Week 11

Pseudo Boolean function, Graph Cut and Quiz 3 discussion*

11.1 Pseudo Boolean function

f:{0,1} $\rightarrow R$
$f(x) = 2x_1 + 3\overline{x_1} + 4x_2 + 2\overline{x_2} + 2\overline{x_1}x_2$
$x_1, x_2 \in \{0, 1\}$

Goal is to minimize f(x)
Graph construction is as follows:

1. Every cut of that graph corresponds to some assignment to variables
2. Minimum cut=Minimum cost assignment

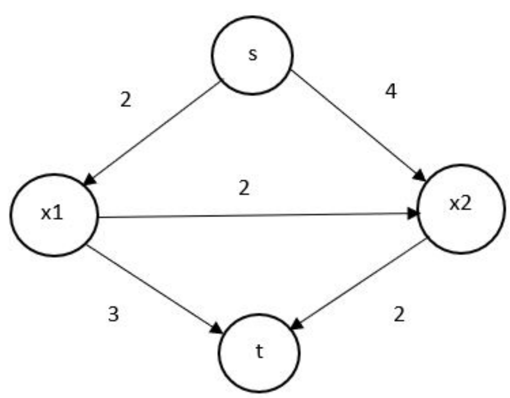

Figure 11.1: Example 1

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Brute force approach:

x1	x2	f(x)
0	0	0+3+0+2+0=5
0	1	0+3+4+0+2=9
1	0	2+0+0+2+0=4
1	1	2+0+4+0+0=6

Figure 11.2: Table

Using Ford Fulkerson algorithm: Push 2 along the path from s to t via x2

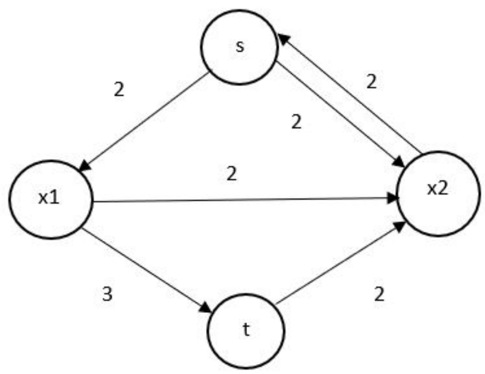

Figure 11.3

Push 2 along the path from s to t via x1

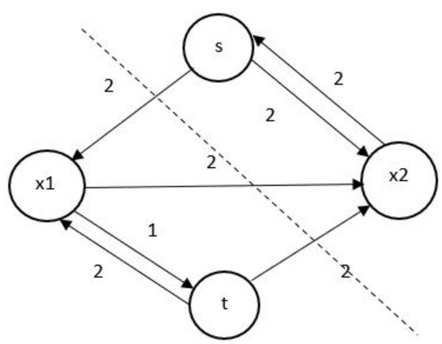

Figure 11.4

Therefore, x2=0, x1=1 and f(x)=4. The min cut=4

11.2 Graph Cut

Let us consider the problem of image segmentation in energy minimization framework. The energy we need to minimize is of following form:

$$E(x) = \sum_i c_i x_i + \sum_{i,j} c_{i,j} x_i (1 - x_j)$$

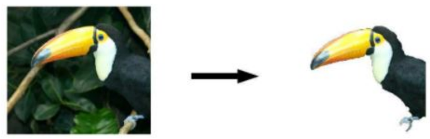

Figure 11.5

What energy functions can be minimized?

- General energy functions: NP hard to minimize, only approximate solutions available.
- Easy energy functions (sub-modular functions) are graph representable and solvable in polynomial time

What is a sub-modular function?
Let f be a function defined over set of boolean variables x = x1, x2, ..., xn, then

- All functions of one boolean variables are sub modular
- A function f of two boolean variable is sub-modular if f(0,0)+f(1,1) $\leqslant f(0,1) + f(1,0)$
- In general a function is sub-modular if all its projection to two variables are sub-modular

Multi-label cases
Solving multi label cases is also possible using Graph cuts. Some of the examples are:

- Find the labels for different objects in an image
- Finding parts of an object
- Stereo correspondence
- Image De-noising
- Image Inpainting

Same as bi-label cases, the goal is to find a labelling that assigns each pixel p ϵP a label $f_p \epsilon L$ where f is both consistent with observed data and piecewise smooth. i.e. we wish to find the global minima of energy function of following form:
$E(f) = E_{data}(f) + E_{smooth}(f)$

The form of $E_{data}(f)$ is typically, $E_{data}(f) = \sum_{p\epsilon P} D_p(f_p)$ where D_p measures the agreement of inferred label from the observed data. The smoothness term is used to impose spatial smoothness. It should have discontinuity preserving property.

Semi-Metric and Metric on space of labels
A function $V(\cdot, \cdot)$ is called a semi-metric on the space of labels $\alpha, \beta \epsilon L$ if it satisfies following two properties:

- $V(\alpha, \beta) = V(\beta, \alpha)$
- $V(\alpha, \beta) = 0 \longleftrightarrow alpha = \beta$

If $V(\cdot, \cdot)$ also satisfies triangle inequality then it is a metric.

Move algorithm for approximate solution
Now we will describe two move algorithms which find out the approximate solution for the energy function.

- First move algorithm is known as $\alpha - \beta$ swap which works when V is a semi-metric.
- Second move algorithm we call as α expansion, it works only when V is metric.

Any labelling f can be uniquely represented by a partition of image pixels $P = \{P_l : l \epsilon L\}$ where P_l is subset of pixels assigned label l.

$\alpha - \beta$ swap: Given a pair of labels α, β a move from partition P to partition P' is called an $\alpha - \beta$ swap if $P_l = P'_l$ for any label $l \neq \alpha, \beta$

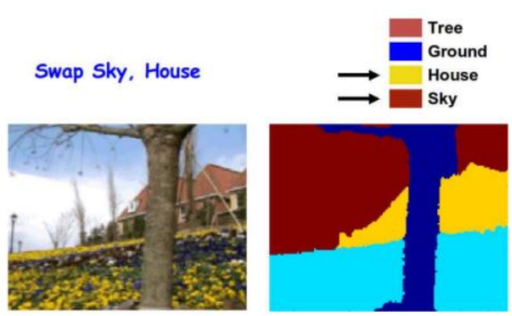

Figure 11.6

α expansion: Given a label α a move from partition P to partition P is called an α expansion if $P_\alpha \subset P'_\alpha$ and $P_l \subset P'_l$ for any label $l \neq \alpha$
Example:

Figure 11.7

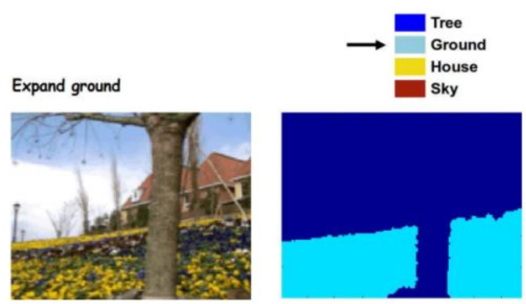

Figure 11.8

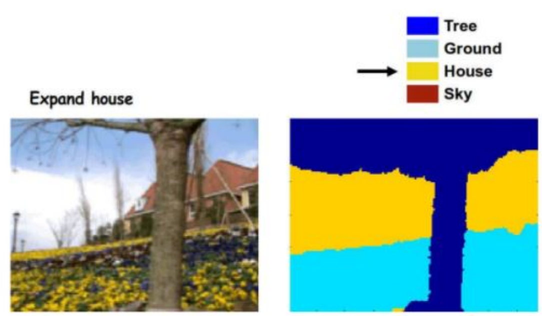

Figure 11.9

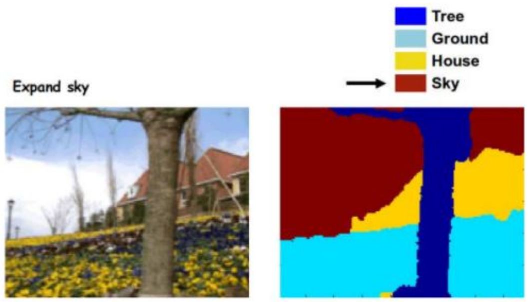

Figure 11.10

11.3 Quiz 3 discussion

Q. Consider the graph shown and compute the following: Size of maximum size matching, maximum size of Independent set, minimum size of vertex cover

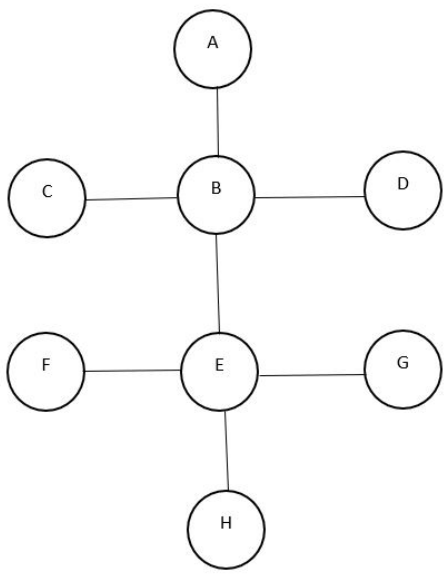

Figure 11.11

Maximum matching={AB, EH} Size of maximum matching=2
Maximum Independent set={A, C, D, F, G, H} Maximum size of Independent set=6
Minimum Vertex cover={B, E} Minimum size of vertex cover=2

Q. The max flow of the given flow network will be ——.

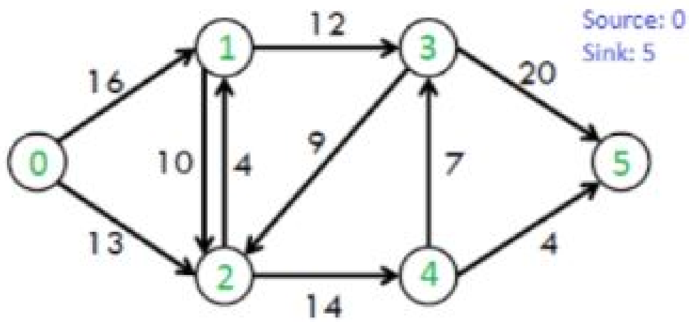

Figure 11.12

Push 12 along the path 0-1-3-5
Push 4 along the path 0-2-4-5
Push 7 along the path 0-2-4-3-5

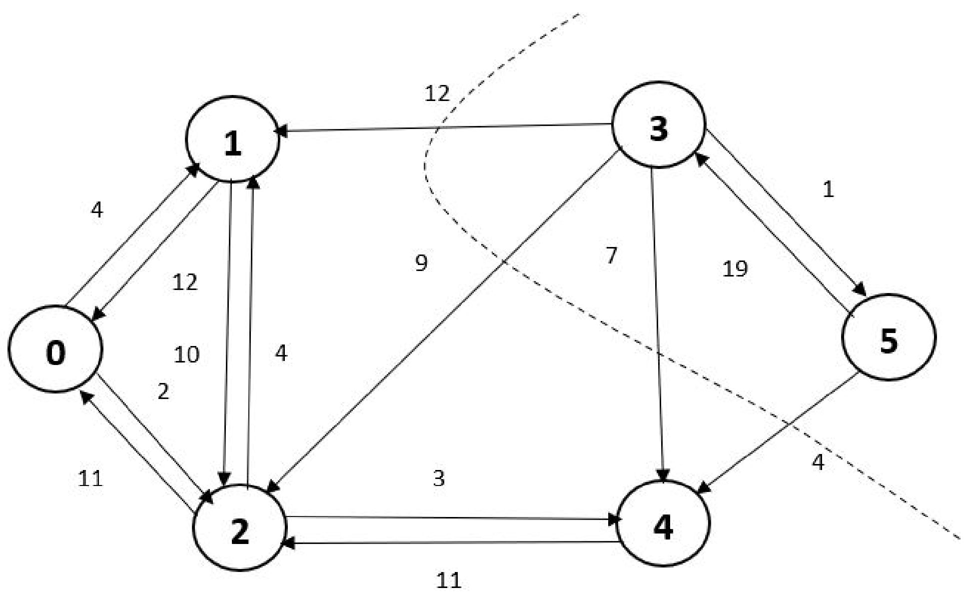

Figure 11.13

Q. Consider the person-task assignment problem. Each cell i,j corresponding to a person i and task j shows how many minutes person i takes to finish task j. Find out the overall cost (including Sita's time) of minimum cost assignment assuming Sita's preference is given highest priority and she chooses to complete task 3.

Person/Task	Task1	Task2	Task3	Task4
Arya	90	20	70	80
Sita	60	40	30	70
Neha	50	80	10	80
Ram	70	60	90	40

Figure 11.14

Since Sita's preference is to be given higher priority thus, we get:

Person/Task	Task1	Task2	Task3	Task4
Arya	90	20	70	80
Sita	60	40	30	70
Neha	50	80	10	80
Ram	70	60	90	40

Figure 11.15

Now, we have

$$
\begin{matrix} & 90 & 20 & 80 \\ A= & 50 & 80 & 80 \\ & 70 & 60 & 40 \end{matrix}
$$

Figure 11.16

According to the Hungarian algorithm, Compute the row minimum and subtract row minimum from the rows, and similarly for column minimum, we get

$$A = \begin{matrix} 70 & 0 & 60 \\ 0 & 30 & 30 \\ 30 & 20 & 0 \end{matrix}$$

Figure 11.17

Check the number of lines required to cover all the zeroes

$$A = \begin{matrix} 70 & 0 & 60 \\ 0 & 30 & 30 \\ 30 & 20 & 0 \end{matrix}$$

Figure 11.18

Here number of lines required to cover all the zeroes=3=n
Now, try to do the assignment:

$$A = \begin{matrix} 70 & \boxed{0} & 60 \\ \boxed{0} & 30 & 30 \\ 30 & 20 & \boxed{0} \end{matrix}$$

Figure 11.19

Thus, Total cost of assignment=20+50+40+30=140

Q. In a minimum cost flow network, which among the following is/are true?
A. Each edge denote capacity of flow.
B. Supply is always greater than demand in the network.
C. The minimum cost of such flow network can be obtained using Network Simplex Method

Ans. Only C