Assignment - 1

1. Let $M(t)$ be a symmetric random walk. Check if $M(t)^2 - t$ is a martingale with respect to filteration $F_t = \sigma(X_1, X_2, \dots, X_t)$.

→ As $M(t)$ is a symmetric random walk.

$$M(t) = \sum_{j=1}^{t} X_j \quad , \quad X_j \begin{cases} +1 & p=0.5 \\ -1 & p=0.5 \end{cases}$$

a) $M(t) = \sum_{j=1}^{t} X_j$ is F_t - measurable since it depends only on X_j, $j \le t$, i.e. on the information available at time t.

t is not a random Variable & it also depends on information available at time t. Hence t is also F_t - measurable.

$\therefore M(t)^2 - t$ only depends on information available at time t, Hence $M(t)^2 - t$ is F_t measurable.

b) As $M(t) = \sum_{j=1}^{t} X_j$

$$|M(t)| = \left| \sum_{j=1}^{t} X_j \right|$$

$$|M(t)| \le \sum_{j=1}^{t} |X_j| \qquad [\text{As } |a+b| \le |a| + |b|]$$

The fact: $\therefore |M(t)| \le t \quad , \quad t \ge 1 \, , \, |M(t)| \ge 0$

Squaring both sides

$$|M(t)|^2 \le t^2$$

but as $|a^2| = |a|^2$

$\therefore \quad |M(t)^2| \leq t^2 \qquad - (1)$

Now,

$|M(t)^2 - t| \leq |M(t)^2| + |-t|$

as $t \geq 0$

$\therefore \quad |M(t)^2 - t| \leq |M(t)^2| + t$

Using (1)

$|M(t)^2 - t| \leq t^2 + t$

$\therefore \quad E\left[|M(t)^2 - t|\right] \leq t^2 + t < \infty$

$\therefore \quad E\left[|M(t)^2 - t|\right] < \infty$

c) $E(M(t)^2 - t \mid F_s) = E\left((M(t)-M(s))^2 + 2M(t)M(s) - M(s)^2 - t \mid F_s\right)$

$$= E\left((M(t)-M(s))^2 \mid F_s\right) + 2E[M(t)M(s) \mid F_s] - E[M(s)^2 \mid F_s] - t$$

At time s, M(s) is observed & no longer random. Also M(t) - M(s) does not depend on info available at time s.

$$\therefore E(M(t)^2 - t \mid F_s) = E[(M(t)-M(s))^2] + 2M(s)E[M(t) \mid F_s] - M(s)^2 - t \quad -(2)$$

As M(t) is a symmetric random walk & we have proved that a symmetric random walk is a martingale (in class notes)

$$\therefore E[M(t) \mid F_s] = M(s)$$

As $E[(M(t)-M(s))^2] = var[M(t)-M(s)]$

$$= var\left[\sum_{j=s+1}^{t} X_j\right]$$

$$= \sum_{j=s+1}^{t} var(X_j) = t-s$$

as $var(X_j) = 1$

$\therefore$ Eq^n (2) becomes

$$E(M(t)^2 - t \mid F_s) = t-s + 2M(s).M(s) - M(s)^2 - t$$

$\therefore E(M(t)^2 - t \mid F_s) = M(s)^2 - s$

Which proves part (c) of martingality property

As $M(t)^2 - t$ satisfies all 3 properties of martingality $M(t)^2 - t$ is a martingale.

2. Compute $E e^{\sigma M(t)}$ for a given constant $\sigma > 0$ and fixed t.

$$E[e^{\sigma M(t)}] = E[e^{\sigma \sum_{j=1}^{t} X_j}]$$

$$= E[\prod_{j=1}^{t} e^{\sigma X_j}]$$

As X_j s are i.i.d. so are $e^{\sigma X_j}$ s

$$\therefore E[e^{\sigma M(t)}] = \prod_{j=1}^{t} E[e^{\sigma X_j}]$$

$$E[e^{\sigma X_j}] = \frac{1}{2} \cdot e^{\sigma} + \frac{1}{2} e^{-\sigma}$$

$$= \frac{1}{2}\left[e^{\sigma} + \frac{1}{e^{\sigma}}\right] = \frac{1}{2}\left[\frac{e^{2\sigma}+1}{e^{\sigma}}\right]$$

$$\therefore E[e^{\sigma M(t)}] = \prod_{j=1}^{t} \frac{1}{2}\left[\frac{e^{2\sigma}+1}{e^{\sigma}}\right]$$

$$\therefore E\left[e^{\sigma M(t)}\right] = \left(\frac{1}{2}\left[\frac{e^{2\sigma}+1}{e^{\sigma}}\right]\right)^{t}$$

3. Give an example of random Variable X such that $E|X| = \infty$

→ Consider a dicrete random variable X that takes on the value 2^n with probability $\frac{1}{2^n}$ for $n = 1, 2, 3, \ldots$

i.e. $P(X = 2^n) = \frac{1}{2^n}$ for $n = 1, 2, 3, \ldots$

As $X > 0$

$$E[|X|] = E(X) = \sum_{n=1}^{\infty} 2^n \cdot \frac{1}{2^n} = \sum_{n=1}^{\infty} 1$$

$\therefore E|X| = \infty$

i.e. $E|X|$ diverges.