Week 5

Kernel, tournaments and king*

Continuation from Lecture 8

5.1 Kernel in a directed graph

Is a set of vertices $S \subseteq V(D)$ such that S indicates no edges and every vertex outsied of S has a succssor in S. Concept of kernel is only defined for directed graph.

Example 1: Let us suppose we take a graph with 4 vertices and 4 edges which are in cycle. Kernel of this directed graph $S_1 = \{1, 3\}$ which are not connected to each other. Similarly $S_2 = \{2, 4\}$ is also a kernel, so there is no unique kernel in this case.
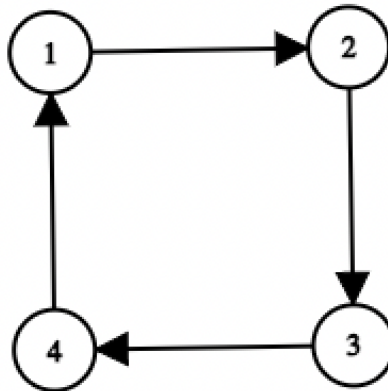

Figure 5.1: Directed Graph to show kernels.

Example 2: Let us suppose we take a graph with 4 vertices and 3 edges which are in cycle. Kernel of this directed graph $S = \{2, 4\}$ which are not connected to each other.
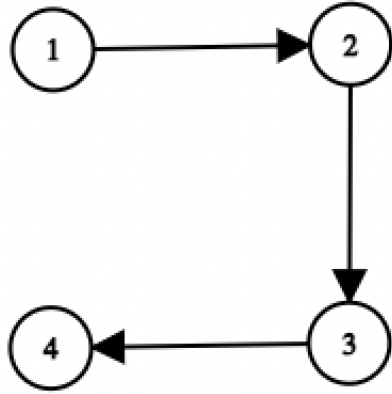

Figure 5.2: Directed Graph to show kernels.

Example 3: Let us suppose we take a graph with 6 vertices and 6 edges which are in cycle. Kernel of this directed graph $S_1 = \{2, 4, 5\}$, $S_2 = \{1, 3, 6\}$ which are not connected to each other.
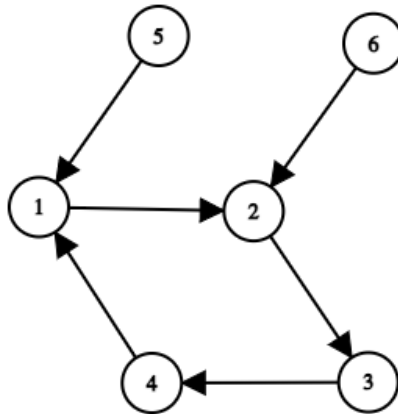

Figure 5.3: Directed Graph to show kernels.

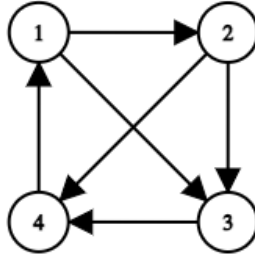**Example 4:** For a complete graph kernel will be null.

Figure 5.4: Fully connected directed graph.

Example 5: A graph with odd cycles will not have kernel. C_n where n is odd will not have kernel because we need to choose something in S to make a kernel and whatever you choose should be chosen a way such that they should be independent and successor of S. In this case every node has only one successor hence if you dont choose any node then we have to choose its successor and simply put we can never have a scenario where kernel can be created.
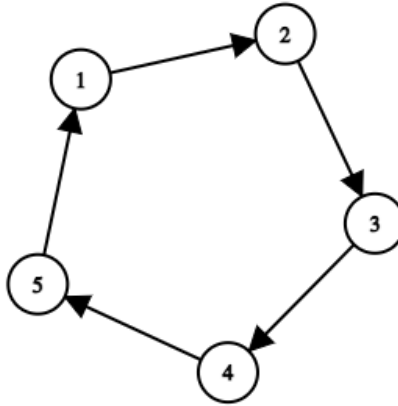

Figure 5.5: Directed graph

Let K be the kernel of C_n, then any $u, v \in K$ should have the following properties

- they should be independent;

- $w \in K$ then w must have successor in K

5.1.1 Outdegree and indegree

Every node in directed graph will have a indegree and outdegree. Number of edges comming into and going out of the graph.

In any directed graph sum of indegrees = sum of all degrees = number of edges in G i.e.

$$\sum_{v \in V(G)} d^+(v) = \sum_{v \in V(G)} d^-(v) = \mid e(G) \mid \tag{5.1}$$

Example 1: For the given digram indegree and outdegree of each node would be

- *Outdegree*: For node A, $d^+(A) = 1$, For node B, $d^+(B) = 1$

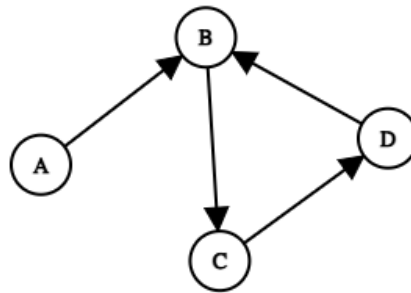- *Indegree*: For node A, $d^-(A) = 0$, For node B, $d^-(B) = 2$

Figure 5.6: Directed graph

5.1.2 Orientation

An orientation of graph G is a directed graph D obtained from G by choosing an orientation $x \to y$ or $y \to x$ for each edge $xy \in E(G)$.
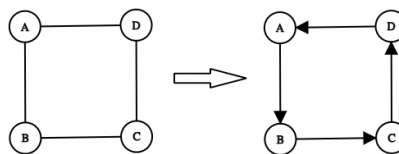

Figure 5.7: Undirected to Directed graph

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5.2 Tournament in graph

Tournament is an orientation of a complete graph.

5.2.1 King of the tournament

King of the tournament is a vertex from where all the other vertices are reachable by a path of length of 2 at most.

Example 1: Let us suppose we have a cricket tournament between different countries and graph should have directions. It can be assumed that $x \to y$ plays a match and x wins against y. In the given example India is the king because all nodes can be reached from India.
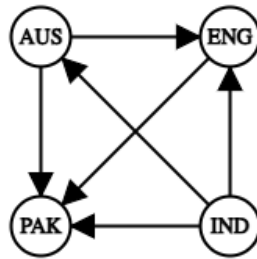

Figure 5.8: King of the tournament

Example 2: A complete graph with 5 vertices and vertices to each other. All vertices are kings because they satisfy the definition of the king. For node B

- $B \to C$

- $B \to D \to A$
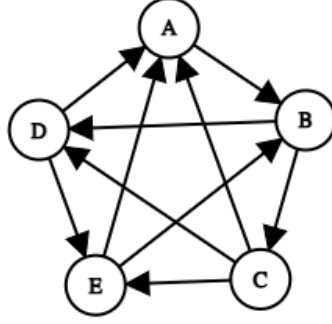
- $B \to D$

- $B \to D \to E$

Figure 5.9: King of the tournament

Every tournament has a king. This can be proved with a contradiction. Suppose u is a node with the highest outgoing degree.

$$N_o(u) + N_I(u) + 1 = \mid V(G) \mid \qquad (5.2)$$

where u is not a king.

Suppose u is not the king. All the nodes is N_I will be connected to all the nodes in N_o and only thing we are not aware of is the direction.

Case 1: If the direction is from $N_I \leftarrow N_o$, then we can say that u is king because we can reach from all the nodes of N_I via just two edges $u \rightarrow N_o \rightarrow N_I$. Hence every tournament must have a king.

Case 2: If the firection is from $N_I \rightarrow N_o$. Lets suppose there is a node in v in $N_I(u)$ and v has outgoing edges in all the nodes i.e. $v \rightarrow N_o$ and $v \rightarrow u$. Number of outgoing edge of node $v = 1+ \mid N_o(u) \mid >$ number of outgong edge of u. This is a contradiction because we have assumed that u is the node which has highest out degree. Hence this case is not applicable.
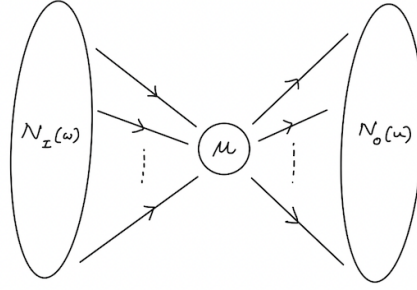
Figure 5.10: King of the tournament

Theorem 5.1. *Prove or disprove if D is a orientation of simple graph with 5 vertices then the vertices of D cannot have distinct out degree.*

Proof. We can create a graph where $x > y$ then $y \to x$ as below. In this way outegree of each of node

- $Node(1) = 4$

- $Node(2) = 3$

- $Node(3) = 2$

- $Node(4) = 1$

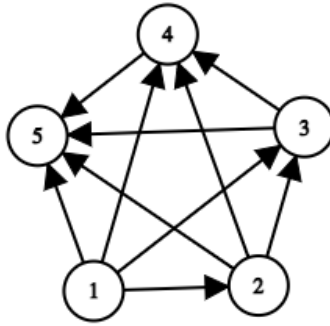Hence the statement given in false because D can have distinct outdegree.

Figure 5.11: Example of simple graph

$\square$

Question   Give example of one real world realtion whose digraphs has no cycles.
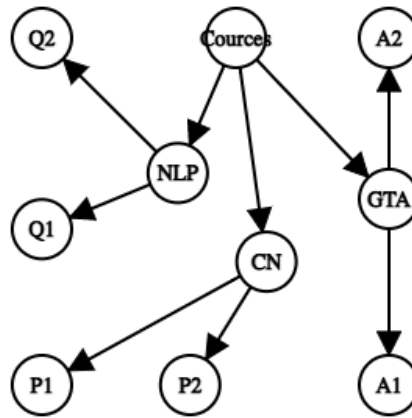

Figure 5.12: Real world example

5.2.2   Tree

In context of directed graph we can define a tree such that if we take any two vertices we will have a unique path between those two vertices. Simply put trees are connected acyclic graphs.
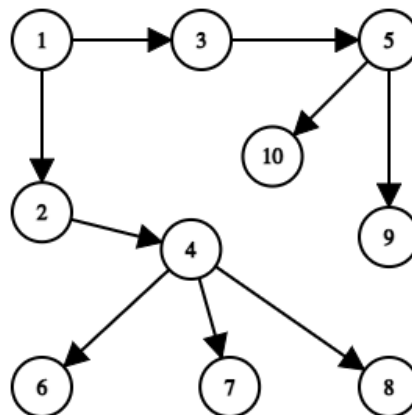

Figure 5.13: Example of tree

Properties of trees

- Deleting a leaf from an n-vertex tree produces an $n - 1$ vertex tree.

- Tree is connected and it has no cycle.

- An n vertex contains $n-1$ edges because if it contains one more it will have a cycle and if it has one less it will not be connected.

- For any pair of vertex (u, v) there exists one and only one path.

- Every edge of a tree is cut-edge.

- Tree is a bipartite graph.

5.3 Math stuff

$$A \xrightarrow{\;\;f\;\;} B$$

$$
\begin{array}{ccc}
a & \longrightarrow & 5 \\
b & \longrightarrow & 6 \\
c & \longrightarrow & z
\end{array}
$$
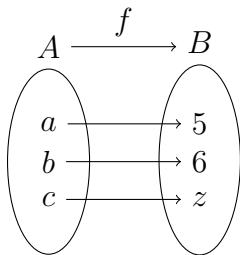
Please make an effort to typeset things nicely. There are quite a few macros in the lpsdp.sty file. Below are illustrated how to do some basic things; please study the LaTeX carefully.

Here's a typical LP in standard/equational form, with an equation number on one of the constraints.

$$
\begin{aligned}
\min \quad & c^\top x \\
\text{s.t.} \quad & Ax = b \\
& x \geq 0 \tag{5.3}
\end{aligned}
$$

Here's a reference to the (5.3) nonnegativity constraint. Some more LPs:

$$
\begin{aligned}
\min \quad & 3x_1 - 5x_2 \\
\text{s.t.} \quad & x_1 + 2x_2 \leq 6 \\
& 2x_1 + x_2 \leq 6 \\
& 2x_1 + 2x_2 \geq 7 \\
& x_1, x_2 \geq 0
\end{aligned}
$$

$$
\begin{aligned}
\text{minimize} \quad & 3x_1 - 5x_2 + 2x_3 - x_4 \\
\text{subject to} \quad & x_1 + 2x_2 - 4x_3 + x_4 \leq 6 \\
& -x_1 + 3x_2 - x_3 - x_4 \geq 7 \\
& x_i \geq 0 \quad \forall i = 1 \ldots 4
\end{aligned}
$$

Let's do some matrices:

$$\begin{pmatrix} 1 & \rho & \rho \\ \rho & 1 & \rho \\ \rho & \rho & 1 \end{pmatrix}, \quad \text{or alternately,} \quad \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}.$$

More generically:

$$A = \begin{bmatrix} | & | & & | \\ A_1 & A_2 & \cdots & A_n \\ | & | & & | \end{bmatrix} = \begin{bmatrix} - & a_1 & - \\ - & a_1 & - \\ & \vdots & \\ - & a_n & - \end{bmatrix}$$

Here's some more random typesetting:

- "P vs. NP, where the former means time $\text{poly}(n)$";

- $\widetilde{O}(f(x))$ is $f(x) \cdot \text{polylog}(f(x))$;

- $g(x) = \begin{cases} \sin(2\theta) & \text{if } \theta \leq \pi, \\ \max\{\cos^2 \theta, \frac{1}{3}\} & \text{if } \theta > \pi. \end{cases}$

Please don't write $max(A)$ when you mean $\max(A)$, or $log(n)$ when you mean $\log(n)$, or "quotes" when you mean "quotes".

A theorem and a proof:

Theorem 5.2. $(a + b)^2 = a^2 + 2ab + b^2$.

Proof. Let for the reader. □

Here's what to do if your proof ends on an equation:

Proof. It's easy:

$$(a + b)^2 = (a + b)(a + b) = (a + b)a + (a + b)b = a^2 + ba + ab + b^2 = a^2 + 2ab + b^2 \quad □$$

Please insert figures liberally. It's probably best if "vector graphics" are in pdf or png format, and "bitmap graphics" are in jpg format, but lots formats are supported. There's a macro defined to make things easy. Inkscape is a pretty reasonable, free program in which to draw figures.[1]

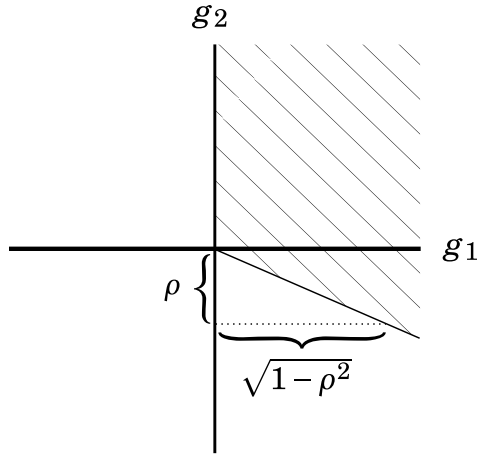[1]Ryan: I admit, I sometimes draw figures in Powerpoint.

Figure 5.14: The region $g_1 > 0, g_2 > -\frac{\rho}{\sqrt{1-\rho^2}}g_1$.

Once you've inserted it, you can refer to it as Figure 5.14.

Finally, if you have citations, see the commented-out stuff in the LaTeX here.