Assignment - 6

1.

$\rightarrow$ $u(t,x) = E_{B_t = x}\, e^{-\gamma \int_t^T B_s^2 ds}$, As $dx_t = B_t$, $a=0, \sigma=1$

Clearly $u(T, x) = 1$

~~Add to~~ Multiply both sides $\exp\left(-\gamma \int_0^t B_s^2 ds\right)$

$$u \cdot e^{-\gamma \int_0^t B_s^2 ds} = E_{B_t = x}\, e^{-\gamma \int_0^T B_s^2 ds}$$

it is a martingale now by lemma 2.

Applying itô's lemma on LHS

$$d\left(e^{-\gamma \int_0^t B_s^2 ds} \cdot u\right) = \left(u_t + a u_x + \frac{1}{2}\sigma^2 u_{xx} - \gamma \cdot u B_t^2\right) e^{-\gamma \int_0^t B_s^2 ds} dt$$

$$+ e^{-\gamma \int_0^t B_s^2 ds} \cdot u_x \, dB_t$$

since it is a martingale the co-efficient of dt should be zero

$$u_t + a u_x + \frac{1}{2} \sigma^2 u_{xx} - \gamma u \cdot B_t^2 = 0$$

as $a = 0$, $\sigma = 1$ & $B_t = x$

$$\boxed{\begin{aligned} & u_t + \frac{1}{2} u_{xx} = \gamma u \cdot x^2 \\ & u(T, x) = 1 \end{aligned}} \quad \text{--- (1)}$$

This is a parabolic partial differential equation with a variable co-efficient in the non-derivative term, which makes it more complex to solve analytically.

We can use finite difference or other numerical methods to find solution of equation 1,

then we will get

$$u(t, x) = E_{B_t = x}\, e^{-\gamma \int_t^T B_s^2 \, ds}$$

2. $u = E_{B_t = x} \int_t^T B_s^2 \, ds$

adding $\int_0^t B_s^2 \, ds$ to both sides,

$$u + \int_0^t B_s^2 \, ds = E_{B_t = x} \int_0^T B_s^2 \, ds$$

The RHS is a martingale by lemma 2

Applying itô's lemma / Feymann kac eqn,

$$u_t + a u_x + \frac{1}{2} \sigma^2 u_{xx} + B_t^2 = 0$$

& $a = 0$, & $\sigma = 1$ as $dX_t = dB_t$ & $x = B_t$

$$\therefore \quad u_t + \frac{1}{2} u_{xx} + x = 0 \quad - (1)$$

& $u(T, x) = 0$

let $u(t, x) = v(t, x) - \frac{x^3}{3}$

eqn (1) becomes $\qquad u_x = v_x - x^2$

$$v_t + 0.5 (v_{xx} - 2x) + x = 0$$

$$V_t + V_{xx} = 0 \quad -(2)$$

$$U(T,x) = 0 = V(T,x) - \frac{x^3}{3}$$

$$\therefore V(T,x) = \frac{x^3}{3}$$

eq^n 2: This is now a heat equation

let $\tau = T - t$

$$\therefore \frac{\partial V}{\partial t} = -\frac{\partial V}{\partial \tau}, \quad \frac{\partial V}{\partial x} = \frac{\partial V}{\partial x}, \quad \frac{\partial^2 V}{\partial x^2} = \frac{\partial^2 V}{\partial x^2}$$

$\therefore$ eq^n (2) becomes:

$$-V_\tau + V_{xx} = 0$$

$$\therefore V_\tau = V_{xx} \quad \& \quad V(0,x) = \frac{x^3}{3}$$
$$-(3)$$

Assumming a seperable Sol^n:

$$V(\tau, x) = T(\tau) X(x)$$

substituting in PDE (3)

$$T'(\tau) X(x) = T(\tau) X''(x)$$

dividing both sides by $T(\tau)X(x)$

let $\frac{T'(\tau)}{T(\tau)} = \frac{X''(x)}{X(x)} = -\lambda$

Where $-\lambda$ is a seperation constant.

As $T'(\tau) = -\lambda T(\tau)$ $\therefore T(\tau) = T_0 e^{-\lambda\tau}$

$X''(x) = -\lambda X(x)$

if $\lambda > 0$: $X(x) = A\cos(\sqrt{\lambda}x) + B\sin(\sqrt{\lambda}x)$

if $\lambda = 0$: $X(x) = Cx + D$

if $\lambda < 0$: $X(x) = Ee^{\sqrt{-\lambda}x} + Fe^{-\sqrt{-\lambda}x}$

Given $v(0,x) = \frac{x^3}{3}$, expand x^3 in terms of

the eigenfunctions & eigen values will depend on the physical boundary conditions (not provided)

assumming $x \in [0, L]$, $X(0) = X(L) = 0$ (Dirichlet). The expansion will involve solving an integral to math $v(0,x) = \sum_n T_0 X_n(x)$, where $X_n(x)$ are eigenfunctions

Convert $v(\tau, x)$ to $v(t,x)$ to get sol^n to eq^n (2)

We can also use numerical methods to solve eq^n (2) (finite difference)

after that we can get

$u(t,x) = v(t,x) - \frac{x^3}{3}$

which is our sol^n i.e. we will get $E_{B_t = x} \int_t^T B_s^2 ds$.