Assignment 5

Girsanov's Theorem & Solving SDEs

1. Solve the following SDE :

$$dX_t = (\beta - \log X_t) X_t \, dt + \sigma X_t \, dB_t ,$$

$X(0) = X_0$, β & σ are constants.

→ let's try transforming the SDE using ito's lema
We assume $X_t > 0$ for all t becaus $\log X_t$ must be defined.

let $Y_t = \log X_t$, by Itô's lemma

$$dY_t = 0.dt + \frac{1}{X_t} dX_t - \frac{1}{2}\frac{1}{(X_t)^2} . dX_t . dX_t \quad — (1)$$

but given,

$$dX_t = (\beta - \log X_t) X_t \, dt + \sigma X_t \, dB_t$$

$$\frac{dX_t}{X_t} = (\beta - \log X_t) dt + \sigma dB_t$$

& $dX_t . dX_t = \sigma^2 (X_t)^2 dt$ ($\because dt\,dt = 0$, $dB_t dt = 0$)

$$\therefore \frac{dX_t . dX_t}{(X_t)^2} = \sigma^2 dt$$

$\therefore$ eq^n (1) becomes

$$dY_t = (\beta - \log X_t) dt + \sigma d B_t - \frac{1}{2}\sigma^2 . dt$$

$$\therefore dY_t = \left(\beta - \frac{1}{2}\sigma^2 - Y_t\right) dt + \sigma \, dB_t$$

but this eq^n is in the form of
Ornstein- Uhlenbeck process

$$\left[\ dx_t = \theta(\mu - x_t)dt + \sigma dW_t\ \right]$$

Solution to it is $\quad x_t = x_0 e^{-\theta t} + \mu(1 - e^{-\theta t}) + \frac{\sigma}{\sqrt{2\theta}} W_{1-e^{-2\theta t}}$

$$\therefore Y_t = e^{-t} \cdot Y_0 + \left(\beta - \frac{1}{2}\sigma^2\right)(1 - e^{-t})$$

$$+ \sigma \int_0^t e^{-(t-s)} dB_s$$

but $\quad Y_t = \log X_t$

$$\therefore X_t = e^{Y_t}$$

$$\therefore \quad X_t = \exp\left\{ e^{-t} \log X_0 + (1 - e^{-t})\left(\beta - \frac{1}{2}\sigma^2\right) + \sigma \int_0^t e^{-(t-s)} dB_s \right\}$$

2. Let $B_t = (B_t^1, B_t^2, \dots, B_t^d)$ be a d-dimensional Brownian motion and

$$R_t = \|B_t\| = \sqrt{(B_t^1)^2 + \dots + (B_t^d)^2}.$$

What is the distribution of the process

$$X_t = \sum_{i=1}^{d} \int_0^t \frac{B_s^i}{R_s} dB_s^i$$

→ The integrand in each term of the sum, $\frac{B_s^i}{R_s}$, represents the i-th component of the brownian motion divided by the Radial distance from the origin in d-dimensional space. This can be seen as the projection of the d-dimensional Brownian motion on the unit sphere scaled by R_s.

* Quadratic variation of X_t.

$$\langle X_t, X_t \rangle = \left\langle \sum_{i=1}^{d} \int_0^t \frac{B_s^i}{R_s} dB_s^i , \sum_{i=1}^{d} \int_0^t \frac{B_s^i}{R_s} dB_s^i \right\rangle$$

Since the integrals are stochastic integrals with respect to orthogonal components (independent B_s^i s) of the brownian motion, the quadratic variation add up to

$$\langle X, X \rangle_t = \sum_{i=1}^{d} \left\langle \int_0^t \frac{B_s^i}{R_s} dB_s^i , \int_0^t \frac{B_s^i}{R_s} dB_s^i \right\rangle$$

$$\langle X, X \rangle_t = \sum_{i=1}^{d} \int_0^t \left(\frac{B_s^i}{R_s}\right)^2 ds = \int_0^t \sum_{i=1}^{d} \left(\frac{B_s^i}{R_s}\right)^2 ds$$

Notice that $\left(\frac{B_s^i}{R_s}\right)^2$ is the squared component of the unit vector pointing in the direction of B_s.

The sum of the squares of these components is 1 (radius of d-dimensional unit sphere)

So,

$$\langle X, X \rangle = \int_0^t 1\, ds = t \quad — (1)$$

Also $X_0 = 0$ as $B_0^i = 0 \ \forall i$'s $\quad — (2)$

~~X_t~~ each term $\left(\frac{B_s^i}{R_s}\right)$ is continuous (as brownian motion) & X_t is a continuous function of brownian motions. Hence it is also continuous

X_t has continuous paths by construction (3)

Also each term in $\sum_{i=1}^{d} \int_0^t \left(\frac{B_s^i}{R_s}\right)^2 ds$ is

an ~~ito~~ (stochastic) integral which are
ito

martingales (Property) — (4)

From (1), (2), (3), (4)

X_t is a continuous local martingale starting at 0 with quadratic variation t, by Levy's theorem, X_t is a standard Brownian motion.

Therefore distribution of X_t is Normal i.e. $X_t \sim N(0, t)$.

3. Find a stochastic differential equation that R_t satisfies.

→ To find a stochastic differential eq^n that $R_t = \| B_t \| = \sqrt{(B_t^1)^2 + \cdots + (B_t^d)^2}$ satisfies, where,

$B_t = (B_t^1, B_t^2, \ldots B_t^d)$ is a d-dimensional Brownian motion, we can use Itô's formula. The process R_t can be thought as the radial part of a multidimensional Brownian motion, and its dynamics can be analyzed by considering the square root of the sum of squares of the components of B_t.

1) We define $R_t = f(B_t)$ where $f(x) = \sqrt{x_1^2 + \cdots x_d^2}$

Using Itô formula for multidimensional case on $f(X_t)$

$$df(X_t) = \sum_{i=1}^{d} \frac{\partial f}{\partial x_i} dX_t^i + \frac{1}{2} \sum_{i,j=1}^{d} \frac{\partial^2 f}{\partial x_i \partial x_j} d\langle X^i, X^j \rangle_t$$

For R_t, $X_t^i = B_t^i$ & $d\langle B^i, B^j \rangle_t = \delta_{ij}\, dt$ (where δ_{ij} is the kronecker delta).

2) The first & second partial derivatives of f are

$$\frac{\partial f}{\partial x_i} = \frac{x_i}{\sqrt{x_1^2 + \cdots + x_d^2}} = \frac{B_t^i}{R_t} \,)$$

$$\frac{\partial^2 f}{\partial x_i^2} = \frac{(R_t)^2 - (B_t^i)^2}{(R_t)^3}$$

The mixed partial derivatives for $i \neq j$

$$\frac{\partial^2 f}{\partial x_i \partial x_j} = -\frac{B_t^i B_t^j}{R_t^3}$$

3. Applying itô's formula

$$dR_t = \sum_{i=1}^{d} \frac{B_t^i}{R_t} dB_t^i + \frac{1}{2} \sum_{i=1}^{d} \frac{(R_t)^2 - (B_t^i)^2}{(R_t)^3} dt,$$

As $\sum_{i=1}^{d} (B_t^i)^2 = R_t^2$ (Radius of d-dimensional Sphere)

$$\therefore dR_t = \sum_{i=1}^{d} \frac{B_t^i}{R_t} dB_t^i + \frac{1}{2} \left(d - \frac{R_t^2}{R_t^2} \right) \frac{dt}{R_t}$$

$$\therefore dR_t = \sum_{i=1}^{d} \frac{B_t^i}{R_t} dB_t^i + \frac{d-1}{2} \frac{dR}{R_t}$$

Since $\sum_{i=1}^{d} \frac{B_t^i}{R_t} dB_t^i$ can be seen as the stochastic integral of a unit vector in the direction of B_t (normalized by R_t) with respect to B_t, it can be represented ~~by~~ as a standard brownian motion.

$$\therefore \; dR_t = dW_t + \frac{d-1}{2} \cdot \frac{dt}{R_t}.$$

This SDE reflects the behaviour of the radial part of a d-dimensional Brownian motion.

The term $\frac{d-1}{2} \cdot \frac{dt}{R_t}$ arises from the curvature effects in higher dimensions, effectively a mean-reverting force proportional to the inverse of the distance from origin.