Week 4

Friday: Quiz-1 Discussion; Vertex Degree and Counting*

4.1 Quiz-1 Discussion

Q1. Suppose U is a uni-cyclic graph of 4 nodes. Which among the following is/are necessarily True?
(I) If one node is deleted it will not remain uni-cyclic.
(II) If one node is added with increase in edge size =1, it will remain uni-cyclic.

(A) None are True

(B) Only (I)

(C) Only (II)

(D) Both (I) and (II)

Answer: (C)

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Explanation:

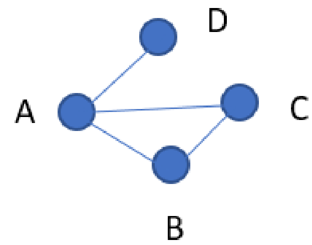

Figure 4.1: Unicyclic graph of 4 nodes

Graph will remain unicyclic even after deleting node D.

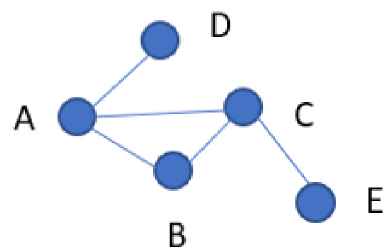

Figure 4.2: Unicyclic graph of 5 nodes

Addition of node E along with an edge C-E makes the graph remain unicylic. **Q2. Which among the following is/are False for a bipartite graph?**

(I) It can not be a unicylic graph

(II) The degree of any node in it can not be more than $max(m, n)$ where m and n are the cardinality of independent sets of vertices.

(A) Both

(B) Only (II)

(C) Only (I)

(D) None

Answer: (B)

Explanation:

Let us understand with some examples:

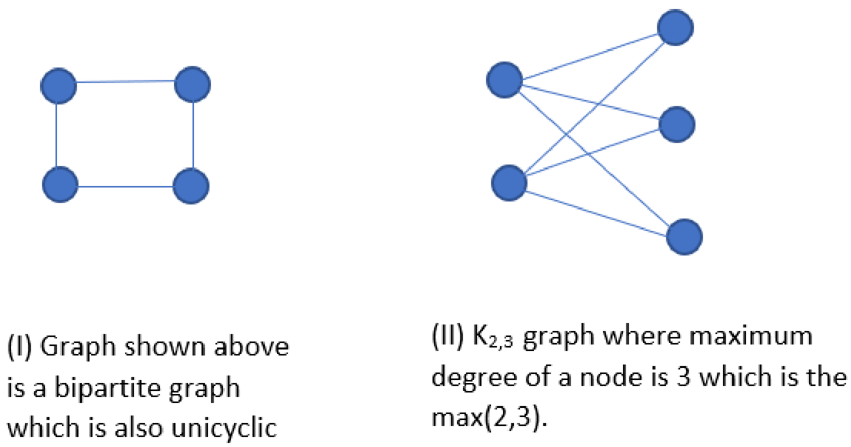

Figure 4.3: India Map

Hence, there exists a graph which is bipartite and unicyclic for the first option and complete bipartite will have have maximum degree nodes in the second option. Maximum degree of a node will be the max of m and n where m,n are two set of vertices. Therefore, option (C) gives the correct answer.

Consider the Indian Map below and answer the questions that follow:

Figure 4.4: Examples of Q2

Q3. Use any recent map of India showing all the states. Consider following states: Chhattisgarh, Madhya Prasdesh, Odisha, Jharkhand, Uttar Pradesh, Maharastra and Rajasthan. Consider each state as a node, further neighbors are connected via an edge. What is the Chromatic Number of this graph?
(Map is as shown in Figure 4.1)

(A) 1

(B) 2

(C) 4

(D) 3

Answer: (D)

Explanation:

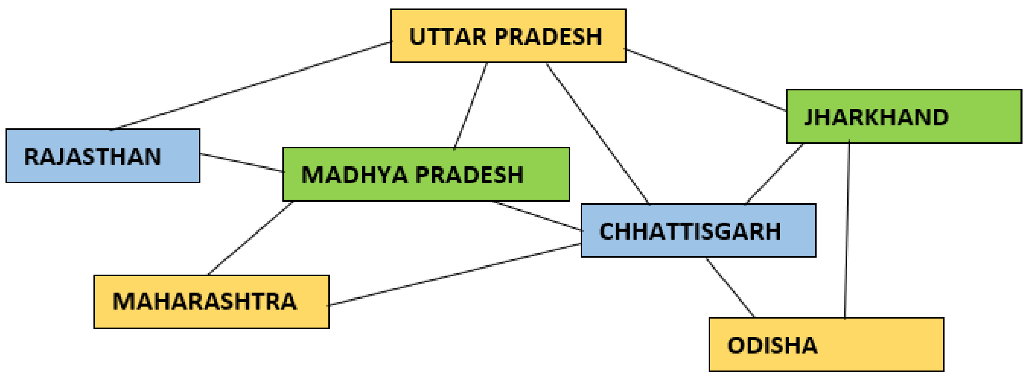

Figure 4.5: Connected graph with chromatic number 3

**Q4. Use any recent map of India showing all the states. Consider following states: Chhattisgarh, Madhya Prasdesh, Odisha, Jharkhand, Uttar Pradesh, Maharastra and Rajasthan. Consider each state as a node, further neighbors are connected via an edge. Is there any cut-edge in this graph?
(Map is as shown in Figure 4.1)**

(A) No

(B) Can not be determined

(C) Yes

(D) Depends on the connection

Answer: (A)

Explanation:

Referring to the graph given in Figure 4.4, every node is connected to atleast two other nodes which makes it impossible to have any cut-edge in the graph.

Q5. Use any recent map of India showing all the states. Consider following states: Chhattisgarh, Madhya Pradesh, Odisha, Jharkhand, Uttar Pradesh, Maharastra and Rajasthan. Consider each state as a node, further neighbors are connected via an edge. How many minimum number of edges in this graph needs to be removed, so that it becomes a biparatite graph?
(Map is as shown in Figure 4.1)

(A) Removing 3 edges

(B) Removing 1 edge

(C) It can never become a bipartite graph

(D) Removing 2 edges

(E) It is already bipartite graph

Answer: (A)

Explanation:

From figure 4.4 we can try removing any one edge or two edges but the graph will not become bipartite in nature. Removing 3 edges from the graph will make the graph bipartite.
For example, removal of edges between Rajasthan-Madhya Pradeh, Madhya Pradesh-Chattisgarh, Chattisgarh-Jharkhand will leave a bipartite graph.

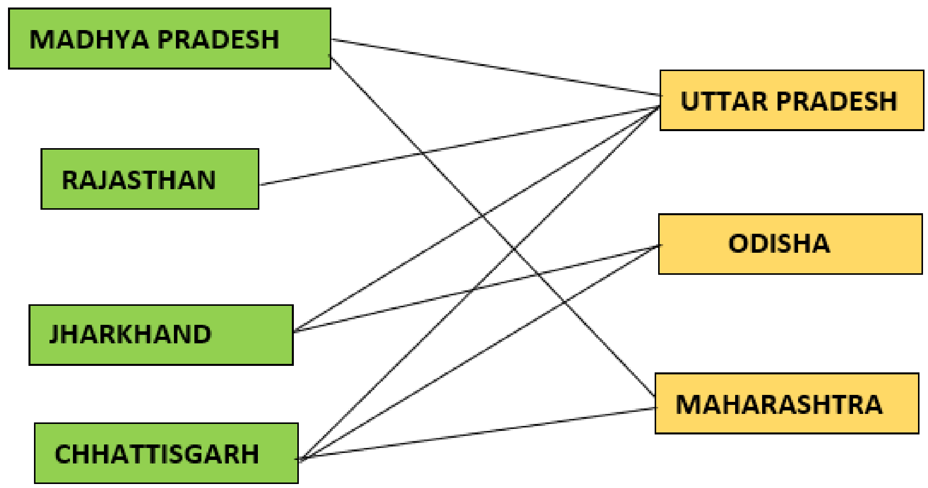

Figure 4.6: Bipartite graph of a few states

Q6. Find out the number of edges in a Wheel Graph W_125

(A) 7750

(B) 124

(C) 15500

(D) 15376

(E) 15625

Answer: Correct answer is 248.

Explanation:

Wheel graph is composed of cycle graph and star graph. There are n-1 edges in each of these graphs which sums up to twice the number of edges in wheel graph. Here, n=125 therefore, total number of edges is 2*(125-1) = 248.

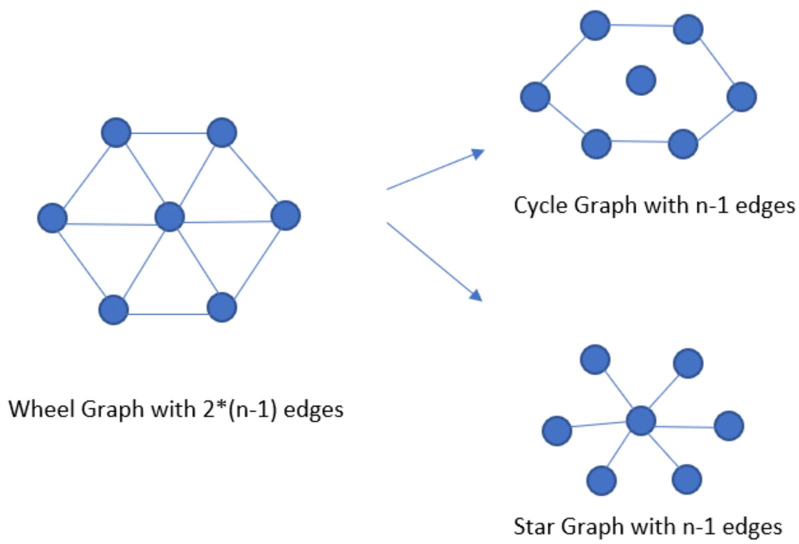

Figure 4.7: Wheel graph and its decomposition

Q7. Which among the following is/are the properties of Biparatite Graph?
(I) It has chromatic number=2
(II) It has even number of edges
(III) It has even number of nodes
(IV) If it is a cycle it can not have odd number of nodes
(V) It is always a tree

(A) Only (I),(II),(IV) and (v)

(B) Only (II) and (V)

(C) Only (I) and (IV)

(D) Only (I) and (V)

(E) All of them

Answer: (C)

Explanation:

(I) Since we have two set of vertices in bipartite graph where vertices of the same set are not interconnected, chromatic number number of such graphs is always 2 (refer figure 4.6).
(II) and (III) Graph shown below has odd number of edges (1) and odd number of vertices (3).

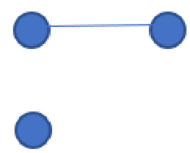

Figure 4.8: Example to disprove (II) and (III)

(IV) Cyclic bipartite graphs will form a cycle with even number of nodes. A cycle with odd number of nodes in bipartite graph is not possible.

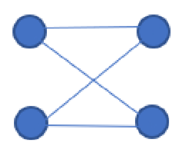

Figure 4.9: Example to support (IV)

(V) Example to prove that the bipartite graphs need not be trees is given below.

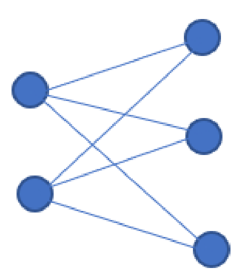

Figure 4.10: Example to support (V)

Q8. Which among the following is correct?
(I) Every Tree is Path
(II) Every Path is a Walk
(III) Every walk is a Path

(A) Only (I) and (II)

(B) Both (III)

(C) Only (I)

(D) Both (I) and (III)

(E) Only (II)

Answer: (E)

Explanation:

(I) Every tree is not a path whereas there exists a unique path between every pair of vertices in tree. (II) Every path is a walk since there is no restriction on the repetition of edges or vertices whereas path does not allow to have repeated vertices. Therefore, path is a subset of walk. (III) Every walk cannot be a path as explained above.

Q9. Sort the following graphs in ascending order based on chromatic number (Assume n is even number and greater than 10):
(I) C_n **(II)** K_n **(III)** W_n

(A) (I) < (II) < (III)

(B) (I) < (III) < (II)

(C) (III) < (II) < (I)

(D) (II) < (III) < (I)

Answer: (B)

Explanation:

Chromatic number of:
(I) C_n - 2 for even number of vertices and 3 for odd number of vertices
(II) K_n - n
(III) W_n - 4 for even number of vertices and 3 odd number of vertices

Chromatic number of the graphs can be arranged as (I) < (III) < (II).
Therefore, option (B) is the correct answer.

Q10. Consider the following two statements:
(I) every vertex of a graph G has degree = 2, then G is a cycle.
(II) If G is a cycle, then every vertex of G has degree=2.

(A) Both (I) and (II) are True

(B) Only (I) is True

(C) Both are False

(D) Only (II) is True

Answer: (D)

Explanation:

Graphs shown below explain the options mentioned above:

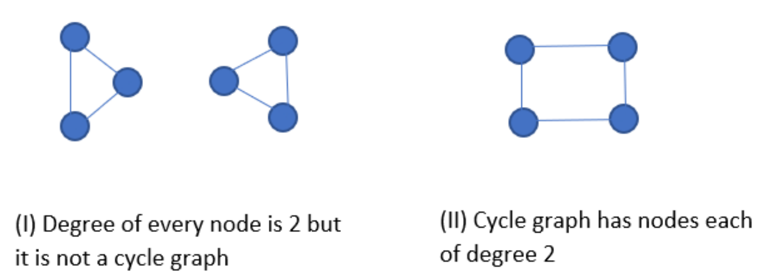

Figure 4.11: Degree of a cycle graph

4.2 Vertex Degree and Counting

Graphical Sequence:

It is a list of non-negative integers (valid sequence of vertices) of appropriate degree values using which a simple graph can be constructed.

Havel-Hakimi

It is an algorithm devised to solve graph realization problem. It is used to determine whether a given finite sequence of non-negative integers is a valid of degree of vertices of any simple graph G having finite number of vertices.

Algorithm:

1. Arrange the list in non-increasing order
2. Let the value at first place be x, subtract 1 from the value of degrees of next x vertices (except the start vertex).
3. If any vertex takes a value less than zero then the given sequence is not valid else, continue from step 1 until a sequence of 0s is formed to declare a valid sequence.

Example:

1. 5 5 5 4 2 1 1 1

   Step 1: Arrange the list in non-increasing order. Given list is already arranged in non-increasing fashion ie. 5 5 5 4 2 1 1 1
   Step 2: Subtract 1 from next 5 vertices except the start vertex. 4 4 3 1 0 1 1
   Step 3: Repeat above two steps until either value becomes negative or obtain a sequence of 0s.
   Non-increasing order : 4 4 3 1 1 1 0
   Subtract 1 from next 4 vertex values: 3 2 0 0 1 0
   Non-increasing order: 3 2 1 0 0 0
   Subtract 1 from next 3 vertex values: 1 0 -1 0 0

Since we have encountered -1, the given sequence is invalid.

2. 5 5 4 3 2 2 2 1

   Step 1: Arrange the list in non-increasing order. Given list is already arranged in non-increasing fashion ie. 5 5 4 3 2 2 2 1
   Step 2: Subtract 1 from next 5 vertices except the start vertex. 4 3 2 1 1 2 1
   Step 3: Repeat above two steps until either value becomes negative or obtain a sequence of 0s.
   Non-increasing order : 4 3 2 2 1 1 1
   Subtract 1 from next 4 vertex values: 2 1 1 0 1 1
   Non-increasing order: 2 1 1 1 1 0
   Subtract 1 from next 2 vertex values: 0 0 1 1 0
   Non-increasing order: 1 1 0 0 0
   Subtract 1 from next 2 vertex value: 0 0 0 0
   Since we have obtained a sequence of 0s, the given list of degrees is a valid graphical sequence.

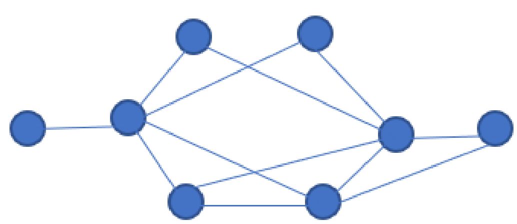

Figure 4.12: Graph of the above valid sequence

Prove or Disprove:

If u and v are only vertices of odd degree in a graph G then there exists a path between u and v in G.
Let G_1 subset of G and G_2 subset of G and $|G_1| + |G_2| = |G|$

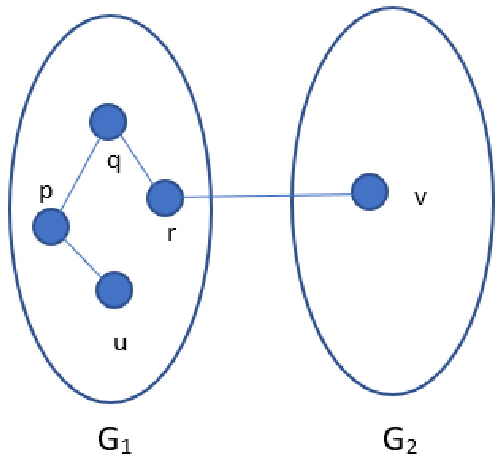

Figure 4.13: Path between two odd vertices of different subgraphs

Now, if G_1 has only one node say u with odd vertex it will violate the Hand-shaking lemma which states:
$\sum d(v) =$ even as the resultant of graph containing only one vertex with odd degree will lead to $\sum d(v) =$ odd.
Similarly, G_2 containing only one node of odd vertex say v, will violate the lemma.

Therefore, by the proof of contradiction it can be deduced that vertices u and v should belong to a same connected graph between which there exists a path. In the given graph, $u - p - q - r - v$ is the path between connecting two vertices.

Determine the maximum number of edges in a bipartite subgraph of Planar graph, Cyclic graph and Complete graph each of n vertices i.e. P_n, C_n and K_n respectively.
Therefore, we have to find a maximal subgraph of G (P_n, C_n or K_n) which is bipartite.

Definitions:

Path Graph- A graph having no overlapping edges.

Cycle graph – A graph in which the traversal starts and ends at the same vertex.

Complete Graph – A graph in which every vertex is adjacent is to each other.
Considering the given statement:

1. Maximal subgraph of P_n which is bipartite is the graph itself i.e. $P_n \subseteq \mathrm{P}_n$. Therefore, maximum number of edges in P_n is $n-1$.

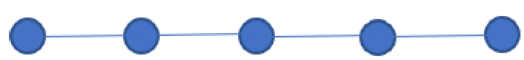

Figure 4.14: Path Grap

2. For maximal subgraph of even length cycle, the graph G itself satisfies the given condition. And if G has odd length cycle then path graph of n vertices will be the maximal bipartite subgraph of G.

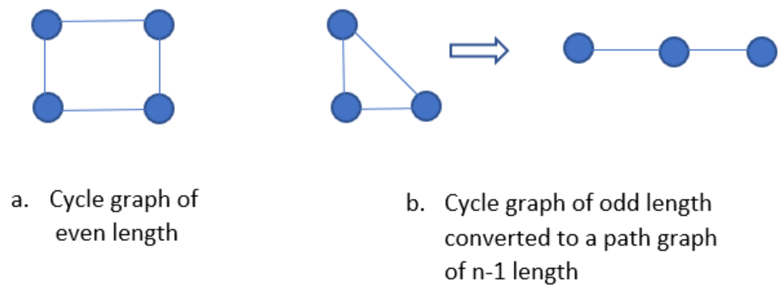

Figure 4.15: Cycle Graph

3. It will be considered as a part of Quiz2

Let l, m and n be non-negative integers such that $l + m = n$. Find necessary and sufficient condition condition on l, m and n such that there exists a simple connected graph of n vertices with l vertices of even degree and m vertices of of degree.

Given a simple connected graph, it imposes a condition on n to be greater than or equal to 1 i.e. $n >= 1$.

To satisfy the Hand-shaking lemma, summation of degree of all the vertices is even.
Therefore, l can be odd or even in number of even degree and m can take only even valued count of odd degree i.e.
l = even or odd
m = even
For example, (l, m, n) = (2,0,2)

Figure 4.16: Graph for (l,m,n) = (2,0,2)

Although the combination satisfies the necessary condition but this will not result in a connected graph.

A Few Announcements:

1. To submit lecture scribing within the mentioned deadline.
2. Problem set has been released to score better grades.
3. Project ideas will be shared the following week.
4. Secret Keys to be shared using which the students can view their quiz scores and know the statistics.
5. Quiz-2 to be held on the 12^{th} of February.

Week 4

Saturday: Vertex Degree and Counting*

4.1 Revisiting a few definitions

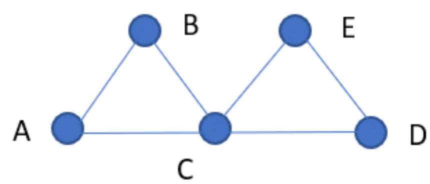

Figure 4.1: Reference figure

Walk – Walk is an ordered sequence of edges and vertices with no constraints on repeating of edges and vertices.

Path – Path is a walk where vertices cannot be repeated and hence, so will not the edges.

Trail – Trail is an open walk where edges cannot repeat but vertices can be repeated.
ACDECB is a trail but not a path.

Cycle – It is a closed walk where all vertices except the vertices at two

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ends, are distinct.
ACDECBA is not a cycle since C is visited twice in the traversal of the graph.

Circuit – Circuit is a closed walk where all edges are distinct (vertices can repeat).
ACDECBA is a circuit.

	Repetition allowed	Edges are not repeated	Vertices are not repeated (also the edges)
Start and end at any vertices	Walk	Trail	Path
Start and end at the same vertex	Closed Walk	Circuit	Cycle

4.2 Discussion

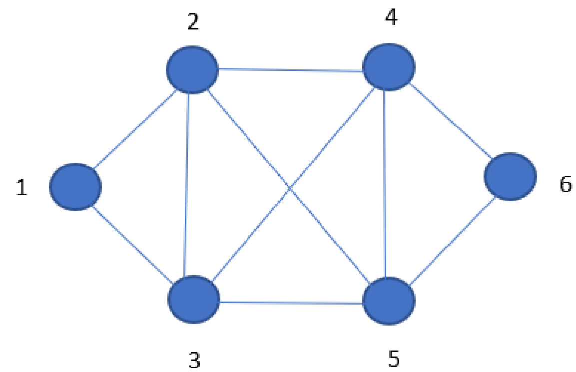

Figure 4.2: Graph for reference

1. 1 2 4 5 2 3 is
   (a) Walk
   (b) Path
   (c) Cycle
   (d) Trail
   (e) Circuit

   Ans.: Walk and Trail

2. 1 2 4 2 3 1 is
   (a) Walk
   (b) Path
   (c) Cycle
   (d) Trail
   (e) Circuit

   Ans.: Walk

3. 1 2 3 1 is
   (a) Walk
   (b) Cycle
   (c) Circuit

   Ans.: All the three (Walk,Cycle and Circuit).

4.3 Directed Graphs

Definitions:

Directed Graph - A directed graph or digraph G is a triplet consisting of a vertex set $V(G)$, an edge set $E(G)$ and a function assigning each edge an ordered direction.

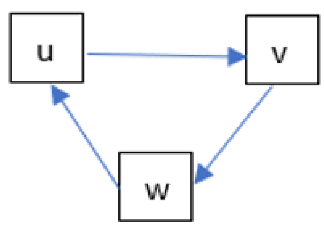

Figure 4.3: Digraph

Terms used in general:

Head - In an edge directed from vertex u to v, u is called the head. It is also known as predecessor. **Tail** - In an edge directed from vertex u to v, v is called the tail. It is also known as successor.

Underlying Graph - Underlying graph of a directed graph D is the graph G obtained by treating each edge of D as unordered pairs.

Figure 4.4: Underlying Graph

Representing Directed Graphs:

There are majorly three ways of representing digraph

1. Adjacency Matrix - Matrix of vertices as rows and columns, values indicating the presence of an edge between i^{th} row and j^{th} column.

2. Incident Matrix - Matrix of vertices as rows and edges as columns, values indicating an outgoing, incoming or no edge between i^{th} row and j^{th} column.

3. Adjacency List - Ordered list of vertices which are connected to each other.

For example,

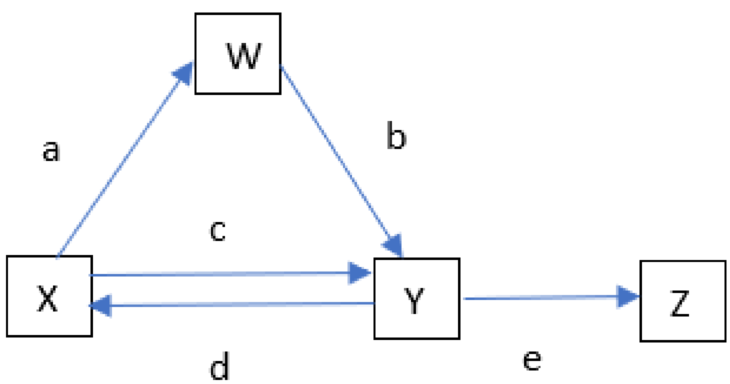

Figure 4.5: Graph for reference

The given graph can be represented in the following ways:

1. Adjacency Matrix:

$$
\begin{array}{c c}
 & \begin{array}{cccc} W & X & Y & Z \end{array} \\
\begin{array}{c} W \\ X \\ Y \\ Z \end{array} &
\begin{pmatrix}
0 & 0 & 1 & 0 \\
1 & 0 & 1 & 0 \\
0 & 1 & 0 & 1 \\
0 & 0 & 0 & 0
\end{pmatrix}
\end{array}
$$

Figure 4.6: Adjacency Matrix representation of figure 4.5

2. Incident Matrix:

$$
\begin{array}{c c}
 & \begin{array}{ccccc} a & b & c & d & e \end{array} \\
\begin{array}{c} W \\ X \\ Y \\ Z \end{array} &
\begin{pmatrix}
-1 & +1 & 0 & 0 & 0 \\
+1 & 0 & +1 & -1 & 0 \\
0 & -1 & -1 & +1 & +1 \\
0 & 0 & 0 & 0 & -1
\end{pmatrix}
\end{array}
$$

Figure 4.7: Incident Matrix representation of figure 4.5

Strongly and Weakly Connected Components:

1. A graph is strongly connected if for each pair of vertices u and v there is a path between these two vertices.
2. A digraph is weakly connected if its underlying graph is connected.

Kernel of digraph:

A kernel in the digraph D is a subset of vertices S such that S induces no edges and every vertex outside S has a successor in S.
For example:

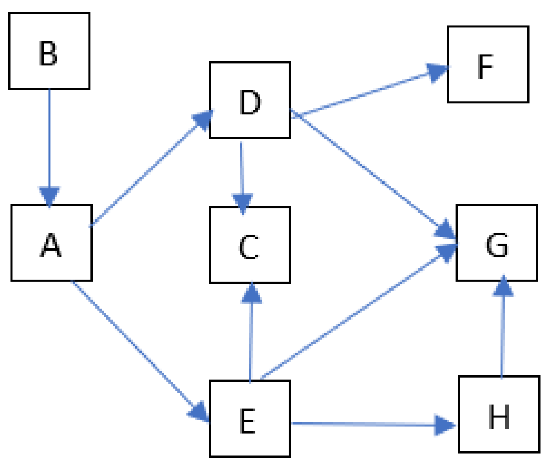

Figure 4.8: Kernel graph

Here, the kernel is {A, F, G, C}.