Quantitative Methods, Assignment Week 5

Problem 1.

We consider the 3-period model seen in the class with the parameters

$S_0 = 4,\ u = 2,\ d = 1/2,\ r = 1/5$

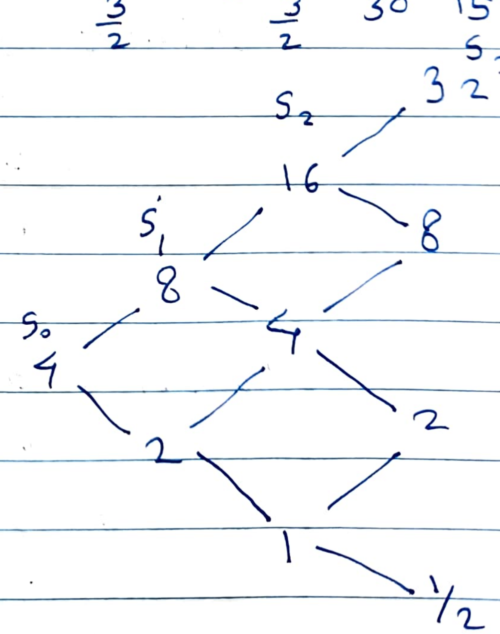
The price of the underlying asset is denoted by S_n for $n = 0, 1, 2, 3$.

1. (10 points) Compute the price V_0, at time 0, of a European put option on the underlying asset S_n with payoff $V_3 = (4 - S_3)^+$ at time 3.

→ The risk neutral probabilities are

$$\tilde{p} = \frac{1+r-d}{u-d} = \frac{1+\frac{1}{5}-\frac{1}{2}}{2-\frac{1}{2}} = \frac{1+\left(-\frac{3}{10}\right)}{\frac{3}{2}} = \frac{\frac{7}{10}}{\frac{3}{2}} = \frac{14}{30} = \frac{7}{15}$$

$$\therefore 1 - \tilde{p} = \frac{8}{15}$$



$V_3(HHH) = (4-32)^+ = 0$

$V_3(HHT) = (4-8)^+ = 0$

$V_3(HTH) = (4-8)^+ = 0$

$V_3(HTT) = 4 - 2 = 2$

$V_3(TTH) = 4 - 2 = 2$

$V_3(TTT) = 4 - \frac{1}{2} = \frac{7}{2}$

$$V_3(THH) = (4-8)^+ = 0$$
$$V_3(THT) = 4-2 = 2$$

$$V_2(HH) = \frac{1}{1+r}\left[\tilde{p}\, V_3(HHH) + (1-\tilde{p})\, V_3(HHT)\right]$$

$$= \frac{1}{1+\frac{1}{5}}\left[\frac{7}{15}\,0 + \frac{8}{15}\,0\right] = 0$$

$$V_2(HT) = \frac{5}{6}\left[\frac{7}{15} V_3(HTH) + \frac{8}{15} V_3(HTT)\right]$$
$$= \frac{5}{6}\left[\frac{7}{15}\cdot 0 + \frac{8}{15}\,2\right]$$
$$= \frac{5}{6} \times \frac{16}{15} = \frac{8}{3\times 3} = \frac{8}{9}$$

$$V_2(TH) = \frac{5}{6}\left[\frac{7}{15} V_3(THH) + \frac{8}{15} V_3(THT)\right]$$
$$= \frac{5}{6}\left[\frac{7}{15}\cdot 0 + \frac{8}{15}\cdot 2\right] = \frac{8}{9}$$

$$V_2(TT) = \frac{5}{6}\left[\frac{7}{15} V_3(TTH) + \frac{8}{15} V_3(TTT)\right]$$
$$= \frac{5}{6}\left[\frac{7}{15}\cdot 2 + \frac{8}{15}\cdot\frac{7}{2}\right]$$
$$= \frac{5}{6}\left[\frac{28+56}{30}\right] = \frac{84}{36} = \frac{7}{3}$$

$$V_1(H) = \frac{5}{6}\left[\frac{7}{15} V_2(HH) + \frac{8}{15} V_2(HT)\right]$$

$$= \frac{5}{6}\left[\frac{7}{15}\cdot 0 + \frac{8}{15}\cdot\frac{8}{9}\right]$$

$$= \frac{\cancel{5} \times 8 \times 8}{6 \times \cancel{15}_{3} \times 9} = \frac{64}{18 \times 9} = \frac{64}{162} = \frac{32}{81}$$

$$V_1(T) = \frac{5}{6}\left[\frac{7}{15} V_2(TH) + \frac{8}{15} V_2(TT)\right]$$

$$= \frac{5}{6}\left[\frac{7}{15}\cdot\frac{8}{9} + \frac{8}{15}\cdot\frac{7}{3}\right]$$

$$= \frac{5}{6}\left[\frac{7 \times 8 + 8 \times 7 \times 3}{15 \times 9}\right]$$

$$= \frac{1}{6}\left[\frac{56 + 168}{3 \times 9}\right] = \frac{224}{27 \times 6} = \frac{112}{81}$$

$$V_0 = \frac{5}{6}\left[\frac{7}{15} V_1(H) + \frac{8}{15} V_1(T)\right]$$

$$= \frac{5}{6}\left[\frac{7}{15}\cdot\frac{32}{81} + \frac{8}{15}\cdot\frac{112}{81}\right]$$

$$= \frac{1}{6}\left[\frac{224 + 896}{3 \times 81}\right] = \frac{1120}{1458} = \frac{560}{729}$$

$$\therefore V_0 = 0.7682$$

2. (5 points) Give also the hedging strategy Δ_0 at time 0.

$$\Delta_0 = \frac{V_1(H) - V_1(T)}{S_1(H) - S_1(T)} = \frac{\frac{32}{81} - \frac{112}{81}}{8 - 2}$$

$$\therefore \Delta_0 = \frac{-80}{81 \times 6} = \frac{-40}{81 \times 3} = \frac{-40}{243}$$

$\therefore$ Hedging stratedy Δ_0 at time 0 $= -0.16461$

3. (10 points) Compute now the price of a lookback call option with fixed strike. Its payoff is given by

$$V_3 = (\max S_n - 4)^+$$

$V_3(HHH) = 32 - 4 = 28$ $\quad \tilde{p} = \frac{7}{15}$

$V_3(HHT) = 16 - 4 = 12$

$V_3(HTH) = 8 - 4 = 4$ $\quad 1 - \tilde{p} = \frac{8}{15}$

$V_3(HTT) = 8 - 4 = 4$

$V_3(TTH) = 4 - 4 = 0$

$V_3(TTT) = 4 - 4 = 0$

$V_3(THH) = 8 - 4 = 4$

$V_3(THT) = 4 - 4 = 0$

$$V_2(HH) = \frac{5}{6}\left[\frac{7}{15} \cdot V_3(HHH) + \frac{8}{15} V_3(HHT)\right]$$

$$= \frac{5}{6}\left[\frac{7}{15} \cdot 28 + \frac{8}{15} \cdot 12\right]$$

$$= \frac{5}{6} \cdot \frac{292}{15} = \frac{292}{6 \times 3} = \frac{146}{9}$$

$$V_2(HT) = \frac{5}{6}\left[\frac{7}{15} V_3(HTH) + \frac{8}{15} V_3(HTT)\right]$$

$$= \frac{5}{6}\left[\frac{7}{15} \cdot 4 + \frac{8}{15} \cdot 4\right]$$

$$= \frac{5}{6} \cdot 4 = \frac{10}{3}$$

$$V_2(TH) = \frac{5}{6}\left[\frac{7}{15} V_3(THH) + \frac{8}{15} V_3(THT)\right]$$

$$= \frac{5}{6}\left[\frac{7}{15} \cdot 4 + \frac{8}{15} \cdot 0\right]$$

$$= \frac{28}{6 \times 3} = \frac{14}{9}$$

$$V_2(TT) = \frac{5}{6}\left[\frac{7}{15} V_3(TTH) + \frac{8}{15} V_3(TTT)\right]$$

$$= \frac{5}{6}\left[\frac{7}{15} \cdot 0 + \frac{8}{15} \cdot 0\right] = 0$$

$$V_1(H) = \frac{5}{6}\left[\frac{7}{15} V_2(HH) + \frac{8}{15} V_2(HT)\right]$$

$$= \frac{5}{6}\left[\frac{7}{15} \cdot \frac{146}{9} + \frac{8}{15} \cdot \frac{10}{3}\right]$$

$$= \frac{5}{6}\left[\frac{1022 + 240}{15 \times 9}\right]$$

$$= \frac{1262}{6 \times 3 \times 9} = \frac{631}{3 \times 3 \times 9} = \frac{631}{81}$$

$$V_1(T) = \frac{5}{6}\left[\frac{7}{15} \cdot V_2(TH) + \frac{8}{15} V_2(TT)\right]$$

$$= \frac{5}{6}\left[\frac{7}{15} \cdot \frac{14}{9} + \frac{8}{15} \cdot 0\right]$$

$$= \frac{5}{6} \cdot \frac{7}{15} \cdot \frac{14}{9} = \frac{98}{6 \times 3 \times 9} = \frac{49}{3 \times 3 \times 9} = \frac{49}{81}$$

$$V_0 = \frac{5}{6}\left[\frac{7}{15} V_1(H) + \frac{8}{15} V_1(T)\right]$$

$$= \frac{5}{6}\left[\frac{7}{15} \cdot \frac{631}{81} + \frac{8}{15} \cdot \frac{49}{81}\right]$$

$$= \frac{1}{6}\left[\frac{4417 + 392}{3 \times 81}\right] = \frac{1 \times 4809}{6 \times 243}$$

$$V_0 = \frac{4809}{1458} = \frac{1603}{486} = 3.2984$$

$\therefore$ Price of the lookback call option with fixed strike $= V_0 = 3.2984$

4. (5 points) Give the hedging strategy Δ_0 at the time 0, for the lookback Option

$$\Delta_0 = \frac{V_1(H) - V_1(T)}{S_1(H) - S_1(T)} = \frac{\frac{631}{81} - \frac{49}{81}}{8-2} = \frac{582}{81 \times 6}$$

$$\therefore \Delta_0 = \frac{194}{81 \times 2} = \frac{194}{162} = 1.1975$$

$\therefore$ Hedging Strategy Δ_0 at time $0 = 1.1975$

Problem 2 (15 points)

Consider the one period binomial tree model seen in class with $d = 0.5$, $u = 1.2$, $r = 0.5$. Is there an arbitrage opportunity in this model? Can you exhibit such a strategy.

$$\begin{array}{ccccc} 0.5 & & 1.2 & & 1.5 \\ d & < & u & < & 1+r \end{array}$$

Here ~~$d = r < u$~~

~~i.e.~~ ~~$r < d < u$~~ is true.

Therefore there is an arbitrage opportunity present.

An investor, with initial wealth $X_0 = 0$ could sell the stock short and invest it in money market. At time 1, His wealth would be either

$$(1+r)S_0 - S_0 d > 0$$

$$(1+r)S_0 - S_0 u \geq 0$$

where

$S_0 =$ price of stock at time 0

$S_0 d =$ price of stock at time 1 (if T occurs)

$S_0 u =$ price of stock at time 1 (if H occurs)

in our case

$(1+r)S_0 - S_0 d = 1.5 S_0 - 0.5 S_0 = S_0 > 0$

$(1+r)S_0 - S_0 u = 1.5 S_0 - 1.2 S_0 = 0.3 S_0 > 0$

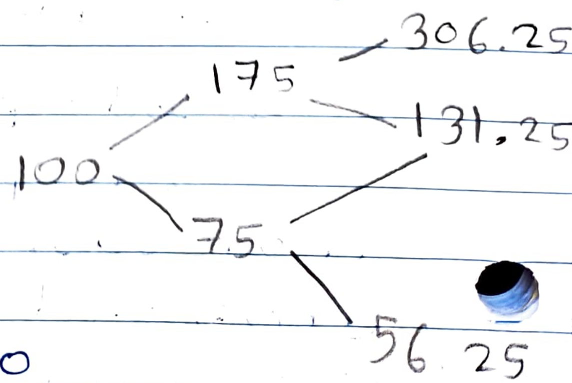
Hence an investor with 0 initial wealth can generate profit without taking any risk $\Rightarrow$ Arbitrage exists.

Problem 3 (55 points)

Consider a two period binomial tree model with both periods of length one year. Suppose that the initial stock price is $S_0 = 100$ US dollars and that, as in the lecture notes, the stock pays no dividend. The simple interest rate is $r = 0.05$ per year; the down factor is $d = 0.75$, and the up factor is $u = 1.75$

1. (10 points) Compute the price V_0, at time 0, of the European call option with strike $K = 100$ and maturity date of two years.

→ as $V_2 = (S_2 - 100)^+$



$V_2(HH) = 206.25$

$V_2(HT) = 31.25$

$V_2(TH) = 31.25$

$V_2(TT) = (56.25 - 100)^+ = 0$

$$\tilde{p} = \frac{1+r-d}{u-d} = \frac{1+0.05-0.75}{1.75-0.75}$$

$$\therefore \tilde{p} = \frac{0.3}{1} \qquad \therefore 1-\tilde{p} = 0.7$$

$$\therefore V_1(H) = \frac{1}{1+0.05}\left[0.3.\,V_2(HH) + 0.7\,V_2(HT)\right]$$

$$= \frac{1}{1.05}\left[0.3 \times 206.25 + 0.7 \times 31.25\right]$$

$$= \frac{1}{1.05}\left[61.875 + 21.875\right]$$

$$V_1(H) = \frac{83.75}{1.05}$$

$$V_1(T) = \frac{1}{1.05}\left[0.3\,V_2(TH) + 0.7\,V_2(TT)\right]$$

$$= \frac{1}{1.05}\left[0.3 \times 31.25 + 0.7 \cdot 0\right]$$

$$= \frac{9.375}{1.05}$$

$$V_0 = \frac{1}{1.05}\left[0.3.\,V_1(H) + 0.7\,V_1(T)\right]$$

$$= \frac{1}{1.05}\left[0.3 \times \frac{83.75}{1.05} + 0.7 \times \frac{9.375}{1.05}\right]$$

$$\therefore V_0 = \frac{1}{1.05 \times 1.05}\left[25.125 + 6.5625\right] = \frac{31.6875}{1.1025}$$

$$\therefore V_0 = \frac{31.6875}{1.1025} = 28.7415$$

2. (10 points) Compute the price V_0, at time 0, of the European put option with strike $K=100$ and maturity date of two years.

$$V_2 = (100 - S_2)^+$$

$$V_2(HH) = (100 - 306.25)^+ = 0$$

$$V_2(HT) = (100 - 131.25)^+ = 0$$

$$V_2(TH) = (100 - 131.25)^+ = 0$$

$$V_2(TT) = (100 - 56.25) = 43.75$$

$$V_1(H) = \frac{1}{1.05}\left[0.3 \cdot V_2(HH) + 0.7\, V_2(HT)\right]$$

$$= \frac{1}{1.05}\left[0.3 \cdot 0 + 0.7 \cdot 0\right] = 0$$

$$V_1(T) = \frac{1}{1.05}\left[0.3\, V_2(TH) + 0.7\, V_2(TT)\right]$$

$$= \frac{1}{1.05}\left[0.3 \times 0 + 0.7 \times 43.75\right]$$

$$= \frac{30.625}{1.05}$$

$$V_0 = \frac{1}{1.05}\left[0.3 \cdot V_1(H) + 0.7 \cdot V_1(T)\right]$$

$$= \frac{1}{1.05}\left[0.3 \times 0 + 0.7 \times \frac{30.625}{1.05}\right]$$

$$V_0 = \frac{1}{1.05}\left[0.3 \times 0 + \frac{21.4375}{1.05}\right]$$

$$\therefore V_0 = \frac{21.4375}{1.1025} = 19.\overline{44}$$

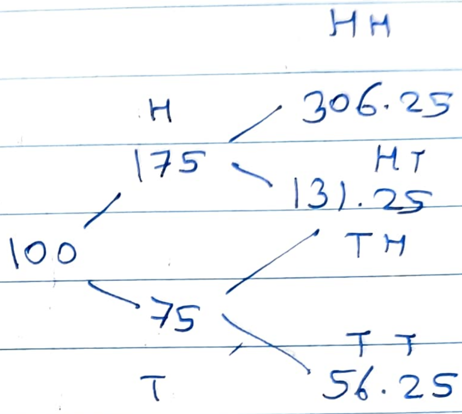
3. (35 points) How can you compute the fair price at time 0 of a chooser option with a maturity date of two years, which gives you the right to choose whether the option is a put or a call after one year? Present an algorithm that leads to the fair price.

→ We will consider previous example as illustration.

Payoff of Call : $(S_2 - 100)^+$

Payoff of Put : $(100 - S_2)^+$

Suppose after one year, H occurs i.e. the price of asset is 175.
Here payoff from call is greater than Put, Hence we choose to convert to call option.



$$\therefore V_2(HH) = (306.25 - 100)^+$$
$$= 206.25$$

$$V_2(HT) = (131.25 - 100)^+$$
$$= 31.25$$

$$\therefore V_1(H) = \frac{1}{1.05}\left[0.3\, V_2(HH) + 0.7\, V_2(HT)\right]$$
$$= \frac{1}{1.05}\left[0.3 \times 206.25 + 0.7 \times 31.25\right]$$
$$= \frac{1}{1.05}\left[61.875 + 21.875\right]$$
$$V_1(H) = \frac{83.75}{1.05}$$

Similarly if after one year, T occurs i.e. price of asset is 75.
Here Payoff from Put is greater than Call option, Hence we choose to convert to put.

$$\therefore V_2'(TH) = (100 - 131.25)^+ = 0$$

$$V_2'(TT) = (100 - 56.25)^+ = 43.75$$

$$\therefore V_1'(T) = \frac{1}{1.05}\left[0.3\, V_2'(TH) + 0.7\, V_2'(TT)\right]$$

$$= \frac{1}{1.05}\left[0.3 \times 0 + 0.7 \times 43.75\right]$$

$$V_1'(T) = \frac{30.625}{1.05}$$

$$\therefore V_0 = \frac{1}{1.05}\left[0.3\, V_1(H) + 0.7\, V_1'(T)\right]$$

$$= \frac{1}{1.05}\left[0.3 \times \frac{83.75}{1.05} + 0.7 \times \frac{30.625}{1.05}\right]$$

$$= \frac{1}{1.05}\left[\frac{25.125 + 21.4375}{1.05}\right]$$

$$V_0 = \frac{46.5625}{1.1025}$$

$$\therefore V_0 = 42.2336$$

As we can see the value of chooser option is higher than similar call or put option which is intuitively correct.

Algorithm for valuing chooser option :

1) Based on the outcome after 1 year, (either $u.S_0$ happens or $d.S_0$ happens), choose between put option or call option. Choose to convert to option with higher payoff / value.

i.e. in our case.

find $V_1(H)$ using $V_2(HH)$ & $V_2(HT)$ with payoff given by call option payoff.

and

find $V_1'(T)$ using $V_2'(TH)$ & $V_2'(TH)$ with payoff given by put option payoff.

2) Now use $V_1(H)$ & $V_1'(T)$ to calculate V_0 normally as we do in pricing normal options.