$P_{data}(x)$

$\downarrow$

$\{x_1, x_2, \ldots, x_N\} \longrightarrow \underline{\underline{P(x)}} \longrightarrow$ Sample new points

$\downarrow$

$z_1, z_2, \ldots, z_k$ ✓

$\downarrow$

X

$$P(x) = \int P(x, z)\, dz$$

$$\underbrace{\min_{P(x)}}\; D_{KL}\left(P_{data}(x) \,\|\, P(x)\right) \qquad P(x) \leftarrow \mu, \Sigma$$

$$D_{KL}\left(P_{data}(x) \,|\, P(x)\right) = \mathbb{E}_{x \sim P_{data}}\left[\log \frac{P_{data}(x)}{P(x)}\right]$$

$$= \mathbb{E}_{x \sim P_{data}}\left[\log P_{data}(x) - \log P(x)\right] \qquad \log \frac{a}{b} = \log a - \log b$$

$$= \mathbb{E}_{x \sim P_{data}}\left[\log P_{data}(x)\right] - \underbrace{\mathbb{E}_{x \sim P_{data}}\left[\log P(x)\right]} \qquad \mathbb{E}[a + b] = \mathbb{E}[a] + \mathbb{E}[b]$$

$$= \mathbb{E}_{x \sim P_{data}}\left[\log P_{data}(x)\right] - \int_X \log P(x)\, P_{data}(x)\, dx$$

$$= \mathbb{E}_{x \sim P_{data}}\left[\log P_{data}(x)\right] - \log P(x) \int P_{data}(x)\, dx$$

$$= \mathbb{E}_{x \sim P_{data}}\left[\log P_{data}(x)\right] - \log P(x)$$

$$\min_{P(x)} D_{KL}\left(P_{data}(x) \,\|\, P(x)\right)$$

$$\underbrace{\min_{P(x)} \underline{\mathbb{E}_{x \sim P_{data}}\left[\log P_{data}(x)\right]}} - \min_{P(x)} \log P(x)$$

$$= \max_{P(x)} \log P(x)$$

$$\log p(x) \qquad\qquad p(x) = \sum_z p(x,z)$$

$$= \log\left(\sum_z p(x,z)\right)$$

$$q(z) \qquad p(z|x)$$

$$= \log\left(\sum_z \frac{p(x,z)}{q(z)} q(z)\right) \qquad g(z) = \frac{p(x,z)}{q(z)}$$

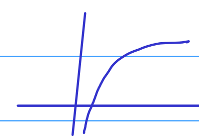
$$q(z) \qquad \mathbb{E}[z] = \sum z\, q(z)$$

$$= \log\left(\mathbb{E}_{q(z)}\left[\frac{p(x,z)}{q(z)}\right]\right) \qquad \mathbb{E}[g(z)] = \sum_z g(z)\, q(z) \ \checkmark$$

$$g(z) = 3z^2 + 2z$$

$$\geq \mathbb{E}_{z\sim q(z)}\left[\log\left(\frac{p(x,z)}{q(z)}\right)\right] \qquad \mathbb{E}[f(z)] \leq f(\mathbb{E}[z]) \qquad f \to \text{concave function}$$

$$= \text{ELBO (Evidence Lower Bound)} \qquad f \to \log$$



$$\text{ELBO} = \mathbb{E}_{z\sim q(z)}\left[\log(p(x,z)) - \log q(z)\right]$$

$$= \mathbb{E}_{z\sim q(z)}\left[\log(p(x,z))\right] - \underbrace{\mathbb{E}_{z\sim q(z)}\left[\log q(z)\right]}$$

$$= \mathbb{E}_{z\sim q(z)}\left[\log(p(x,z))\right] - \sum_z q(z)\log q(z)$$

$$= \mathbb{E}_{z\sim q(z)}\left[\log(p(x,z))\right] + H(q)$$

$$D_{KL}\left(q(z)\,||\,p(z|x)\right) = \mathbb{E}_{z\sim q(z)}\left[\log\frac{q(z)}{p(z|x)}\right]$$

$$= \sum_z q(z)\log\frac{q(z)}{p(z|x)}$$

$$= \sum_z q(z)\log q(z) - \sum_z q(z)\log(p(z|x))$$

$$= -H(q) - \sum_z q(z)\log(p(z|x)) \qquad p(z|x) = \frac{p(z,x)}{p(x)}$$

$$= -H(q) - \sum_z q(z)\log p(z,x) + \sum_z q(z)\underline{\log p(x)} \qquad \log\left(\frac{p(z,x)}{p(x)}\right)$$

$$= \log p(x,z) - \log p(x)$$

$$D_{KL}\left(q(z)\,||\,p(z|x)\right) = -H(q) - \sum_z q(z)\log(p(x,z)) + \log p(x)$$

$$\log p(x) = D_{KL}\left(q(z)\,||\,p(z|x)\right) + \underbrace{H(q) + \sum_z q(z)\log(p(x,z))}_{\text{ELBO}}$$

$$\underline{\log p(x)} = \underline{\text{ELBO}} + \underline{D_{KL}\left(q(z)\,||\,p(z|x)\right)}$$

$$\log p(x) \geq \text{ELBO}$$

$x = 1024 \times 1024 = 2^{10} \times 2^{10} = 2^{20}$

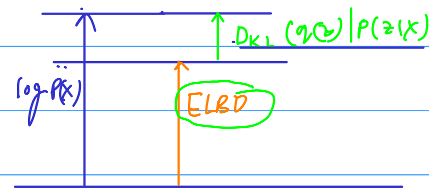
$\sum_{x}$

$\log p(x) = \text{ELBO}$ iff $D_{KL}(q(z) \,||\, p(z|x)) = 0$

$\Rightarrow q(z) = p(z|x)$

$p(z|x) = \dfrac{p(x,z)}{p(x)}$

$q_\phi(z)$



$\max \log p(x) \Rightarrow \max \text{ELBO}$
$\Rightarrow \min D_{KL}(q(z) \,||\, p(z|x))$

$\max \log p(x) \iff \max \; \mathbb{E}_{q}[\log p(x|z)]$ such that $q(z) = p(z|x)$

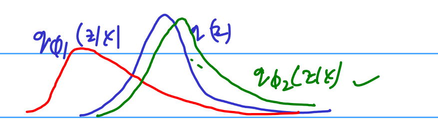
$q(z) = \sum_{x} p(x,z)$

$q_\phi(z) = \mathcal{N}(0, \sigma^2 I)$

$q_\phi(z|x) = \mathcal{N}(\mu_{z|x}, \Sigma_{z|x})$

$p(z|x)$

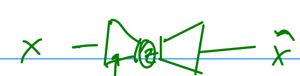
$q_\phi(z|x) \sim p(z)$



$q_\phi(z|x) = \mathcal{N}(\mu_{z|x}, \Sigma_{z|x})$

$p(x|z) = \mathcal{N}(\mu_{x|z}, \Sigma_{x|z})$

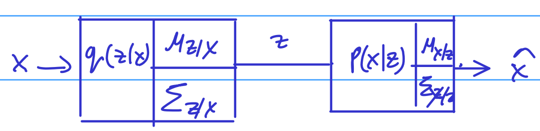
$q(z) = \mathcal{N}(0, \sigma^2 I)$



$x \rightarrow z \quad z \rightarrow \hat{x}$

$q(z|x) \quad p(x|z)$

$x \xrightarrow{q(z|x)} z \xrightarrow{p(x|z)} \hat{x}$



Variational Auto-Encoders