Quantitative methods; Assignment Chapter 6

Problem 1 (30 points)

Consider the process $\{X(t), t \geq 0\}$ defined by

$$X(t) = \begin{cases} 1 & \text{if } t \leq Y \\ 0 & \text{if } t > Y \end{cases}$$

where Y is a uniformly distributed random variable on the interval $(0, 1)$

1. (5 points) Compute, for $t \in [0,1]$, the first-order probability mass of X, $f(t, x)$.

→ Using C.D.F. of uniform distribution.

$$P(Y \geq t) = \frac{1-t}{1-0} = 1-t$$

$$P(Y < t) = \frac{t-0}{1-0} = t$$

∴ Probability mass of X

$$f(t, x) = \begin{cases} 1-t & \text{when } x = 1 \\ t & \text{when } x = 0 \\ 0 & \text{otherwise} \end{cases}$$

2. What is first-order probability mass of X for $t > 1$?

→ for $t > 1$ → implies $Y < t$ → always ∴ the probability mass of X is ~~f(t,x) = 0~~

$$X(t) = \begin{cases} 1 & p = 0 \\ 0 & p = 1 \end{cases}$$

3. (5 points) Compute the expectation and the variance of x at time t, for $t \in (0,1)$.

$$E[x(t)] = \sum x_i P_i$$

$$= 1 \cdot (1-t) + 0 \cdot t$$

$$E[x(t)] = 1 - t$$

$$\text{var}[x(t)] = E[x^2(t)] - (E[x(t)])^2$$

$$E[x^2(t)] = \sum x_i^2 P_i$$

$$= 1^2 (1-t) + 0^2 t$$

$$= 1 - t$$

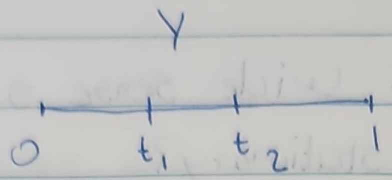
$$\therefore \text{var}[x(t)] = 1 - t - (1-t)^2$$

$$= 1 - t - (1 - 2t + t^2)$$

$$= t - t^2$$

4. (5 points) Compute the autocovariance function of X, $C_X(t_1, t_2)$ for $t_1, t_2 \in (0,1)$; with $t_1 \le t_2$.

$$C_x(t_1, t_2) = E[x(t_1)\, x(t_2)] - E[x(t_1)]\, E[x(t_2)]$$



When $X(t_2) = 1 \Rightarrow Y \geq t_2 \geq t_1 \Rightarrow X(t_1) = 1$

$\therefore P = 1 - t_2$ & $X(t_2) \cdot X(t_1) = 1 \cdot 1 = 1$.

When $X(t_1) = 0 \rightarrow Y < t_1 \leq t_2 \Rightarrow X(t_2) = 0$

$\therefore p = t_1$ & $X(t_2) \cdot X(t_1) = 0 \cdot 0 = 0$

When $t_1 < Y < t_2$ $\quad X(t_2) = 0$ & $X(t_1) = 1$

$\therefore$ * $P = \dfrac{t_2 - t_1}{1 - 0} = t_2 - t_1$ & $X(t_2) \cdot X(t_1) = 0 \cdot 1 = 0$

$\therefore$ PMF of $X(t_1) X(t_2)$ is

$$= \begin{cases} 1 & p = 1 - t_2 \\ 0 & p = t_1 + t_2 - t_1 = t_2 \end{cases}$$

$$\therefore E[X(t_1) X(t_2)] = 1 \cdot (1 - t_2) + 0 \cdot t_2$$
$$= 1 - t_2$$

$$\therefore C_X(t_1, t_2) = 1 - t_2 - (1 - t_1)(1 - t_2)$$
$$= 1 - t_2 - [1 - t_2 - t_1 + t_1 t_2]$$
$$= -t_1 t_2 + t_1$$

$$\therefore C_X(t_1, t_2) = t_1 - t_1 t_2$$

5. (5 points) Is X wide sense stationary? is X strict sense stationary?

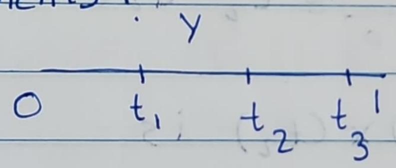
→

As $E[X(t)] = 1-t$ does depend on t

Hence the process is not wide sense stationary, which in term means it is not strict sense stationary.

6. (5 points) What is the distribution of an increment of X? Does X have stationary increments?

→



~~P = 1-t_2~~ When $Y \in [t_2, 1)$ $X(t_2) - X(t_1) = 1 - 1 = 0, P = 1 - t_2$

P= $Y \in [0, t_1]$ $X(t_2) - X(t_1) = 0 - 0 = 0, P = t_1$

$Y \in (t_1, t_2)$ $X(t_2) - X(t_1) = 0 - 1 = -1$ $P = t_2 - t_1$

The distribution of an increment from t_1 to t_2 is

$$\therefore X(t_2) - X(t_1) = \begin{cases} 0 & , P = 1 - t_2 + t_1 \\ -1 & , P = t_2 - t_1 \end{cases}$$

Similarly from t_2 to t_3 is

$$X(t_3) - X(t_2) = \begin{cases} 0 & , P = 1 - t_3 + t_2 \\ -1 & , P = t_3 - t_2 \end{cases}$$

~~As p changes, increments do not follow same distribution. Hence~~

~~x does not have stationary increments.~~

If time steps are equally spaced
i.e.

$$t_2 - t_1 = t_3 - t_2$$

then the ~~pr.~~ increments of x ~~follo~~ are stationary.

~~the~~

Problem 2 (30 points)

Consider the stochastic process

$$X(t) = e^{-Yt}, \quad \text{for } t \geq 0$$

where Y is a random variable with uniform distribution on the interval $(0,1)$.

1. (5 points) Calculate the first-order cumulative distribution function $F(t,x)$ of the process $\{X(t), t > 0\}$ for $x \in (0, +\infty)$

→ CDF of Y

$$F_Y(y) = \begin{cases} 0 & \text{for } y \leq 0 \\ y & 0 < y < 1 \\ 1 & \text{for } y \geq 1 \end{cases}$$

As $Y \in (0,1) \Rightarrow X \in (e^{-t}, 1)$

As e^{-t} is montonically decreasing

$$F_X(x) = P(X \leq x)$$
$$= P(e^{-Yt} \leq x)$$
$$= P(-Yt \leq \ln x)$$
$$= P(Y \geq -\tfrac{1}{t}\ln x)$$
$$= 1 - P(Y < -\tfrac{1}{t}\ln x)$$
$$= 1 - \frac{\left[-\frac{1}{t}\ln x - 0\right]}{1-0}$$

$$F_X(x) = 1 + \frac{1}{t}\ln x$$

$$F(t,x) = F_X(x) = \begin{cases} 0 & \text{for } x \le e^{-t} \\ 1 + \frac{1}{t}\ln x & \text{for } e^{-t} < x < 1 \\ 1 & \text{for } x \ge 1 \end{cases}$$

$, t > 0$

2. (5 points) Deduce the first-order density function of the process $\{X(t), t > 0\}$ on $(0, +\infty)$

→ $f_X(x)$ = first order density function $= F_X'(x)$

$$\therefore f_X(x) = \begin{cases} 0 & \text{for } x \le e^{-t} \\ \frac{1}{xt} & \text{for } e^{-t} < x < 1 \\ 0 & \text{for } x > 1 \end{cases}$$

3. (5 points) Compute $E[X(t)]$, for $t > 0$

→ $E[X(t)] = E[e^{-Yt}]$

$$= \int_0^1 e^{-Yt} \, \frac{1}{} \, dy$$

PDF of uniform distribution bet $(0,1)$

$$E[X(t)] = \left.\frac{e^{-yt}}{-t}\right|_0^1 = \frac{-1}{t}\left[e^{-t} - e^{0}\right] = \frac{-1}{t}(e^{-t} - 1)$$

$\therefore E[X(t)] = \frac{1 - e^{-t}}{t}$

4. (5 points) Compute the second-order cumulative function $F(t_1, t_2, x_1, x_2)$ of the process $\{X(t), t > 0\}$ for $(x_1, x_2) \in (0, +\infty)^2$

As

$f_X(x_1) = \frac{1}{x_1 t_1}$ $\quad e^{-t} < x_1 < 1$

$f_X(x_2) = \frac{1}{x_2 t_2}$ $\quad e^{-t} < x_2 < 1$

$$F(t_1, t_2, x_1, x_2) = \int_{e^{-t_2}}^{x_2} \int_{e^{-t_1}}^{x_1} \frac{1}{x_1 t_1} \frac{1}{x_2 t_2} \, dx_1 \, dx_2$$

$$= \int_{e^{-t_2}}^{x_2} \frac{1}{x_2 t_2} \frac{1}{t_1} \Big[\ln x_1 \Big]_{e^{-t_1}}^{x_1} dx_2$$

$$= \int_{e^{-t_2}}^{x_2} \frac{1}{t_1 t_2} \frac{1}{x_2} \Big[\ln x_1 + t_1 \Big] dx_2$$

$$\therefore F(t_1, t_2, x_1, x_2) = \int_{e^{-t_2}}^{x_2} \frac{1}{x_2 t_1 t_2}\, dx_2 \, (t_1 + \ln x_1)$$

$$= \frac{(t_1 + \ln x_1)}{t_1 t_2} \left[\ln x_2 \right]_{e^{-t_2}}^{x_2}$$

$$F(t_1, t_2, x_1, x_2) = \frac{(t_1 + \ln x_1)(t_2 + \ln x_2)}{t_1 t_2}$$

5. (5 points) Compute the ~~process~~ $\cancel{\{x(t), t \geq 0\}}$ ~~defin~~ autocovariance function $C_x(t, t+s)$ for $s, t > 0$

$$\rightarrow \quad C_x(t, t+s) = E[x(t)\, x(t+s)] - E[x(t)]\, E[x(t+s)]$$

$$x(t) \cdot x(t+s) = e^{-Yt} \cdot e^{-Y(t+s)}$$

$$= e^{-Y(2t+s)}$$

$$\therefore E\left[e^{-Y(2t+s)}\right] = \frac{1 - e^{-(2t+s)}}{2t+s} = E[x(t)\, x(t+s)]$$

$$E[x(t)] = \frac{1 - e^{-t}}{t}, \quad E[x(t+s)] = \frac{1 - e^{-(t+s)}}{t+s}$$

$$\therefore C_x(t, t+s) = \frac{1 - e^{-(2t+s)}}{2t+s} - \frac{(1 - e^{-t})}{t} \frac{(1 - e^{-(t+s)})}{t+s}$$

6. (5 points) Is the process X wide sense stationary, strict sense stationary?

→ As

$E[X(t)] = \frac{1 - e^{-t}}{t}$ depends on t

Hence the process is not wide sense stationary.

Consequently it is not strict sense stationary.

Problem 3 (40 points)

Consider the process $\{X(t), t \geq 0\}$ defined by

$$X(t) = N(t) - \lambda t,$$

where N is a Poisson process with rate $\lambda > 0$

1. (5 points) Compute, for $t_1, t_2, n_1, n_2 > 0$, the second-order probability mass of N, $G(t_1, t_2; n_1, n_2)$

→ As $N(t_1)$ is a poisson process & $N(0) = 0$, $N(t_1) - N(0)$ will follow poisson distribution with mean λt_1

$\therefore P[N(t_1) - N(0)] = P[N(t_1)] = \frac{e^{-\lambda t_1} (\lambda t_1)^{n_1}}{n_1 !}$

$\forall n_1 = 0, 1, \ldots$

Similarly

$P[N(t_2)] = \frac{e^{-\lambda t_2} (\lambda t_2)^{n_2}}{n_2 !}, \forall n_2 = 0, 1, \ldots$

$\therefore$ Second order probability mass of N,

$$G(t_1, t_2; n_1, n_2) = \sum_{n_2=0}^{n_2} \sum_{n_1=0}^{n_1} \frac{e^{-\lambda t_1} (\lambda t_1)^{n_1}}{n_1!} \frac{e^{-\lambda t_2} (\lambda t_2)^{n_2}}{n_2!}$$

$\therefore$ [struck-out equation]

$$G(t_1, t_2, n_1, n_2) = \sum_{n_2=0}^{n_2} \sum_{n_1=0}^{n_1} \frac{e^{-\lambda(t_1+t_2)} \lambda^{n_1+n_2} \cdot t_1^{n_1} \cdot t_2^{n_2}}{n_1! \, n_2!}$$

2. (5 points) Give the expectation and the variance of $X(t)$.

a)

$E[X(t)] = E[N(t) - \lambda t]$

$\therefore E[X(t)] = E[N(t)] - \lambda t$
$= E[N(t) - N(0)] - \lambda t$

As $N(t)$ ~~fo~~ is a poisson process
$E[N(t) - N(0)] = \lambda t$

$\therefore E[X(t)] = \lambda t - \lambda t = 0$

$\therefore E[X(t)] = 0$

b) $Var[X(t)] = Var[N(t) - \lambda t]$
$= Var[N(t)]$ as λt is deterministic

As $Var[N(t) - N(0)] = \lambda t$

$\therefore Var[X(t)] = Var[N(t) - N(0)] = \lambda t$

$\therefore Var[X(t)] = \lambda t$

3. (5 points) Compute $E[X(t_1)X(t_2)]$, for $0 < t_1 \leq t_2$

$$X(t_1)X(t_2) = [N(t_1) - \lambda t_1][N(t_2) - \lambda t_2]$$
$$= N(t_1)N(t_2) - \lambda t_2 N(t_1) - \lambda t_1 N(t_2) + \lambda^2 t_1 t_2$$

As $N(0)=0$

$$E[N(t_1)N(t_2)] = E[(N(t_1)-N(0))(N(t_2)-N(t_1)+N(t_1))]$$

$$= E[(N(t_1)-N(0))(N(t_2)-N(t_1))] + E[(N(t_1)-N(0))(N(t_1))]$$

As $N(t_1)-N(0)$ & $N(t_2)-N(t_1)$ are non overlapping ~~inter~~ increments, by definition they are independent.

$$\therefore E[(N(t_1)-N(0))(N(t_2)-N(t_1))] = E[N(t_1)-N(0)]\,E[N(t_2)-N(t_1)] \quad -(1)$$

Also $E[(N(t_1)-\overset{0}{N(0)})N(t_1)] = E[N^2(t_1)]$

As $\mathrm{var}[N(t_1)] = E[N^2(t_1)] - (E[N(t_1)])^2$

$$\therefore \lambda t_1 = E[N^2(t_1)] - (\lambda t_1)^2$$

$$\therefore E[N^2(t_1)] = \lambda t_1(1+\lambda t_1) \quad -(2)$$

from 1 & 2 & $E[N(t_2)-N(t_1)] = \lambda(t_2-t_1)$ & $E[N(t_1)-N(0)] = \lambda t_1$

$$E[N(t_1)N(t_2)] = E[N(t_1)-N(0)]\,E[N(t_2)-N(t_1)] + E[N^2(t_1)]$$

$$\therefore E[N(t_1)N(t_2)] = \lambda t_1 \cdot \lambda(t_2-t_1) + \lambda t_1(1+\lambda t_1)$$

$$= \lambda t_1[\lambda t_2 - \lambda t_1 + 1 + \lambda t_1]$$

$\therefore E[N(t_1)N(t_2)] = \lambda t_1 (1+\lambda t_2)$

As $E[X(t_1)X(t_2)] = E[N(t_1)N(t_2)]$
$- \lambda t_2 E[N(t_1)] - \lambda t_1 E[N(t_2)]$
$+ \lambda^2 t_1 t_2$

$\therefore E[X(t_1)X(t_2)] = \lambda t_1 (1+\lambda t_2)$
$- \lambda t_2 \lambda t_1 - \lambda t_1 \lambda t_2$
$+ \lambda^2 t_1 t_2$

$= \lambda t_1 + \lambda^2 t_1 t_2 - \lambda^2 t_1 t_2$
$- \lambda^2 t_1 t_2 + \lambda^2 t_1 t_2$

$\therefore E[X(t_1)X(t_2)] = \lambda t_1$

4. (5 points) Is X wide sense stationary? is X strict sense stationary?

AS $E[X(t)] = 0$ it does not depend on time

$R_X(t, t+h) = E[X(t)X(t+h)] = \lambda t$

As R_X depends on t X is not wide sense stationary. Which in term means X is not strict sense

Stationary either.

5. (5 points) is X a martingale? Justify your answer.

→ As $\cancel{X(0) - N(0) - \lambda \cdot 0 = 0 - 0 = 0}$

$\cancel{E[X(t) \mid X(0) = 0] = \sum x \cdot P[X(t) = x \mid X(0) = 0]}$

As $x(t) -$

$\cancel{= \sum x(t) \cdot P\{x(t) = x\}}$

Suppose $t_1 < t_2$

$$X(t_2) - X(t_1) = N(t_2) - N(t_1) - \lambda(t_2 - t_1)$$

& $\quad X(t_1) = N(t_1) - \lambda t_1$

$$E[X(t_2) \mid X(t_1)] = E[N(t_2) - \lambda t_2 \mid N(t_1) - \lambda t_1]$$

$$= E[N(t_2) - N(t_1) + N(t_1) - \lambda t_2 \mid N(t_1) - \lambda t_1]$$

$$= E[N(t_2) - N(t_1) \mid N(t_1) - \lambda t_1]$$
$$+ E[N(t_1) - \lambda t_2 \mid N(t_1) - \lambda t_1]$$

We can drop λt_1 in the condition as it is deterministic.

$\therefore E[X(t_2)|X(t_1)] = E[N(t_2)-N(t_1)|N(t_1)-\overset{0}{\overset{\shortparallel}{N(0)}}]$
$+ E[N(t_1)-\lambda t_2 | N(t_1)]$

by def^n of poisson process $N(t_2)-N(t_1)$ is independent of $N(t_1)-N(0)$

$\therefore E[X(t_2)|X(t_1)] = E[N(t_2)-N(t_1)]$
$+ N(t_1) - \lambda t_2$

$= E[N(t_2)] - E[N(t_1)] + N(t_1) - \lambda t_2$

$E[X(t_2)|X(t_1)] = \lambda t_2 - \lambda t_1 + N(t_1) - \lambda t_2$
$= N(t_1) - \lambda t_1$

but $X(t_1) = N(t_1) - \lambda t_1$

$\therefore E[X(t_2)|X(t_1)] = X(t_1)$

Hence X is a martingale.

6. (15 points) Is $t \to \frac{X(t)}{t}$ mean ergodic? Justify your answer rigorously.

$\to$ $Y = \frac{X(t)}{t} = \frac{N(t)}{t} - \lambda$

$$\langle Y \rangle_0 = \frac{1}{M+1} \sum_{n=0}^{M} Y$$

$$E[\langle Y \rangle_0] = \frac{1}{M+1} \sum_{n=0}^{M} E[Y]$$

$$= \frac{1}{M+1} \sum_{n=0}^{M} E\left[\frac{N(t)}{t} - \lambda\right]$$

$$E\left\{\frac{N(t)}{t} - \lambda\right] = \frac{1}{t} E[N(t)] - \lambda$$

$$= \frac{\lambda t}{t} - \lambda = \lambda - \lambda = 0$$

$$\therefore E[\langle Y \rangle_0] = \frac{1}{M+1} \sum_{n=0}^{M} 0 = 0$$

but the process Y has mean

$$E[Y] = E\left[\frac{X(t)}{t}\right] = \frac{1}{t} E[X(t)] = 0$$

$$\therefore E[\langle Y \rangle_0] = E[Y] = 0$$

Hence temporal mean converges to mean of the process in mean squares.

therefore the process is mean ergodic.