Stochastic Calculus, Final Exam

1. Let B_t be a standard brownian motion.
Is $X_t = e^{\int_0^t B_s dB_s - \frac{1}{2}\int_0^t B_s^2 dB_s}$ a martingale.

→ let $Y_t = \int_0^t B_s \, dB_s - \frac{1}{2}\int_0^t B_s^2 \, dB_s$

then $X_t = \exp(Y_t)$.

1) Applying itô lemma on Y_t:

let $x = B_t$ $\therefore dx = dB_t$ $Y = f(x)$

$f_t = 0$, $f_x = B_t - \frac{1}{2}B_t^2$

$\therefore$ $f_{xx} = 1 - B_t$

$$dY = f_t \, dt + f_x \, dx + \frac{1}{2} f_{xx} \, dx \, dx$$

$$= 0 + \left(B_t - \frac{1}{2}B_t^2\right) dB_t + \frac{1}{2}(1 - B_t)\, dt$$

$$\therefore dY \, dY = \left(B_t - \frac{1}{2}B_t^2\right)^2 dt \quad — (1)$$

as $dB_t dt = dt dt = 0$ (Rules of stochastic Calculus)
& $dB_t dB_t = dt$

2) Applying itô lemma on X_t i.e. $\exp(Y_t)$

$$\therefore dX_t = 0\,dt + X_t \cdot dY_t + \frac{1}{2} X_t \, dY_t \cdot dY_t$$

$$\therefore dX_t = X_t \cdot \left(\left(B_t - \frac{1}{2}B_t^2\right) dB_t + \frac{1}{2}(1 - B_t)\,dt \right)$$

$$+ \frac{1}{2} X_t \left(B_t - \frac{1}{2}B_t^2\right)^2 dt$$

$$\therefore dX_t = \frac{1}{2} X_t \left[(1 - B_t) + \left(B_t - \frac{1}{2}B_t^2\right)^2 \right] dt$$

$$+ X_t\left(B_t - \frac{1}{2}B_t^2\right) dB_t$$

$$dX_t = \frac{1}{2} X_t \left[1 - B_t + B_t^2 - B_t^3 + \frac{1}{4}B_t^4 \right] dt$$

$$+ X_t \left(B_t - \frac{1}{2}B_t^2 \right) dB_t$$

This is a Stochastic differential equation with non zero drift term. Hence X_t is <u>not</u> a martingale due to the presence of ~~non ze~~ deterministic trend.

2. Let B_t be a standard Brownian Motion. What is the probability than on the interval $[0,1]$ it reaches level $a>0$ but the final value $B(1)$ is smaller than $-a$. What is the limit of your answer as $a \to 0$?

→ This probability can be split into two events:

1. B_t reaches the level a at some time $t_a \in [0,1]$
2. $B(1) < -a$

∴ ~~Our Required Probability~~

~~$= P(B(1) < -a \mid B_t$ reaches the level a at $t \in [0,1])$~~

1. Probability of B_t reaching a level a:

let τ_a be the first time the Brownian motion hits the point $a>0$:

$$\therefore \tau_a = \inf\{t \geq 0,\ W(t) \geq a\}$$

Computing the distribution of τ_a, $P[\tau_a \leq t]$

As $P[B_t \geq a] = P[B_t \geq a \mid \tau_a \leq t]\, P[\tau_a \leq t]$

$\qquad + P[B_t \geq a \mid \tau_a > t]\, P[\tau_a > t]$

We notice that the second term in the right hand side is necessarily equal to 0 because Brownian motion cannot be above a if it has not hit a yet by time t $(\tau_a > t)$.

$\therefore P[B_t \geq a] = P[B_t \geq a \mid \tau_a \leq t] \, P[\tau_a \leq t]$

As Reflection principle states - knowing that the Brownian motion has hit a sometime before time t, the Brownian motion is likely to be above a than below a, at time t.

$$\therefore P[B_t \geq a \mid \tau_a \leq t] = \frac{1}{2}$$

$$\therefore P[\tau_a \leq t] = 2 \, P[B_t \geq a]$$

$$= 2 \cdot \frac{1}{\sqrt{2\pi t}} \int_a^\infty e^{-\frac{x^2}{2t}} \, dx$$

let $\frac{x}{\sqrt{t}} = y$

$$\therefore \frac{dx}{\sqrt{t}} = dy$$

$$\therefore P[\tau_a \leq t] = \frac{2}{\sqrt{2\pi}} \int_{a/\sqrt{t}}^\infty e^{-y^2/2} \, dy \quad \text{--- (1)}$$

$$= 2\left(1 - \phi\left(\frac{a}{\sqrt{t}}\right)\right)$$

where ϕ is the standard normal cumulative distribution function.

By symmetry, the distribution of τ_a for $a < 0$ is the same a the distribution of τ_{-a} & implies in general

$$P[\tau_a \leq t] = 2\left(1 - \phi\left(\frac{|a|}{\sqrt{t}}\right)\right)$$

We need to compute the probability of the event where B_t hits a at least once in the interval $[0,1]$ and also ends up at a value less than $-a$ at time $t=1$.

We denote this event by

$$\{\tau_a \leq 1\} \cap \{B(1) < -a\}, \quad \tau_a = \inf\{t \geq 0 : B_t \geq a\}$$

is the first hitting time.

The reflection principle states that if B_t hits a at some time τ_a, then the process after τ_a can be thought of as starting anew from a but reflected about a. Thus, the probability that B_t hits a & later crosses down $-a$ is equivalent to the probability that a Brownian motion starting from 0 ends up below $-2a$ at time $t=1$, because once a is hit, reflecting about a means considering $B_t - a$ as the new Brownian motion starting at 0.

The brownian motion's increments independence property & symmetry gives

$$P[B(1) < -2a \mid \tau_a \leq 1] = P[B(1) < -2a]$$
$$= P[B(1) > 2a]$$

Since $B(1)$ is normally distributed as $N(0,1)$.

$$\therefore P[B(1) > 2a] = 1 - \phi(2a) \qquad - (2)$$

where ϕ is CDF of standard normal distribution.

Considering the Entire Event

The probability that B_t hits a at least once in $[0,1]$ & ends up at $B(1) < -a$ is the joint probability of hitting a & then going below $-a$ after hitting it.

$$P[\tau_a \leq 1] \times P[B(1) < -2a \mid \tau_a \leq 1]$$

from eq^n (1)

$$P[\tau_a \leq 1] = \frac{2}{\sqrt{2\pi}} \int_a^{\infty} e^{-y^2/2}\, dy$$

$$= \cancel{\frac{2}{\sqrt{2\pi}} \int} \; 2(1 - \phi(a))$$

from eq^n (2)

$$P[B(1) < -2a \mid \tau_a \leq 1] = 1 - \phi(2a)$$

$$\therefore P[\tau_a \leq 1 \cap B(1) < -a] = 2(1 - \phi(a))(1 - \phi(2a))$$

as $a \to 0$, $\phi(a)$ & $\phi(2a) \to \phi(0) = \frac{1}{2}$

$$\lim_{a \to 0} \therefore P[\tau_a \leq 1 \cap B(1) < -a] = 2\left(1 - \frac{1}{2}\right)\left(1 - \frac{1}{2}\right)$$

$$= \frac{1}{2}$$

3. Let $B_t = (B_t^{(1)}, B_t^{(2)})$ be a two-dimensional Brownian Motion. Using Itô's lemma check if

$X_t = \log\left((B_t^{(1)})^2 + (B_t^{(2)})^2 \right)$ is a martingale.

What is quadratic variation of process X_t on interval $[0,T]$?

→ Itô's formula in 2-d case:

$$df(t, X(t), Y(t)) = f_t(t, X(t), Y(t))dt$$

$$+ f_x(t,X,Y)\,dX + f_y(t,X,Y)\,dY$$

$$+ \frac{1}{2} f_{xx}(t,X,Y)\,dX.dX + \frac{1}{2} f_{yy}(t,X,Y)\,dY\,dY$$

$$+ f_{xy}(t,X,Y)\,dX\,dY$$

Applying it to our case:

$$df = 0\,dt + \frac{2 \cdot B_t^{(1)}}{(B_t^{(1)})^2 + (B_t^{(2)})^2}\,dB_t^{(1)}$$

$$+ \frac{2 \cdot B_t^{(2)}}{(B_t^{(1)})^2 + (B_t^{(2)})^2}\,dB_t^{(2)} + \frac{1}{2} f_{xx}\,dB_t^{(1)}dB_t^{(1)}$$

$$+ \frac{1}{2} f_{yy}\,dB_t^{(2)}dB_t^{(2)} + f_{xy}\,dB_t^{(1)}dB_t^{(2)}$$

— (1)

As

$$f_x = \frac{2x}{x^2+y^2}, \quad f_{xy} = -\frac{2x}{(x^2+y^2)^2}\cdot 2y = \frac{-4xy}{(x^2+y^2)^2}$$

$$f_{xx} = \frac{(x^2+y^2)(2) - 2x(2x)}{(x^2+y^2)^2} = \frac{2x^2+2y^2-4x^2}{(x^2+y^2)^2}$$

$$f_{xx} = \frac{2(y^2-x^2)}{(x^2+y^2)^2} = \frac{2\left((B_t^{(2)})^2 - (B_t^{(1)})^2\right)}{\left((B_t^{(1)})^2 + (B_t^{(2)})^2\right)^2}$$

Similarly $f_{yy} = \dfrac{2\left((B_t^{(1)})^2 - (B_t^{(2)})^2\right)}{\left((B_t^{(1)})^2 + (B_t^{(2)})^2\right)^2}$

Substituting in eq^n (1) & $dB_t^{(1)}dB_t^{(1)} = dt$
& $dB_t^{(2)}dB_t^{(2)} = dt$

$$df = \frac{2B_t^{(1)}}{(B_t^{(1)})^2 + (B_t^{(2)})^2}\cdot dB_t^{(1)} + \frac{2B_t^{(2)}}{(B_t^{(1)})^2 + (B_t^{(2)})^2}\, dB_t^{(2)}$$

$$+ \frac{1}{\cancel{2}}\cdot \cancel{2}\frac{\left((B_t^{(2)})^2 - (B_t^{(1)})^2\right)}{\left((B_t^{(1)})^2 + (B_t^{(2)})^2\right)^2}\, dt$$

$$+ \frac{1}{\cancel{2}}\cdot \cancel{2}\frac{\left((B_t^{(1)})^2 - (B_t^{(2)})^2\right)}{\left((B_t^{(1)})^2 + (B_t^{(2)})^2\right)^2}\, dt + f_{xy}\, dB_t^{(1)} dB_t^{(2)}$$

Both dt terms cancel each other

$$\therefore df = \frac{2B_t^{(1)}}{(B_t^{(1)})^2 + (B_t^{(2)})^2}\cdot dB_t^{(1)}$$

$$+ 2\frac{B_t^{(2)}}{(B_t^{(1)})^2 + (B_t^{(2)})^2}\cdot dB_t^{(2)} + f_{xy}\, dB_t^{(1)} dB_t^{(2)}$$

$dB_t^{(1)} dB_t^{(2)} = ?$

independent brownian motions

As $E\left[dB_t^{(1)} dB_t^{(2)}\right] = E\left[dB_t^{(1)}\right] E\left[dB_t^{(2)}\right]$

$= 0 \cdot 0 = 0 \quad -(2)$

$Var\left[dB_t^{(1)} dB_t^{(2)}\right] = E\left[\left(dB_t^{(1)} dB_t^{(2)}\right)^2\right]$

$- \left(E\left[dB_t^{(1)} dB_t^{(2)}\right]\right)^2$

$= E\left[\left(dB_t^{(1)}\right)^2\right] E\left[\left(dB_t^{(2)}\right)^2\right]$

$- 0$

$= dt \cdot dt - 0$

but $dt \cdot dt = 0$

$\therefore Var\left[dB_t^{(1)} dB_t^{(2)}\right] = 0 \quad - (3)$

From (2) & (3) $\quad dB_t^{(1)} dB_t^{(2)} = 0$

(necessarily zero)

$$\therefore df = 2\left[\frac{B_t^{(1)} \cdot dB_t^{(1)}}{(B_t^{(1)})^2 + (B_t^{(2)})^2} + \frac{B_t^{(2)} dB_t^{(2)}}{(B_t^{(1)})^2 + (B_t^{(2)})^2}\right]$$

As this SDE has no drift term (dt term)
, it is a martingale.

$\therefore X_t = \log\left((B_t^{(1)})^2 + (B_t^{(2)})^2\right)$ is a martingale.

As $dB_t^{(1)} dB_t^{(2)} = dB_t^{(2)} dB_t^{(1)} = 0$

$$df.df = 4\left[\frac{(B_t^{(1)})^2 dB_t^{(1)} dB_t^{(1)} + (B_t^{(2)})^2 dB_t^{(2)} dB_t^{(2)}}{\left((B_t^{(1)})^2 + (B_t^{(2)})^2\right)^2}\right]$$

$$df\,df = 4\left[\frac{(B_t^{(1)})^2\,dt \quad (B_t^{(2)})^2\,dt}{\left((B_t^{(1)})^2 + (B_t^{(2)})^2\right)^2}\right]$$

$\therefore$ Quadratic variation of X_t over $[0,T]$

$$\int_0^T df\,df = 4\int_0^T \frac{\left((B_t^{(1)})^2 + (B_t^{(2)})^2\right)}{\left((B_t^{(1)})^2 + (B_t^{(2)})^2\right)^2}\,dt$$

Quadratic Variation of X_t $= 4\int_0^T \frac{1}{(B_t^{(1)})^2 + (B_t^{(2)})^2}\,dt$

4. Let B_t, $t \geq 0$ be a standard Brownian motion starting from zero. Fix $t < T$ and a constant α. Compute $P(B_t > B_T + \alpha)$

As $\quad B_T = B_t + (B_T - B_t)$

$$\therefore P(B_t > B_T + \alpha) = P(B_t > B_t + (B_T - B_t) + \alpha)$$

$$= P(0 > (B_T - B_t) + \alpha)$$

$$= P(B_T - B_t < -\alpha)$$

As $B_T - B_t$ is increment of a standard brownian motion

$$B_T - B_t \sim N(0, T-t)$$

$$\frac{B_T - B_t}{\sqrt{T-t}} \sim N(0,1)$$

$$\therefore P(B_t > B_T + \alpha) = P\left(\frac{B_T - B_t}{\sqrt{T-t}} < -\frac{\alpha}{\sqrt{T-t}}\right)$$

$$= \phi\left(\frac{-\alpha}{\sqrt{T-t}}\right)$$

Where ϕ is CDF of standard normal distribution.

By symmetry,

$$\therefore P(B_t > B_T + \alpha) = \phi\left(\frac{-\alpha}{\sqrt{T-t}}\right) = 1 - \phi\left(\frac{\alpha}{\sqrt{T-t}}\right)$$

5. Let B_t be a standard brownian motion & $X_t = \int_0^t B_s \, dB_s$. What is the expectation & variance of X_t ? Calculate exactly $P(X_t > 0)$.

→ Let

$$f(t,x) = \frac{1}{2}x^2 \quad , \quad x = B_t \quad , \quad dx = dB_t$$

Applying Itô's formula,

$$df = f_t \, dt + f_x \, dB_t + \frac{1}{2} f_{xx} \, dt$$

$$= 0 \cdot dt + B_t \, dB_t + \frac{1}{2} \cdot 1 \cdot dt$$

$$\therefore df = B_t \, dB_t + \frac{1}{2} dt$$

integrating both sides from 0 to t.

$$\int_0^t d\left(\frac{1}{2}B_s^2\right) = \int_0^t \left(B_s \, dB_s + \frac{1}{2} ds\right)$$

$$\therefore \frac{1}{2}B_t^2 - \underbrace{\frac{1}{2}B_0^2}_{0} = \int_0^t B_s \, dB_s + \frac{t}{2}$$

$$\therefore \int_0^t B_s \, dB_s = \frac{1}{2}B_t^2 - \frac{t}{2}$$

$$\therefore X_t = \frac{1}{2}B_t^2 - \frac{t}{2}$$

$$\therefore E[X_t] = \frac{1}{2}E[B_t^2] - \frac{t}{2} = \frac{1}{2} \cdot t - \frac{t}{2} = 0$$

As $B_t \sim N(0,t)$

$$\mathrm{Var}(X_t) = \mathrm{Var}\left(\frac{1}{2}B_t^2 - \frac{t}{2}\right)$$

$$= \mathrm{Var}\left(\frac{1}{2}B_t^2\right)$$

$$= \frac{1}{4}\cdot \mathrm{Var}(B_t^2)$$

$$= \frac{1}{4}\left[\underbrace{E(B_t^4)}_{4^{th}\text{ moment of normal }(0,t)} - \left(E[B_t^2]\right)^2\right]$$

$$= \frac{1}{4}\left[3t^2 - (t)^2\right]$$

$$= \frac{1}{4}\cdot 2t^2 = \frac{t^2}{2}$$

$\therefore \mathrm{Var}(X_t) = \frac{t^2}{2}$, $E[X_t] = 0$

As $X_t = \frac{1}{2}B_t^2 - \frac{t}{2}$

~~X_t is a linear combination of square of normal random variable.~~
~~$\therefore X_t$ follows χ^2 distribution with 1 degree of freedom.~~

As $B_t \sim N(0,t)$, then B_t^2 follows a chi-square distribution with 1 degree of freedom scaled by t. i.e. $B_t^2 \sim t\chi_1^2$

$$P(X_t > 0) = P\left(\frac{1}{2}B_t^2 - \frac{t}{2} > 0\right)$$

$$= P\left(\frac{B_t^2}{2} > \frac{t}{2}\right)$$

$$= P(B_t^2 > t)$$

$$= P(B_t > \sqrt{t}) + P(B_t < -\sqrt{t})$$

$B_t \sim N(0, t)$

by symmetry $P(B_t > \sqrt{t}) = P(B_t < -\sqrt{t})$

$$\therefore P(X_t > 0) = 2P(B_t > \sqrt{t})$$

$$= 2 \cdot (1 - P(B_t < \sqrt{t}))$$

$$= 2\left(1 - P\left(\frac{B_t}{\sqrt{t}} < 1\right)\right)$$

As $\frac{B_t}{\sqrt{t}} \sim N(0, 1)$

$$\therefore P(X_t > 0) = 2(1 - \phi(1))$$

$$= 2(1 - 0.8413)$$

$$P(X_t > 0) = 0.3174$$