Indian Institute of Technology- Jodhpur

Graph Theory and Applications(GTA)
COURSE CODE: CSL7410

Lecture Scribing Assignment: Week 3

Done by:
Aditya Mishra
M20MA201

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Week 3

Traversing through the Graph and Fundamental Theorem of Graph Theory*

It is very important to discuss about the basic ideas before moving towards to the Fundamental concepts and theorem of the graph theory and these basic ideas are related to the theorems of Path,Walk, Cycle and Euilerian Graph and Circuit for traversing to the Graph.after that we will be discussing about the vertex degree and counting.By the help of these concepts and ideas we will jump upon the most fundamental theorem of Graph theory that is Handshaking lemma.

3.1 Some Theorems and Lemma

In this Section we will learn about the Properties of walk,cycle and path related to each other and these are the building pillars of this course and then we will discuss about the concept of Euilerian circuit and graph.

3.1.1 Theorem

Statement : Every Closed odd walk contain an odd Cycle.

Proof: Let l is the number of edges in the graph. Proof is by Induction method. for $l = 1$, obviously true (trivially true).

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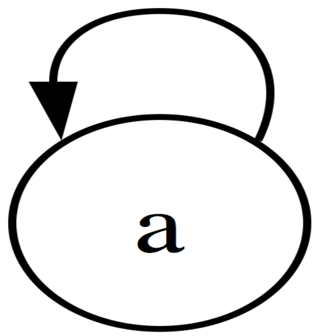
*Lecturer: Anand Mishra. Scribe: Aditya Mishra (M20MA201).

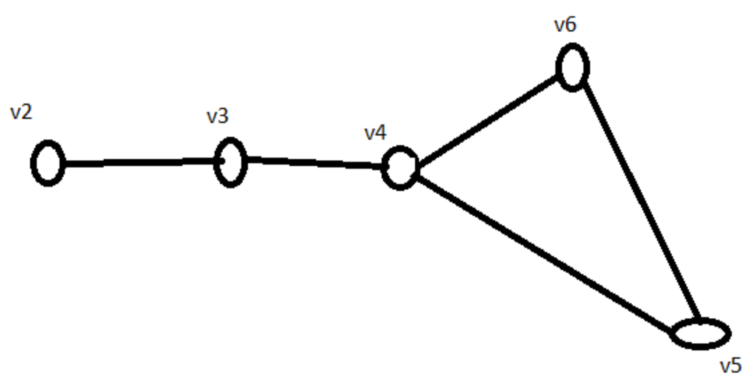


Suppose it is true for $l < L$, then we have to show that this is also true for L.

Case 1: If there is no repetition of vertex in walk then, closed walk = a closed cycle (very trivial again).

Case 2: If there is repetition in vertices in the walk and let us suppose v is vertex which repeats Break the walk into two $v - v$ walk (sayw_1 and w_2) since $|w_1| + |w_2|$ =odd, that means either w_1, or w_2 is odd walk. And surely they both less then L . from induction one of them (odd walk one) contain odd cycle.

Induction Hypothesis is: For any walk of size less than L, the fallowing is true:-
Every closed walk contain an odd cycle.



closed walk $v_2v_3v_4v_5v_6v_4v_3v_2$ (W) (There are multiple virtex are repeating)

$w_1 = v_2v_3v_4v_4v_3v_2$

$w_2 = v_4v_6v_5v_4$

$|w_1| + |w_2| = |w| = \text{odd}$

so there is two possibility that

$\Rightarrow |w1|$ is odd or $|w2|$ is odd.

Let's for simplicity take $|w2|$ is odd

that means, we can say w_2 is an odd closed walk.

and $|w_1| + |w_2| < L$

and our Hypothesis was any thing which is less than l in size in any dosed odd walk does contain an odd cycle.

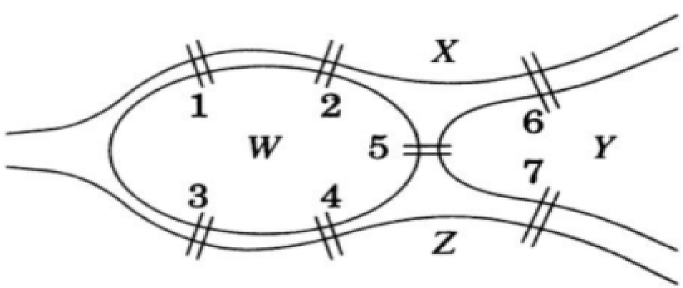
$\Rightarrow W_2$ contains odd cycle.

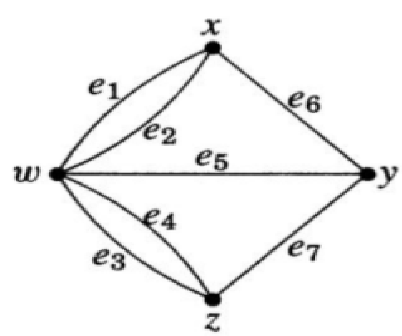
$\Rightarrow W$ contains odd cycle (because W_2 is is part

Hence proved.

3.1.2 Eulerian Circuit

let us travel to Konigsberg again!





Eulerian Graph

1. A graph is Eulerian if it has a closed path containing all the edges.
2. We call a closed path a circuit when we do not specify the first vertex but keep the list in cyclic order.
3. An Eulerian circuit in a graph is a circuit containing all the edges.

3.1.3 Lemma

Statement: If every vertex of graph has degree at least 2 then G has a Cycle.

Proof by contrapositive method :

P = every vertex has degree at least 2.

Q = G has a cycle.

so, P* = There exist a vertex of graph has degree less than 2.

Q* = G has no cycle $\implies$ so G is collection of trees(Forest).

Let T be a component of G

we have to shoe P $\implies Q \equiv P* \implies Q*$

Case 1: T is Trivial graph.
$\implies \exists$ vertex that has degree =1.

Case 2: T is non trivial graph.
$\implies$ T must have at least 2 end vertices .(By the property of tree.)

$\implies$ end vertices have degree = 1.

$\implies \exists$ a vertex that has degree =1.

hence $P* \implies Q* \equiv P \implies Q$

3.1.4 Theorem

Statement : A Graph G is Eulerian iff it has at most One non trivial component and all it vertices have even degree..

Proof : Necessary Condition:- Assume a Graph G is Eulerian $\implies$ It has closed trail containing all the edges $\implies$ There will be an incoming and outgoing even degree. (It has to have one non-trivial component, as more than one components closed trail might not be possible).

Proof: Suppose G is Eulerian

$\implies$ G has closed walk containing all edges

$\implies$ There will be incoming and outgoing edges for all the vertices.

$\implies$ every node has even degree

Since G is a closed path, it cannot have more than one nontrivial component.
Suppose G has at most one non-trivial component and each node of G has even degree if number of edges $(m) = 0$ then G is eulerian

Assume this is true for all graphs having above properties and less than m edges (Induction Hypothesis)

Consider a Graph containing one non trivial component has degree even.

$\implies$ Each vertex of G' has at least two degree

$\implies$ G contains cycle.

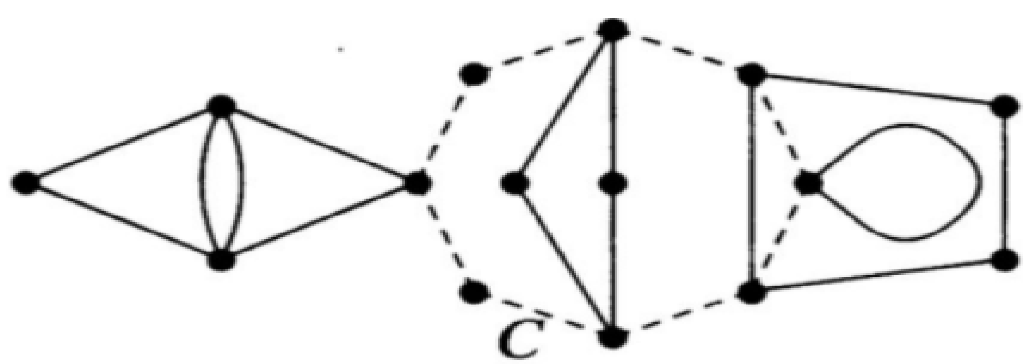
Say this is C.

Remove E(C) from G' to construct G"

1.) G " has less than m edges.
2.) Each vertex of G " has even degree.

Each of multiple possible component will have above two properties and therefore all of them are eulerian.

Example G is a graph which has M edges and each vertex has even degree.



Sufficient Condition:- We combine these Eulerian cycles with C to construct an Eulerian circuit as Fallows: Traverse C untill a component of G ' appear. then traverse Eulerian cycle of that component, come back to C and repeat this. (why you can do this?)

3.2 Vertex Degree and Counting

3.2.1 Degree of Vertex:

Degree of vertex v in a graph G written as d(v) is number of edges incidents to v, except that each loop at v count twice The maximum and minimum number of degrees are denoted by $\Delta(G)$ and $\delta(G)$ respectively.

Que: What will be $\Delta(G)$ and $\delta(G)$ for K-regular graph?
Ans: K for each case.

3.2.2 Order and Size of a Graph:

The order of a graph $G, n(G)$ is number of vertices in The size of a graph $C_2, e(G)$ is number of edges.

Que: what is the order and size of complete graph K_n ?
Ans:

$$\begin{aligned} \text{order } &= n \\ \text{size } &= \tfrac{n(n-1)}{2} \end{aligned}$$

Que: for complete bipartite graph $K_{m,n}$?
Ans:

$$\text{Order} = (m+n)$$

$$\text{Size} = m * n$$

3.2.3 Handshaking Lemma (1st theorem of Graph theory)

(Degree - Sum formula) If G is a graph then

$$\sum_{v \in V(G)} d(v) = 2e(G)$$

Proof by Induction :

if $e((2) = 1$

$$, \sum_{v \in v(a)} d(v) = 1 + 1$$

$$= 2$$

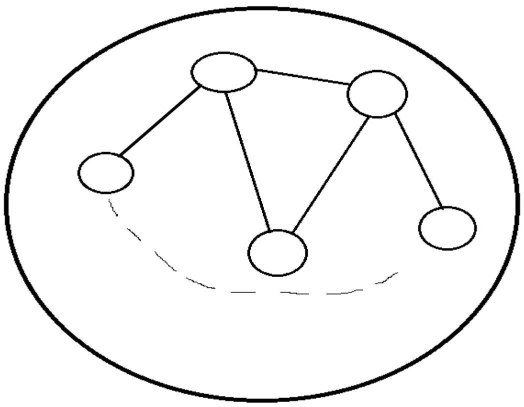
$$= 2 \times \text{ no g edges}$$

So it is true for $e(a) = 1$ Let assume $e(G) = k$, then also $\sum_{v(v(a)} d(v) = 2 \times k$

We have to prove that if $e(a') = k + 1$, then also

$$\sum_{v \in v(a)} d(v) = 2 \times (k + 1)$$

Here, now we will construct the graph. from Graph G', we will construct G'



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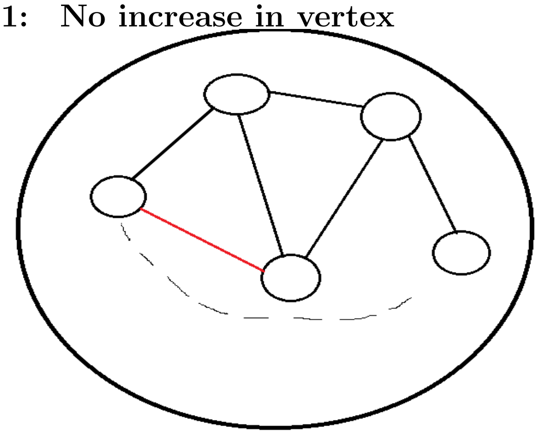
now for the graph G' there are some cases.

Case 1: No increase in vertex

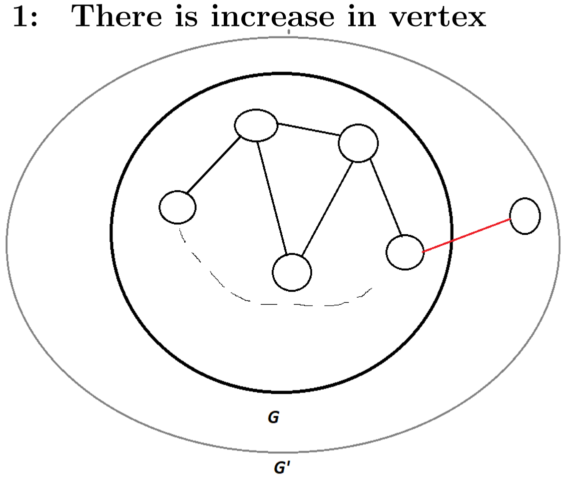


G'

$$\sum_{v \in V(G)} d(v) = 2 \times k$$

$$\begin{aligned}\sum_{v \in v(G')} d(v) &= \sum_{v \in v(G)} d(v) + 2 \\ &= 2k + 2 \\ &= 2(k+1)\end{aligned}$$

Case 1: There is increase in vertex



$$\sum_{v \in V(G)} d(v) = 2 \times k$$

$$\begin{aligned}\sum_{v \in v(G')} d(v) &= \sum_{v \in v(G)} d(v) + 2 \\ &= 2k + 2 \\ &= 2(k+1)\end{aligned}$$

Now we can prove the statement: "A Graph cannot have exactly one node with odd degree"

with the help of this lemma.

Let us consider there are k vertices.

$$\begin{aligned}\text{Let us consider } (K-1) \times \textit{even} \text{ degree } &+ 1 \times \textit{odd} \text{ degree} \\ = \text{even } &+ \text{ odd}\end{aligned}$$

$$\begin{aligned}&= \text{even } + \text{ odd} \\ &= \text{odd } \neq 2 \times \textit{edges}\end{aligned}$$

and therefore there is not possible that exactly one node with odd degree.