Week 4

Lecture 7 and Lecture 8*

Graphs are diagrammatic representations composed of edges and vertices. The study of the relationships between the edges and vertices is the purpose of graph theory. To be able to understand the various applications of graphs, a thorough understanding of the structural properties and nature of different types of graphs is crucial.

4.1 Discussion on Quiz 1

A basic understanding of the concept of graphs and their various properties, the different types and their variations are important to grasp the essence of graph theory. Based on previous discussions during lectures the Quiz 1 aimed to touch on the main ideas of the types of graphs and their properties. During the discussion on the quiz questions, the following topics were discussed:

- Concept of unicyclic graphs: The question presented two conditions where a node is deleted or added. The first condition states that when a node is deleted from a unicyclic graph, then the graph will no longer be unicyclic. This condition is not always true. Fore example, if we consider a graph like in Figure 4.1-A, once the node in red is deleted, the graph remains unicyclic. Thus, the statement is false. The second statement in the question states that if a node is added to an existing unicyclic graph, with an increase in edge size by 1, then the graph will still be unicyclic. As visible from the Figure 4.1-B, the graph remains unicyclic when the node in green is added, and thus the second statement is correct.

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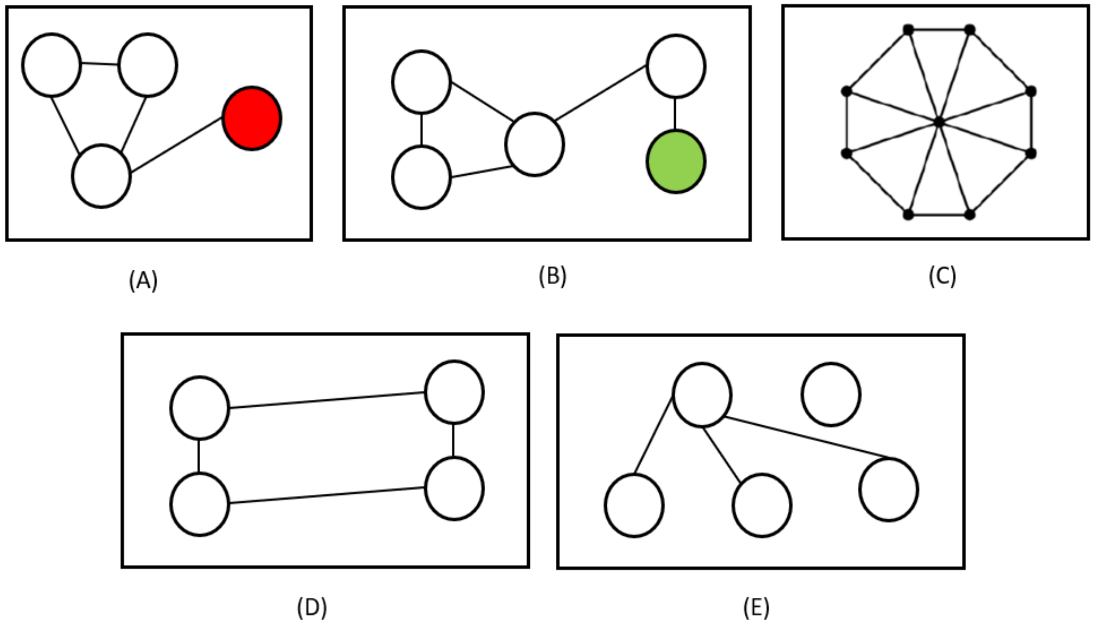

Figure 4.1: A and B are unicyclic graphs. C is a Wheel graph having 9 nodes and 16 edges. D and E are bipartite graphs.

- Wheel graph: A wheel graph consists of one node have n-1 edges and all other nodes being connected to this central node and its neighbours. A wheel graph is shown in figure 4.1-C. For a wheel graph with n nodes, the total number of edges would be equal to (n-1) for the central node plus (n-1) for the remaining nodes. Thus, when the question asked for the number of edges in a wheel graph having 125 nodes, the answer would be 2 * (125-1), which gives us 248 edges

- Unicycles in bipartite graphs: A bipartite graph is one where the nodes may be arranges in separate sets such that no edges exist between nodes in the same set and only with nodes in the other set. The first statement to be analysed about bipartite graphs is that it cannot be unicyclic. The answer is false. As is visible from Figure 4.1-D, a bipartite graph as shown can be a unicyclic graph. The second statement says that the degree of any node in it cannot be more than max (m, n) where m and n are the cardinality of independent sets of vertices. This statement is true, as from the Figure 4.1-E, the maximum degree we can get is 3.

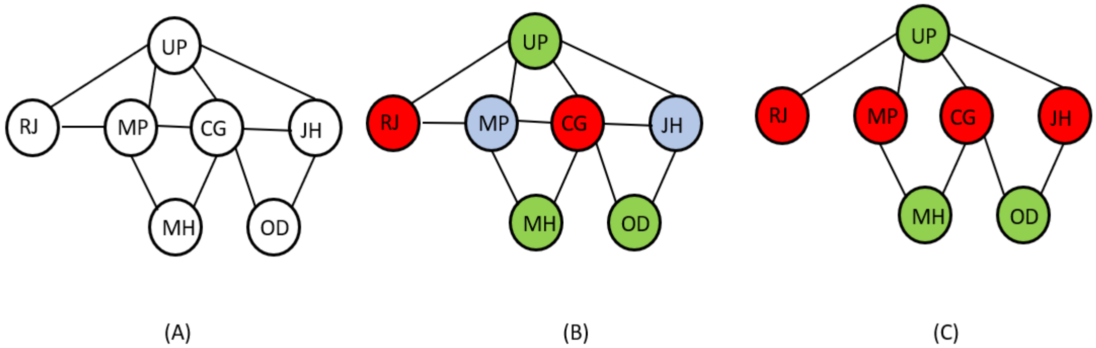

Figure 4.2: Graph of states Rajasthan (RA), Odisha (OD), Madhya Pradesh (MP), Chhattisgarh (CH), Uttar Pradesh (UP), Jharkhand(JH), Maharashtra (MH)

- Chromatic number and cut-edge: Given a map of India, when a graph is generated for the states Chhattisgarh, Madhya Pradesh, Odisha, Jharkhand, Uttar Pradesh, Maharashtra and Rajasthan, it looks like that in Figure 4.2-A. The chromatic number is the number of colours which need to be used to colour the nodes such that no two adjacent nodes have the same shade of colour. For the given graph, as visible from Figure 4.2-B, the chromatic number is three- where Madhya Pradesh and Jharkhand have the same colour, Rajasthan and Chhattisgarh have the same colour and Maharashtra, Uttar Pradesh and Odisha have the same colour.

  For a cut edge, we need to check if there exists such an edge, which when removed creates a disconnected graph, that it is will create two or more graphs. On analysing the graph, we see that there exists no such edge as removal of any one edge will still leave the graph connected in some way. Thus, there are no cut edges in the graph.

  Another question asked for the number of edges which may be removed to make the graph bipartite. One way of approaching this is to check the number of edges which can be removed such that the graph will have a chromatic number of two. On studying the graph, we see that the edges between Rajasthan-Madhya Pradesh, Madhya Pradesh-Chhattisgarh and Chhattisgarh-Jharkhand can be removed to get the required graph as visible in Figure 4.2-C. Thus, the number of edges to be removed is three.

- Properties of bipartite graphs: A number of options provided for bipartite graphs were to be analysed as true or false. These options include:

  - It has chromatic number=2

    This statement is true as in a bipartite graph, the nodes can be arranged into two sets such that connected nodes appear in different sets. Thus, these two sets can be indicated with two different colours, so the graph will have a chromatic number of 2.

- It has even number of edges.

  This statement is false. There exists a bipartite graph with odd number of edges where there are two sets of nodes, one with 2 nodes and another with 3 and only one edge exists between the two sets.

- It has even number of nodes.

  As explained in the previous point, there can exist a bipartite graph with odd number of nodes. For example, a bipartite graph containing 5 nodes, can be separated into two sets having two and three nodes respectively. So, this statement is false.

- If it is a cycle it cannot have odd number of nodes.

  For a bipartite graph which is a cycle, there must be even number of nodes. We know that a bipartite graph has chromatic number 2. Thus, every alternate node in the cycle formed must be of the same colour. However, for odd number of nodes, two adjacent nodes end up being of the sae colour and an edge exists between them. Thus, does not fulfil the criteria of a bipartite graph and this the statement is True- if a bipartite graph is a cycle, it cannot have odd number of nodes.

- It is always a tree.

  A bipartite graph is a graph which contains no odd cycles. A tree is a connected graph which contains no cycles at all. Thus, we can state that every tree is bipartite in nature. However, every bipartite graph is not a tree. Thus, the given statement is false.

• Walk and Path: A path in a graph is one where there is no repetition of vertices or edges. Whereas, a walk in a graph is one where both vertices as well as edges may be repeated. This question on this concept calls for the validation of the following statements:

  - Every Tree is Path.

    This statement is false as to traverse a tree, both nodes and edges need to revisited. Thus, a tree can never be a path.

  - Every Path is a .

    This statement is true as a path does not violate the conditions of a walk. It is simply a more restricted case of a walk. So, we can say that every path is a walk.

  - Every walk is a Path.

    This statement is false. We know that a walk allows the flexibility of repeating nodes and edges. However, a path does not allow it. So, every walk cannot be a path. Only a walk where no nodes or edges have been repeated can qualify as a path.

- Chromatic number of Cyclic, Complete and Wheel graphs: The question here calls for the chromatic number of the three mentioned graphs in ascending order.

  For a cyclic graph, when the number of nodes, n is even, then the chromatic number is 2, else for odd number of nodes, it is 3. For a complete graph having n nodes, the chromatic number is n. For a wheel graph, the chromatic number will be four for even number of nodes and three for odd number of nodes. Thus, when arranging the chromatic number of the three mentioned graphs in ascending order, the order we get is Cycle – Wheel – Complete.

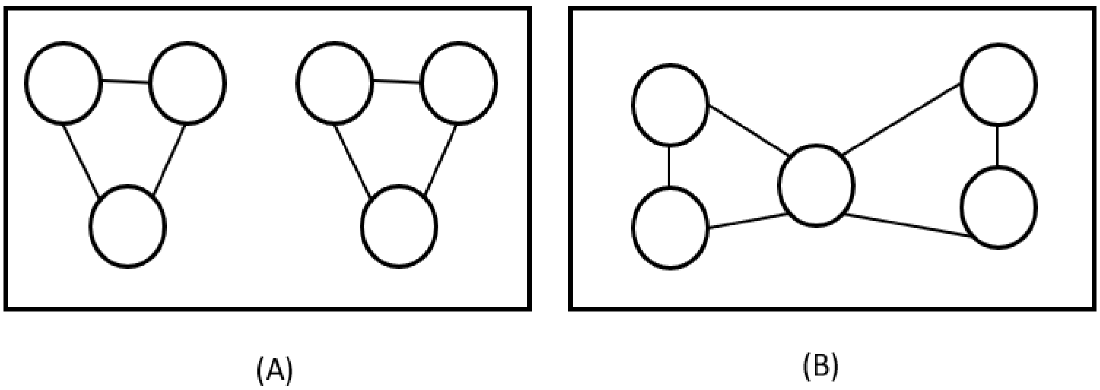

Figure 4.3: Cycle in a Graph

- Cycle in a graph: This question provides two statements to be analysed as true or false. The first statement states that every vertex of a graph G has degree = 2, then G is a cycle. This statement is false, as it is possible to have a graph having six nodes, where they form two disconnected triangles each of 3 nodes and 3 edges as shown in Figure 4.3-A . So, this graph contains two cycles but they are not a single connected cycle. Thus, this statement is false.

  The second statement says If G is a cycle, then every vertex of G has degree=2. If we consider a graph again having 5 nodes, as shown in Figure 4.3-B and every node in the cycle is connected to its two adjacent nodes we see that the degree of all nodes is not equal to 2. Thus, the given statement is false.

4.2 Vertex Degree and Counting

The degrees of vertices of a graph are an important parameter to study the nature and properties of a graph. This first topic which comes up is when given a graphical sequence – a series of degrees of nodes of a graph, how can one determine whether the given sequence is valid or not?

Havel Hakimi Theorem. The Havel Hakimi Theorem answers the following question: Given a finite list of non-negative integers in non-increasing order, is there a simple graph such that its degree sequence is exactly this list. The solution to this question using the theorem, is provided through a recursive process where the sequence is analysed step by step. The process involves the following steps:

Consider the sequence below:
5 5 4 3 2 2 2 1
Calculate the sum of the given sequence.
Sum = 24
If even, then arrange the list in descending order.
5 5 4 3 2 2 1
Remove the first value, say x and subtract x from the next x values in the sequence.
4 3 2 1 1 2 1
Rearrange the sequence in descending order.
4 3 2 2 1 1 1
Repeat the above steps.
2 1 1 0 1 1
Rearranging:
2 1 1 1 1 0
Removing 2
0 0 1 1 0
Rearranging
1 1 0 0 0
Removing 1
0 0 0 0

Thus, we see that the given sequence is valid as we get all zeroes at the end. Considering another example, we see the following sequence:
5 5 5 4 2 1 1 1
4 4 3 1 0 1
4 4 3 1 1 1 0 (rearranged)
3 2 0 0 1 0
3 2 1 0 0 0 (rearranged)
Here we see that 3 cannot be deducted further and thus, the given sequence is not valid.

Prove or Disprove: If u and v are the only vertices of odd degree in a given graph G, then G contains a path u-v. The proof may be attempted through proof by contradiction. We assume that the given statement isn't true.

Thus, we consider two components of the graph G - G1 and G2 which are not connected. We assume u is a part of G1 and v a part of G2. As G1 and G2 are both graphs, they must follow the graph properties. As G1 has only one node u with odd degree, as per graph properties, it must be connected to another node. Similarly, in G2, v is the only node having odd degree. That too must be connected to another node. Thus, they are not following the

Handshaking lemma which states that every graph must have an even number of odd degree nodes. This is not being followed in G1 and G2.

Thus, as it is a contradiction to the statement, we can say that, u and v must be a part of the same connected component and thus, there has to be a path existing between u and v in graph G so justify their degrees. Thus, the given statement is proved true.

Maximum number of edges in a bipartite subgraph of a Path, Cycle and Complete graph having n nodes: To calculate the maximum number of edges, we consider the maximum bipartite subgraph of each of the mentioned graphs. For a path, if the the path considered has odd number of edges, the whole path itself is bipartite. Thus, the maximum number of edges in the path having n nodes is n-1. In case a path contains even number of edges, then we exclude one edge to get the maximum bipartite graph and the number of edges if 1 less than the number of nodes in the obtained bipartite graph.

For a cycle containing n nodes, if n is even, then the cycle is bipartite and the maximum number of edges will be n. However, for a cycle having odd number of nodes, one edge must be removed to make it bipartite, and this the maximum number of edges is n-1.

For a complete graph having n edges, if the maximal bipartite subgraph contains m nodes where there are a number of nodes in one set and b in another. Then the maximum number of edges will be the product of a and b or ab.

Problem statement: Let l, m, and n be such that they are non-negative integers. Given that

- n is the sum of l and m.

Find a necessary and sufficient condition on l and m such that there exists a connected simple n-vertex graph with l vertices of even degree and m vertices of odd degree.

The first consideration we make is the l and m are non-negative so they can or may be zero. However, n cannot be equal to zero. So, the first condition is that

- n must be greater then or equal to 1.

Next we see that l can be either even or odd. As there are l vertices of even degree, l can also be zero as there can always be zero vertices having even degree. Similarly, l can also be odd as an odd number of nodes multiplied by an even number of degrees still gives an even number. Thus,

- l can be both even or odd.

However, in case of m, we cannot have an odd number of nodes having odd degrees as the product of an odd number of nodes with an odd number of degrees will give us an odd product, which is not possible. Thus, m can only be odd. So,

- m can only be odd.

Now, an exception to these conditions could be where l is 2, m is 0 and n is 2. Though l and m sum to n, the graph formed will be one having two nodes with self loops, which is not a simple graph. Thus, this combination of l, m, and n is an exception.

4.3 Clarifying walk, path, trail, cycle, and circuits

There are numerous ways of traversing a graph. Some of the ways used are walks, trails and paths.

Walks: It is a route followed in a graph - an ordered sequence of vertices and edges such that both the nodes as well as edges can be repeated. Walks can be either an open walk or a closed walk. A walk is said to be open if the starting and ending vertices are not the same. A closed walk is one where the starting and ending nodes are the same.

Paths: They can be considered a special case of walks where neither nodes nor vertices can be repeated. Thus, the starting and ending node is never the same. s are the same.

Trails: A trail is an open walk containing a sequence of vertices where edges cannot be repeated but vertices can.

Cycle: A cycle is a closed walk which has a series of nodes in a graph such that no edges are repeated and no vertex can be repeated excepting the starting and ending node.

Circuit: A circuit is a closed walk where vertices can be repeated but the edges cannot.

To summarise the above definitions, we can skim through the following points:

- Walks can start and end anywhere in a graph and both nodes and vertices can be repeated.
- An open walk must start and end at different nodes and may reuse both nodes and vertices.
- A closed walk must start and end at the same nodes and may reuse nodes and vertices.
- A path is an open walk where no noes and vertices are repeated.
- A trail may start and end anywhere in the graph but nodes cannot be reused.
- A circuit starts and ends with the same node but does not reuse any edges.
- Cycles start and end at the same node but do not repeat any edges or vertex except the starting and ending node.
- A cycle is a closed path.
- A circuit is a closed trail.

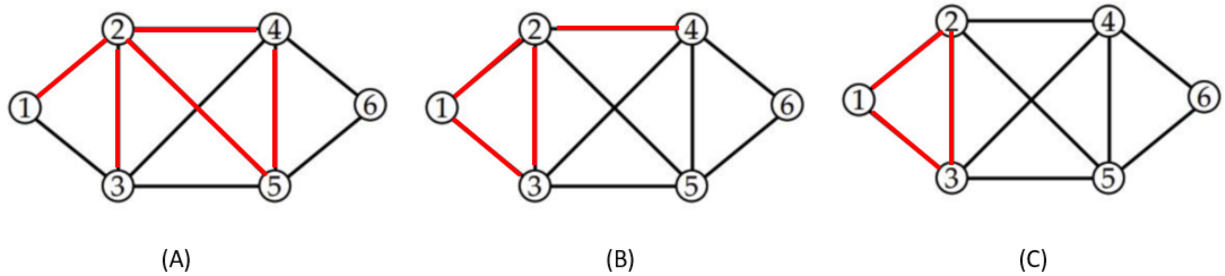

Figure 4.4: Graph having 6 nodes and 10 edges.

For the graph given in Figure 4.4-A,

- 124523 is an open walk. It is not a path or cycle as the vertex 2 is repeated. It is a trail as it is an open walk and no edges are repeated. It is not a circuit as it is not a closed trail.

For the graph given in Figure 4.4-B,

- 124231 is a closed walk. It is not a path or cycle as certex 2 and edge between vertices 2 and 4 have been repeated. It is not a circuit or a trail as an edge has been repeated.

For the graph given in Figure 4.4-C,

- 1231 is a closed walk as well as a cycle. It is a closed trail as no edge has been repeated.It is also a circuit as it forms a closed trail.

4.4 Directed Graphs

Directed graphs or a digraph is a type of graph which consists of a set of nodes and set of edges. However, what makes it different is that the edges in this type of graph provide a direction of movement in the graph. Each edge has a node from which it connects the movement in the graph to another node.

Digraphs are very useful for transportation networks and are actively used in social network analysis. When we have a directed graph, then the accessibility of nodes and possible paths from one node to all other nodes gets restricted as the direction permitted by the edges must be considered. Thus, every traversal through a digraph is guided by an order of directed edges.

Underlying graph of a given directed graph is the graph obtained by treating each of the edges as unordered pairs. This means the underlying graph is the graph obtained from a directed graph where the edges do not define a specific direction.

Representation of directed graphs: Directed graphs can be represented using adjacency and incident matrices. If we consider a directed graph having n nodes and e edges, then an adjacency matrix will have size v x v. The elements of the matrix are denoted by the number of edges which connected a node with the other nodes. Thus, if the i-j th element is 1, it denotes that there exists one edge between node I and node j. An incident matrix is of the order v x e. This matrix tells us whether an edge is entering the node or leaving it. For an edge j leaving the node i, the i-j th element of the incident matrix will be +1. For an edge j entering the node i, the i-j th element of the incident matrix will be -1. If the edge j does not connect node i with any other node, then the i-jth element will be 0.

Strongly and Weakly Connected Digraphs: A digraph is said to be strongly connected if for each pair of vertices u,v there exists a path connecting u and v. A digraph is said to be weakly connected if the underlying graph of the digraph is connected.

Kernel of a digraph: For a given digraph, a kernel in it is a subset of vertices such that this subset induces no edges and every vertex outside s has a successor in s.

Applications of Graphs are innumerable. Social network analysis is a type of graph network analysis where a number of networks (like Facebook,airport network, railway network, etc) can be generated and analysed to not only understand the crucial aspects of these systems but also improvise and improve them further. In the field of epidemiology, graphs generated to analyse the spread of diseases worldwide help understand which areas require most attention and provide basis for a plan of action to prevent further spread of infection. In biology, the networks generated to replicate molecular structures help identify scope for breakthroughs in the field of medicine. On similar grounds, there are many other applications of graph theory and much scope for understanding and analysing them better.

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