Assignment 2

1. Let B_t be a standard Brownian Motion. In class we showed $dB_t\, dB_t = dt$. Using similar notation, show that $dB_t\, dt = 0$.

As

a) $$dB_t\, dt = \lim_{\|\pi\| \to 0} \sum_{i=0}^{n} (B_{t_i} - B_{t_{i-1}})(t_i - t_{i-1})$$

$$E[dB_t\, dt] = \lim_{\|\pi\| \to 0} \sum_{i=0}^{n} (t_i - t_{i-1}) E[B_{t_i} - B_{t_{i-1}}]$$

but $E[B_{t_i} - B_{t_{i-1}}] = 0 \quad \forall\ i = 0 \ldots n$

$$\therefore E[dB_t\, dt] = 0 \quad \text{— (1)}$$

b) $$\operatorname{var}[dB_t\, dt] = \lim_{\|\pi\| \to 0} \sum_{i=0}^{n} \operatorname{var}[(B_{t_i} - B_{t_{i-1}})(t_i - t_{i-1})]$$

as $B_{t_i} - B_{t_{i-1}}$ $\forall\ i = 0 \ldots n$ are i.i.d.

$$\operatorname{var}[(B_{t_i} - B_{t_{i-1}})(t_i - t_{i-1})] = (t_i - t_{i-1})^2 \operatorname{var}[B_{t_i} - B_{t_{i-1}}]$$

$$= (t_i - t_{i-1})^2 \cdot (t_i - t_{i-1})$$

$$= (t_i - t_{i-1})^3$$

$$\therefore \operatorname{var}[dB_t\, dt] = \lim_{\|\pi\| \to 0} \sum_{i=0}^{n} (t_i - t_{i-1})^3$$

$$= \lim_{\|\pi\| \to 0} \sum_{i=0}^{n} (t_i - t_{i-1})^2 \cdot (t_i - t_{j-1})$$

$$\therefore var[dB_t\, dt] \le \lim_{\|\pi\| \to 0} \sum_{i=0}^{n} \|\pi\|^2 (t_i - t_{i-1})$$

$$\le \lim_{\|\pi\| \to 0} \|\pi\|^2 \cdot T$$

$var[dB_t\, dt] \le 0$

but Variance is always ≥ 0

$\therefore var[dB_t\, dt] = 0$ — (2)

From (1) & (2)

$E[dB_t\, dt] = 0$ & $var[dB_t\, dt] = 0$

$\therefore$ by def^n of L^2 convergence

$$dB_t\, dt = 0$$

2. Show that the brownian motion B_t has unbounded first variation, i.e,

$$\lim_{\|\Pi\| \to 0} \sum_{j=0}^{n-1} |B_{t_{j+1}} - B_{t_j}| = \infty,$$

where Π is the partition as defined in class.

→ let $X \sim N(0, \sigma^2)$

$$E[X \mid X>0] = \frac{\int_0^\infty x \cdot \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2\sigma^2}x^2} dx}{P(X>0)}$$

$$= \frac{1}{\sqrt{2\pi}\sigma} \int_0^\infty 2x \cdot e^{-\frac{1}{2\sigma^2}x^2} dx$$

let $x^2 = u$ $\therefore$ $2x\,dx = du$

As $x \to 0$, $u \to 0$, $x \to \infty$, $u \to \infty$

$$\therefore E[X \mid X>0] = \frac{1}{\sqrt{2\pi}\sigma} \int_0^\infty e^{-\frac{1}{2\sigma^2} \cdot u} du$$

$$= \frac{1}{\sqrt{2\pi}\sigma} \left[\frac{e^{-\frac{1}{2\sigma^2}u}}{-\frac{1}{2\sigma^2}} \right]_0^\infty$$

$$= \frac{1}{\sqrt{2\pi}\sigma} \left[2\sigma^2 \right] = \sqrt{\frac{2}{\pi}} \cdot \sigma$$

by symmetry

$$\therefore E[X \mid X<0] = -\sqrt{\frac{2}{\pi}}\sigma$$

$$\therefore E[-X \mid X<0] = \sqrt{\frac{2}{\pi}}\sigma$$

$$\therefore E[|X|] = E[E[|X| \mid X]]$$

$$= \frac{1}{2} E[|X| \mid X>0] + \frac{1}{2} E[|X| \mid X<0]$$

$$= \frac{1}{2} E[X \mid X>0] + \frac{1}{2} E[-X \mid X<0]$$

$$= \frac{1}{2}\sqrt{\frac{2}{\pi}}\sigma + \frac{1}{2}\sqrt{\frac{2}{\pi}}\sigma = \sqrt{\frac{2}{\pi}}\sigma$$

$$\therefore E[|B_{t_{j+1}} - B_{t_j}|] = \sqrt{\frac{2}{\pi}}(t_{j+1} - t_j)$$

$$\therefore E\left[\lim_{\|\pi\|\to 0} \sum_{j=0}^{n-1} |B_{t_{j+1}} - B_{t_j}|\right]$$

$$= \lim_{\|\pi\|\to 0} \sum_{j=0}^{n-1} E[|B_{t_{j+1}} - B_{t_j}|]$$

$$= \lim_{\|\pi\|\to 0} \sum_{j=0}^{n-1} \sqrt{\frac{2}{\pi}}(t_{j+1} - t_j)$$

$$= \lim_{\|\pi\|\to 0} \sqrt{\frac{2}{\pi}} \sum_{j=0}^{n-1} t_{j+1} - t_j$$

$$= \sqrt{\frac{2}{\pi}} \cdot T \quad [\because \text{telescopic sum}] \quad -(1)$$

$$\mathrm{var}\left[\lim_{\|\pi\|\to 0} \sum_{j=0}^{n-1} |B_{t_{j+1}} - B_{t_j}|\right] = \lim_{\|\pi\|\to 0} \sum_{j=0}^{n-1} \mathrm{var}|B_{t_{j+1}} - B_{t_j}|$$

$$\mathrm{var}[|B_{t_{j+1}} - B_{t_j}|] = E[(B_{t_{j+1}} - B_{t_j})^2] - (E[|B_{t_{j+1}} - B_{t_j}|])^2$$

$$= (t_{j+1} - t_j) - \frac{2}{\pi}(t_{j+1} - t_j)^2$$

$$\therefore var\left[\lim_{\|\pi\|\to 0}\sum_{j=0}^{n-1}|B_{t_{j+1}}-B_{t_j}|\right]$$

$$= \lim_{\|\pi\|\to 0}\sum_{j=0}^{n-1}(t_{j+1}-t_j)-\frac{2}{\pi}(t_{j+1}-t_j)^2$$

$$= \lim_{\|\pi\|\to 0} T - \frac{2}{\pi}\sum_{j=0}^{n-1}(t_{j+1}-t_j)^2$$

$$\geq \lim_{\|\pi\|\to 0} T - \frac{2}{\pi}\|\pi\| \cdot \sum_{j=0}^{n-1}(t_{j+1}-t_j)$$

$$\geq \lim_{\|\pi\|\to 0} T - \frac{2}{\pi}\|\pi\| \cdot T$$

$$\geq \lim_{\|\pi\|\to 0} T\left(1-\frac{2}{\pi}\cdot\|\pi\|\right)$$

$$\geq T \quad —(2)$$

$\therefore$ From (1) & (2), For any time period $[0,T]$

$$E\left[\lim_{\|\pi\|\to 0}\sum_{j=0}^{n-1}|B_{t_{j+1}}-B_{t_j}|\right] = \sqrt{\frac{2}{\pi}}\cdot T$$

$$Var\left[\lim_{\|\pi\|\to 0}\sum_{j=0}^{n-1}|B_{t_{j+1}}-B_{t_j}|\right] \geq T$$

This implies $\lim_{\|\pi\|\to 0}\sum_{j=0}^{n-1}|B_{t_{j+1}}-B_{t_j}|$ can potentially ~~take~~ blow up. i.e. Brownian motion has unboundeded first variation

3. Show that $(dB_t)^n = 0$ for $n \geq 3$

$\rightarrow$ $(dB_t)^p = \lim\limits_{\|\pi\| \to 0} \sum\limits_{i=1}^{m} (B_{t_i} - B_{t_{i-1}})^p, \quad p = 3, 4 \ldots$

a)

As $B_{t_i} - B_{t_{i-1}} \sim N(0, t_i - t_{i-1})$

p^{th} moment of a normal distribution is

$\hat{m}_p = \sigma^p (p-1)!$ if p is even

$= 0$ if p is odd

$E[(dB_t)^p] = \lim\limits_{\|\pi\| \to 0} \sum\limits_{i=1}^{m} E(B_{t_i} - B_{t_{i-1}})^p$

if p is even & $p \geq 3$

$E[(dB_t)^p] = \lim\limits_{\|\pi\| \to 0} \sum\limits_{i=1}^{m} \sigma_i^p (p-1)!$

where $\sigma_i = t_i - t_{i-1}$

$E[(dB_t)^p] = \lim\limits_{\|\pi\| \to 0} \sum\limits_{i=1}^{m} (t_i - t_{i-1})^p (p-1)! \quad -(1)$

$\leq \lim\limits_{\|\pi\| \to 0} \sum\limits_{i=1}^{m} \|\pi\|^p (p-1)!$

≤ 0

but from (1) $E[(dB_t)^p]$ is always +ve $\therefore E[(dB_t)^p] = 0$

if p is odd & $p \geq 3$

$$E\left[(dB_t)^p\right] = \lim_{\|\Pi\| \to 0} \sum_{i=1}^{m} E(B_{t_i} - B_{t_{i-1}})^p$$

$$= \lim_{\|\Pi\| \to 0} \sum_{i=1}^{m} 0 = 0$$

$$E\left[(dB_t)^p\right] = 0 \quad \forall\ p \geq 3 \quad \text{———} \quad (2)$$

b) $$Var\left[(dB_t)^p\right] = \lim_{\|\Pi\| \to 0} \sum_{i=1}^{m} Var\left[(B_{t_i} - B_{t_{i-1}})^p\right]$$

$$var\left[(B_{t_i} - B_{t_{i-1}})^p\right] = E\left[(B_{t_i} - B_{t_{i-1}})^{2p}\right] - \left(E\left[(B_{t_i} - B_{t_{i-1}})^p\right]\right)^2$$

but

$$E\left[(B_{t_i} - B_{t_{i-1}})^p\right] = 0 \quad \forall\ p\text{s}$$

similarly

$$\& \ E\left[(B_{t_i} - B_{t_{i-1}})^{2p}\right] = 0$$

$$\therefore var\left[(B_{t_i} - B_{t_{i-1}})^p\right] = 0$$

$$\therefore var\left[(dB_t)^p\right] = \lim_{\|\Pi\| \to 0} \sum_{i=1}^{m} Var\left[(B_{t_i} - B_{t_{i-1}})^p\right]$$

$$= \sum_{i=1}^{m} 0 = 0 \quad \text{—} \ (3)$$

From (2) & (3), $(dB_t)^p = 0$ for $p \geq 3$

replacing p by n $\underline{\underline{(dB_t)^n = 0}}$ for $n \geq 3$