Lecture (23-24)
Graph Coloring and Interval graphs
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March 2022

1 Graph Coloring - Definition

A k-coloring of a graph G is described as labeling of the graph G, such that:

$V(G) \rightarrow S$, where $S = \{C_1, C_2, ..., C_k\}$ is a set of k-colors.

A k-coloring is said to be proper if adjacent vertices have different labels. The chromatic number of a graph is the least number of colors needed to make a coloring. This ensures that the graph is a proper k-color-able. A coloring of graph is used to solve many practical problems like scheduling problems, cell phone traffic problems and coloring of the maps such that no two regions have the same color sharing the same boundary.

2 Greedy Algorithm for graph coloring

- Step-1: Choose any vertex and color it.
- Step-2: Do the following for remaining $|V| - 1$ vertices.
- Step-3: Choose any non-colored vertex and color it with the least numbered color that has not been used on any previously colored vertex adjacent to it. if all previously used colors appear on the adjacent vertex then assign a new color to it from the given set of colors e.g. $S = \{R, G, B, Y, C\}$

3 Fully connected sub graph and Clique

For a fully connected graph G the clique is the fully connected subset of G with all its vertices are adjacent to each other. The clique size is denoted by $W(G)$

For every graph the chromatic number is always greater than or equal to its clique size.

i.e. $X(G) \geq W(G)$

Proof: The chromatic number X(G) of graph G is always equal to the maximum of the chromatic numbers of its k-component graphs.
$X(G) = Max\{X(G_1), X(G_2), ...X(G_k)\}$
where $X(G_1), X(G_2), ...X(G_k)$ are k-components of G

4 Bounds on chromatic number $X(G)$

The chromatic number can not be more than the size of the vertex set V(G).
The trivial upper bound is given by $X(G) \leq |V(G)|$
The trivial lower bound is given by $X(G) \geq 0$
Therefore, $X(G) \leq \Delta G + 1$, where Δ is the maximum degree of G.

For

- Star Graph: $\Delta G = n - 1$, $X(G) \leq n$ or $X(G) = 2$
- Complete Graph: $\Delta G = n$, $X(G) \leq n + 1$ or $X(G) = n$
- Wheel Graph: $\Delta G = n - 1$, $X(G) \leq n$ or $X(G) = 3(even), 4(odd)$
- Cycle Graph: $\Delta G = 2$, $X(G) \leq 3$ or $X(G) = 2(even), 3(odd)$

5 Welsh Powell Algorithm

Welsh Powell algorithm is used to give the minimum colours for labeling a graph and is also used to find the chromatic number of a graph.

It states that if G has a degree sequence of $d_1 \geq d_2 \geq d_3 ... \geq d_n$ then

$X(G) \leq 1 + max, min\{d_i, i - 1\}$

6 Interval Graphs and Scheduling

Interval graphs are related to some time-stamp related events or some common object sharing events. In this case the overlapping events form the edges between them and non-overlapping events don't. It should be noted here that not every graph is an interval graph.
e.g. C_4 is not an interval graph whereas Star graph is an interval graph.

A graphs that has no induced sub-graphs as C_4 and its complement graph has transitive orientation is called as interval graph. And there are other heuristic approaches like shortest event first, earliest finish first etc. to solve for the interval graph problems.