Assignment 4
Itô's formula

1. Let x_t be an Itô process satisfying
$$dx_t = a(t,\omega)dt + \sigma(t,\omega)dB_t$$

Prove that x_t is a martigale if and only if $a(t,\omega) = 0$.

→ In integral form →

$$X_T = X_0 + \int_0^T a(t,\omega)dt + \int_0^T \sigma(t,\omega)dB_t \quad -(1)$$

Taking conditional expectation w.r.t. filteration F_s

$$E[X_T | F_s] = X_0 + \int_0^S a(t,\omega_s)dt + E\left[\int_S^T a(t,\omega)\,dt \,\Big|\, F_s\right] + \int_0^S \sigma(t,\omega_s)dB_t + E\left[\int_S^T \sigma(t,\omega)dB_t\right]$$

but $X_S = X_0 + \int_0^S a(t,\omega_s)dt + \int_0^S \sigma(t,\omega_s)dB_t$

$T \geq s > 0$

$$\therefore E[X_T | F_s] = X_s + E\left[\int_s^T a(t,\omega)\,dt \,\Big|\, F_s\right] + \underbrace{E\left[\int_s^T \sigma(t,\omega)\,dB_t \,\Big|\, F_s\right]}_{\text{expectation of ito integral } = 0}$$

For X_T to be martingale

$$E[X_T | F_s] = X_s$$

$$\therefore E\left[\int_s^T a(t,\omega)\,dt \,\Big|\, F_s\right] = 0$$

as expectation is a linear operator

$$\int_s^T E[a(t,\omega) | F_s]\,dt = 0 \quad -(2)$$

if ~~$a(t,\omega)$ is deterministic i.e. $a(t,\omega) = a(t)$~~

$$\therefore \cancel{E[a(t,\omega) | F_s] = a(t,\omega) = a(t)}$$

$$\therefore \cancel{\int_s^T a(t)\,dt = 0}$$

~~this only happens when $a(t) = 0$~~

~~i.e. $a(t,\omega) = a(t) = 0$~~

Since the choice of s and T is arbitrary, the only way to ensure that the integral in eqn (2) is always zero is that $a(t,\omega)$ is itself always zero.

2. Use Itô's formula to prove that the following stoachastic processes are martingales:

(1) $X_t = e^{\frac{t}{2}} \cos B_t$

let $f = e^{\frac{t}{2}} \cos X, \; X = B$

$f_x = - e^{\frac{t}{2}} \sin x \quad , f_{xx} = - e^{\frac{t}{2}} \cos x$

$f_t = \frac{1}{2} e^{\frac{t}{2}} \cos x$

by itô's formula

$$df = f_t \, dt + f_x \, dx + \frac{1}{2} f_{xx} (dx)^2$$

$$df = \frac{1}{2} e^{\frac{t}{2}} \cos x \, dt + \left(- e^{\frac{t}{2}} \sin x \right) dB_t$$
$$+ \frac{1}{2} \left(- e^{\frac{t}{2}} \cos x \right) dB_t \, dB_t$$

but $dB_t \, dB_t = dt$

$\therefore$ 1^{st} & 3^{rd} term cancel each other

$$\therefore df = - e^{\frac{t}{2}} \sin x \, dB_t$$

in integral form

$$f(T) - f(0) = - \int_0^T e^{\frac{t}{2}} \sin x \, dB_t$$

$$e^{\frac{T}{2}} \cos B_T - e^0 \cos 0 = - \int_0^T e^{\frac{t}{2}} \sin B_t \, dB_t$$

Taking conditional Expectation on both sides w.r.t fiteration F_s, $T \geq s > 0$

$$E\left[e^{\frac{T}{2}}\cos B_T - 1 \mid F_s\right] = -E\left[\int_0^T e^{\frac{t}{2}}\sin B_t \, dB_t \mid F_s\right]$$

$$= -E\left[\int_0^s e^{\frac{t}{2}}\sin B_t \, dB_t \mid F_s\right]$$

$$- E\left[\int_s^T e^{\frac{t}{2}}\sin B_t \, dB_t \mid F_s\right]$$

1^{st} term is deterministic
2^{nd} term is ito integral Hence its expectation is 0

$$\therefore E\left[e^{\frac{T}{2}}\cos B_T \mid F_s\right] - 1 = -\int_0^s e^{\frac{t}{2}}\sin B_t \, dB_t$$

$$E\left[e^{\frac{T}{2}}\cos B_T \mid F_s\right] = 1 - \int_0^s e^{\frac{t}{2}}\sin B_t \, dB_t$$

realization until s

but ~~E~~ $E\left[e^{\frac{s}{2}}\cos B_s \mid F_s\right]$

$$= 1 - \int_0^s e^{\frac{t}{2}}\sin B_t \, dB_t$$

$$\therefore e^{\frac{s}{2}}\cos B_s = 1 - \int_0^s e^{\frac{t}{2}}\sin B_t \, dB_t$$

$$\therefore E\left[e^{\frac{T}{2}}\cos B_T \mid F_s\right] = e^{\frac{s}{2}}\cos B_s$$

which implies $e^{\frac{t}{2}}\cos B_t$ is a martingale.

② $X_t = e^{\frac{t}{2}} \sin B_t$

let $x = B_t$ $dx = dB_t$

$f = e^{\frac{t}{2}} \sin x$, $f_x = e^{\frac{t}{2}} \cos x$, $f_{xx} = -e^{\frac{t}{2}} \sin x$
, $f_t = \frac{1}{2} e^{\frac{t}{2}} \sin x$

by ito's formula,

$$df = f_t\, dt + f_x\, dx + \frac{1}{2} f_{xx} (dx)^2$$

$$\therefore df = \frac{1}{2} e^{\frac{t}{2}} \sin B_t\, dt + e^{\frac{t}{2}} \cos x\, dB_t + \frac{1}{2} \times -e^{\frac{t}{2}} \sin B_t\, dt$$

$$df = e^{\frac{t}{2}} \cos B_t\, dB_t$$

integrating both sides

$$f = e^{\frac{t}{2}} \sin B_t - e^{0} \sin 0 = \int_0^t e^{\frac{t}{2}} \cos B_t\, dB_t$$

$$\therefore e^{\frac{t}{2}} \sin B_t = \underbrace{\int_0^t e^{\frac{t}{2}} \cos B_t\, dB_t}_{\text{Ito integral}}$$

R.H.S is an ito integral & by property of ito integral, which says that ito
of brownian motion
integral is a martingale.
of brownian motion

as $E\left[\int_0^t e^{\frac{t}{2}} \cos B_t\, dB_t \,\middle|\, F_s \right]$

$$= \int_0^s e^{\frac{t}{2}} \cos B_s\, dB_s + \underbrace{E\left[\int_s^t e^{\frac{t}{2}} \cos B_t\, dB_t \right]}_{= 0}$$

(3) $x_t = (B_t + t)e^{-B_t - \frac{t}{2}}$

let $x = B_t + \frac{t}{2} \Rightarrow dx = dB_t + \frac{dt}{2}$

$(dx)^2 = dt$

$\therefore f = (x + \frac{t}{2})e^{-x}$, $f_t = \frac{1}{2}e^{-x}$

$f_x = -xe^{-x} + e^{-x} - \frac{t}{2}e^{-x}$

$f_{xx} = -[-xe^{-x} + e^{-x}] - e^{-x} + \frac{t}{2}e^{-x}$

$f_{xx} = xe^{-x} - 2e^{-x} + \frac{t}{2}e^{-x}$

by itô's formula

$df = \frac{1}{2}e^{-x}dt + \left[-xe^{-x} + e^{-x} - \frac{t}{2}e^{-x}\right]\left[dB_t + \frac{dt}{2}\right]$

$+\frac{1}{2}\left[xe^{-x} - 2e^{-x} + \frac{t}{2}e^{-x}\right]dt$

$df = [-xe^{-x} + e^{-x} - \frac{t}{2}e^{-x}]dB_t$

$+ \frac{1}{2}e^{-x}dt - \frac{1}{2}xe^{-x}dt + \frac{1}{2}e^{-x}dt$

$- \frac{t}{4}e^{-x}dt + \frac{1}{2}xe^{-x}dt - e^{-x}dt$

$+ \frac{t}{4}e^{-x}dt$

$\therefore df = [-xe^{-x} + e^{-x} - \frac{t}{2}e^{-x}]dB_t$

$\therefore df = g(x,t)\,dB_t \quad -(1)$

where

$$g(x,t) = -xe^{-x} + e^{-x} - \frac{t}{2}e^{-x}$$

· integrating eq^n (1)

$$\int df = (B_t + t)e^{-B_t - \frac{t}{2}} = \cancel{\int_0^t g(B_t,t)\,dB_t}$$

$$= \underbrace{\int_0^t g(B_t + \tfrac{t}{2}, t)\, dB_t}_{\text{ito integral wrt brownian motion}}$$

but by property of ito integral wrt brownian motion, as it is martingale RHS is a martingale.
Now it follows that LHS $= (B_t + t)e^{-B_t - \frac{t}{2}}$ is also a martingale.

Q.3. Assume that $f(x)$ is twice continuously differentiable. Find all functions f such that $f(B_t)$ is a martingale. Hint: apply Itô lemma to $f(B_t)$

→ let $x = B_t$, $dx = dB_t$

$$g(x) = f(x) = f(B_t)$$

$$g_t = 0, \quad g_x = f_x, \quad g_{xx} = f_{xx}$$

applying itô's lemma to dg,

$$dg = 0\,dt + f_x\,dB_t + \frac{1}{2}f_{xx}\,dt$$

$\therefore dg = f_x \, dB_t + \frac{1}{2} f_{xx} dt$

integrating both sides from o to t

$$g(t) - g(0) = \int_0^t f_x \, dB_t + \frac{1}{2}\int_0^t f_{xx} dt$$

$$f(t) - f(0) = \underbrace{\int_0^t f_x \, dB_t}_{\text{ito integral}} + \frac{1}{2}\int_0^t f_{xx} dt$$

if f_x is continuous
which is given

$$\underbrace{f(t) = f(0) + \int_0^t f_x(B_t) dB_t}_{\text{const + ito integral is a martingale}} + \frac{1}{2}\int_0^t f_{xx}(B_t) dt$$

[martingale + martingale
= martingale]

if we want $f(t)_t$ to be a martingale
we need $\frac{1}{2}\int_0^t f_{xx} \, dt$ to be a martingale.

i.e. All those functions f are martingale
Where $t \geq s > 0$

$$E\left[\int_0^t f_{xx}(B_t) dt \,\middle|\, F_s\right] = \int_0^s f_{xx} dt$$

but this can only happen when

$$E\left[\int_0^t f_{xx}\,dt - \int_0^s f_{xx}\,dt \,\Big|\, F_s\right] = 0$$

$$\therefore E\left[\int_s^t f_{xx}\,dt \,\Big|\, F_s\right] = 0$$

but As B_t does not depend on F_s bet^n the interval s to t [by def^n]

We can drop the conditional

$$\therefore E\left[\int_s^t f_{xx}(B_t)\,dt\right] = 0$$

As the choice of s & t is arbitrary

$$\text{for } \int_s^t E[f_{xx}(B_t)]\,dt = 0$$

$f_{xx}(B_t)$ should itself be equal to zero.

$\therefore$ All the functions $f(x)$, which are twice continuously differentiable, $f(B_t)$ is a martingale only if $f_{xx}(B_t)$ is equal to 0.