* Lecture 2.1 ABC of Graph. *

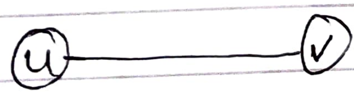
What is a Graph?

A Graph (Aka network) $G=(V,E)$ consists set of two vertices (aka nodes) denoted by V, or by $V(G)$ and set of edges E, or $E(G)$.

Edge Types:

Directed :- ordered pair of vertices. Represented as (u,v) directed from vertex u to v.

Undirected unordered pair of vertices. Represented as $\{u,v\}$. Disregards any sense of direction and treats both end vertices interchangeably.

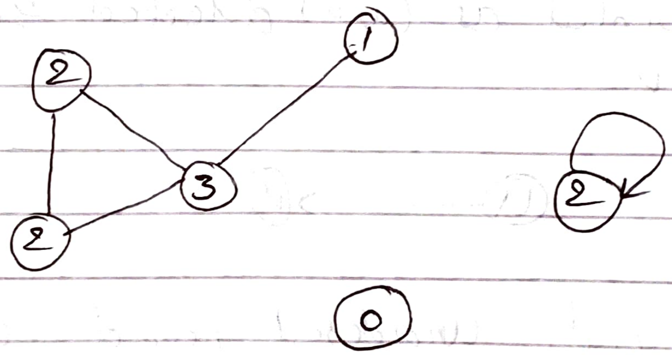


Loop A loop is an edge whose endpoints are equal i.e, an edge joining to a vertex itself is called a loop. Represented as $\{u,u\}=\{u\}$

$\{u,u\} = \{u\}$

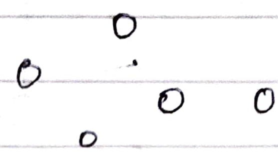
Multiple Edges Two or more edges joining the same pair of vertices

Degree of a node:- The number of edges connected to a node is called it's degree.



20 - Classes of Graph

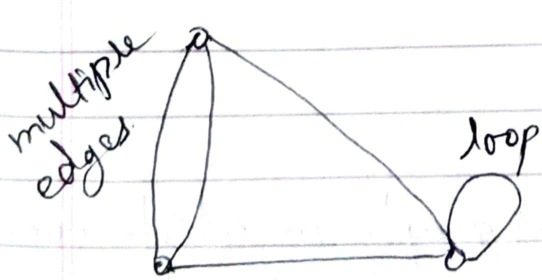
Null Graph:
A null graph is a graph in which there are no edges between its vertices. A null graph is also called empty graph.



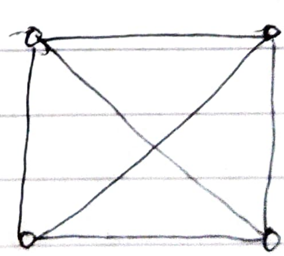
A null graph with n vertices is denoted by N_n.

Trivial Graph: A Graph that has only one vertex.

Simple Graph: A simple graph is the undirected graph with no parallel edges and no loop's.

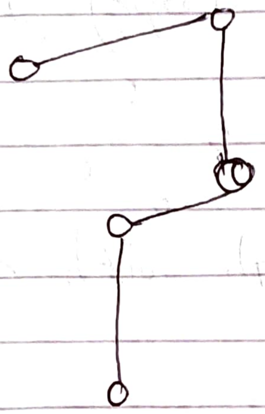


Not a simple graph

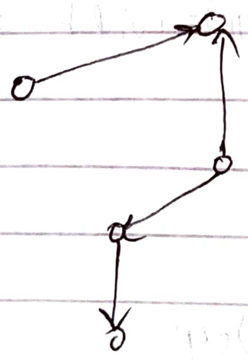


simple graph

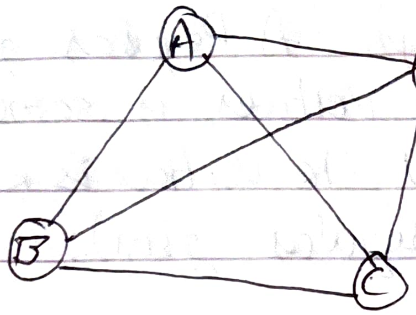
Undirected Graph An undirected graph is a graph whose edges are not directed.



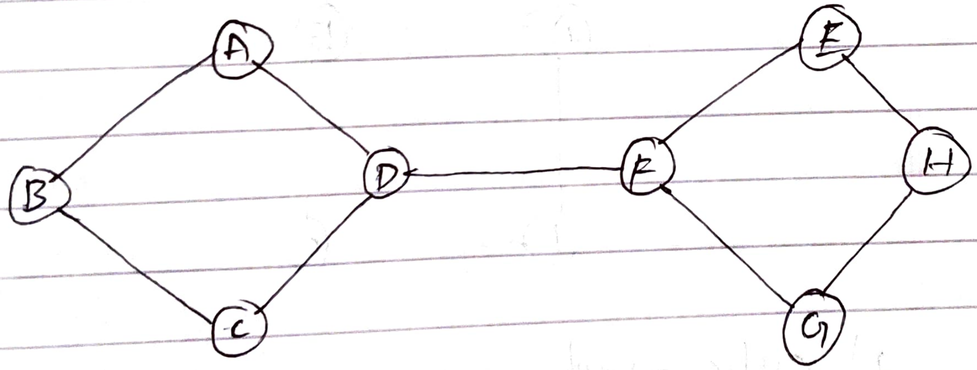
Directed Graph A directed graph is a graph in which the edges are directed by arrows.
Directed graph is also known as digraphs.



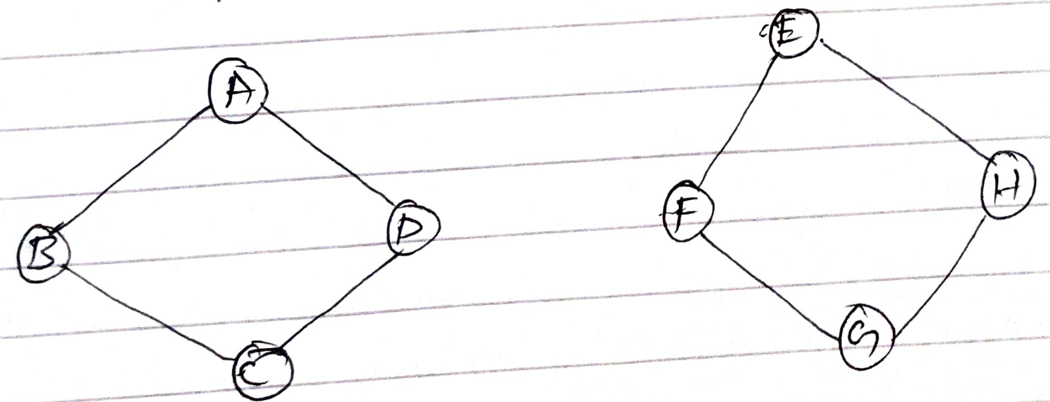
complete graph A graph in which every pair of vertices is joined by exactly one edge is called complete graph.
It contain's all possible edges.



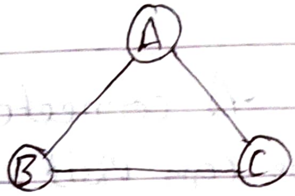
Connected Graph: A connected graph is a graph in which we can visit from any one vertex to any other vertex.
In a connected graph, at least one edge (or) path exists between pair of vertices.



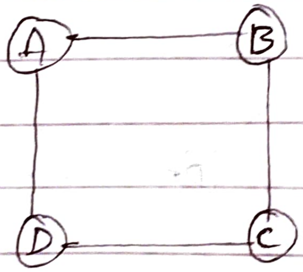
Disconnected Graph: A disconnected graph is a graph in which no path exists between at least a pair of vertices.



Regular Graph A Regular graph in which degree of all vertices is same. If the degree of all the vertices is k, then it is called k-regular graph.

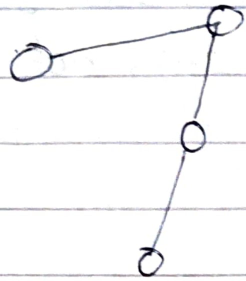


Cyclic Graph A graph containing at least one cycle in it is known as a cyclic graph.



A cylic graph possessing exactly one (undirected, simple) cycle is called a unicyclic graph.

Acyclic graph → A graph containing no cycle is known as a acyclic graph.



True/False

① A simple graph which has 'n' vertices, the degree of every vertex is at most 'n'

② unicyclic graph is a regular graph.

③ Facebook graph is an example of undirected graph.

④ Airline graph is an example of directed graph

⑤ In a disconnected graph atleast one node always have zero degree.

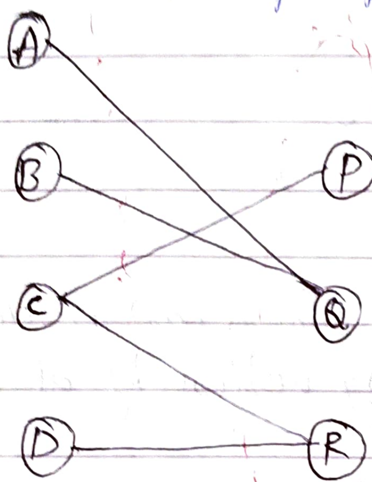
⑥ There always exist an edge in a connected graph ~~thi~~ which when removed the graph becomes disconnected.

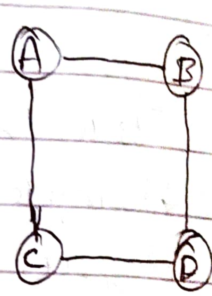
* Answer is later pages under Text book notes *

Bipartite Graph

A bipartite graph is a graph in which the vertex set can be partitioned into two sets such that edges only go between sets, not within them.

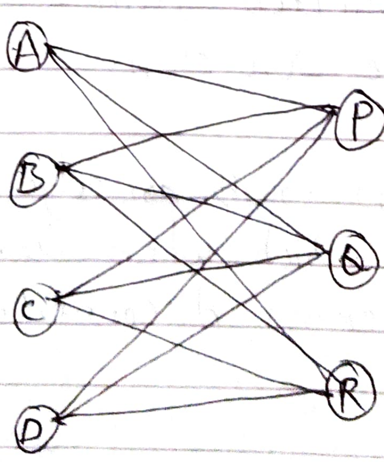
A simple path is bipartite graph



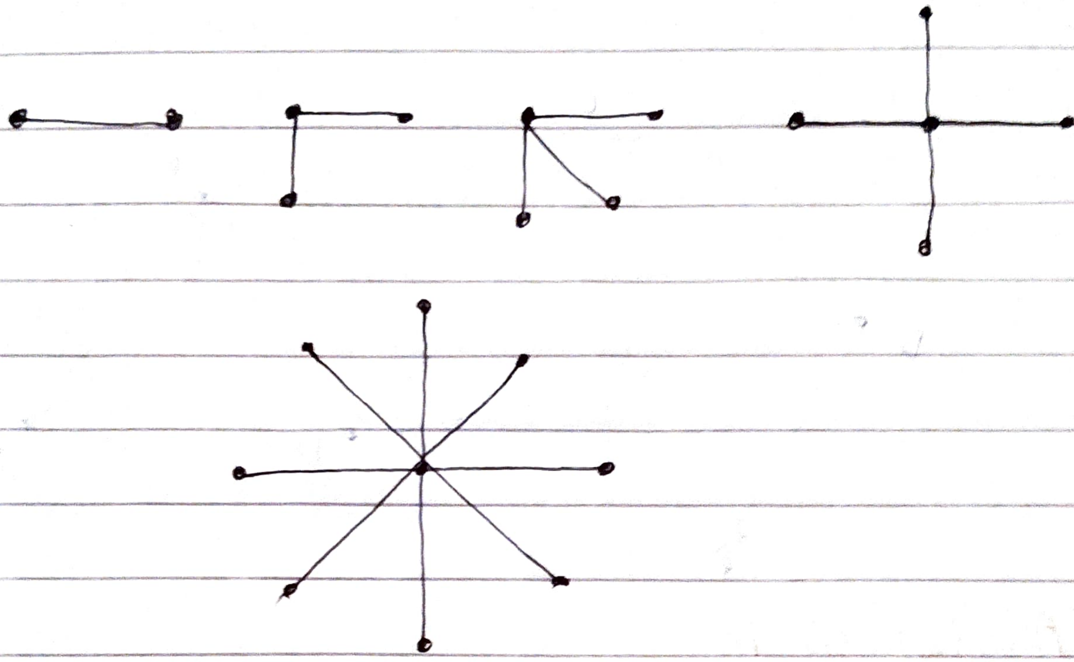


Complete Bipartite Graph

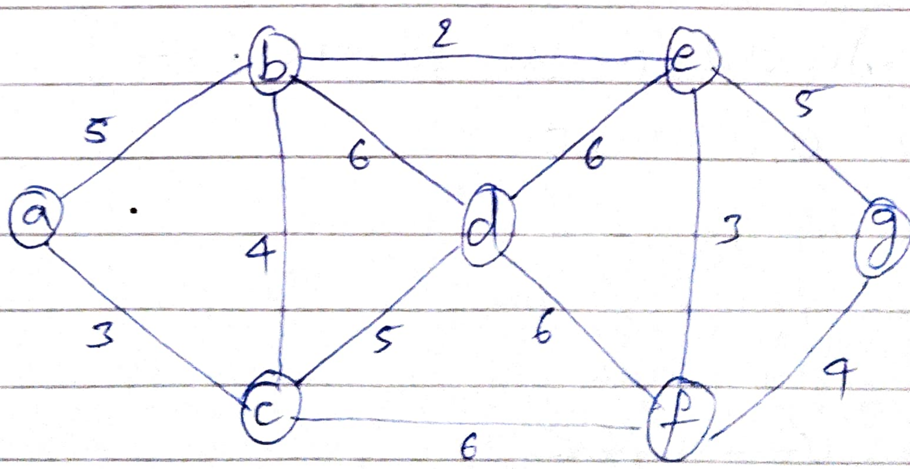
A complete bipartite graph is a bipartite graph in which each vertex in the first set is joined to each vertex in the second set by exactly one ~~e~~ edge.



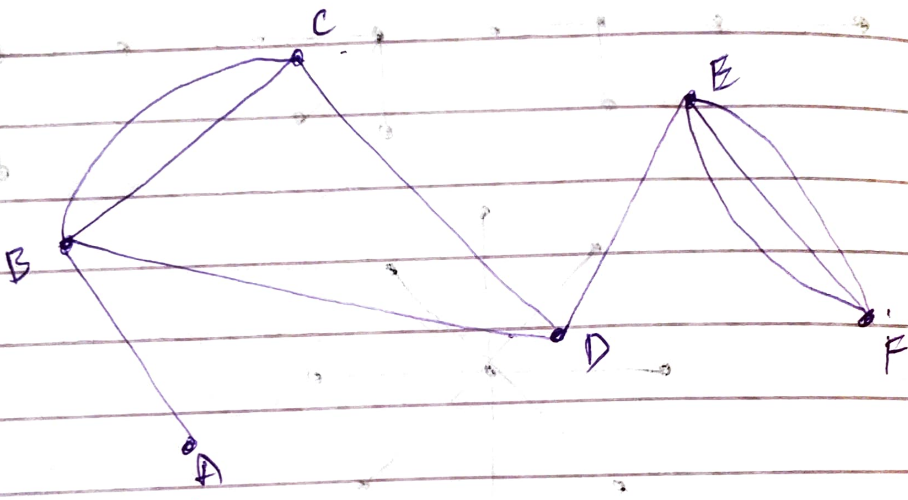
Star Graph A star graph is a complete bipartite graph in which $n-1$ vertices have degree 1 and a single vertex have degree $n-1$



Weighted Graph A weighted graph is a graph whose edges have been labeled with some weights or numbers



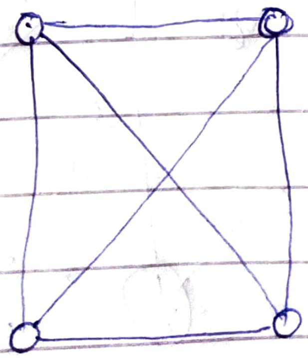
Multi Graph A Graph in which there are multiple edges between any pair of vertices (or) there are edges from a vertex to itself (loop) is called a multi-graph.

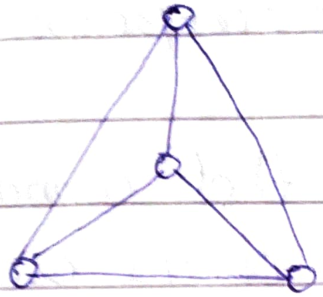


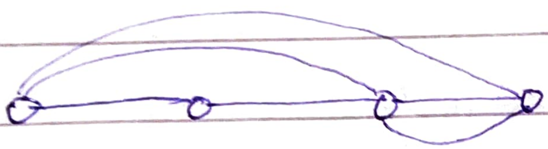
Planner graph

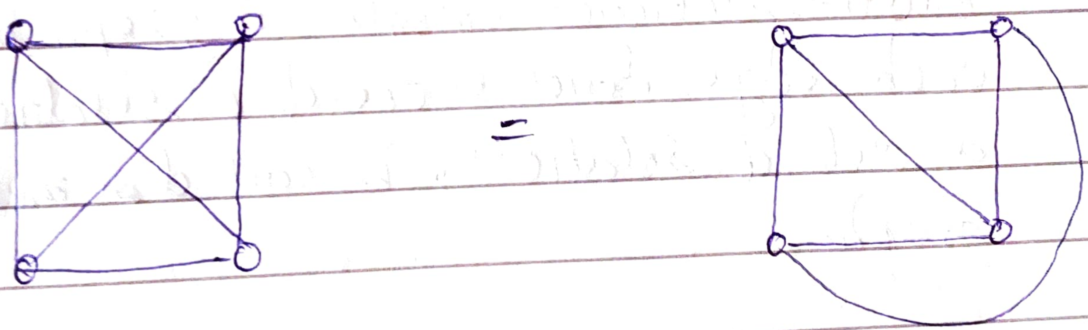
A planner graph is a graph that we can draw in a plane in such a way that no two edges of it cross each other except at a vertex to which they are incident.

All of below are planner. How?



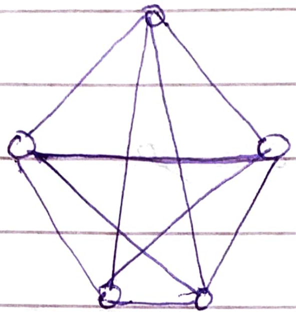




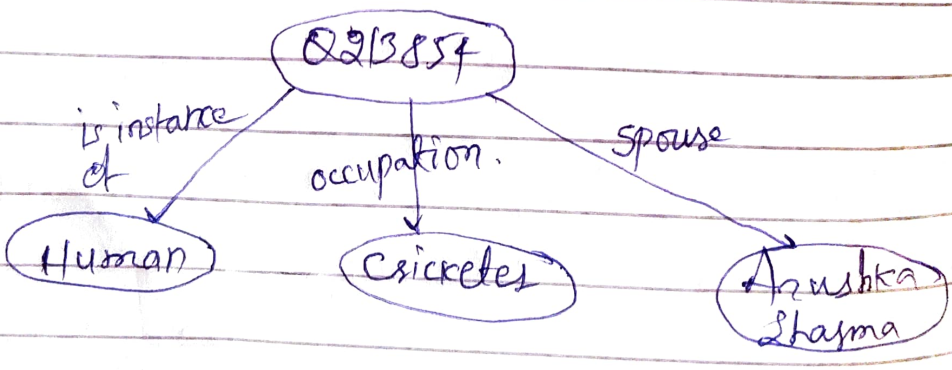


Non-planner graph A graph that is not a planner graph is called a non-planner graph.

In other words, a graph that cannot be drawn without at least one pair of it's crossing edges is known as non-planner graph.



Multi-relational Graph A graph in which each edge denotes one of a relations from a set of relations, is called multi-relational graph.

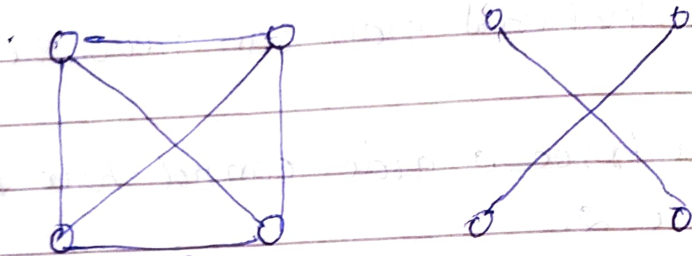


* Lecture - 3.1 *

Subgraph- A subgraph of a graph 'G' is a graph H such that

$$V(H) \subseteq V(G)$$
$$E(H) \subseteq E(G)$$

We say $H \subseteq G$ and G contains H.

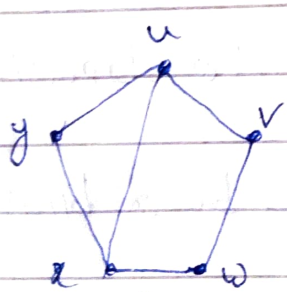


G

Independent set and Clique

Independent (stable) set :- A set of pair-wise non-adjacent vertices.

Clique:- A set of pair-wise adjacent vertices.



G

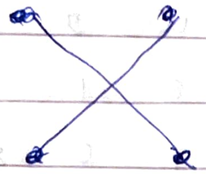
Independent set : {u, w} & {y, v}, {y, w} of size 2.

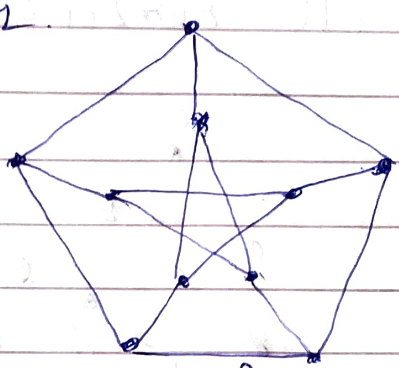
Clique : {u, x, y} of size 3.
{u, v}, {v, w}, {w, x}, {y, u} also clique.

Why 4 clique is not there?

chromatic number

The chromatic number $\chi(G)$ of a graph G, is minimum number of colors needed to label the vertices so that the adjacent vertices receive different color.

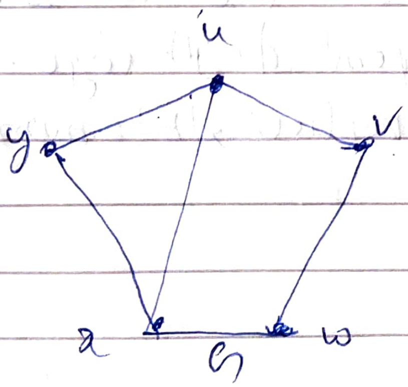


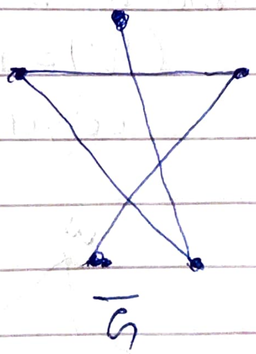


Assignment: How to find chromatic number?

Complement of a Graph:-

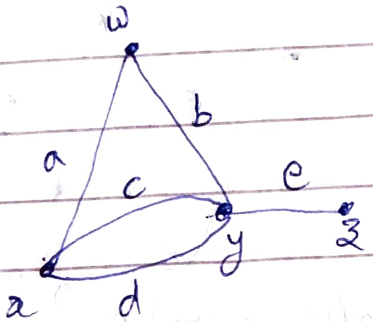
The complement $\bar{G}$ of a simple graph G is the simple graph with vertex set $V(G)$ defined by $uv \in E(\bar{G})$ if and only if $uv \notin E(G)$.





*Representing graphs *

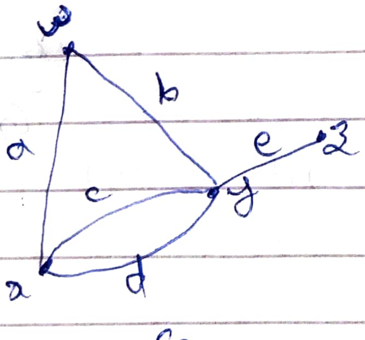
Adjacency graph An adjacency matrix of a graph G denoted as $A(G)$ is $|V| \times |V|$ size square matrix whose i-j th element denote number of edges between i th and j th vertices ($|V|$ = number of vertices).



$$A(G) = \begin{array}{c|cccc} & w & x & y & z \\ \hline w & 0 & 1 & 1 & 0 \\ x & 1 & 0 & 2 & 0 \\ y & 1 & 2 & 0 & 1 \\ z & 0 & 0 & 1 & 0 \end{array}$$

Disadvantage :- space & sparse matrix.

Incident matrix An incident matrix of a graph G denoted as $M(G)$ is $|V| \times |E|$ size matrix whose i-j th element = 1. If i th matrix is an endpoint of j th edge.
($|V|$ = number vertices, $|E|$ = number of edges).



G

$$M(G) = \begin{array}{c|ccccc} & a & b & c & d & e \\ \hline w & 1 & 1 & 0 & 0 & 0 \\ x & 1 & 0 & 1 & 1 & 0 \\ y & 0 & 1 & 1 & 1 & 1 \\ z & 0 & 0 & 0 & 0 & 1 \end{array}$$

If Graph is fully connected then it will be $n \times {}^{n}C_{2}$

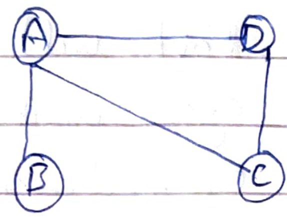
List ~~As~~ a linked list of neighbour's.

$A \to B \to C \to D$

$B \to A$

$C \to A \to D$

$D \to A \to C$

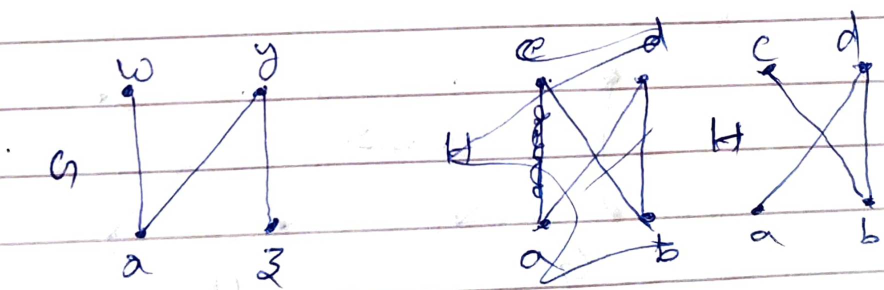


Isomorphism

An isomorphism from a simple graph G to a simple graph H is a bijection $f: V(G) \to V(H)$ such that

$$uv \in E(G) \text{ iff } f(u)f(v) \in E(H)$$

We say G is isomorphic to H and denote it as $G \cong H$.



Why?

$w \to a \to y \to z$

$c \to b \to d \to a$

$f(w) = c$

$f(a) = b$

$f(y) = d$

$f(z) = a$

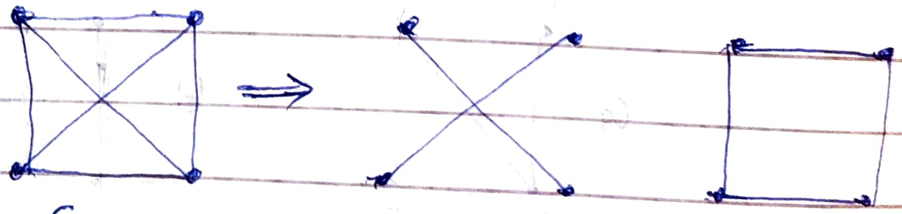
A few notation's

Path with 'n' vertices – P_n
Cycle with 'n' vertices – C_n
Complete graph with n vertices – K_n
Bipartite graph between two independent set $K_{m,n}$

Null graph N_n
Wheel Graph – W_n
Hypercube Graph – Q_n

Decomposition and special graphs

A decomposition of a graph is a list of subgraphs such that each edge appears in exactly one subgraph in the list



G H₁ H₂

$H_1 \cup H_2 = G$

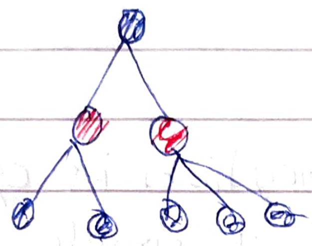
What will be the chromatic number for an empty graph having 'n' vertices?

a) 0 b) 1 ✓ c) 2 d) n

empty graph

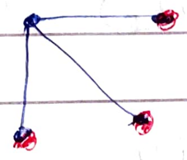
② What will be the chromatic number for a tree having more than 1 vertex?

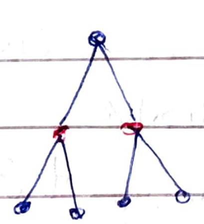
a) 0 b) 1 c) 2 ✓ d) varies with type tree



③ The chromatic number of star graph with 3 vertices is greater than that of a tree with same number of vertices.

a) True b) False ✓





* Lecture - 3.2 *

① Isomorphism

② Isomorphism is equivalence relation on set of simple graph.

③ Decomposition

Statement - Isomorphism is equivalence relation on set of simple graph.

~~Proof~~

*An equivalence relation is a relation that is reflective, symmetric & transitive

Theorem - Graph isomorphism is an equivalence relation.

Proof Let's G_1, G_2 & G_3 are graph's.

$G_1 = (V_1, E_1)$

$G_2 = (V_2, E_2)$

$G_3 = (V_3, E_3)$

(i) Reflexive

$\forall G, G_1 \cong G_1$

As per definition bijunction function $f : V_1 \rightarrow V_1$

(ii) Symmetric

if $G_1 \cong G_2$ then $G_2 \cong G_1$

~~$G_1 \cong G_2 \Rightarrow \exists$ a bijection f~~

$G_1 \cong G_2 \Rightarrow \exists$ a bijection function f such that $f : V_1 \rightarrow V_2$

$\Rightarrow \exists$ a bijection function f^{-1} such that $f^{-1} : V_2 \rightarrow V_1$

$\Rightarrow G_2 \cong G_1$

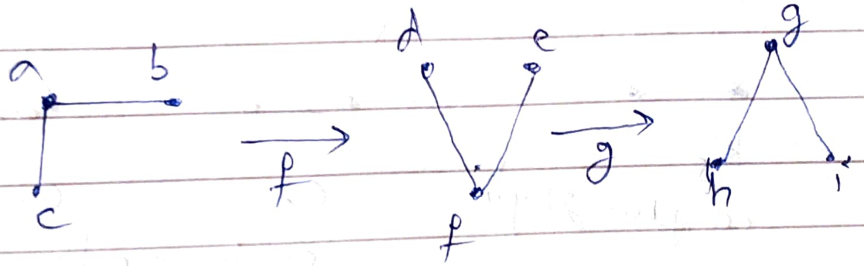
(iii) Transivity

If $G_1 \cong G_2$ and $G_2 \cong G_3$ then $G_1 \cong G_3$

$G_1 \cong G_2$, $G_2 \cong G_3 \Rightarrow \exists f, g$ bijection such that
$f : V_1 \to V_2$, $g : V_2 \to V_3$

$\Rightarrow \exists$ fog a bijection
$fog : V_1 \to V_3$

$\Rightarrow G_1 \cong G_3$



$f(b) = d$ $\quad$ $g(d) = h$
$f(a) = f$ $\quad$ $g(f) = g$
$f(c) = e$ $\quad$ $g(e) = i$

$gof(a) = g(f(a)) = g(f) = g$

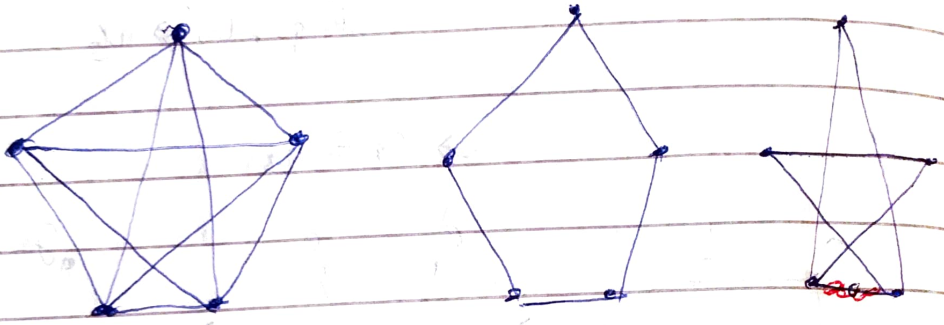
~~Reflexitive property~~ – The isomorphism relation ~~is an equivalence relation~~

Reflexection Property – The identity permutation on $V(G)$ is an isomorphism from G to itself. thus $G \cong G$

* A graph is self-complementary if it is isomorphic to it's complement. A Decomposition of a graph is a list of subgraphs such that each edge appears in exactly one subgraph in the list.

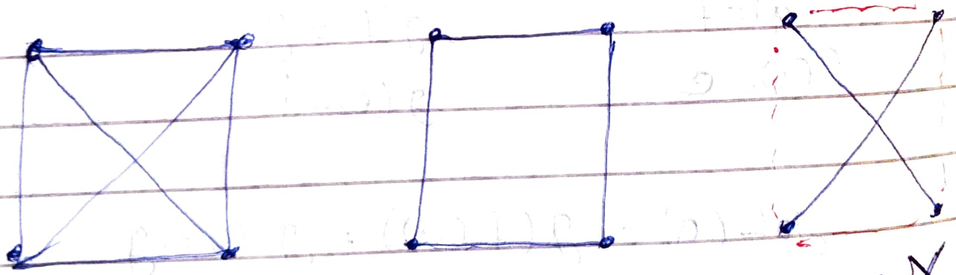
1/2 Decomposition

- K_5 be decomposed into C_5 two
- Can K_4 be decomposed into two C_4s?
- Any n-Vertex and it's complement decompose K_n.
- $K_{1,n-1}$ and K_{n-1} decompose K_n



K_5 complete graph

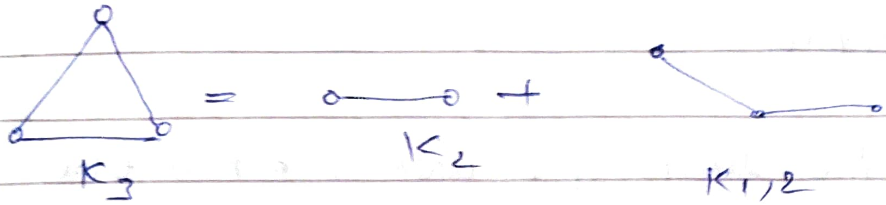
C_5 C_5 cycle graph's



K_4 C_4 C_4 ✗

It's not a cycle.

Q) Can we decompose K_n into K_1, K_{n-1} and K_{n-1}?



Isomorphism

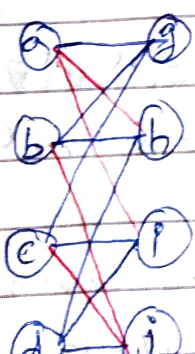
⇒ In graph theory, an isomorphism of graphs G and H is bijunction between the vertex sets of G and H.

$$f: V(G) \to V(H)$$

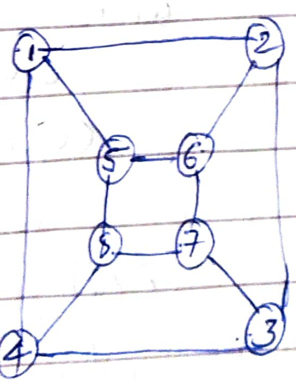
such that any two vertices u and v of 'G' are adjacent in 'G' if and only if f(u) and f(v) are adjacent in H.

⇒ Graph isomorphism is an equivalence relation on graphs and such it partitions the class of all graphs into equivalence classes.

Graph G



Graph H



An Isomorphism between G & H.

$f(a)=1$

$f(b)=6$

$f(c)=8$

$f(d)=3$

$f(g)=5$

$f(h)=2$

$f(i)=4$

$f(j)=7$

* Lecture-9 *

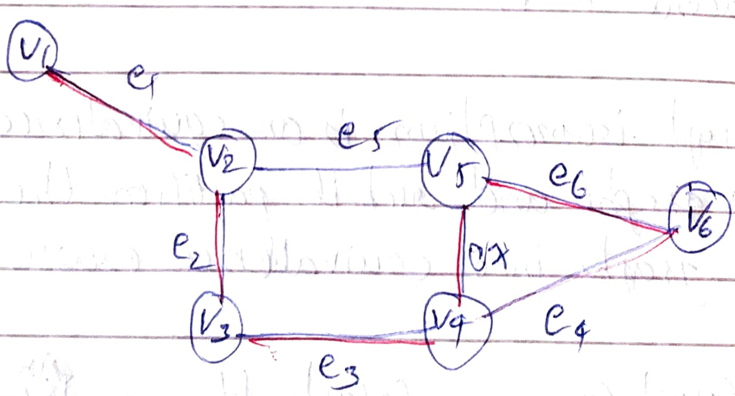
Walk - A walk in a graph is a sequence of edges such that each edge (except the first one) start with a vertex where the previous edges ended.

The length of a walk is a number of edges in it.

A path is a walk where all edges are distinct.

A simple path is a path where all vertices are distinct.

Example



walks

$(e_1, e_2, e_3, e_4, e_6, e_7)$

$(e_1, e_5, e_6, e_4, e_7, e_6, e_4, e_3)$

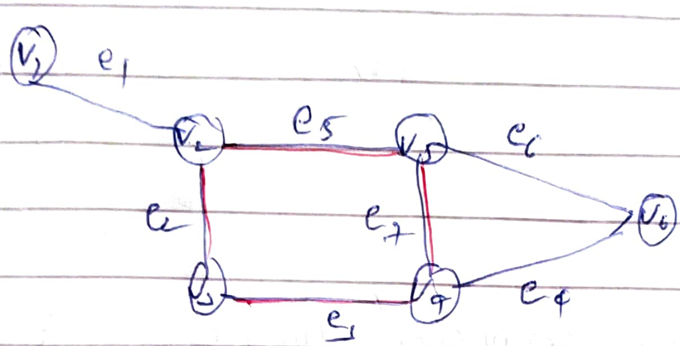
Path $(e_1, e_5, e_7, e_4, e_6)$

∴ edeg's not repeating

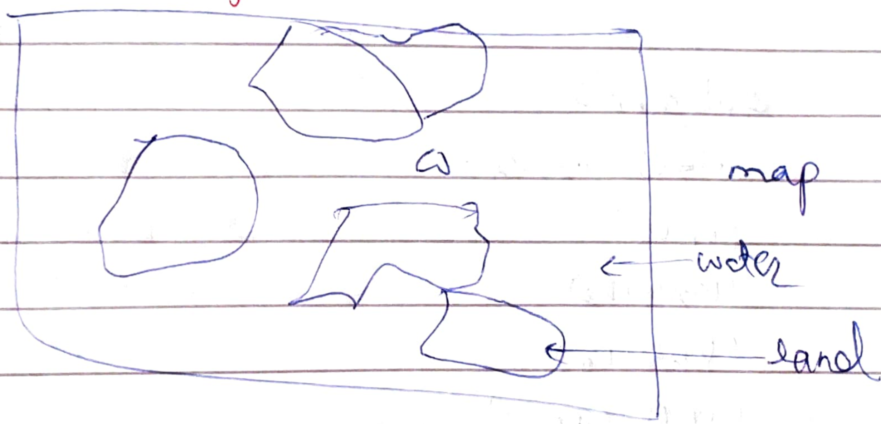
Simple path - (e_1, e_5, e_7, e_4)

$(V_1, V_2, V_5, V_4, V_6)$

cycle- A cycle is a path in which first and the last vertices are same.

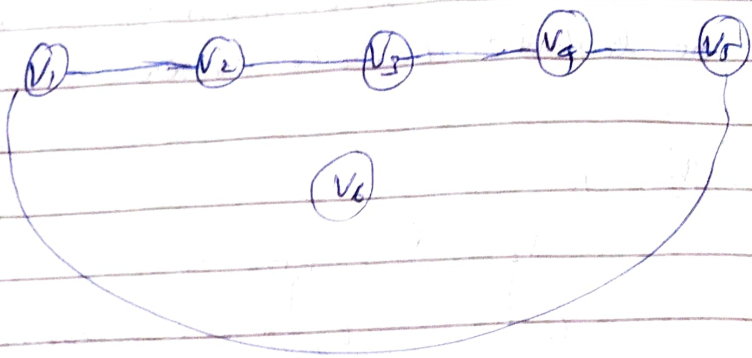


Connected components -



Connected & disconnected graph- A graph is connected if there is a path between any two pair of vertices in that graph. otherwise it is disconnected.

Connected component of G is maximal connected subgraph of G. (In other words, those connected subgraphs which are not contained in larger connected subgraph of G)



maximal connected subgraph's - 2

$(V_1, V_2, V_3, V_4, V_5)$

(V_6)

subgraph's

~~V_1, V_2, V_3, V_4, V~~

(V_1, V_2)

(V_1, V_2, V_3)

(V_1, V_2, V_4)

(V_1, V_2, V_4, V_5)

(V_6)

⋮

proposition

statement - Every graph with 'n' vertices and k edges has at least n-k components.

How many maximum node's using 'k' edges

$2k$

node's 'n' = $2k$