– Utkarshbhanu G. Andurkar
(ua2085)

Quantitative Methods
Assignment - 1

Problem 1: Let $X_1, X_2, \ldots X_n$ be n independent Poison random variables with parameter $\lambda_1, \lambda_2, \ldots \lambda_n$.

1. Determine the distribution of $X_1, X_2 + \ldots + X_n$ and its parameter. Hint: use the moment generating function of Point distribution with parameter λ

$$\phi(t) = E[e^{tX}] = e^{\lambda(e^t - 1)}$$

Calculating the moment generating function for X_i,

$$\phi(t) = E[e^{tX_i}] = e^{\lambda_i(e^t-1)} \quad (1) \quad [\because \text{From Hint}]$$

Let $Y = \sum_{i=1}^{n} X_i$

Calculating moment generating function for Y

$$\phi(t) = E[e^{tY}]$$

$$= E[e^{t\sum_{i=1}^{n} X_i}]$$

$$= E\left[\prod_{i=1}^{n} e^{tX_i}\right]$$

$$= \prod_{i=1}^{n} \left[E[e^{tX_i}]\right] \qquad \left[\text{As all } X_i\text{'s are independent Random variables}\right]$$

$$\phi(t) = \prod_{i=1}^{n} e^{\lambda_i (e^t - 1)} \quad \text{[From eq}^n \text{ 1]}$$

$$= \exp\left(\sum_{i=1}^{n} \lambda_i (e^t - 1) \right) \quad [$$

$$= \exp\left((e^t - 1) \sum_{i=1}^{n} \lambda_i \right)$$

let $\sum_{i=1}^{n} \lambda_i = \lambda$ be a new parameter.

$$\therefore \phi(t) = \exp(\lambda(e^t - 1))$$

$$\phi(t) = e^{\lambda(e^t - 1)} \quad \text{— (2)}$$

But equation (2) is moment generating function of a poisson random variable with parameter $= \lambda$.

$\therefore$ Y follows poisson distribution with parameter λ.

i.e $Y = \sum_{i=1}^{n} X_i$, where X_i s are independent

poisson random variables,

then Y follows a poisson distribution with parameter $\lambda = \lambda_1 + \lambda_2 + \cdots + \lambda_n = \sum_{i=1}^{n} \lambda_i$.

Problem 2

A portfolio of n loans is modeled by using correlated Bernouilli variables as follows. For $i = 1 \dots n$, we denote by X_i the variable defined for each loan i as

$$X_i = \sqrt{\rho}\, Y + \sqrt{1-\rho}\, \varepsilon_i$$

where Y, which is a common factor to all the loans, has a standard normal distribution, $\rho \in (0,1)$ is the correlation co-efficient between two loons, and ε_i are independent random variables with a standard normal distribution. Moreover we assume that ε_i is independent of Y, for all i. You can view the variable X_i as the value of the ith borrower's assets. Finally, we define the Bernouilli variable Y_i as

$$Y_i = \mathbb{I}\{X_i < x^*\},$$

where $\mathbb{I}$ is the indicator function and x^* is the level below which loan i defaults. In other words, Y_i takes the value 1 if the loan i defaults and a value 0 if it does not.

1. Is the distribution of X_i normal? Justify your answer.

→ As $X_i = \sqrt{\rho}\, Y + \sqrt{1-\rho}\, \varepsilon_i$,

where ρ is a constant (correlation co-efficient) and Y and ε_i are standard normally distributed & independent of each other.

it is a fact that the linear combination of two independent random variables having a normal distribution also has a normal distribution.

Therefore X follows normal distribution.
We can prove this fact using moment generating function as moment generating function has unique form for normal distribution.

⋆As $Y \sim (0,1)$ & $\varepsilon_i \sim (0,1)$ $\therefore X_i \sim N(0, \rho + 1 - \rho)$ $\therefore X_i \sim N(0,1)$

2. Compute $E[X_i]$

→ $E[X_i] = E\left[\sqrt{\rho}\, Y + \sqrt{1-\rho}\, \varepsilon_i\right]$
as Y & ε_i are independent R.V.

$\therefore E[X_i] = \sqrt{\rho}\, E[Y] + \sqrt{1-\rho}\, E[\varepsilon_i]$

but $Y \sim N(0,1)$ and $\varepsilon_i \sim N(0,1)$

$\therefore E[Y] = 0$, $E[\varepsilon_i] = 0$

$\therefore E[x_i] = \sqrt{\rho} \cdot 0 + \sqrt{1-\rho} \cdot 0 = 0$

$\therefore E[x_i] = 0$

3. Compute $\mathrm{var}[x_i]$.

$\mathrm{var}[x_i] = \mathrm{var}[\sqrt{\rho}\, Y + \sqrt{1-\rho}\; \varepsilon_i]$
$= \rho \cdot \mathrm{var}[Y] + (1-\rho)\,\mathrm{var}[\varepsilon_i]$

as

$\mathrm{var}[ax+by] = a^2 \mathrm{var}\, X + b^2 \mathrm{var}\, Y + 2ab\, \mathrm{cov}(X,Y)$

but cov is 0 in our case.

Also $\mathrm{var}[Y] = 1$, $\mathrm{var}[\varepsilon_i] = 1$ as $Y \sim N(0,1)$ & $\varepsilon_i \sim N(0,1)$

$\therefore \mathrm{var}[x_i] = \rho \cdot 1 + (1-\rho) \cdot 1$

$\therefore \mathrm{var}[x_i] = \rho + 1 - \rho = 1$

4. Verify that

$\text{cov}(X_i, X_j) = \rho$, for all i, j such that $i \neq j$

$$\text{cov}(X_i, X_j) = \text{cov}(\sqrt{\rho}\,Y + \sqrt{1-\rho}\,\varepsilon_i, \sqrt{\rho}\,Y + \sqrt{1-\rho}\,\varepsilon_j)$$

$$= \text{cov}(\sqrt{\rho}\,Y, \sqrt{\rho}\,Y) + \text{cov}(\sqrt{\rho}\,Y, \sqrt{1\rho}\,\varepsilon_j)$$
$$+ \text{cov}(\sqrt{1-\rho}\,\varepsilon_i, \sqrt{\rho}\,Y) + \text{cov}(\sqrt{1-\rho}\,\varepsilon_i, \sqrt{1-\rho}\,\varepsilon_j)$$

$$\therefore \text{cov}(X_i, X_j) = \sqrt{\rho}\cdot\sqrt{\rho}\,\text{cov}(Y, Y) + \sqrt{\rho}\sqrt{1-\rho}\,\text{cov}(Y, \varepsilon_j)$$
$$+ \sqrt{1-\rho}\sqrt{\rho}\,\text{cov}(\varepsilon_i, Y)$$
$$+ \sqrt{1-\rho}\sqrt{1-\rho}\,\text{cov}(\varepsilon_i, \varepsilon_j)$$

but $\text{cov}(Y, Y) = \text{var}(Y) = 1$ [as $Y \sim N(0,1)$]

also as Y & ε_i are independent of R.V.

$\text{cov}(\varepsilon_i, Y) = \text{cov}(Y, \varepsilon_i) = 0$

Similarly $\text{cov}(Y, \varepsilon_j) = \text{cov}(\varepsilon_j, Y) = 0$

$$\therefore \text{cov}(X_i, X_j) = \rho \cdot 1 + \sqrt{\rho}\sqrt{1-\rho}\cdot 0$$
$$+ \sqrt{1-\rho}\sqrt{\rho}\cdot 0$$
$$+ (1-\rho)\,\text{cov}(\varepsilon_i, \varepsilon_j)$$

for all $i \neq j$, ε_i & ε_j are independent.
(Given)

$\therefore$ $\text{cov}(\varepsilon_i, \varepsilon_j) = 0$

$\therefore \operatorname{cov}(X_i, X_j) = \rho + 0 + 0 + (1-\rho) \cdot 0$

$$\boxed{\therefore \operatorname{cov}(X_i, X_j) = \rho}$$

5. Write the probability p_i of default of the i^{th} loan in terms of the standard normal cumulative distribution function N and give $E[Y_i]$. Does p_i depend on i?

→ As we know from problem 1s solution $X_i \sim N(0,1)$

$$\therefore P_i = P(-\infty < X_i < x^*) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x^*} e^{-\frac{1}{2}x_i^2}\, dx_i$$

but as definite integral only depend on the function and limits of integration but not on variable of integration.

Therefore

$$P_i = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x^*} e^{-\frac{1}{2}x^2}\, dx$$

$\boxed{\text{As we can see the value of } P_i \text{ does not depend on } i.}$

As $Y_i = \mathbb{I}\{x_i < x^*\}$,

$$\therefore E[Y_i] = \left[\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} e^{-\frac{1}{2}x_i^2}\,dx_i\right] y_i$$

but $y_i = 1$ for $x_i < x^*$ & 0 otherwise

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x^*} e^{-\frac{1}{2}x_i^2}\,dx_i \cdot 1 + \frac{1}{\sqrt{2\pi}}\int_{x^*}^{+\infty} e^{-\frac{1}{2}x_i^2}\,dx_i \cdot 0$$

$$\therefore E[Y_i] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x^*} e^{-\frac{1}{2}x_i^2}\,dx_i$$

$$E[Y_i] = p_i = P(-\infty < X_i < x^*)$$

$$\therefore E[Y_i] = p_i = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x^*} e^{-\frac{1}{2}x_i^2}\,dx_i = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x^*} e^{-\frac{1}{2}x^2}\,dx$$

6. Compute var$[Y_i]$

→ As Y_i is a Bernouilli random variable

$$Y_i \begin{cases} 0 & 1-p_i \\ 1 & p_i \end{cases}$$

$$E[Y_i] = p_i \cdot 1 + (1-p_i)\,0 = p_i$$

$$E[Y_i^2] = p_i \cdot (1)^2 + (1-p_i)\cdot(0)^2 = p_i$$

$\therefore Var[Y_i] = E[Y_i^2] - (E[Y_i])^2$
$= p_i - p_i^2$

$var[Y_i] = p_i(1 - p_i)$

where $p_i = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x^*} e^{-\frac{1}{2}x^2} dx$

7. We consider the random variable L representing the number of defaults in the portfolio. Write it in the terms of the Bernouilli variables Y_i, for $i = 1 .. n$.

→ As L is total number of defaults in the portfolio

$$L = \sum_{i=1}^{n} Y_i \quad ; \quad Y_i < \begin{matrix} 1 & p_i \\ 0 & 1 - p_i \end{matrix}$$

i.e. L is a simple random walk.

8. Compute the expected loss.

→ Expected loss $= E[L]$

$= E[\sum_{i=1}^{n} Y_i]$

$= \sum_{i=1}^{n} E[Y_i]$

$\therefore$ Expected loss $= \sum_{i=1}^{n} p_i = \frac{n}{\sqrt{2\pi}} \int_{-\infty}^{x^*} e^{-\frac{1}{2}x^2} dx$

9. Discuss the flaw of the model.

→ 1) The model defines the fixed level x^* for all the $1 \dots n$ loans, below which the loans will default. This level should be variable for each individual loan i, based on both the amount (value) of loan and the value X_i of assets of i^{th} borrower.

2) The model can be further improved by taking into consideration other impacting factors like creditworthiness of i^{th} borrower, cash assets & illequid assets, financial discipline of i^{th} borrower etc while calculating the value of level x^* or x_i^*.

Problem 3:

We consider the following linear factor model: we observe the daily log returns r_i of N assets. We suppose that the returns are driven by P common factors $Y_1, Y_2, \dots, Y_p$ and a white noise process ε_i that is specific to each asset. We assume that the $p \times 1$ vector $Y = (Y_1, Y_2, \dots, Y_p)^t$ has a multinormal distribution with mean $p \times 1$ vector $\mu = (\mu_j)_{j=1 \dots p}$ and $p \times p$ covariance matrix $\Sigma = (\Sigma_{ij})_{i,j=1 \dots p}$.

Furthermore $\varepsilon_1, \ldots, \varepsilon_N$ are i.i.d. with a standard normal distribution. We also assume that the variables $\varepsilon_1, \ldots \varepsilon_N$ are independent of the variables $Y_1, \ldots, Y_p$. The model is written as

$$r_i = \sum_{(1 \le j=1)}^{p} \beta_{i,j} Y_j + \sigma_i \varepsilon_i, \text{ where } i = 1 \ldots N,$$

where $\beta_{i,j}$ and σ_i are given.

We also consider the $N \times P$ matrix,

$\beta = (\beta_{i,j})_{i=1..N,\, j=1..p}$ and the $N \times 1$ vector

$\sigma = (\sigma_i)_{i=1,\ldots N}$.

1. Compute the vector R of expected returns
$$R = (E[r_1], E[r_2], \ldots, E[r_N])^t$$
in terms of the matrix β and the vector μ, assumming that they are known.

$$r_i = \sum_{j=1}^{p} \beta_{i,j} Y_j + \sigma_i \varepsilon_i$$

$$\therefore E[r_i] = E\left[\sum_{j=1}^{p} \beta_{i,j} Y_j + \sigma_i \varepsilon_i\right]$$

$$= \sum_{j=1}^{p} \beta_{i,j} E[Y_j] + \sigma_i E[\varepsilon_i]$$

As $E[Y_j] = \mu_j$ & $\varepsilon_i \sim N(0,1)$ $\therefore E[\varepsilon_i] = 0$

$$\therefore E[r_i] = \sum_{j=1}^{P} \beta_{i,j}\, \mu_j + \sigma_i \cdot 0$$

$$\therefore E[r_i] = \sum_{j=1}^{P} \beta_{i,j}\, \mu_j$$

As $\mu = (\mu_j)_{j=1..P}$ which is a $P \times 1$ vector

& $R = (E[r_1], E[r_2], \cdots, E[r_N])^t$

& $\beta = (\beta_{i,j})_{i=1..N, j=1..P}$

$$\therefore \underset{(N\times 1)}{R} = \underset{(N\times P)}{\beta} \cdot \underset{(P\times 1)}{\mu}$$

$$\therefore \boxed{R = \beta \cdot \mu}$$

2. Compute the $N \times N$ covariance matrix C of the vector of returns $(r_i)_{i=1..N}$ in terms of the matrix β, the matrix Σ and the $N\times N$ diagonal matrix $Diag(\sigma)$, whose diagonal is given by the vector σ.

→ Let us first calculate covariance of i^{th} & k^{th} return.

$$cov(r_i, r_k) = cov\left(\sum_{j=1}^{P} \beta_{i,j} Y_j + \sigma_i \varepsilon_i ,\ \sum_{j=1}^{P} \beta_{k_j} Y_j + \sigma_k \varepsilon_k \right)$$

As $\forall Y_j, \varepsilon_i, \forall i, j$ ε_i & Y_j are independent (Given assumption)

$$\therefore cov\left(\sum_{j=1}^{P} \beta_{ij} Y_j, \sigma_k \varepsilon_k\right) = cov\left(\sigma_i \varepsilon_i, \sum_{j=1}^{P} \beta_{kj} Y_j\right) = 0$$

Also as all ε_i, $i=1\cdots N$ are i.i.d.

$\therefore cov(\varepsilon_i, \varepsilon_k)=0$, for $i \neq k$

$$\therefore cov(r_i, r_k) = cov\left(\sum_{j=1}^{p} \beta_{ij} Y_j, \sum_{j=1}^{p} \beta_{kj} Y_j\right) + \sigma_i \cdot \sigma_k \, cov(\varepsilon_i, \varepsilon_k)$$

a) for $i \neq k$, $cov(\varepsilon_i, \varepsilon_k) = 0$

$$\therefore cov(r_i, r_k) = cov(\beta_{i1} Y_1 + \beta_{i2} Y_2 + \cdots \beta_{ip} Y_p, \beta_{k1} Y_1 + \beta_{k2} Y_2 + \cdots \beta_{kp} Y_p) + 0$$

$$= (\beta_{i1} \beta_{k1} cov(Y_1, Y_1) + \beta_{i1} \beta_{k2} cov(Y_1, Y_2) + \cdots \beta_{i1} \beta_{kp} cov(Y_1, Y_p))$$
$$+ \cdots + (\beta_{ip} \beta_{k1} cov(Y_p, Y_1) + \beta_{ip} \beta_{k2} cov(Y_p, Y_2) + \cdots + \beta_{ip} \beta_{kp} cov(Y_p, Y_p))$$

$$= \sum_{j=1}^{p} \beta_{i1} \beta_{kj} cov(Y_1, Y_j) + \cdots + \sum_{j=1}^{p} \beta_{ip} \beta_{kj} cov(Y_p, Y_j)$$

$$= \sum_{\lambda=1}^{p} \sum_{j=1}^{p} \beta_{i\lambda} \beta_{kj} cov(Y_\lambda, Y_j)$$

b) for $i = k$

$$cov(r_i, r_i) = \sum_{\lambda=1}^{p} \sum_{j=1}^{p} \beta_{i\lambda} \beta_{ij} cov(Y_\lambda, Y_j) + \sigma_i^2 cov(\varepsilon_i, \varepsilon_i)$$

but $cov(\varepsilon_i, \varepsilon_i) = var(\varepsilon_i) = 1$ as $\varepsilon_i \sim N(0,1)$

$$\therefore cov(r_i, r_i) = \sum_{\lambda=1}^{p} \sum_{j=1}^{p} \beta_{i\lambda} \beta_{ij} cov(Y_\lambda, Y_j) + \sigma_i^2$$

Therefore the $N \times N$ covariance Matrix C is

$$C_{(N\times N)} = \beta_{(N\times p)} \; \Sigma_{(p\times p)} \; \beta^{t}_{(p\times N)} + \mathrm{Diag}(\sigma) \cdot (\mathrm{Diag}(\sigma))^{t}$$

where $\beta = (\beta_{ij})_{i=1\cdots N, j=1\cdots p}$

$\beta^{t} = (\beta_{ij})_{i=1\cdots p, j=1\cdots N}$ [transpose of β]

Σ is covariance matrix. $\Sigma = (\Sigma_{ij})_{i=1\cdots p, j=1\cdots p}$

$\mathrm{Diag}(\sigma)$ is a diagonal matrix whose diagonal is given by the vecto σ.

$(\mathrm{Diag}(\sigma))^{t}$ is transpose of $\mathrm{Diag}(\sigma)$

$$\boxed{\therefore C = \beta \Sigma \beta^{t} + \mathrm{Diag}(\sigma) \cdot \mathrm{Diag}(\sigma)^{t}}$$