Quantitative methods
Assignment Week 7

Problem 1 (15 points)

Consider an investor who is holding one share of a stock whose price is evolving according to a standard Brownian motion process, ie.

$$S(u) = S(0) + \sigma W(u), \; u \geq 0$$

where $\sigma > 0$ is the volatility coefficient. This investor purchased the stock at a price $S(0) > 0$ at time 0 and decides to sell the stock if it reaches the price $S(0)$ ~~> 0 at time~~ $(1+\Delta)$ where $\Delta > 0$.

1. (5 points) What is the cumulative distribution function of the hitting time $\tau_{S(0)(1+\Delta)}$?

→ as

$$\tau_{S(0)(1+\Delta)} = \inf \{ u \geq 0, \; S(u) \geq S(0)(1+\Delta) \}$$

$$P[S(u) \geq S(0)(1+\Delta)] =$$

$$P[S(u) \geq S(0)(1+\Delta) \mid \tau_{S(0)(1+\Delta)} \leq u] \, P[\tau_{S(0)(1+\Delta)} \leq u]$$

$$+ P[S(u) \geq S(0)(1+\Delta) \mid \tau_{S(0)(1+\Delta)} > u] \, P[\tau_{S(0)(1+\Delta)} > u]$$

Second term is necessarily zero as $S(u)$ cannot be above $S(0)(1+\Delta)$ if it has not hit $S(0)(1+\Delta)$ yet by time u $(\tau_{S(0)(1+\Delta)} > u)$.

let $S(0)(1+\theta) = a$

$\therefore \tau_{S(0)(1+\theta)} = \tau_a$

The above eq^n becomes.

$$p[S(u) \geq a] = p[S(u) \geq a \mid \tau_a \leq u]\, p[\tau_a \leq u]$$

as reflection principle gives.

$$p[S(u) \geq a \mid \tau_a \leq u] = \frac{1}{2}$$

$$\therefore p[S(u) \geq a] = \frac{1}{2}\, p[\tau_a \leq u]$$

$$\therefore p[\tau_a \leq u] = 2\, p[S(u) \geq a]$$

$$= 2p[S(0) + \sigma W(u) \geq a], \sigma > 0$$

$$= 2\, p\left[W(u) \geq \frac{a - S(0)}{\sigma}\right]$$

let $b = \dfrac{a - S(0)}{\sigma}$ — (1)

$$\therefore p[\tau_a \leq u] = 2\, p[W(u) \geq b] \quad \text{— (2)}$$

from notes $p[W(u) \geq b] = 1 - \phi\left(\dfrac{|b|}{\sqrt{u}}\right)$

Where ϕ is the standard normal cumulative distribution function

$$\therefore\ P[T_a \le u] = 2\left[1 - \phi\left(\frac{|b|}{\sqrt{u}}\right)\right]$$

substituting value of b from eq^n (1)

$$P[T_a \le u] = 2\left(1 - \phi\left(\frac{|a - S(0)|}{\sqrt{u}\,\sigma}\right)\right)$$

~~2. (5 points) Give also the density of the distribution of the hitting time $T_{S(0)(1+\Delta)}$.~~

as $a - S(0) = S(0)(1+\Delta) - S(0) = S(0).\Delta$

$\therefore |a - S(0)| = |S(0).\Delta|$ as both $S(0) > 0$ & $\Delta > 0$

$\therefore |a - S(0)| = S(0).\Delta$

$$\therefore\ P[T_a \le u] = 2\left(1 - \phi\left(\frac{S(0).\Delta}{\sqrt{u}\,\sigma}\right)\right)$$

2. (5 points) Give also the density of the distribution of the hitting time $\tau_{S(0)(1+\Delta)}$.

→ As C.D.F. of hitting time

$$F_{\tau_{S(0)(1+\Delta)}}(u) = 2\left(1 - \phi\left(\frac{S(0).\Delta}{\sqrt{u}\,\sigma}\right)\right)$$

∴ Correspond P.D.F is

$$f_{\tau_{S(0)(1+\Delta)}}(u) = F'(u)$$

$$= \phi'\left(\frac{S(0).\Delta}{\sqrt{u}\,\sigma}\right) . u^{-\frac{3}{2}} \times \frac{S(0).\Delta}{\sigma}$$

$$\therefore f_{\tau(S(0)(1+\Delta))}(u) = \frac{\Delta S(0)}{\sigma} . \phi'\left(\frac{\Delta . S(0)}{\sigma\sqrt{u}}\right) . u^{-\frac{3}{2}}$$

3. (5 points) What is the distribution of the hitting time $\tau_{S(0)(1-\delta)}$, i.e. of the first time at which the asset price falls below $S(0)(1-\delta)$ where $0 < \delta < 1$?

let $a = S(0)(1-\delta)$, $0 < \delta < 1$)

As in previous examples.

$$P[S(u) \le a] = P[S(u) < a \mid T_a \le u] P[T_a \le u]$$
$$+ P[S(u) \le a \mid T_a > u] P[T_a > u]$$

second term being 0 as $S(u)$ cannot be $\le a$ with first hitting a.

$$\therefore P[S(u) \le a] = P[S(u) \le a \mid T_a \le u] P[T_a \le u]$$

by reflection principle.

~~$P[S(u) \le a]$~~ =

$$P[S(u) \le a \mid T_a \le u] = \frac{1}{2}$$

$$\therefore P[T_a \le u] = 2P[S(u) \le a]$$

as $S(u) \sim N(S(0), \sigma^2 u)$

$$I = \therefore P[T_a \le u] = \frac{2}{\sigma\sqrt{2\pi u}} \int_{-\infty}^{a} e^{-\frac{(x - S(0))^2}{2 \cdot \sigma^2 \cdot u}} dx$$

let $y = \frac{x - S(0)}{\sigma\sqrt{u}}$ when $x \to -\infty, y \to -\infty$

$x \to a, y \to \frac{a - S(0)}{\sigma\sqrt{u}} = b$

$\therefore dy = \frac{dx}{\sigma\sqrt{u}}$

$$\therefore I = \frac{2}{\sqrt{2\pi}} \int_{-\infty}^{b} e^{-y^2/2} \, dy$$

$\therefore I = 2\,\phi(b)$

$$\therefore I = 2\,\phi\left(\frac{a - S(0)}{\sigma\sqrt{u}}\right)$$

where ϕ is standard normal cumulative distribution function.

as $a - S(0) = S(0)(1-\delta) - S(0) = -S(0).\delta$

$$\therefore I = 2\,\phi\left(\frac{-\delta.S(0)}{\sigma\sqrt{u}}\right)$$

$$\therefore P[\tau_{S(0)(1-\delta)} \leq u] = 2\,\phi\left(\frac{-\delta.S(0)}{\sigma\sqrt{u}}\right)$$

which is the cumulative distribution function of $\tau_{S(0)(1-\delta)}$

Problem 2 (20 points)

Consider a brownian motion W. ~~Som~~ Show that the following process is also a standard Brownian motion by using the definition in the lecture notes

$$B(t) = c\, W(t/c^2), \text{ for all } t \geq 0.$$

1) $B(0) = c\, W(0/c^2)$

$= c\,W(0)$

as $W(0) = 0$

$\therefore \quad B(0) = 0 \qquad - (1)$

2) As $W(t)$ is continuous in variable t, $cW(t/c^2)$ is also continuous in variable t.

$\therefore$ $B(t)$ is continuous in variable t. $- (2)$

3)
$$P[B(t_{j+1}) - B(t_j) \mid B(t_j) - B(t_{j-1})]$$

$$= P\left[c\,W\left(\frac{t_{j+1}}{c^2}\right) - c\,W\left(\frac{t_j}{c^2}\right) \middle| \cancel{B}\, cW\left(\frac{t_j}{c^2}\right) - cW\left(\frac{t_{j-1}}{c^2}\right)\right]$$

As c is a constant $\rightarrow$ fixed $\mapsto$ it is a deterministic quantity.

Hence we can drop it from inside of the Probability

$$\therefore P[B(t_{j+1}) - B(t_j) \mid B(t_j) - B(t_{j-1})]$$

$$= P[W(t_{j+1}) - W(t_j) \mid W(t_j) - W(t_{j-1})]$$

but as W has independent increments, we can drop the condition

$$\therefore RHS = P[W(t_{j+1}) - W(t_j)]$$

We can reintroduce the constant c by similar reasoning

$$\therefore RHS = P\left[cW\left(\frac{t_{j+1}}{c^2}\right) - cW\left(\frac{t_j}{c^2}\right)\right]$$

$$\therefore P[B(t_{j+1}) - B(t_j) \mid B(t_j) - B(t_{j-1})]$$

$$= P[B(t_{j+1}) - B(t_j)]$$

Hence $B(t)$ also has independent increments for all $0 = t_0 < t_1 < t_2 < \ldots < t_n$ — (3)

as $B(t_{j+1}) - B(t_j) = c\left[W\left(\frac{t_{j+1}}{c^2}\right) - W\left(\frac{t_j}{c^2}\right) \right]$

as ~~let~~ $W(t) \sim N(0, t)$

$\therefore \quad cW\left(\frac{t}{c^2}\right) \sim N\left(0, \frac{t}{c^2}\right)$

as $B(t_{j+1}) - B(t_j)$ is linear combination of normal variables it is also normal with mean 0 & variance $\frac{t_{j+1} - t_j}{c^2}$

Hence $B(t)$ has stationary increments ~~(a)~~

& ~~var~~ $B(t_{j+1}) - B(t_j)$ is normally distributed with mean 0 &

variance $\frac{t_{j+1} - t_j}{c^2}$ for all $j = 0, \dots, n-1$

$\to (4)$

from equations 1, 2, 3 and 4 $B(t)$ satisfies all the conditions in the definition of a standard brownian motion.

Hence the proce $B(t) = cW(t/c^2)$ for all $t > 0$ is also a standard brownian motion.

Problem 3 (15 points)

Consider a standard Brownian motion W. Is the process $t \to W(ct^2)$, where c is a positive constant, a standard Brownian motion? Justify your answer.

→ 1) as for $W(ct^2)$ at $ct^2 = 0$

$W(0) = 0$

2) As for all

$0 = t_0 < t_1 < t_2 < \dots < t_n$,

$W(t_1) - W(t_0)$, $W(t_2) - W(t_1)$ ... , $W(t_n) - W(t_{n-1})$ are independent
[As no t_k's overlap]
Similarly for all
$0 = ct_0^2 < ct_1^2 < ct_2^2 < \dots < ct_k^2$

$W(ct_1^2) - W(ct_0^2)$, $W(ct_2^2) - W(ct_1^2)$, ... $W(ct_n^2) - W(ct_{n-1}^2)$
are independent as

$ct_{k+1}^2 - ct_k^2$ & $ct_k^2 - ct_{k-1}^2$ do not overlap.

as $W(t)$ is normally distributed with mean 0 & variance 1.

$$P[W(t) = x] = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$$

Similarly, $W(ct_k^2)$ will also be normally distributed but $W(ct_k^2)$ will not have variance of 1.
The variance will be $c^2 t_k^2$.
Hence $W(ct_k^2)$ will not be a standard brownian motion.

~~but~~ it will not be ~~a~~ brownian motion also as it ~~has~~ does not have stationary increments as

$W(ct_1^2) - W(ct_0^2)$, $W(ct_2^2) - W(ct_1^2)$,
$\vdots$ $W(ct_n^2) - W(ct_{n-1}^2)$ ~~will~~
~~follow a normal distribution with~~
~~mean 0 & variance~~

As $W(t_1) - W(t_0), W(t_2) - W(t_1), \dots$ ~~where k = 1, 2, ... n~~ are stationary for all

$$t_0 = 0 < \overset{t}{t_1} < \overset{2t}{t_2} < \overset{3t}{t_3} < \dots \overset{nt}{t_n},$$

$$\therefore t_0^2 = 0 < \overset{t^2}{t_1^2} < \overset{4t^2}{t_2^2} < \overset{9t^2}{t_3^2} < \dots < \overset{n^2t^2}{t_n^2},$$

as $W(ct_1^2) - W(ct_0^2) \sim N(0, c^2(t^2 - 0)) = N(0, c^2t^2)$
$W(ct_2^2) - W(ct_1^2) \sim N(0, c^2(4t^2 - t^2)) = N(0, 3c^2t^2)$
$\vdots$
$W(ct_{n+1}^2) - W(ct_n^2) \sim N(0, c^2((n+1)^2 - n^2)t^2)$
$= N(0, c^2(2n+1)t^2)$

this shows that even though the increments are still normally distributed, the variance of increments is not the same for all increments.
This violates the condition for stationarity of increments.

Hence $t \to W(ct^2)$ is not a brownian motion.
Consequently it is not a standard brownian motion either.

Problem 4 (20 points)

Compute the mean and autocovariance function of the process X defined as

$$X(t) = \int_0^t W(s)\,ds$$

1) Mean :

$$E[X(t)] = E\left[\int_0^t W(s)\,ds\right]$$

by linearity of mean

$$E[X(t)] = \int_0^t E[W(s)]\,ds$$

as $E[W(s)] = 0$ as $W \sim N(0,1)$

$$\therefore E[X(t)] = \int_0^t 0 \cdot ds = 0$$

$$\therefore E[X(t)] = 0$$

★ as $\int_0^t W(s)\,ds = \lim_{n\to\infty} \sum_{i=1}^{n} \frac{t}{n} W(s_i)$

$$\therefore E\left[\int_0^t W(s)\,ds\right] = \lim_{n\to\infty} \sum_{i=1}^{n} \frac{t}{n} E[W(s_i)]$$

$$= \int_0^t E[W(s)]\,ds$$

2) Autocovariance :

$$C_X(t_1, t_2) = E[X(t_1)\,X(t_2)] - E[X(t_1)]\,E[X(t_2)]$$

$$E[X(t_1)X(t_2)] = E\left[\int_0^{t_1} W(s)ds \cdot \int_0^{t_2} W(s)ds\right]$$

if $0 < t_1 \le t_2$

$$\therefore \int_0^{t_2} W(s)ds = \int_0^{t_1} W(s)ds + \int_{t_1}^{t_2} W(s)ds$$

$$\therefore E[X(t_1)X(t_2)] = E\left[\int_0^{t_1} W(s)ds \left[\int_0^{t_1} W(s)ds + \int_{t_1}^{t_2} W(s)ds\right]\right]$$

$$= E\left[\left(\int_0^{t_1} W(s)ds\right)^2 + \int_0^{t_1} W(s)ds \cdot \int_{t_1}^{t_2} W(s)ds\right]$$

$$= E\left[\left(\int_0^{t_1} W(s)ds\right)^2\right] + E\left[\int_0^{t_1} W(s)ds\right] \cdot E\left[\int_{t_1}^{t_2} W(s)ds\right]$$

(by independence)

$$\therefore E\left[x(t_1)x(t_2)\right] = E\left[\left(\int_0^{t_1} w(s)\,ds\right)^2\right]$$

$$+ \int_0^{t_1} E(w(s))\,ds \cdot \int_{t_1}^{t_2} E(w(s))\,ds$$

$$E\left[x(t_1)x(t_2)\right] = E\left[\left(\int_0^{t_1} w(s)\,ds\right)^2\right] + \int_0^{t_1} 0 \cdot ds$$

$$\cdot \int_{t_1}^{t_2} 0 \cdot ds$$

$$E\left[x(t_1)x(t_2)\right] = E\left[\left(\int_0^{t_1} w(s)\,ds\right)^2\right]$$

$$\therefore C_x(t_1, t_2) = E\left[\left(\int_0^{t_1} w(s)\,ds\right)^2\right]$$

$$- E\left[\int_0^{t_1} w(s)\,ds\right] E\left[\int_0^{t_2} w(s)\,ds\right]$$

As $E\left[\int_0^{t_i} w(s)\,ds\right] = \int_0^{t_i} E\left[w(s)\right]ds = 0$

$$\therefore C_x(t_1, t_2) = E\left[\left(\int_0^{t_1} w(s)\,ds\right)^2\right]$$

As

$$C_X(t_1, t_2) = E\left[\left(\int_0^{\min(t_1,t_2)} W(s)\,ds\right)^2\right]$$

let $\min(t_1, t_2) = t$

as $d(tW_t) = W_t\,dt + t\,dW_t$.

$$\therefore \int_0^t W_s\,ds = tW_t - \int_0^t s\,dW_s$$

$$= \int_0^t (t-s)\,dW_s,$$

Which can be treated as an ito integral (parametrized)

$$\therefore E\left(\int_0^t W(s)\,ds\right) = 0$$

$$\& \ Var\left(\int_0^t W(s)\,ds\right) = \int_0^t (t-s)^2\,ds = \frac{1}{3}t^3$$

$$\text{but } E\left[\left(\int_0^t W(s)\,ds\right)^2\right] = \text{Variance} - \text{Expectation}$$

$$\therefore E\left[\left(\int_0^t W(s)\,ds\right)^2\right] = \frac{1}{3}t^3 - 0 = \frac{1}{3}t^3$$

$$\therefore C_x(t_1, t_2) = \frac{1}{3}\left(\min(t_1, t_2)\right)^3$$

Problem 5: (30 points)

We consider the Geometric Brownian Motion model for a stock price:

$$d\log S(t) = \left(\mu - \frac{1}{2}\sigma^2\right)dt + \sigma\, dW(t).$$

We then define the log return over the interval $[t, t+\Delta]$

$$r(t,\Delta) = \log S(t+\Delta) - \log S(t).$$

Integrating the first equation over $[t, t+\Delta]$ yields

$$\log S(t+\Delta) - \log S(t) = \left(\mu - \frac{1}{2}\sigma^2\right)\Delta + \sigma\left(W(t+\Delta) - W(t)\right).$$

In other words, the log return r can be written

$$r(t,\Delta) = \left(\mu - \frac{1}{2}\sigma^2\right)\Delta + \sigma\left(W(t+\Delta) - W(t)\right)$$

Suppose that we are given a set of daily data for which the above model is a good fit with $\mu = 0.1$ per year and $\sigma = 0.2$ per year. Note that $\Delta = 1$ day $= \frac{1}{252}$ years. We wish to estimate μ, assuming that σ has already been estimated (i.e. $\sigma = 0.2$). Since the random walk model is

Stationary, ergodic and has a finite variance, which allows us to apply the Central Limit Theorem, we can safely estimate μ by computing a time average. Unfortunately, the next questions show that obtaining an accurate value for μ requires very long time series that are never available in practice.

1. (5 points) What is the distribution of $r(t, \Delta)$? in particular, give its mean & variance.

→

$$E[r(t,\Delta)] = E\left[\left(\mu - \frac{1}{2}\sigma^2\right)\Delta + \sigma\left(W(t+\Delta) - W(t)\right)\right]$$

$$= \left(\mu - \frac{1}{2}\sigma^2\right)\Delta + \sigma\left(E\left(W(t+\Delta) - W(t)\right)\right)$$

as increments of brownian motion are normally distributed with mean 0

$$\therefore E[r(t,\Delta)] = \left(\mu - \frac{1}{2}\sigma^2\right)\Delta + 0$$

$$\therefore E[r(t,\Delta)] = \left(\mu - \frac{1}{2}\sigma^2\right)\Delta$$

$$Var[r(t,\Delta)] = Var\left[\left(\mu - \frac{1}{2}\sigma^2\right)\Delta + \sigma(W(t+\Delta) - W(t))\right]$$

as $Var(a+bx) = Var(bx)$

$$= Var[\sigma(W(t+\Delta) - W(t))]$$

$$= \sigma^2 \underbrace{Var[W(t+\Delta) - W(t)]}_{\Delta}$$

Δ (variance of increment of a brownian motion)

$$\therefore Var[r(t,\Delta)] = \sigma^2\Delta$$

As $r(t,\Delta)$ is a linear combination of an increment of a brownian motion and the increments are normally distributed, then $r(t,\Delta)$ is also normally distributed with mean $\left(\mu - \frac{1}{2}\sigma^2\right)\Delta$ and Variance $\sigma^2\Delta$.

2. (5 points) Write the estimator $\hat{\mu}$ of the parameter μ for a number of samples equal to N.

We derive the estimate using Central Limit theorem.

→ The upper estimate is

$$\left(\mu - \frac{1}{2}\sigma^2\right)\Delta \leq \left(\hat{\mu} - \frac{1}{2}\sigma^2\right)\Delta + Z_{1-\frac{\alpha}{2}} \frac{\sqrt{\text{variance}}}{\sqrt{N}}$$

$$\therefore \mu \leq \hat{\mu} + Z_{1-\frac{\alpha}{2}} \frac{\sigma\sqrt{\Delta}}{\sqrt{N}}$$

where $Z \sim N(0,1)$ & α is the significance level

Similarly lower estimate

$$\mu \geq \hat{\mu} - Z_{\alpha/2}\, \sigma \sqrt{\frac{\Delta}{N}}$$

As given

$\hat{\mu} = 0.1$ per year

$\sigma = 0.2$ per year

let $\Delta = 1$ year

$$\therefore \hat{\mu} - Z_{\alpha/2}\, \sigma \sqrt{\frac{\Delta}{N}} \leq \mu \leq \hat{\mu} + Z_{\alpha/2}\, \sigma \sqrt{\frac{\Delta}{N}}$$

$$\therefore 0.1 - Z_{\alpha/2} \cdot 0.2 \sqrt{\frac{1}{N}} \leq \mu \leq 0.1 + Z_{\alpha/2} \cdot 0.2 \sqrt{\frac{1}{N}}$$

$\therefore$ $\cancel{0.1 - Z_{\alpha/2}}$

$$\therefore 0.1 - \frac{0.2}{\sqrt{N}} Z_{\alpha/2} \leq \mu \leq 0.1 + \frac{0.2}{\sqrt{N}} Z_{\alpha/2}$$

or ~~$\hat{\mu} = \frac{0.1}{252}$ per day~~

~~$\sigma = \frac{0.2}{252}$ per day~~

~~$\Delta = 1$ day~~

3) (5 points) What is the convergence rate of the above estimator?

→ Central limit theorem implies $N^{-\frac{1}{2}}$ order of convergence ($\sqrt{N}$ rate of convergence).

As we derived our estimate using Central limit theorem, our estimate also has $\sqrt{N}$ rate of convergence.

4) (15 points) If one wants to determin a 95% confidence interval of the form $[\hat{\mu}-0.01, \hat{\mu}+0.01]$, how many years of data do you need?

→ Here, $\alpha = 100-95 = 5\% = 0.05$

$\frac{\alpha}{2} = 0.025$, $z_{\alpha/2} = 1.96$

also $z_{\frac{\alpha}{2}} \cdot 0.2\sqrt{\frac{1}{N}} = 0.01$

$\therefore \sqrt{N} = \dfrac{z_{\frac{\alpha}{2}} \times 0.2}{0.01} = \dfrac{1.96 \times 0.2}{0.01}$

$\therefore \sqrt{N} = 39.2$

$\therefore N = 1536.64$ years.

Hence we will need 1536.64 years of data for our estimate.