Chapter 7

Graph Theory and Algorithms*

7.1 Quiz-1 Discussion

This section presents the detailed answers of all the questions from Quiz-1.[1]

7.1.1 Question 1

Find out the number of edges in a Wheel Graph W_{125}

□ 124 □ 7750 □ 15376 □ 15625 □ 15500

Answer to question 1

What is a Wheel graph? A wheel graph W_n of order n, is a graph that contains a cycle of order $n-1$ and for which every graph vertex in the cycle is connected to one other graph vertex known as the hub. The edges of a wheel which include the hub are called spokes. The *wheel* was invented by the eminent graph theorist *W. T. Tutte*. For $n \geq 4$, the wheel W_n is defined as the graph join $K_1 + C_{(n-1)}$, where K_1 is the singleton graph and C_n is the cycle graph. Some examples of wheel graph are shown in the figure 7.1 [2] [4]
Number of edges in $C_{(n-1)} = (n-1)$
and every vertex of the cycle is connected to a single vertex(hub). Thus, we have additional $(n-1)$ edges.
Thus, total number of edges in a wheel graph is : $(n-1) + (n-1) = 2 \times (n-1)$
Now, given wheel graph is W_{125}. Number of edges in W_{125} are:

$$(2 \times (n-1)) = 2 \times (125-1) = 248$$

Note None of the given options are correct. The correct answer is : W_{125} has 248 edges.

[1] Quiz-1 was held on 22^{nd} January 2022

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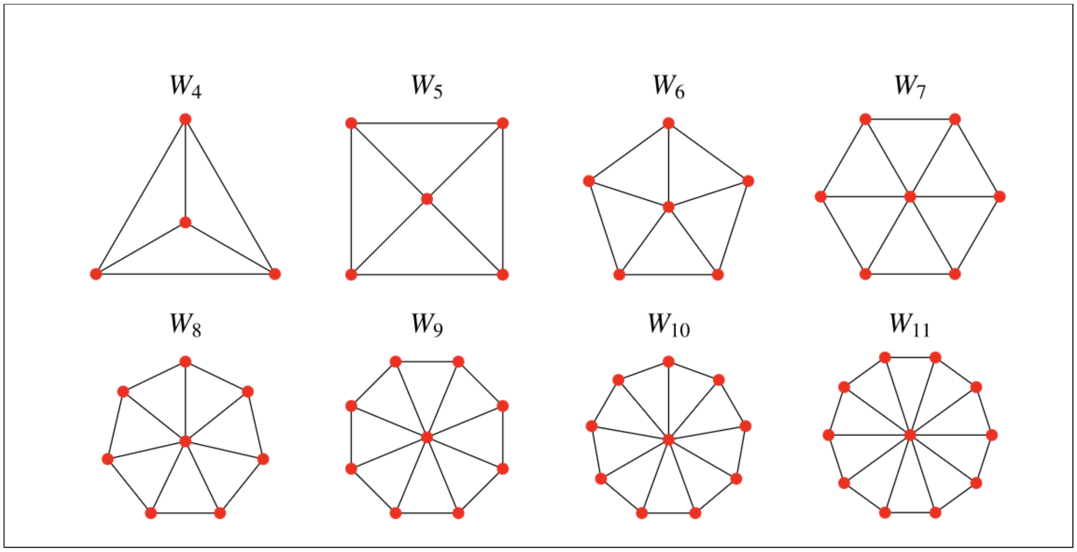


Figure 7.1: Examples of Wheel graph

7.1.2 Question 2

Consider the following two statements:

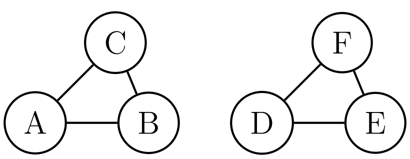
I. Every vertex of a graph G has degree $= 2$, then G is a cycle.

II. If G is a cycle, then every vertex of G has degree $= 2$.

- ☐ Both (I) and (II) are True.
- ☑ Only (II) is True.
- ☐ Only (I) is True.
- ☐ Both are False.

Answer to question 2

Let us consider the following graph G.



Every vertex in G has degree 2 but G is not cyclic. This demonstrates a counter-example to statement-I. Thus *statement-I is false.*

If G is a cycle graph, then every vertex of G will have degree 2. Thus *statement-II is true.*

7.1.3 Question 3

Which among the following is(are) the properties of Bipartite Graph?

I. It has chromatic number=2.

II. It has even number of edges.

III. It has even number of nodes.

IV. If it is a cycle it can not have odd number of nodes.

V. It is always a tree.

☑ Only (I) and (IV)

☐ Only (II) and (V)

☐ Only (I) and (V)

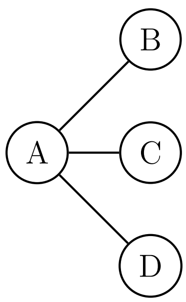
☐ Only (I), (II), (IV) and (V)

☐ All of them

Answer to question 3

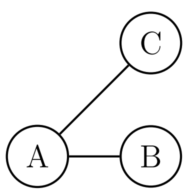
From the properties of a bipartite graph we know that :*The chromatic number of a bipartite graph is 2.*
Thus, *statement-I is true.*

Let us consider the following graph:



This graph is a bipartite graph and has 3 edges. This demonstrates a counter-example for statement-II.
Thus, *statement-II is false.*

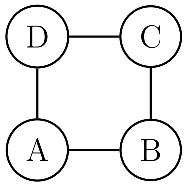
Let us consider the following graph:



This graph is a bipartite graph and has 3 nodes. This demonstrates a counter-example for statement-III.
Thus, *statement-III is false.*

From the properties of a bipartite graph we know that :*A graph is bipartite if and only if it has no odd cycle.* Let G be a graph which is a cycle. Then, G can be bipartite only if it is an even cycle and thus has even number of nodes.
Thus, *statement-IV is true.*

Let us consider the following graph:



This graph is a cycle and a bipartite graph bipartite graph with bi-partitions as $\{A, C\}$ and $\{B, D\}$. This demonstrates a counter-example for statement-V.
Thus, *statement-V is false.*

7.1.4 Question 4

Suppose U is a unicyclic graph of 4 nodes. Which among the following is/are necessarily True?

I. If one node is deleted it will not remain unicyclic.

II. If one node is added with increase in edge size = 1, it will remain unicyclic.

- ☐ Only (I)
- ☑ Only (II)
- ☐ Both (I) and (II)
- ☐ None are True

Answer to question 4

Let us consider the following graph G
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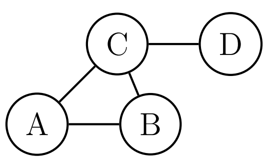


G is a unicyclic graph. It will remain unicyclic even after vertex D is removed. Thus, G demonstrates a counter-example for statement-I.
Hence, statement-I is false.

Notice that: adding one new vertex with an addition of one new edge will not have any effect on the cycle. Thus, the graph will remain unicyclic.
Hence, statement-II is true.

7.1.5 Question 5

Which among the following is/are False for a bipartite graph?

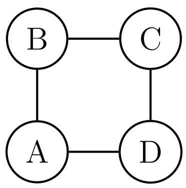
I. It can not be a unicyclic graph.

II. The degree of any node in it can not be more than $G = \max(m, n)$ where m and n are the cardinality of independent sets of vertices.

- ☑ Only (I)
- ☐ Only (II)
- ☐ Both (I) and (II)
- ☐ None are True

Answer to question 5

Recall the theorem : *A graph is bipartite if and only if it has no odd cycle.*
This suggests that we can have an even unicyclic graph which is a bipartite graph. Let us consider the following graph G:



Notice that, G is a bipartite graph with bi-partitions as $\{A, C\}$ and $\{B, D\}$. G is a unicyclic graph with cycle length 4. Thus, it demonstrates a counter-example for statement-I.
Thus,*statement-I is false.*

Let us assume that G is a bipartite graph with bi-partitions G_1 and G_2 with $|G_1| = m$ and $|G_2| = n$.
A vertex in G_1 will have the maximum degree only if it is connected with every vertex in G_2. Thus, maximum degree of any vertex in G_1 is $|G_2|(= n)$.
Similarly, a vertex in G_2 will have the maximum degree only if it is connected with every vertex in G_1. Thus, maximum degree of any vertex in G_2 is $|G_1|(= m)$.

Hence, the maximum degree of any vertex in bipartite graph $G = \max(m, n)$
Thus,*statement-II is true.*
Note We need to find the false statements.

7.1.6 Question based on India-map

For this section, please refer to the figure 7.3

Question 6

Use any recent map of India showing all the states. Consider following states: Chhattisgarh, Madhya Prasdesh, Odisha, Jharkhand, Uttar Pradesh, Maharastra and Rajasthan. Consider each state as a node, further neighbors are connected via an edge. Is there any cut-edge in this graph? (please refer to the figure 7.3)

- ☐ Yes
- ☑ No
- ☐ Can not be determined
- ☐ Depends on the connection

Question 7

Use any recent map of India showing all the states. Consider following states: Chhattisgarh, Madhya Prasdesh, Odisha, Jharkhand, Uttar Pradesh, Maharastra and Rajasthan. Consider each state as a node, further neighbors are connected via an edge. How many minimum number of edges in this graph needs to be removed, so that it becomes a bipartite graph? (please refer to the figure 7.3)

- ☐ It can never become a bipartite graph.
- ☐ Removing two edges
- ☐ It is already bipartite graph.
- ☐ Removing 1 edge
- ☑ Removing three edges.

Question 8

Use any recent map of India showing all the states. Consider following states: Chhattisgarh, Madhya Prasdesh, Odisha, Jharkhand, Uttar Pradesh, Maharastra and Rajasthan. Consider each state as a node, further neighbors are connected via an edge. What is the Chromatic Number of this graph? (please refer to the figure 7.3)

- ☐ 1
- ☐ 2
- ☑ 3
- ☐ 4

Answer to question 6,7,8

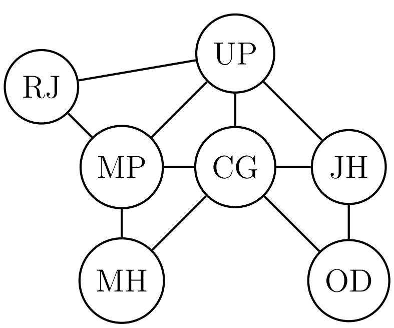
The map of India shown in figure 7.3 will be used for Question 6,7 and 8. [1]
We will use the following notations:

State	Notation
Chhattisgarh	CG
Madhya Prasdesh	MP
Odisha	OD
Jharkhand	JH
Uttar Pradesh	UP
Maharastra	MH
Rajasthan	RJ

Table 7.1: Notations for states

Figure 7.2: Map of India for Question 6,7 and 8

As given, each state is a node. If two states are neighbors then they are connected via an edge. We observe the states listed in table 7.1 and construct the graph.



Notice that, this graph will remain connected even after we remove any one edge from this graph. Thus, *there are no cut-edge in this graph.*
The scheme to find the chromatic number of this graph is demonstrated in table 7.2.
Chromatic number of this graph is 3.

Selected vertex	Non-neighbor vertices (which are not yet colored)	Assigned color	Remaining vertices (to be colored)
CG	RJ	Red	{UP,MP,JH,OD,MH}
UP	MH and OD	Green	{MP,JH}
MP	JH	Blue	{}

Table 7.2: Finding the chromatic number of graph in question-8

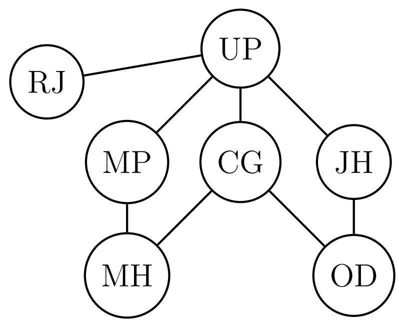
If we want to make the graph into a bipartite graph we need to remove some edges.
Recall that : *the chromatic number of a bipartite graph is 2.*
We want to remove such edges which will make the chromatic number of this graph as 2.

Let us remove the edges joining the vertices :

i. CG and JH

ii. CG and MP

iii. MP and RJ



Thus, we get the new graph. This graph has chromatic number 2 and thus it is a bipartite graph.
Thus, *we need to remove 3 edges to make the given graph bipartite.*

7.1.7 Question 9

Sort the following graphs in ascending order based on chromatic number (Assume n is even number and greater than 10):

I. C_n II. K_n III. W_n

- ☐ (I) < (II) < (III)
- ☐ (III) < (II) < (I)
- ☑ (I) < (III) < (II)
- ☐ (II) < (III) < (I)

Answer to question 9

Graph	Chromatic number
C_n	2 (if n is even) 3 (if n is odd)
K_n	n
$W_n (n \geq 3)$	4 (if n is even) 3 (if n is odd)

Table 7.3: Chromatic number of C_n, K_n and W_n

The chromatic number of C_n, K_n and W_n are listed in the table 7.3. As given, n is an even number (greater than 10). Thus:

i. chromatic number of C_n is 2

ii. chromatic number of K_n is n

iii. chromatic number of W_n is 4

Thus, we get *(I) < (III) < (II)*.

7.1.8 Question 10

Which among the following is correct?

I. Every Tree is Path.

II. Every Path is a Walk.

III. Every walk is a Path

- ☐ Only (I)
- ☑ Only (II)
- ☐ Only (III)
- ☐ Only (I) and (II)
- ☐ Only (I) and (III)

Answer to question 10

From the definitions of tree,path and walk, we know that:
Every tree is not a path because walk can have repeated edges or vertices whereas tree can not. Thus, *statement-I is false.*
Every path is a walk by definition of path. Thus, *statement-II is true.*
Every walk is not a path because walk can have repeated edges or vertices whereas path can not. Thus, *statement-III is false.*

7.2 Vertices, degree and counting

Definition 7.1. The ***degree sequence*** of a graph is the list of vertex degrees, usually written in non-increasing order as $d_1 \geq d_2 \geq \cdots \geq d_n$ [5]

Definition 7.2. A ***graphic sequence*** is a list of non-negative numbers that is the degree sequence of some simple graph. A simple graph with degree sequence d *realizes* d. [5]

Proposition 7.3. *The non-negative integers $d_1, \cdots, d_n$ are the vertex degrees of some graph if and only if $\sum d_i$ is even. [5]*

Proof. Necessity

Let G be a graph with vertices $v_1, v_2, \cdots, v_n$ and corresponding degree of the vertices as $d_1, \cdots, d_n$. As per handshaking lemma, we get:

$$\sum_{i=1}^{n} d_i = 2e(G)$$

where $e(G)$ is the number of edges in graph G.
Hence, $\sum d_i$ is even.

Sufficiency

Let $\sum d_i$ is even.
We need to construct a graph, with vertex set $v_1, v_2, \cdots, v_n$ with $d(v_i) = d_i \ \forall i$.
Now, as $\sum d_i$ is even that implies, the number of odd values is even.
Let us construct an arbitrary pairing of the vertices in $\{v_i : d_i \text{ is odd}\}$. For each such pair of vertices we form an edge such that these two vertices are the endpoints of that edge.
The remaining degree needed at each vertex is even and non-negative.We construct $\lfloor \frac{d_i}{2} \rfloor$ loops at $v_i \ \forall i$.
This completes the construction of the graph G with vertex degrees $v_1, v_2, \cdots, v_n$. □

Theorem 7.4. *(Havel [1955], Hakimi [1962])*
For $n > 1$, an integer list d of size n is graphic if and only if d' is graphic, where d' is obtained from d by deleting its largest element Δ and subtracting 1 from its Δ next largest elements. The only 1-element graphic sequence is $d_1 = 0$. [5]

Proof. Notice that, *For* $n = 1$, the statement is trivially true.

For $n > 1$

Sufficiency

Let us consider an integer list d such that $d = \{d_1 \geq d_2 \geq \cdots \geq d_n\}$ and let G' be a simple graph with degree sequence d'.
We add a new vertex adjacent to vertices in G' with degrees $\{(d_2-1), (d_3-1) \cdots, (d_{\Delta+1}-1)\}$.
These d_i are the Δ largest elements of d after (one copy of) Δ itself.
However, $\{(d_2-1), (d_3-1) \cdots, (d_{\Delta+1}-1)\}$ need not be the Δ largest numbers in d'.

Necessity

Let us assume the following:
G be a simple graph realizing d and it produces a simple graph G' realizing d'.
$d_w(G) = \Delta$
$S = \{d_2, d_3, \cdots, d_{\Delta+1}\}$ then $|S| = \Delta$
Let S be the set of Δ vertices in G with degree $\{d_2, d_3, \cdots, d_{\Delta+1}\}$ and are obtained from d by deleting its largest element Δ and subtracting 1 from its Δ next largest elements.

Case-1 : If $N(w) = S$, we delete w to obtain G'
Case-2 : If $N(w) \neq S$, that implies, some vertex of S is missing from $N(w)$.
In this case, we modify G to increase $|N(w) \cap S|$ without changing any vertex degree.
Since $|N(w) \cap S|$ can increase at most Δ times, repeating this converts G into another graph G^* that realizes d and has S as the neighborhood of w.
From G^* we delete w to obtain the desired graph G' realizing d'.

To find the modification when $N(w) \neq S$, we choose $x \in S$ and $z \notin S$ so that $w \leftrightarrow z$ and $w \not\leftrightarrow x$. We want to add wx and delete wz, but we must preserve vertex degrees.
Since $d(x) \geq d(z)$ and already w is a neighbor of z but not x, there must be a vertex y adjacent to x but not to z. Now we delete $\{wz, xy\}$ and add $\{wx, yz\}$ to increase $|N(w) \cap S|$. $\square$

Theorem 7.5. *A partition $\Pi = (d_1, d_2, \cdots, d_p)$ of an even number into p parts with $(p-1) \geq d_1 \geq d_2 \geq \cdots \geq d_p$ is graphical if and only if the modified partition $\Pi = ((d_2-1), (d_3-1), \cdots, d_p)$ is graphical. [4]*

Corollary 7.6 (*Havel-Hakimi Algorithm*)**.** *A given partition $\Pi = (d_1, d_2, \cdots, d_p)$ with $(p-1) \geq d_1 \geq d_2 \geq \cdots \geq d_p$ is graphical if and only if the following procedure results in a partition with every summand zero. [4]*

1. Determine the modified partition Π' *as in the statement of Theorem 7.5*

2. Reorder the terms of Π' *so that they are non-increasing and call the resulting partition* Π_1.

3. Determine the modified partition Π'' *of* Π_1*, as in step-1, and the reordered partition* Π_2.

4. Continue the process as long as non-negative summands can be obtained.

If a partition obtained at an intermediate stage is known to be graphical, stop, since Π *itself is then established as graphical.*

7.2.1 Numerical examples on Havel-Hakimi Algorithm

Which of the following are graphic sequences? Provide a construction or a proof of impossibility for each.

a. (5,5,4,3,2,2,2,1)

b. (5,5,4,4,2,2,1,1)

c. (5,5,5,3,2,2,1,1)

d. (5,5,5,4,2,1,1,1)

We will use the Havel-Hakimi Algorithm describe Havel-Hakimi Algorithm in 7.6.

Solution for Part-a

Given sequence is $\Pi_0 = (5, 5, 4, 3, 2, 2, 2, 1)$
Notice that: sum of all the elements in the given sequence is 24 which is even. Thus, we proceed to apply Havel-Hakimi algorithm which is described below:

Iteration-1
Gievn sequence is in non-increasing order. The largest degree is 5. Thus, we omit the first element and subtract 1 from the next 5 largest elements of this sequence. The modified sequence is:
$\Pi_1 = (5-1, 4-1, 3-1, 2-1, 2-1, 2, 1) = (4, 3, 2, 1, 1, 2, 1)$ but this is not in non-increasing order. So, we re-arrange it to be in a non-increasing order $\Pi'_1 = (4, 3, 2, 2, 1, 1, 1)$

Iteration-2
The largest degree is 4. Thus, we omit the first element and subtract 1 from the next 4 largest elements of this sequence. The modified sequence is:
$\Pi_2 = (3-1, 2-1, 2-1, 1-1, 1, 1) = (2, 1, 1, 0, 1, 1)$ but this is not in non-increasing order. So, we re-arrange it to be in a non-increasing order $\Pi'_2 = (2, 1, 1, 1, 1, 0)$

Iteration-3

The largest degree is 2. Thus, we omit the first element and subtract 1 from the next 2 largest elements of this sequence. The modified sequence is:
$\Pi_3 = (1-1, 1-1, 1, 1, 0) = (0, 0, 1, 1, 0)$ but this is not in non-increasing order. So, we re-arrange it to be in a non-increasing order $\Pi'_3 = (1, 1, 0, 0, 0)$

Iteration-4

The largest degree is 1. Thus, we omit the first element and subtract 1 from the next 1 largest elements of this sequence. The modified sequence is:
$\Pi_4 = (1-1, 0, 0, 0) = (0, 0, 0, 0)$
Notice that Π_4 is a trivial graph and thus it is graphical. Hence, Π_0 is also graphical.

Summary of solution

Following the above explanation we can solve the other parts. A summary of the solution is shown in tables 7.11 and 7.9

(5,5,4,3,2,2,2,1)
(4,3,2,1,1,2,1)
(4,3,2,2,1,1,1)
(2,1,1,0,1,1)
(2,1,1,1,1,0)
(0,0,1,1,0)
(1,1,0,0,0)
(0,0,0,0) $\rightarrow$ graphical

Table 7.4: Part-a

(5,5,4,4,2,2,1,1)
(4,3,3,1,1,1,1)
(2,2,0,0,1,1)
(2,2,1,1,0,0)
(1,0,1,0,0)
(1,1,0,0,0)
(0,0,0,0) $\rightarrow$ graphical

Table 7.5: Part-b

Table 7.6: Havel-Hakimi algorithm for part a and b

(5,5,5,3,2,2,1,1)
(4,4,2,1,2,1,1)
(4,4,2,2,1,1,1)
(3,1,1,0,1,1)
(3,1,1,1,1,0)
(0,0,0,1,0)
(1,0,0,0,0) $\rightarrow$ graphical

Table 7.7: Part-c

(5,5,5,4,2,1,1,1)
(4,4,3,1,0,1,1)
(4,4,3,1,1,1,0)
(3,2,0,0,1,0)
(3,2,1,0,0,0) $\rightarrow$ does not exist[a]

Table 7.8: Part-d

[a] A simple graph with this degree list does not exist because if there is a vertex of degree 3 that will require three other vertices with non-zero degree.

Table 7.9: Havel-Hakimi algorithm for part c and d

7.2.2 Question 11

Prove or disprove: If u and v are the only vertices of odd degree in a graph G,then G contains a $u - v$ path.

Answer to question 11

Given that : Graph G has exactly two vertices of odd degree namely u and v.
Recall the handshaking lemma (degree-sum formula). If G is a graph then

$$\sum_{v \in V(G)} d(v) = 2e(G)$$

and as a corollary to handshaking lemma we have: s*Every graph has an even number of vertices of odd degree.*
Now, let us claim: Given graph G is connected.
Proof of claim: Proof by contradiction.
Let us assume G is disconnected. A graph having exactly two vertices of odd degree must contain a path from one to the other. The degree of a vertex in a component of G is the same as its degree in G. If the vertices of odd degree are in different components, then those components are graphs with odd degree sum. Every component of a graph is a graph in itself and this contradicts the handshaking lemma in every component of the graph.
Thus, our assumption is wrong. Hence, G is connected and there exits a $u - v$ path in G.

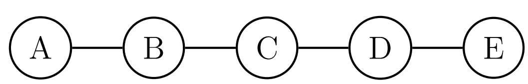
7.2.3 Question 12

Determine the maximum number of edges in a bipartite subgraph of P_n of C_n and K_n.

Answer to question 12

The path graph P_n is a tree with two nodes of vertex degree 1, and the other $n - 2$ nodes of vertex degree 2. A path graph is therefore a graph that can be drawn so that all of its vertices and edges lie on a single straight line. [2]

Example of path graph with 5 vertices:



Notice that : P_n is a bipartite graphs.
Thus, the largest bipartite subgraph of P_n is P_n itself.
Hence, number of edges in in the largest bipartite subgraph of the largest bipartite subgraph of P_n is $(n - 1)$

Notice that : If n is even, then C_n itself a bipartite graph. Thus, if n is even, then the largest bipartite subgraph of C_n is C_n itself. In this case, the number of edges in C_n is n.

Recall the theorem: *A graph is bipartite if and only if it contains no odd cycle.*
Thus, when n is odd, C_n is *not* a bipartite graph.To make it bipartite we need to remove an edge and after removing an edge C_n gets converted into P_n. Thus, if n is odd, then the largest bipartite subgraph of C_n is P_n and number of edges in P_n is $(n-1)$.

$$\textit{Number of edges in the largest bipartite subgraph of } C_n = \begin{cases} n & \text{when } n \text{ is even} \\ (n-1) & \text{when } n \text{ is odd} \end{cases}$$

The largest bipartite subgraph of K_n is $K_{\lfloor \frac{n}{2} \rfloor, \lfloor \frac{n}{2} \rfloor}$ which has $\lfloor \frac{n^2}{4} \rfloor$edges.
Hence,*number of edges in the largest bipartite subgraph of* $K_n = \lfloor \frac{n^2}{4} \rfloor$

7.2.4 Question 13

Let l, m and n be non-negative integers with $l + m = n$. Find necessary and sufficient conditions on l, m, n such that there exists a connected simple n-vertex graph with l vertices of even degree and m vertices of odd degree.

Answer to question 13

Notice that : As we want to have a n-vertex graph, thus, $n \geq 1$ for the graph to exist.

Necessity

Let us assume that G is a simple connected n-vertex graph with l vertices of even degree and m vertices of odd degree.
Given that: l, m and n be non-negative integers with $l + m = n(\geq 1)$.

From the corollary of handshaking lemma we know that :*Every graph has an even number of vertices of odd degree.*
Thus, m must be even except for $(l, m, n) = (2, 0, 2)$.

Consider the case of $(l, m, n) = (2, 0, 2)$. This implies, G has 2 vertices and both of them are of even degree.
The only simple-connected graph of 2 vertices is P_2. It has 2-vertices and both of them have degree 1. Thus, $l = 0$, $m = 2$ and $n = l + m = 0 + 2 = 2$. Hence, $(1, m, n) = (0, 2, 0)$
Thus we can conclude that, $(l, m, n) = (2, 0, 2)$ is *not* possible.

Sufficiency

Let us construct such a graph G. We take a cyclic graph G of length l. This has l vertices of even degree.
Now, we add m vertices of degree one with a common neighbor on the cycle.That vertex of the cycle has even degree because m is even.

7.3 Revisiting walk, path, trail, cycle, and circuits

Definition 7.7. A ***walk*** is a list $v_0, e_1, v_1, e_2, v_2 \cdots, v_{k-1}, e_k, v_k$ of vertices and edges such that, for $1 \leq i \leq k$, the edge e_i, has endpoints v_{i-1} and v_i.

Notice that, none of the vertices or edges are repeated in a walk.

Definition 7.8. A ***path*** is a walk where vertices can not repeat.

Notice that, as vertices can not repeat in a path, this implies that the edges can not repeat in path.

Definition 7.9. A ***trail*** is an open walk where edges can not repeat, but vertices can repeat.

Definition 7.10. A ***cycle*** is a closed walk where all vertices except end-point vertices are distinct.

Notice that, as all the vertices (except the end-points) of a cycle graph are distinct, thus it follows that edges of a cycle graph will be distinct.

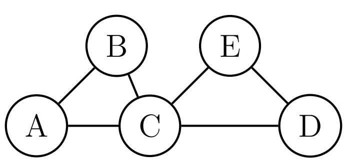
Definition 7.11. A ***circuit*** is a closed walk where all edges are distinct.

Notice that, as the only requirement for a closed walk is to have distinct edges and no conditions on vertices, thus, vertices can repeat in a closed walk.

These definitions can be understood from the summary table 7.10 [3]

Characteristic	Any reuse	Don't reuse edges	Don't reuse vertices (or edges)
Start and end anywhere	walk	trail	path
Start and end at the same place	closed walk	circuit	cycle

Table 7.10: Summary of definitions of walk, path, trail, cycle, and circuits



i. ACDECB is a trail but not a path

ii. ACDECBA is not a cycle

iii. ACDECBA is a circuit

Table 7.11: Example of path,cycle,circuit

7.4 Directed Graphs

Definition 7.12. A ***directed graph*** or ***digraph*** G is a triple consisting of a vertex set $V(G)$, an edge set $E(G)$, and a function assigning each edge an ordered pair of vertices. The first vertex of the ordered pair is the tail of the edge, and the second is the head; together, they are the endpoints. We say that an edge is an edge from its tail to its head. [5]

If there is an edge from u to v, then v is a successor of u, and u is a predecessor of v. We write $u \to v$ for *there is an edge from u to v*.

Definition 7.13. The ***underlying graph*** of a digraph D is the graph G obtained by treating the edges of D as unordered pairs; the vertex set and edge set remain the same, and the endpoints of an edge are the same in G as in D, but in G they become an unordered pair. [5]

Representation of directed graphs We can represent a digraph G using the ***adjacency matrix*** denoted as $A(G)$.
$A(G) = m_{i,j}$ where $m_{i,j}$ denotes the number of edges from vertex v_i to vertex v_j.

Another method of representing a digraph is to use the ***incidence matrix*** which is denoted as $M(G)$. Let $M(G)$ be the incidence matrix of a loop-less digraph G and $M(G) = m_{i,j}$.

$$m_{i,j} = \begin{cases} +1 & \text{if } v_i \text{ is the tail of } e_j \\ -1 & \text{if } v_i \text{ is the head of } e_j \end{cases}$$

Example: [5]

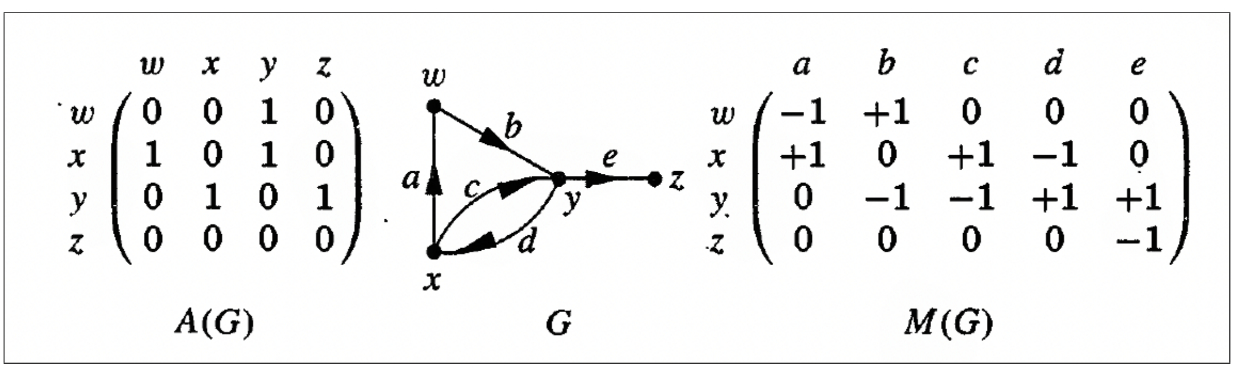


Figure 7.3: Example of adjacency and incidence matrix for a directed graph

Definition 7.14. A digraph is ***weakly connected*** if its underlying graph is connected. [5]

Definition 7.15. A digraph is ***strongly connected*** or strong if for each ordered pair u, v of vertices, there is a path from u to v. [5]

Example:
The 2-vertex digraph consisting only of the edge xy has an (x, y)-path but no (y, x)-path. Also, it is not strongly connected.

Definition 7.16. The ***strong components*** of a digraph are its maximal strong subgraphs.[5]

Example:
As a digraph, an n vertex path has n strong components, but a cycle has only one.

Definition 7.17. A ***kernel*** in the digraph D is a set $S \subseteq V(D)$ such that S induces no edges and every vertex outside S has a successor in S.[5]

Theorem 7.18. *(Richardson [1953])*
Every digraph having no odd cycle has a kernel.[5]

Proposition 7.19. *In a digraph G: [5]*

$$\sum_{v \in V(G)} d^+(v) = e(G) = \sum_{v \in V(G)} d^-(v)$$

Proof. Every directed edge in the digraph G will have exactly one head. Thus, their sum must be same as number edges in G.
Similarly, every directed edge will have exactly one tail and their sum must be same as $e(G)$. □

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