t)$$

$$\therefore d(W^2(t)) = 2W(t)\,dW(t) + dW(t)\,dW(t)$$

but $dW(t)\,dW(t) = dt$ (rule of stoch. calculus)

$$\boxed{\therefore d\,W^2(t) = 2W(t)\,dW(t) + dt}$$

2. (5 points) $W^3(t)$

Here our choice of
$X(t) = W(t)$

$f(t, X(t)) = (X(t))^3 = W^3(t)$

$\therefore f_t(t, X(t)) = 0$

$f_x(t, X(t)) = 3(X(t))^2 = 3(W(t))^2$

$f_{xx}(t, X(t)) = 6X(t) = 6W(t)$

$\therefore$ Using ito's formula

$df(t, X(t)) = f_t\,dt + f_x\,dX(t) + \frac{1}{2} f_{xx}\,dX(t)\,dX(t)$

$= 0 + 3(X(t))^2 dX(t) + \frac{1}{2} \cdot 6\,X(t)\,dX(t)\,dX$

$\therefore d(W^3(t)) = 3(W(t))^2 dW(t) + 3W(t)\,dW(t)\,dW(t)$

but $dW(t)dW(t) = dt$

$\therefore d(W^3(t)) = 3(W(t))^2 dW(t) + 3W(t)dt$

$\therefore d(W^3(t)) = 3W^2(t)dW(t) + 3W(t)dt$

3. (5 points) $\exp(-rt)W(t)$

Our choice is $X(t) = W(t)$ ∴ $dX(t) = dW(t)$

$f(t, X(t)) = e^{-rt} X(t) = e^{-rt} W(t)$

$f_t(t, X(t)) = e^{-rt} X(t) \times -r = -re^{-rt} X(t)$

$f_x(t, X(t)) = e^{-rt}$

$f_{xx}(t, X(t)) = 0$

Using ito's formula,

$df(t, X(t)) = f_t dt + f_x dX(t) + \frac{1}{2} f_{xx} dX(t) dX(t)$

$= -re^{-rt} X(t) + e^{-rt} dX(t) + 0$

$\therefore df(t, X(t)) = -re^{-rt} W(t) + e^{-rt} dW(t)$

$\therefore d(\exp(-rt)W(t)) = e^{-rt}[-rW(t) + dW(t)]$

4. (5 points) $\exp\{t^2 - W(t)\}$

$\rightarrow$ $X(t) = t^2 - W(t)$

$\therefore dX(t) = 2t\,dt - dW(t)$

$\therefore f = e^{X(t)}$

$\therefore f_t(t, X(t)) = 0$

$f_x(t, X(t)) = e^{X(t)}$

$f_{xx}(t, X(t)) = e^{X(t)}$

by ito's formula

$d\exp\{t^2 - W(t)\} = f_t\,dt + f_x\,dX(t)$

$+ \frac{1}{2} f_{xx}\,dX(t)\,dX(t)$

$\therefore d(\exp\{t^2 - W(t)\}) = 0 + e^{X(t)}dX(t)$

$+ e^{X(t)}dX(t)\,dX(t)$ —(1)

$dX(t).dX(t) = (2t\,dt - dW(t))(2t\,dt - dW(t))$

$= 4t^2 dt\,dt - 2t\,dW(t)\,dt$

$- 2t\,dW(t)\,dt + dW(t)\,dW(t)$

By rules,

as $dw(t)\,dw(t) = dt$

$dw(t)\,dt = 0$, $dt \cdot dt = 0$

$\therefore dx(t)\,dx(t) = 0 - 0 - 0 + dw(t)\,dw(t)$

$\therefore dx(t)\,dx(t) = dt$

putting in eq^n (1)

$$d\left(\exp\{t^2 - w(t)\}\right) = e^{x(t)}\left(2t\,dt - dw(t)\right) + \frac{1}{2}e^{x(t)} \cdot dt$$

$$\therefore d\left(\exp\{t^2 - w(t)\}\right) = 2te^{x(t)}dt - e^{x(t)}dw(t) + \frac{1}{2}e^{x(t)} \cdot dt$$

$$\therefore d\left(\exp\{t^2 - w(t)\}\right) = 2t\exp\{t^2 - w(t)\}\,dt - \exp\{t^2 - w(t)\}\,dw(t) + \frac{1}{2}\exp\{t^2 - w(t)\}\,dt$$

$$\therefore d\left(\exp\{t^2 - w(t)\}\right) = \left(2t + \frac{1}{2}\right)\exp\{t^2 - w(t)\}\,dt - \exp\{t^2 - w(t)\}\,dw(t)$$

$$\therefore d\left\{\exp\left\{t^2 - W(t)\right\}\right\} = \exp\left\{t^2 - W(t)\right\} \times \left(\left\{2t + \frac{1}{2}\right\}dt - dW(t)\right)$$

Problem 2 (16 points)

1. (5 points) Compute the differential of $W^4(t)$, where W is a standard brownian motion.

$x(t) = W(t)$

$f(t, x(t)) = (x(t))^4$

$\therefore f_t(t, x(t)) = 0$

$f_x(t, x(t)) = 4(x(t))^3$

$f_{xx}(t, x(t)) = 12(x(t))^2$

Using Itô's formula,

$$df(t, x(t)) = f_t\,dt + f_x\,dx(t) + \frac{1}{2} f_{xx}\,dx(t)dx(t)$$

$$\therefore dW^4(t) = 0 + 4(x(t))^3 dx(t) + 6(x(t))^2 dx(t)dx(t)$$

$$dW^4(t) = 4(W(t))^3 dW(t) + 6(W(t))^2 dW(t)dW(t)$$

$$\therefore dW^4(t) = 4(W(t))^3 dW(t)$$
$$+ 6(W(t))^2 dt \quad 4)$$

2. Integrate the above formula on $[0, T]$

→ Integrating the eq^n,

$$W^4(T) = W^4(0) + \int_0^T 4W^3(s)\,dW(s)$$

$$+ \int_0^T 6W^2(s)\,ds$$

$$\therefore W^4(T) = 0 + 4\int_0^T W^3(s)\,dW(s) + 6\int_0^T W^2(s)\,ds$$

$$\therefore W^4(T) = 4\int_0^T W^3(s)\,dW(s) + 6\int_0^T W^2(s)\,ds$$

3. (6 points) Take the expectation of the left and right hand sides and deduce $E[W^4(T)]$.

$$\therefore E[W^4(T)] = 4E\left[\int_0^T W^3(s)\,dW(s)\right] + 6E\left[\int_0^T W^2(s)\,ds\right] \quad -(1)$$

1) $I(t) = \int_0^T W^3(s)\,dW(s) = \int_0^T W^3(u)\,dW(u)$

if $I(t) = \int_0^t \sigma(u)\,dW(u)$

in our case

$\sigma(u) = W^3(u)$

As $I(0) = \int_0^0 W^3(s)\,dW(s) = 0$

$I(t)$ is an ito integral

& $I(t)$ is a martingale with respect to brownian motion (Given in class notes)

$$\therefore E[I(t)] = E\left[\int_0^T W^3(s)\,dW(s)\right] = I(0)$$

$$\therefore E\left[\int_0^T W^3(s)\,dW(s)\right] = 0 \quad -(2)$$

i) $E\left[\int_0^T W^2(s)\,ds\right] = ?$

let

$f = t W^2(t)$

choose $X(t) = W(t)$, $dX(t) = dW(t)$

$\therefore f = t(X(t))^2$

$\therefore f_t = (X(t))^2 = (W(t))^2$

$f_x = 2t X(t) = 2t W(t)$

$f_{xx} = 2t$

by ito's formula.

$$df = f_t\,dt + f_x\,dx + \frac{1}{2} f_{xx}\,dx\,dx$$

$$\therefore d\left[tW^2(t)\right] = (W(t))^2 dt + 2t W(t)\,dW(t) + \frac{1}{2} \cdot 2t \cdot dW(t)\,dW(t)$$

$$d\left[tW^2(t)\right] = W^2(t)\,dt + 2t\,W(t)\,dW(t) + t\,dt$$

Integrating both sides from 0 to T

$$TW^2(T) - 0 = \int_0^T W^2(s)\,ds + \int_0^T 2s\,W(s)\,dW(s) + \left[\frac{T^2}{2} - 0\right]$$

Taking expectation to bothsides,

$$E[TW^2(T)] = E\left[\int_0^T W^2(s)\,ds\right] + 2E\left[\int_0^T sW(s)\,dW(s)\right] + \frac{T^2}{2}$$

As $\int_0^T (sW(s))\,dW(s)$ is an ito integral

it's expectation is 0.

$$\therefore TE[W^2(T)] = E\left[\int_0^T W^2(s)\,ds\right] + 2\times 0 + \frac{T^2}{2}$$

As W is a brownian motion

$$E[W^2(T)] = T - 0 = T$$

$$\therefore T.T = E\left[\int_0^T W^2(s)\,ds\right] + \frac{T^2}{2}$$

$$\therefore E\left[\int_0^T W^2(s)\,ds\right] = \frac{T^2}{2} \quad \text{--- (3)}$$

(iii) As

$$E[W^4(T)] = 4E\left[\int_0^T W^3(s)\,dW(s)\right] + 6E\left[\int_0^T W^2(s)\,ds\right]$$

from eq^n 2 & 3. substituting

$$E[W^4(T)] = 4\times 0 + 6\times \frac{T^2}{2}$$

$$\therefore E[W^4(T)] = 3T^2$$

Problem 3 (17 points)

We consider the following mean-reverting model for the spread $S(t)$ of two co-integrated stocks:

$$dS(t) = -\lambda S(t)dt + \sigma dW(t), \quad S(0) = s > 0,$$

where $\lambda > 0$, $\sigma > 0$

1. (5 points) Show by using Ito-dublin formula that

$$S(t) = e^{-\lambda t}\left(s + \sigma\int_0^t e^{\lambda u}\,dW(u)\right),$$

is the solution of above differential equation.

$$S(t) = e^{-\lambda t}\left(s + \sigma \int_0^t e^{\lambda u}\, dW(u)\right)$$

Choose $x(t) = \sigma \int_0^t e^{\lambda u}\, dW(u)$

$$f(t, x(t)) = e^{-\lambda t}(s + x)$$

$$f_t = -\lambda e^{-\lambda t}(s + x)$$

$$f_x = e^{-\lambda t}$$

$$f_{xx} = 0$$

$$\therefore dx(t) = \sigma e^{\lambda t}\, dW(t)$$

$$\therefore dx(t)\, dx(t) = \sigma^2 e^{2\lambda t}\, dW(t)\, dW(t)$$
$$= \sigma^2 e^{2\lambda t}\, dt$$

By Ito-doublin formula,

$$dS(t) = df(t, x(t)) = f_t\, dt + f_x\, dx(t) + \frac{1}{2} f_{xx}\, dx(t)\, dx(t)$$

$$= -\lambda e^{-\lambda t}(s + x)\, dt + e^{-\lambda t} \cdot \sigma e^{\lambda t}\, dW(t) + 0$$

$$= -\lambda e^{-\lambda t}\left(s + \sigma \int_0^t e^{\lambda u}\, dW(u)\right) + \sigma\, dW(t)$$

$dS(t) = -\lambda S(t)\,dt + \sigma\,dW(t)$

we started with given $S(t)$ & reached the required differential equation.
Hence we have proved that $S(t)$ is the solution to the given given differential equation.

2. (6 points) Give the distribution of $S(t)$ and specify its mean and variance.

→ As

$$S(t) = e^{-\lambda t}\Big(S + \sigma \underbrace{\int_0^t e^{\lambda u}\,dW(u)}_{\text{ito integral}}\Big)$$

Here the integrand of ito integral is deterministic, Hence by the theorem from Class notes
The ito integral has mean 0 and

$$\text{Variance} = \int_0^t e^{(2\lambda u)}\,du = 2\lambda(e^{2\lambda t} - 1)$$

Hence $S(t)$ is also normally distributed with

mean $= Se^{-\lambda t}$ and

$$\text{Variance} = e^{-2\lambda t} \cdot \sigma^2 \cdot 2\lambda(e^{2\lambda t} - 1)$$

$\therefore$ Variance of $S(t) = 2\lambda\sigma^2\left(1 - \frac{1}{e^{2\lambda t}}\right)$

3. (4 points) Give $\lim_{t \to +\infty} E[S(t)]$, and interpret the result in plain english (1 sentence).

→ $\lim_{t \to \infty} E[S(t)] = \lim_{t \to \infty} \frac{S}{e^{\lambda t}} = 0$

This means the spread $S(t)$ of two cointegrated assets reverts back to long term mean $(t \to \infty)$ of zero.

4. (2 points) Does the variance $S(t)$ increase or decrease over time.

→ The variance $= 2\lambda\sigma^2\left(1 - \frac{1}{e^{2\lambda t}}\right)$

At time increases $e^{2\lambda t}$ increases consequently $\frac{1}{e^{2\lambda t}}$ decreases, which implies $1 - \frac{1}{e^{2\lambda t}}$ increases which ~~says~~ mean that variance increases with time (as $2\lambda\sigma^2$ remains the same, with time its coefficient increases)

Question 4 (23 points) Let $W(t)$ be a brownian motion, and define

$$B(t) = \int_0^t \text{Sign}(W(s))\, dW(s)$$

where

$$\text{Sign}(x) = \begin{cases} 1 & \text{if } x \geq 0 \\ -1 & \text{if otherwise.} \end{cases}$$

1. (5 points) Show that $B(t)$ is a Brownian Motion.

→

To show that $B(t)$ is a brownian motion it needs to satify the following properties.

① Initial condition

$$B(0) = \int_0^0 \text{sign}(W(s))\, dW(s) = 0$$

This is trivially true since the integral from 0 to 0 is zero.

② Independent increments:

The increments of $B(t)$ are dependent on the increments of $W(t)$, which is a brownian motion with independent increments.

Since sign ($W(s)$) only depends on the current value of $W(s)$ and not on past values of $W(s)$. Hence, increments of $B(t)$ are independent.

③ Stationary increments

The increments of $B(t)$ depend on the increments of $W(t)$ which are stationary. Since the sign function does not introduce any time dependency other than through $W(t)$, the increments of $B(t)$ are also stationary.

④ Normally distributed increments:

The increments of $W(t)$ are normally distributed (as W is brownian motion). The sign function being deterministic does not affect the normality of the increments but affects the variance, Hence the increments of $B(t)$ are also normally distributed.

⑤ Continuity

$B(t)$ is continuous since it is an integral of a continuous function times a continuous process $dW(s)$.

Since $B(t)$ satisfies all the properties of a Brownian motion, it itself is a brownian motion.

2. (5 points) Use Ito's product rule to compute $d[B(t)\, W(t)]$. Integrate both sides of the resulting eq^n, take expectations and deduce that $E[B(t)\, W(t)] = 0$

- Using Itô rule & product rule of stochastic calculus

$$d[B(t)\, W(t)] = B(t)\, dW(t) + W(t)\, dB(t) + dB(t)\, dW(t)$$

Given

$$B(t) = \int_0^t \text{Sign}\,(W(s))\, dW(s)$$

$$\therefore dB(t) = \text{sign}\,(W(t))\, dW(t)$$

$$\Rightarrow d\{B(t)\, W(t)] = B(t)\, dW(t) + W(t)\, \text{sign}\,(W(t))\, dW(t) + \text{sign}\,(W(t))\, dW(t)\, dW(t)$$

as $dW(t)\, dW(t) = dt$.

$$\therefore d[B(t)\, W(t)] = B(t)\, dW(t) + W(t)\, \text{sign}(W(t))\, dW(t) + \text{sign}(W(t))\, dt$$

Integrating both sides from 0 to t.

$$\int_0^t d(B(s)W(s)) = \int_0^t B(s)\,dW(s) + \int_0^t W(s)\,sign(W(s))\,dW(s)$$
$$+ \int_0^t sign(W(s))\,ds$$

Taking expectation of both sides

$$E\left[\int_0^t d(B(s)W(s))\right] = E\left[\int_0^t B(s)\,dW(s)\right]$$
$$+ E\left[\int_0^t W(s)\,sign(W(s))\,dW(s)\right]$$
$$+ E\left[\int_0^t sign(W(s))\,ds\right]$$

first 2 expectations are expectations of an ito integral.

$$\therefore E\left[\int_0^t d(B(s)W(s))\right] = E\left[\int_0^t sign(W(s))\,ds\right]$$

since $W(s)$ is symmetric around zero, $sign(W(s))$ will take $+1$ & -1 value with equal probaballity

$$\therefore E[sign(W(s))] = 0$$

$$\therefore E\left[\int_0^t d(B(s)W(s))\right] = \int_0^t E[\text{sign}(w(s))]\,ds$$

$$= \int_0^t 0 \cdot ds$$

$$\therefore E\left[B(t)W(t)\right] = 0$$

3. (3 points) verify that.

$$dw^2(t) = 2W(t)dW(t) + dt$$

choose $x(t) = W(t)$ $\therefore dx(t) = dW(t)$

$$f(t, x(t)) = (W(t))^2 = 2(x(t))^2$$

i) $f_x = 2x(t) = 2W(t)$

ii) $f_{xx} = 2$

iii) $f_t = 0$

by ito's lemma,

$$df(t, x(t)) = f_t\,dt + f_x\,dx(t) + \frac{1}{2} f_{xx}\,dx(t)dx(t)$$

$$\therefore dW^2(t) = 0 + 2W(t) \cdot dW(t) + \frac{1}{2} \cdot 2 \cdot dW(t)dW(t)$$

$\therefore dW^2(t) = 2W(t)dW(t) + 1 \cdot dW(t)\,dW(t)$

by $dW(t)\,dW(t) = dt$ [rule of stoch. calcul.

$\therefore dW^2(t) = 2W(t)dW(t) + dt$

4. (5 points) Use ito product's rule to compute $d[B(t)\,W^2(t)]$. deduce that $E[B(t)W^2(t)]$ is not equal to $E[B(t)]E[W^2(t)]$.

→ Applying the product rule

$$d(B(t)\,W^2(t)) = B(t)dW^2(t) + W^2(t)dB(t) + dB(t)dW^2(t) \quad -(1)$$

We know, $dB(t) = \text{sign}(W(t))\,dW(t)$

and $dW^2(t) = 2W(t)\,dW(t) + dt$ [From prev questions]

Since $(dW(t))^2 = dt$, we can simplify

$\Rightarrow$ ~~$d(B(t)W^2(t)) = 2W(t)B(t)dW(t) + B(t)$~~

$W^2(t)\,\text{sign}(W(t))\,dW(t)$

$+ \text{sign}(W(t))$

i) $B(t)\, dw^2(t) = 2\, W(t)\, B(t)\, dW(t) + B(t)\, dt$

ii) $w^2(t)\, dB(t) = W^2(t)\, \text{sign}(w(t))\, dw(t)$

iii) $dB(t)\, dw^2(t) = (\text{sign}(w(t))\, dw(t)) \cdot (2\, w(t)\, dw(t) + dt)$

as $dW(t)\, dt = 0$ & $dW(t) dW(t) = dt$.

$\therefore dB(t)\, dw^2(t) = 2\, w(t)\, \text{sign}(w(t))\, dt$

$\therefore$ eq^n (i) becomes.

$$d(B(t) W^2(t)) = 2 B(t)\, w(t)\, dW(t) + B(t)\, dt + w^2(t)\, \text{sign}(W(t))\, dW(t) + 2\, W(t)\, \text{Sign}(w(t))\, dt$$

integrating both sides from 0 to t.

$$B(t)\, W^2(t) = 2 \int_0^t B(s)\, w(s)\, dW(s) + \int_0^t B(s)\, dt + \int_0^t w^2(s)\, \text{sign}(w(s))\, dw(s) + 2 \int_0^t w(s)\, \text{sign}(w(s))\, ds$$

taking expectation on both sides

$$E[B(t)W^2(t)] = 2E\left[\underbrace{\int_0^t B(s)W(s)\,dW(s)}_{\text{ito integral}}\right]$$

$$+ E\left[\int_0^t B(\xi)\,d\xi\right]$$

$$+ E\left[\underbrace{\int_0^t W^2(\xi)\,\text{sign}(W(\xi))\,dW(\xi)}_{\text{ito integral.}}\right]$$

$$+ 2E\left[\int_0^t W(\xi)\,\text{sign}(W(\xi))\,ds\right]$$

As expectation of ito integral = 0

$$\therefore E[B(t)W^2(t)] = 2\times 0 + \int_0^t E[B(s)]\,ds$$

$$+ 0 + 2\int_0^t E[W(s)\,\text{sign}(W(s))]\,ds$$

$$\therefore E[B(t)W^2(t)] = \int_0^t 0\,ds + 2\int_0^t E[W(s)\,\text{sign}(W$$
$$ds$$

as ~~E~~ $W(s)\,\text{sign}(W(s))$ will always be positive
So its expectation will not be zero

$\therefore E[B(t)W^2(t)] = 2\int_0^t E[W(s)\,\text{sign}(W(s))]\,ds$

as $E[B(t)]E[W^2(t)] = 0 \times t = 0$

but $E[B(t)W^2(t)] \neq 0$

$\therefore E[B(t)W^2(t)] \neq E[B(t)]E[W^2(t)]$

5. (5 points) Conclude.

→ From the above calculation we can conclude that $B(t)$ and $W^2(t)$ are not independent. This is intuitively true as the source of randomness in $B(t)$ is the standard brownian motion $W(t)$ which is also the source of randomness in $W^2(t)$.

Problem 5 (24 points)

In the Hull-white interest rate model, the interest rate is given by the stochastic differential equation

$$dR(t) = (a(t) - b(t)R(t))dt + \sigma(t)d\tilde{W}(t),$$

where $a(t)$, $b(t)$, $b(t)$, $\sigma(t)$ are nonrandom positive functions ~~and~~ $\tilde{W}$ is a standard Brownian motion under a risk-neutral probability measure. Assume that the initial condition is given at time t by $R(t) = r$. The goal of this excercise is to solve this equation explicitly

1. (5 points) First of all, compute, by using Itô-Doeblin formula,

$$d\left(e^{\int_0^u b(v)dv} R(u)\right).$$

let $X(u) = R(u)$ $\quad \therefore dX(u) = dR(u)$

Choose

$$f(u, X(u)) = e^{\int_0^u b(v)dv} \cdot X(u)$$

$$f_u = e^{\int_0^u b(v)dv} \, b(u) \cdot X(u)$$

$$= b(u) e^{\int_0^u b(v)dv} \cdot R(u)$$

$$f_x = e^{\int_0^u b(v)dv}$$

$$f_{xx} = 0$$

Using ito's formula.

$$d\left(e^{\int_0^u b(v)dv} R(u)\right) = f_u\,du + f_x\,dx(u) + f_{xx}\,dx(u)dx(u)$$

$$= b(u)\, e^{\int_0^u b(v)dv} \cdot R(u)\,du + e^{\int_0^u b(v)dv}\, dR(u)$$

$$= e^{\int_0^u b(v)dv} \left[b(u)R(u)\,du + dR(u) \right] \quad -(1)$$

As $b(u)R(u)du + dR(u) = b(u)R(u)du + (a(u) - b(u)R(u))\,du + \sigma(u)\,d\tilde{W}(u)$

$\therefore\ b(u)R(u)du + dR(u) = a(u)\,du + \sigma(u)\,d\tilde{W}(u)$

putting in eq^n (1)

$$d\left(e^{\int_0^u b(v)dv} R(u)\right) = e^{\int_0^u b(v)dv} \left[a(u)du + \sigma(u)\,d\tilde{W}(u) \right]$$

2. (5 points) Integrate both sides of the formula found above from t to T and use the initial condition $R(t) = r$ to find the formula.

$$e^{\int_0^T b(v)dv} R(T) = r e^{\int_0^t b(v)dv} + \int_t^T e^{\int_0^u b(v)dv} a(u)du + \int_t^T e^{\int_0^u b(v)dv} \sigma(u) d\tilde{W}(u).$$

→ integrating the prev. eq^n from t to T

$$\int_t^T d\left(e^{\int_0^u b(v)dv} R(u)\right) = \int_t^T e^{\int_0^u b(v)dv} \left[a(u)du + \sigma(u) d\tilde{W}(u)\right]$$

$$\therefore e^{\int_0^T b(v)dv} R(T) = e^{\int_0^t b(v)dv} \cdot R(t) + \int_t^T e^{\int_0^u b(v)dv} \cdot a(u)du + \int_t^T e^{\int_0^u b(v)dv} \sigma(u) d\tilde{w}(u).$$

putting $R(t) = r$

$$\therefore e^{\int_0^T b(v)dv} \cdot R(T) = r e^{\int_0^t b(v)dv} + \int_t^T e^{\int_0^u b(v)dv} \cdot a(u)du + \int_t^T e^{\int_0^u b(v)dv} \cdot \sigma(u) d\tilde{w}(u)$$

Hence proved.

3. (5 points) Solve the above formula for $R(T)$. This yields an explicit formula for $R(T)$

- dividing ~~the~~ both sides by $e^{\int_0^T b(v)\,dv}$ which is a constant.

$$\therefore R(T) = r\, e^{\int_0^t b(v)dv - \int_0^T b(v)dv} + \int_t^T e^{\int_0^u b(v)dv - \int_0^T b(v)dv} \cdot a(u)\,du + \int_t^T e^{\int_0^u b(v)dv - \int_0^T b(v)dv} \cdot \sigma(u)\,d\tilde{W}(u)$$

$$\therefore R(T) = r\, e^{-\int_t^T b(v)dv} + \int_t^T e^{\int_0^u b(v)dv - \int_0^T b(v)dv} \cdot a(u)\,du + \int_t^T e^{\int_0^u b(v)dv - \int_0^T b(v)dv} \cdot \sigma(u)\,d\tilde{W}(u)$$

$$R(T) = r e^{-\int_t^T b(v)dv} + \int_t^T e^{-\int_u^T b(v)dv} \cdot a(u)\,du + \int_t^T e^{-\int_u^T b(v)dv} \cdot \sigma(u)\,d\tilde{W}(u)$$

4. (4 points) Give the distribution of $R(T)$

→ As $R(T)$ = Some deterministic term
+ ito integral, with deterministic process

& ito integral which has deterministic integrand is normally distributed
Hence $R(T)$ is also normally distributed.

5. (5 points) Give also its mean and variance.

→ As mean of ito integral is 0.

mean of $R(T)$ = deterministic term.

$$\therefore \text{Mean}(R(T)) = r e^{-\int_t^T b(v)\,dv} + \int_t^T e^{\left(\int_0^u b(v)\,dv - \int_0^T b(v)\,dv\right)} \cdot a(u)\,du$$

Similarly

$$= r e^{-\int_t^T b(v)\,dv} + \int_t^T e^{-\int_u^T b(v)\,dv} \cdot a(u)\,du$$

Variance of $(R(T))$ = Variance of ito integral

$$= \int_t^T e^{-2\int_u^T b(v)\,dv} \sigma^2(u)\,du$$