Assignment 3
ITÔ Integral

1. Let $\sigma(t)$ be deterministic function of time, β be constant and define

$$X(T) = \int_0^T \sigma(t) e^{-\beta(T-t)} dB_t .$$

Find the expectation and variance of $X(T)$. What is the distribution of $X(T)$?

→ By definition of ito integral

a) $$X(T) = \lim_{\|\pi\| \to 0} \sum_{i=1}^{n} \underbrace{\sigma(t_i) e^{-\beta(T-t_i)}}_{\text{deterministic function}} (B_{t_{i+1}} - B_{t_i})$$

$$E[X(T)] = \lim_{\|\pi\| \to 0} \sum_{i=1}^{n} \sigma(t_i) e^{-\beta(T-t_i)} E[B_{t_{i+1}} - B_{t_i}]$$

As $E[B_{t_{i+1}} - B_{t_i}] = 0$ [Expected value of increment of brownian motion]

$$\therefore E[X(T)] = \lim_{\|\pi\| \to 0} \sum_{i=1}^{n} \sigma(t_i) e^{-\beta(T-t_i)} \cdot 0$$

$$E[X(T)] = 0$$

b) $Var(X(T)) = E[(X(T))^2] - [E(X(T))]^2$

$= E[(X(T))^2]$

$$E[(X(T))^2] = E\left[\left(\int_0^T \sigma(t) e^{-\beta(T-t)} dB_t\right)^2\right]$$

by itô's isometry

$$= E\left[\int_0^T \left(\sigma(t) e^{-\beta(T-t)}\right)^2 dt\right]$$

$$E[(X(T))^2] = E\left[\int_0^T \sigma^2(t) e^{-2\beta(T-t)} dt\right]$$

$$\therefore Var(X(T)) = E\left[\int_0^T \sigma^2(t) e^{-2\beta(T-t)} dt\right]$$

$$\therefore Var(X(T)) = \int_0^T \sigma^2(t) . e^{-2\beta(T-t)} . dt$$ [As everything is deterministic

c) As

$$X(T) = \lim_{\|\pi\| \to 0} \sum_{i=1}^{n} \sigma(t_i) e^{-\beta(T-t_i)} (B_{t_{i+1}} - B_{t_i}) \quad -(1)$$

& $B_{t_{i+1}} - B_{t_i}$ is normally distributed.

i.e. $B_{t_{i+1}} - B_{t_i} \sim N(0, t_{i+1} - t_i)$

as they are independent increments of Brownianmotion.

Hence the RHS of (1) is sum of scaled independent increments of brownian motion,

with scaling factor $= \sigma(t_i) e^{-\beta(T-t_i)}$
i.e. sum of scaled i.i.d. normally distributed random variables.
Hence $X(T)$ also follows normal distribution with mean 0 & variance $E\left[\int_0^T \sigma^2(t) e^{-2\beta(T-t)} dt\right]$

2. For given functions $\mu(t,\omega)$ and $\sigma(t,\omega)$ Itô process X_t is defined as

$$X_t = X_0 + \int_0^t \mu(s,\omega) ds + \int_0^t \sigma(s,\omega) dB_s(\omega).$$

What is the quadratic variation of process X_t on a fixed interval $[0,T]$?
In your derivation you can assume that functions μ and σ are simple functions, that is piecewise constants.

t

As μ & σ are simple functions

$$\int_0^t \mu(s,\omega)ds = \sum_i \mu_i (s_{i+1}-s_i)$$

$$\int_0^t \sigma(s,\omega)dB_s = \sum_j \sigma_j (B_{j+1}-B_j)$$

$$\therefore \langle X,X\rangle = \lim_{\|\pi\|\to 0} \sum_k \Big[X_0 + \sum_{i=0}^{k} \mu_i (s_{i+1}-s_i) + \sum_{j=0}^{k} \sigma_j (B_{j+1}-B_j) - X_0 - \sum_{i=0}^{k-1} \mu_i (s_{i+1}-s_i) - \sum_{j=0}^{k-1} \sigma_j (B_{j+1}-B_j) \Big]^2$$

$$= \lim_{\|\pi\|\to 0} \sum_k \left(\mu_k (s_{k+1}-s_k) + \sigma_k (B_{k+1}-B_k)\right)^2$$

$$= \lim_{\|\pi\|\to 0} \sum_k \mu_k^2 (s_{k+1}-s_k)^2 + 2\sum_k \mu_k \sigma_k (s_{k+1}-s_k)(B_{k+1}-B_k)$$

$$+ \sum_k \sigma_k^2 (B_{k+1}-B_k)^2 \qquad - (0)$$

a) As $\pi = \max_k (s_{k+1}-s_k)$

$$\lim_{\|\pi\|\to 0} \sum_k \mu_k^2 (s_{k+1}-s_k)^2 \leq \lim_{\|\pi\|\to 0} \pi \sum_k \mu_k^2 (s_{k+1}-s_k)$$

$$\leq \lim_{\|\pi\|\to 0} \pi \int_0^T \mu^2 ds \leq 0$$

$$= 0 \qquad - (1)$$

As sum of +ves less than equal to 0 should be zero.

b) $\lim_{\|\pi\| \to 0} \sum_k \mu_k \sigma_k (B_{k+1} - B_k)(s_{k+1} - s_k)$

$$= \lim_{\|\pi\| \to 0} \sum_k \mu_k \sigma_k \, dB \, ds$$

but by rules of stochastic calculus

$$dB_s ds = dB_t dt = 0$$

$$\therefore \lim_{\|\pi\| \to 0} \sum_k \mu_k \sigma_k (B_{k+1} - B_k)(s_{k+1} - s_k)$$

$$= \lim_{\|\pi\| \to 0} \sum_k \mu_k \sigma_k \cdot 0 = 0 \quad -(2)$$

$= 0 \quad -(2)$

c) $\lim_{\|\pi\| \to 0} \sum_k \sigma_k^2 (B_{k+1} - B_k)^2$

As $B_{k+1} - B_k = dB$

$(B_{k+1} - B_k)^2 = dB \cdot dB$

but $dB \cdot dB = dt = t_{k+1} - t_k$ (by rule of stochastic calculus)

$\therefore \lim_{\|\pi\| \to 0} \sum_k \sigma_k^2 (B_{k+1} - B_k)^2$

$= \lim_{\|\pi\| \to 0} \sum_k \sigma_k^2 (t_{k+1} - t_k)$

by def^n of reimann integral

$= \int_0^T \sigma^2 \, dt = \int_0^T \sigma^2(s,\omega) \, ds \quad -(3)$

Substituting (1), (2), (3) in (0)

$$\langle X, X \rangle = 0 + 2 \cdot 0 + \int_0^T \sigma^2(s, \omega)\, ds$$
over [0,T]

$\therefore \langle X, X \rangle$ = Quadratic variation of X_t over fixed interval $[0,T]$

$$= \int_0^T \sigma^2(s, \omega)\, ds$$

3. Compute $E e^{-\int_0^t \sqrt{s}\, dB_s}$

let $X = -\int_0^t \sqrt{s}\, dB_s$

$$E[X] = -E \int_0^t \sqrt{s}\, dB_s$$

$$= -E\left[\lim_{\|\pi\| \to 0} \sum_k \sqrt{s_k}\,(B_{s_{k+1}} - B_{s_k})\right]$$

$$= -\lim_{\|\pi\| \to 0} \sum_k \sqrt{s_k}\, \underbrace{E[(B_{s_{k+1}} - B_{s_k})]}_{0}$$

$$E[X] = -1 \times 0 = 0$$

$$Var[X] = Var\left[-\int_0^t \sqrt{s}\, dB_s\right]$$

$$= Var\left[\int_0^t \sqrt{s}\, dB_s\right]$$

$$= E\left[\left(\int_0^t \sqrt{s}\, dB_s\right)^2\right]$$

by Itô's isometry

$$Var[X] = E\left[\int_0^t s\, ds\right]$$

$$Var[X] = E\left[\frac{t^2}{2} - \frac{0}{2}\right] = \frac{t^2}{2}$$

As
$$X = -\int_0^t \sqrt{s}\, dB_s$$

X follows a normal distribution by the same argument as in question (1).

$\therefore$ e^X follows lognormal distribution

$$E\, e^X = E\, e^{-\int_0^t \sqrt{s}\, dB_s} = \text{mean of lognormal distribution}$$

As $X \sim N(\mu, \sigma^2)$, $Y = e^X$

$Y \sim \text{Lognormal}(\mu, \sigma^2)$

$$E[Y] = E[e^X] = e^{\mu + \frac{\sigma^2}{2}}$$

$$\therefore E\, e^{-\int_0^t \sqrt{s}\, dB_s} = e^{0 + \frac{t^2}{4}}$$

$$\therefore E\, e^{-\int_0^t \sqrt{s}\, dB_s} = e^{\left(\frac{t^2}{4}\right)}$$