Week 5

Lecture 9 and 10*

Continuation from Lecture 8.

5.1 Lecture 9: Kernel, tournaments and king

5.1.1 Kernel in a directed graph

Is a set of vertices $S \subseteq V(D)$ such that S indicates no edges and every vertex outsied of S has a succssor in S. Concept of kernel is only defined for directed graph.

Example 1: Let us suppose we take a graph with 4 vertices and 4 edges which are in cycle. Kernel of this directed graph $S_1 = \{1, 3\}$ which are not connected to each other. Similarly $S_2 = \{2, 4\}$ is also a kernel, so there is no unique kernel in this case.

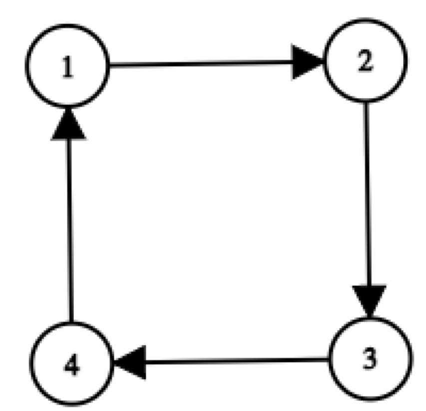

Figure 5.1: Directed Graph to show kernels.

*Lecturer: Anand Mishra. Scribe: Utkarsh Thusoo.

Example 2: Let us suppose we take a graph with 4 vertices and 3 edges which are in cycle. Kernel of this directed graph $S = \{2, 4\}$ which are not connected to each other.

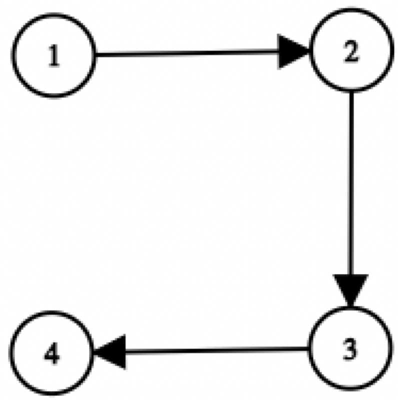

Figure 5.2: Directed Graph to show kernels.

Example 3: Let us suppose we take a graph with 6 vertices and 6 edges which are in cycle. Kernel of this directed graph $S_1 = \{2, 4, 5\}$, $S_2 = \{1, 3, 6\}$ which are not connected to each other.

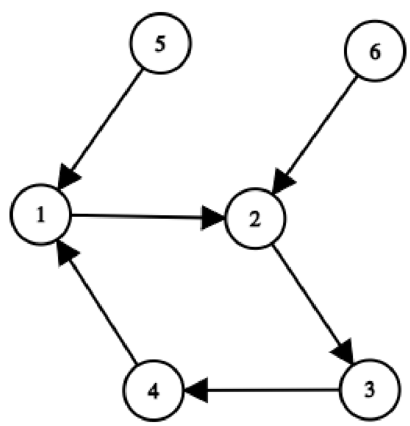

Figure 5.3: Directed Graph to show kernels.

Example 4: For a complete graph kernel will be null.

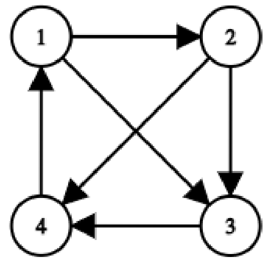

Figure 5.4: Fully connected directed graph.

Example 5: A graph with odd cycles will not have kernel. C_n where n is odd will not have kernel because we need to choose something in S to make a kernel and whatever you choose should be chosen a way such that they should be independent and successor of S. In this case every node has only one successor hence if you dont choose any node then we have to choose its successor and simply put we can never have a scenario where kernel can be created.

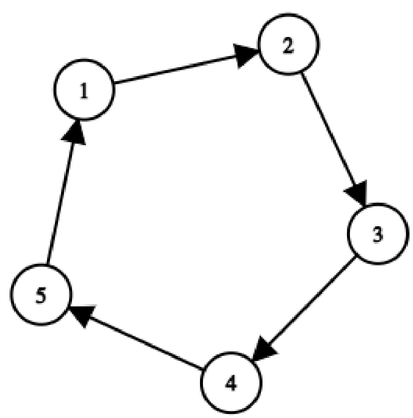

Figure 5.5: Directed graph

Let K be the kernel of C_n, then any $u, v \in K$ should have the following properties

- they should be independent;
- $w \in K$ then w must have successor in K

5.1.2 Outdegree and indegree

Every node in directed graph will have a indegree and outdegree. Number of edges comming into and going out of the graph.

In any directed graph sum of indegrees = sum of all degrees = number of edges in G i.e.

$$\sum_{v \in V(G)} d^{+}(v) = \sum_{v \in V(G)} d^{-}(v) = \mid e(G) \mid \tag{5.1}$$

Example 1: For the given digram indegree and outdegree of each node would be

- *Outdegree*: For node A, $d^{+}(A) = 1$, For node B, $d^{+}(B) = 1$
- *Indegree*: For node A, $d^{-}(A) = 0$, For node B, $d^{-}(B) = 2$

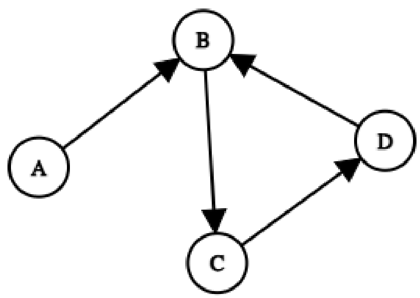

Figure 5.6: Directed graph

5.1.3 Orientation

An orientation of graph G is a directed graph D obtained from G by choosing an orientation $x \to y$ or $y \to x$ for each edge $xy \in E(G)$.

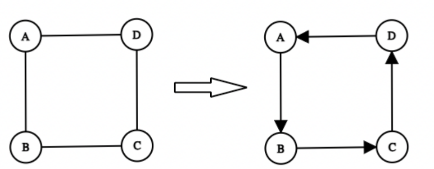

Figure 5.7: Undirected to Directed graph

5.1.4 Tournament in graph

is an orientation of a complete graph.

5.1.5 King of the tournament

King of the tournament is a vertex from where all the other vertices are reachable by a path of length of 2 at most.

Example 1: Let us suppose we have a cricket tournament between different countries and graph should have directions. It can be assumed that $x \to y$ plays a match and x wins against y. In the given example India is the king because all nodes can be reached from India.

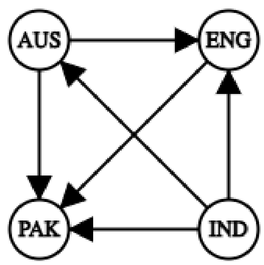

Figure 5.8: King of the tournament

Example 2: A complete graph with 5 vertices and vertices to each other. All vertices are kings because they satisfy the definition of the king. For node B

- $B \to C$
- $B \to D \to A$
- $B \to D$
- $B \to D \to E$

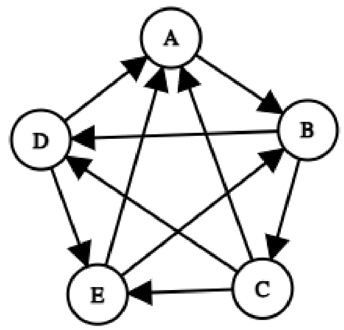

Figure 5.9: King of the tournament

Every tournament has a king. This can be proved with a contradiction. Suppose u is a node with the highest outgoing degree.

$$N_o(u) + N_I(u) + 1 = \mid V(G) \mid \tag{5.2}$$

where u is not a king.

Suppose u is not the king. All the nodes is N_I will be connected to all the nodes in N_o and only thing we are not aware of is the direction.

Case 1: If the direction is from $N_I \leftarrow N_o$, then we can say that u is king because we can reach from all the nodes of N_I via just two edges $u \rightarrow N_o \rightarrow N_I$. Hence every tournament must have a king.

Case 2: If the firection is from $N_I \rightarrow N_o$. Lets suppose there is a node in v in $N_I(u)$ and v has outgoing edges in all the nodes i.e. $v \rightarrow N_o$ and $v \rightarrow u$. Number of outgoing edge of node $v = 1+ \mid N_o(u) \mid >$ number of outgong edge of u. This is a contradiction because we have assumed that u is the node which has highest out degree. Hence this case is not applicable.

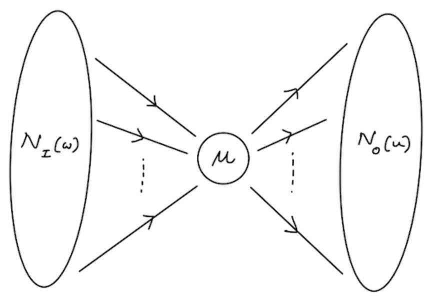

Figure 5.10: King of the tournament

Theorem 5.1. *Prove or disprove if D is a orientation of simple graph with 5 vertices then the vertices of D cannot have distinct out degree.*

Proof. We can create a graph where $x > y$ then $y \to x$ as below. In this way outegree of each of node

- $Node(1) = 4$
- $Node(2) = 3$
- $Node(3) = 2$
- $Node(4) = 1$

Hence the statement given in false because D can have distinct outdegree.

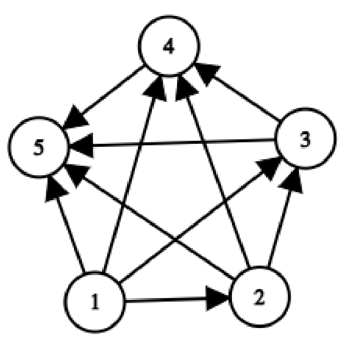

Figure 5.11: Example of simple graph

□

Question Give example of one real world realtion whose digraphs has no cycles.

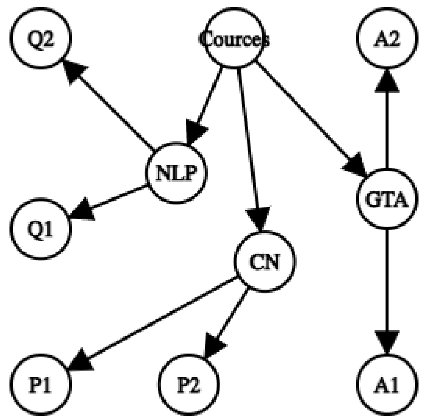

Figure 5.12: Real world example

5.1.6 Tree

In context of directed graph we can define a tree such that if we take any two vertices we will have a unique path between those two vertices. Simply put trees are connected acyclic graphs.

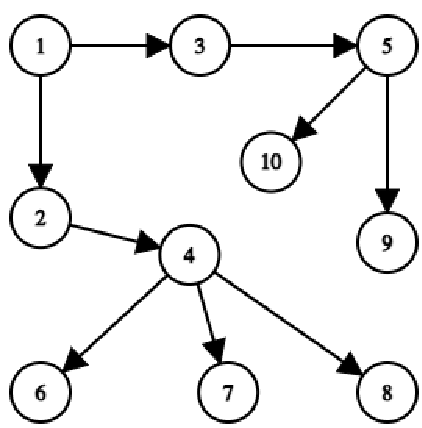

Figure 5.13: Example of tree

Properties of trees

- Deleting a leaf from an n-vertex tree produces an $n-1$ vertex tree.

- Tree is connected and it has no cycle.
- An n vertex contains $n-1$ edges because if it contains one more it will have a cycle and if it has one less it will not be connected.
- For any pair of vertex (u, v) there exists one and only one path.
- Every edge of a tree is cut-edge.
- Tree is a bipartite graph.

5.2 Lecture 10: Common definitions

5.2.1 Distance in a graph

If graph G has a $u-v$ path, then the distance between u to v is written as $d(u-v)$ is the least length of $u-v$ path. This distance is the shortest distance. For the given graph

$$d(u, v) = 1 \tag{5.3}$$

Similarly,

$$d(u, w) = \infty \tag{5.4}$$

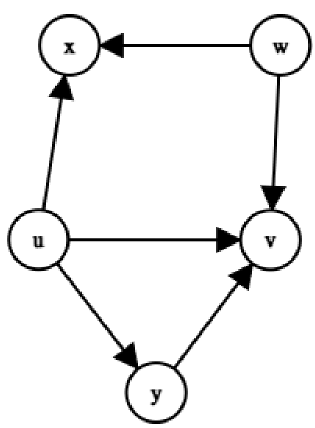

Figure 5.14: Example of tree

5.2.2 Diameteter of a graph

is the maximum of distanes between any two pair's of vertices. Diameter of graph Figure 5.14. is ∞

5.2.3 Eccentricity

of a vertex u is the maximum distance it has with any node in the graph.

5.2.4 Radius

denoted as $rad(G)$ is the minimum of eccentricity of all the nodes.

Example 1: For the given graph in 5.15

- $d(B, K) = 4$
- Diameter of the graph $Dia(G) = 4$
- Eccentricity $e(A, B, D, E) = 4$, $e(F) = 2$, $e(C, G, H, I) = 3$, $e(I, J, K, L) = 4$.
- Radius $rad(G) = 2$

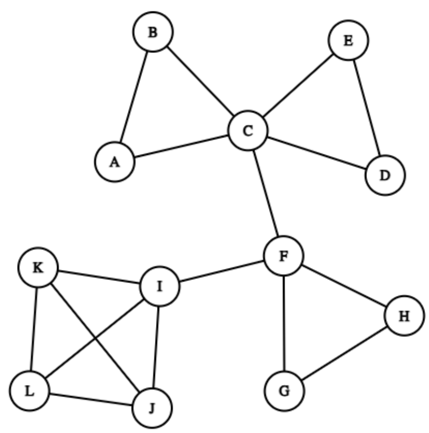

Figure 5.15: King of the tournament

Example 2: For the given graph in 5.16

- $d(A, B) = 1$, $d(A, C) = 2$, $d(A, D) = 3$, $d(A, E) = 2$, $d(A, F) = 3$.
- Diameter of the graph $Dia(G) = 3$.
- Eccentricity $e(A, B, C, D, E, F) = 3$.
- Radius $rad(G) = 3$

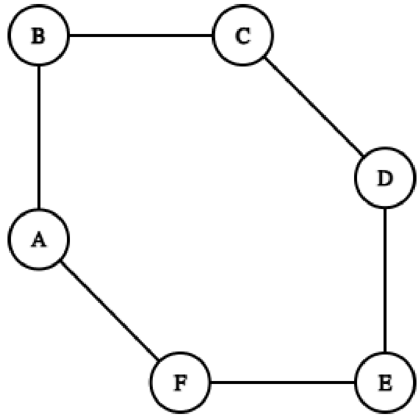

Figure 5.16: King of the tournament

Theorem 5.2. *If G is a simple graph then $diam(G) \geq 3$ implies $diam(G^c) \leq 3$*

Proof. $diam(G) \geq 3$ refers to the fact that there are some nodes that do not share neighbours.

$$\exists u, v \in V(G) \tag{5.5}$$

s.t. they do not have common neighbour in G. This is because if they have a common neighbour then they can be reached by just two nodes i.e. their diameter will never be greater than 3. Hence structure 5.17 does not exists in G.

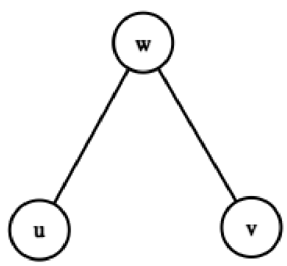

Figure 5.17

Also,

$$\exists u, v \in V(G^c) \tag{5.6}$$

s.t. they have a common neighbour in G and u, v are non adjacent. □

Problem: Computer diameter and radius of $K_{m,n}$

- Eceentricity of each node $E(A) = 2$ and $E(B) = 2$
- $diam(K_{m,n}) = 2$
- $rad(K_{m,n}) = 2$