Indian Institute of Technology- Jodhpur

Graph Theory and Applications(GTA-2)
COURSE CODE: CSL7410

Lecture Scribing Assignment: Week 6

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Contents

1 Distance

Definition: If G has a u, v-path, then the distance from u to v, written $d_G(u, v)$ or simply d(u,v), is the least length of a u, v-path. If G has no such path, then $d(u, v) = \infty$

1.1 eccentricity

Definition: The eccentricity of a vertex u,denoted by $\in (u)$ is $max_{v \in V(G)} d(u, v)$
The radius of a graph G, written rad G,is $min_{u \in V(G)} \in (u)$

2 Diameter

Definition: Diameter is maximum of distances between any two pair of vertices

Theorem-:

If G is a simple graph then diam(G) $\geq 3 \Rightarrow diam(G^c) \leq 3$.

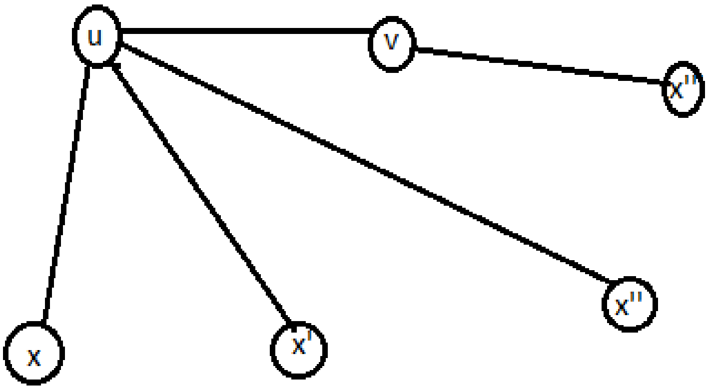
Proof

$diam(G) \geq 3$
$\Rightarrow$ (a) $\ni$ u and v $\in$ V(G) such that

u and v are not adjecent
(b) u and v have no common neighbour

$\Rightarrow$ uv does not exist in G
$\forall$ x$\in$ V(G)-(u,v) means ux or vx does not exist

$\Rightarrow$ uv exist in G^c and ux or vx will also exist



$\forall$ x$\in$ V(G)-(u,v)

Therefore
$diam(G^c) \leq 3$

3 Problem

Statement Let G be a simple graph with diameter(G)$\geq$ 4 prove that $diam(g^c) \leq$ 2.
Proof
We will prove it by contrapositively.
i.e.
P$\equiv$ $diam(G) \geq 4$

Q$\equiv$ $diam(G^c) \leq 2$

equivalent to above
$\sim Q \Rightarrow \sim P$

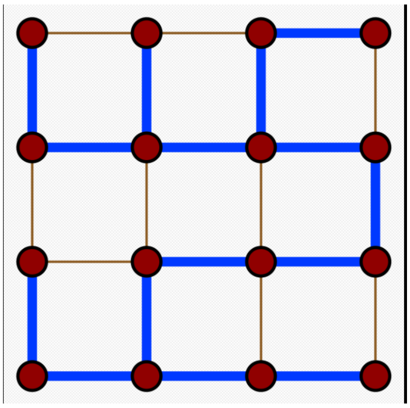
where
$\sim Q \equiv diam(G^c) \geq 2$
$\sim P \equiv diam(G) \leq 3$
Since by previous theorem
diam(G) $\geq 3 \Rightarrow diam(G^c) \leq 3$

So $diam(G^c) \geq 3 \Rightarrow diam((G^c)^c) \leq 3$

$\Rightarrow diam(G) \leq 3$
hence proved.

4 Spanning Tree

Definition: a spanning tree T of an undirected graph G is a subgraph that is a tree which includes all of the vertices of G.In general, a graph may have several spanning trees, but a graph that is not connected will not contain a spanning tree. If all of the edges of G are also edges of a spanning tree T of G, then G is a tree and is identical to T.



A spanning tree (blue heavy edges) of a grid graph.

minimum spanning tree- In a connected weighted graph of possible communication links, all spanning trees have n-1 edges; we seek one that minimizes or maximizes the sum of the edge weights.

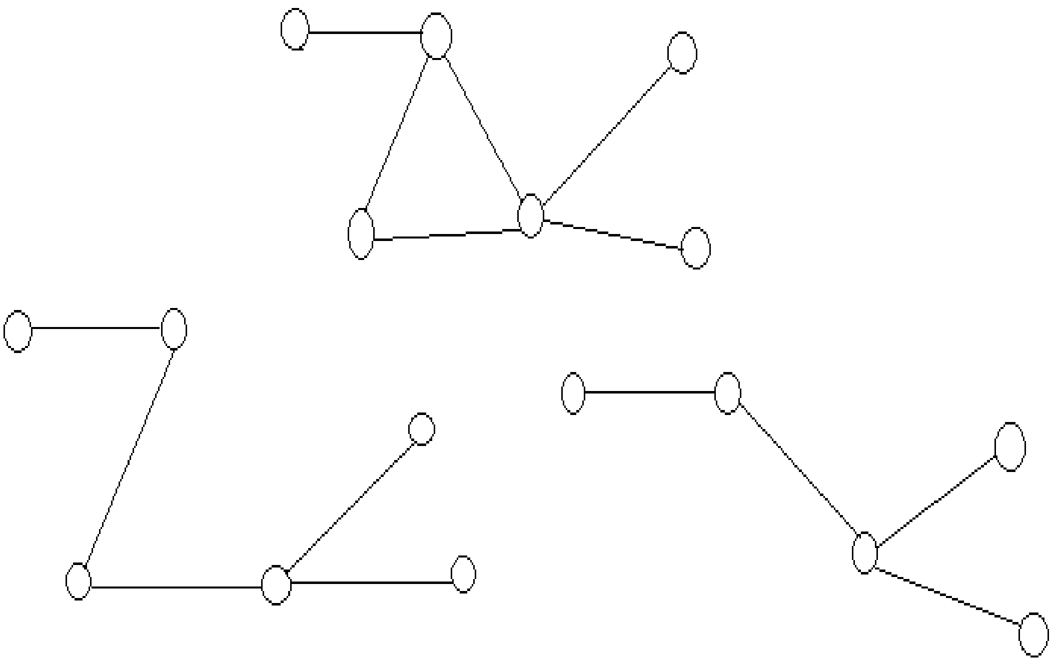


figure: minimum spanning tree

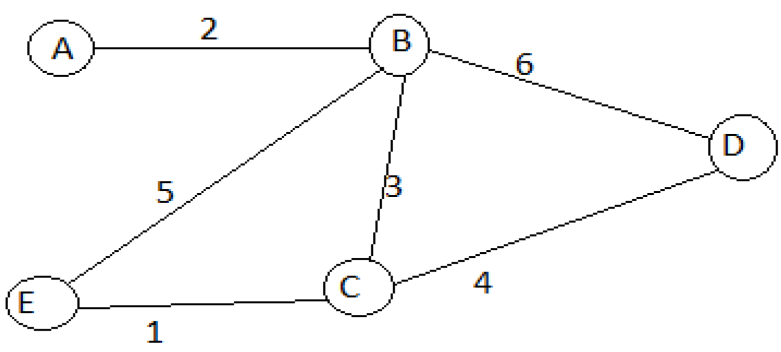
5 Prim's Algorithms

Prim's algorithm is a greedy algorithm that finds a minimum spanning tree for a weighted undirected graph. This means it finds a subset of the edges that forms a tree that includes every vertex, where the total weight of all the edges in the tree is minimized. The algorithm operates by building this tree one vertex at a time, from an arbitrary starting vertex, at each step adding the cheapest possible connection from the tree to another vertex.
The idea behind Prim's algorithm is simple, a spanning tree means all vertices must be connected. So the two disjoint subsets of vertices must be connected to make a Spanning Tree. And they must be connected with the minimum weight edge to make it a Minimum Spanning Tree.

The algorithm may informally be described as performing the following steps:

1. Initialize a tree with a single vertex, chosen arbitrarily from the graph.
2. Grow the tree by one edge: of the edges that connect the tree to vertices not yet in the tree, find the minimum-weight edge, and transfer it to the tree.
3. Repeat step 2 (until all vertices are in the tree).

Example



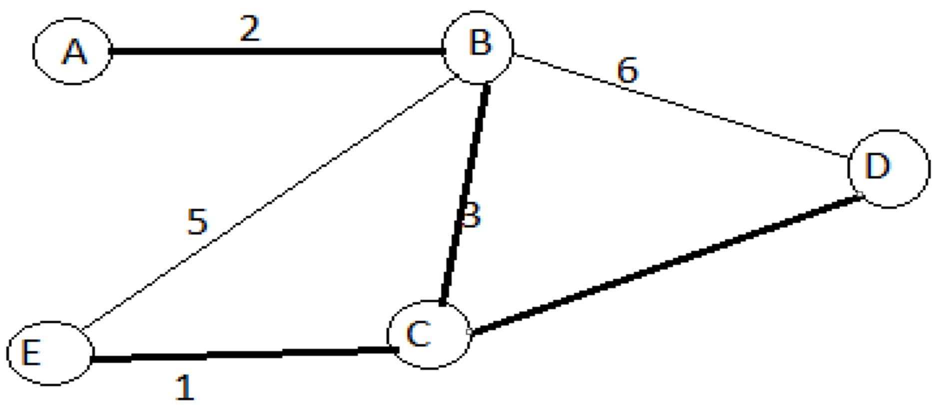
visited=(A B C E D)
so 2+3+1+4=10
Time complexity of the algorithm is $O(|V|^2)$

6 Kruskal's Algorithm

Kruskal's algorithm finds a minimum spanning forest of an undirected edge-weighted graph. If the graph is connected, it finds a minimum spanning tree. (A minimum spanning tree of a connected graph is a subset of the edges that forms a tree that includes every vertex, where the sum of the weights of all the edges in the tree is minimized. For a disconnected graph, a minimum spanning forest is composed of a minimum spanning tree for each connected component.)

1. create a forest F (a set of trees), where each vertex in the graph is a separate tree
2. create a set S containing all the edges in the graph
3. while S is nonempty and F is not yet spanning remove an edge with minimum weight from S if the removed edge connects two different trees then add it to the forest F, combining two trees into a single tree

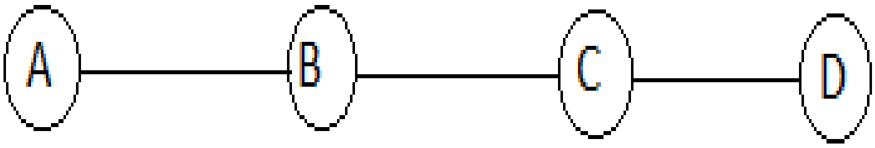
Order of this algorithm is O(|E|log|E|)



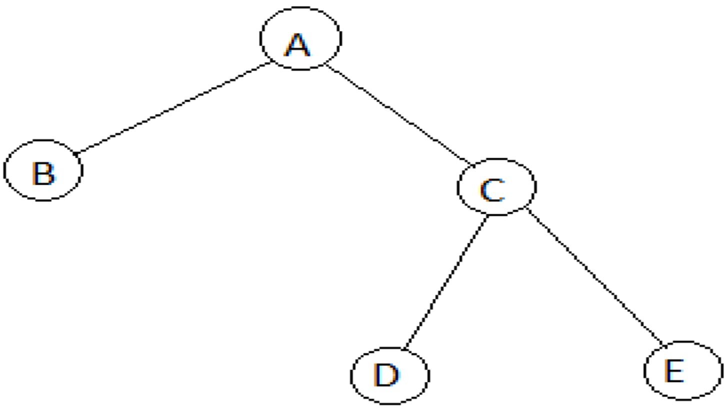
Since 1+2+3+4=10

7 Matching and Covers

Definition: A matching in a graph G is a set of non loop edges with no shared endpoints.The vertices incident to the edges of a matching M are saturated by M;the others are unsaturated.A perfect matching in a graph is a matching that staurates every vertix.



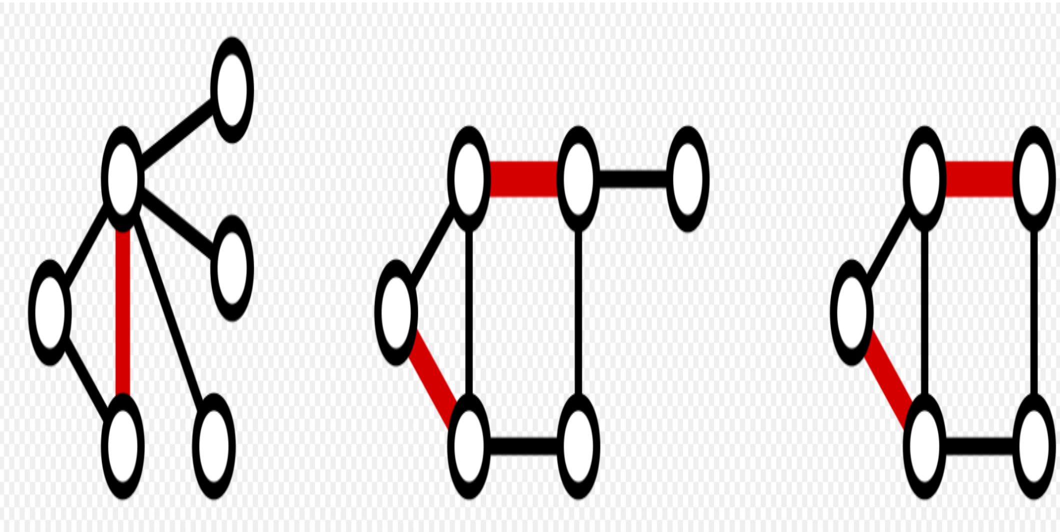
Matching = (AB,CD)=M_1
=(BC)=M_2



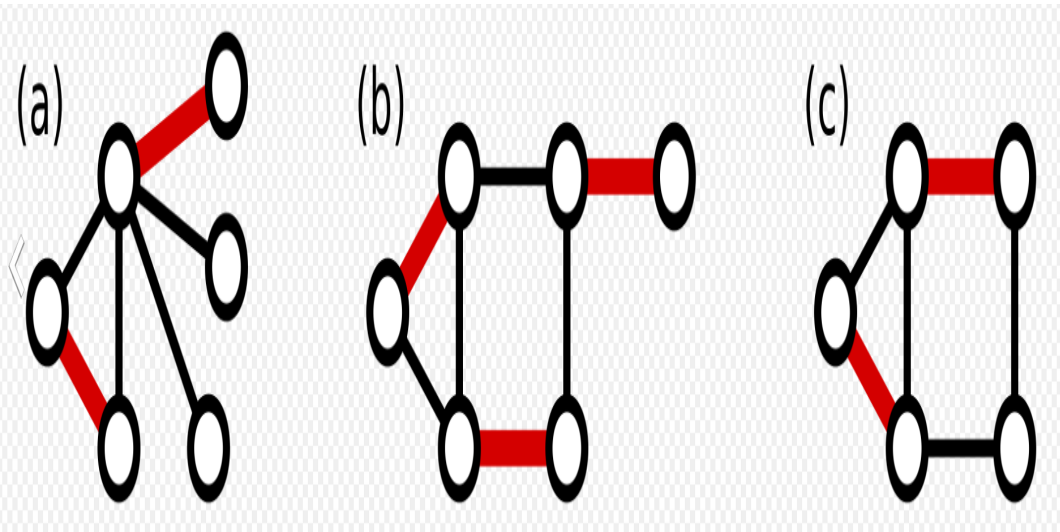
Matching=
M_1=(AB,CD)
M_2=(AB,CE)
M_3=(AC)
M_4=(CD)
M_5=(AC)

7.1 Maximum and Maximal matching

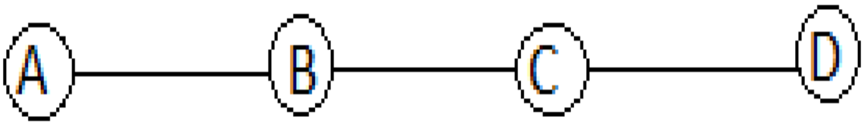
A maximal matching in a graph is a matching that can't be enlarge by adding an edge .A maximum mathing is a matching of maximum size among all the matching in the graph.



$figure : Maximalmatching$



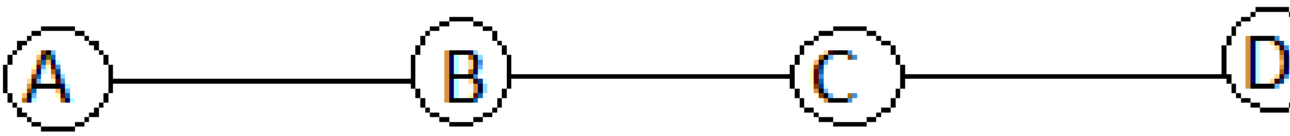
$figue : Maximummatching$

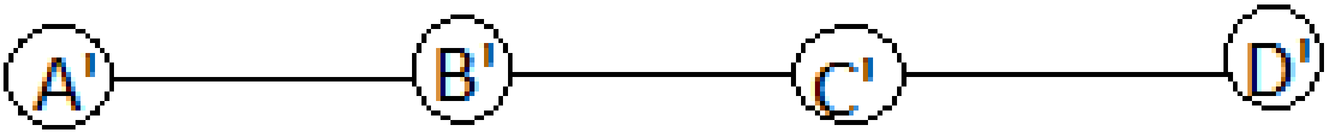


Maximum matching=(AB,CD)
Maximal matching=(BC)

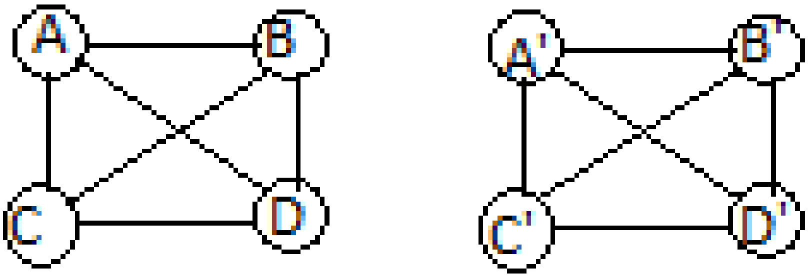
7.2 Perfect Matching of a graph

A perfect matching of a graph is a matching (i.e., an independent edge set) in which every vertex of the graph is incident to exactly one edge of the matching. A perfect matching is therefore a matching containing n/2 edges (the largest possible), meaning perfect matchings are only possible on graphs with an even number of vertices. A perfect matching is sometimes called a complete matching or 1-factor.
PM of a graph=$no.of PM_{c_1} * no.of PM_{c_2} * no.of PM_{c_3} .... * no.of PM_{c_k}$

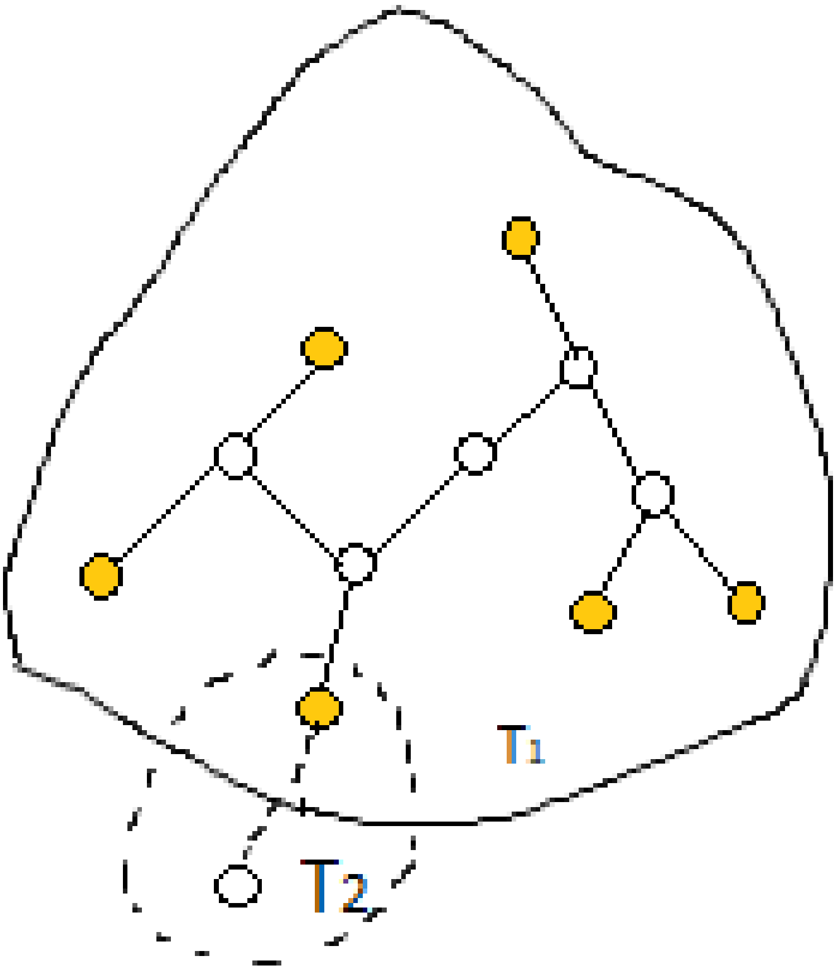




$PM(C_1) = (AB, CD) = C_1 \Rightarrow$ no. of perfect matching in $C_1 = 1$
$PM(C_2) = (A'B', C'D') = C_2 \Rightarrow$ no. of perfect matching in $C_2 = 1$



In C_1 (AB,CD)
(AD,BC)
(AC,BD)
In C_2
(A'B',C'D')
(A'D',B'C')
(A'C',B'D')
so number og PM in graph=3*3=9
Theorem: Prove that every tree has atmost one perfect matching.
Proof by induction:
base condition :- a tree having n=1 node has number of perfect matching =0
n=2, no.of PM=1
Induction hypo.:-
Suppose that the theorem is true for $n \leq knodes$



No.of perfect matching in $T_1 \leq 1$
No.of perfect matching in $T_2 \leq 1$
Therefore no. of perfect matching in T $\leq$ no.of PM in T_1*no. of PM in T_2
no. of perfect matching in T $\leq 1 * 1$
no. of perfect matching in T ≤ 1

8 Hall Marriage Theorem

Hall's marriage theorem, proved by Philip Hall, is a theorem with two equivalent formulations:

The combinatorial formulation deals with a collection of finite sets. It gives a necessary and sufficient condition for being able to select a distinct element from each set.
The graph theoretic formulation deals with a bipartite graph. It gives a necessary and sufficient condition for finding a matching that covers at least one side of the

graph.

Theorem An X,Y bigraph G has a matching that saturates X if and only if |N(S)| $\leq |S|$ for all S $\subset X$

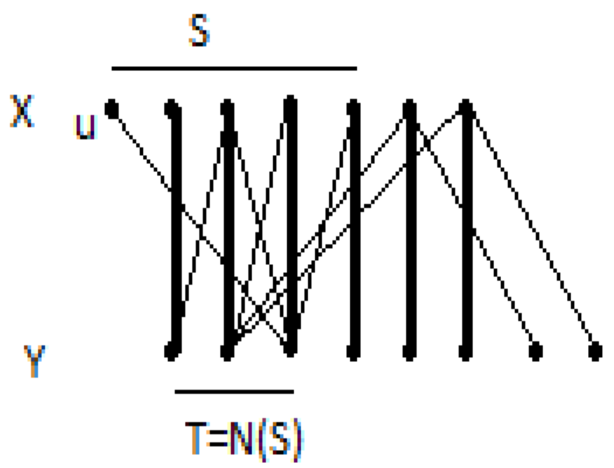
$Here|N(S| \subset$ Y is a set of neighbours of elements in S.

Proof The |S| vertices matched to S must lie in N(S).

Sufficiency. To prove that Hall's Condition is sufficient, we prove the contrapositive. If M is a maximum matching in G and M does not saturate X, then

we obtain a set S⊂X such that IN(S)|<|S|. Let u A∈ X be a vertex unsaturated by M. Among all the vertices reachable from u by M-alternating paths in

G, let S consist of those in X, and let 7 consist of those in Y.



We claim that M matches T with S-(u). The M-alternating paths from u reach Y along edges not in M and return to X along edges in M. Hence every vertex of S-u is reached by an edge in M from a vertex in T. Since there is no M-augmenting path, every vertex of T is saturated; thus an M-alternating T = N(S) path reaching y∈T extends via M to a vertex of S. Hence these edges of M yield a bijection from T to S-(u), and we have |T|=|S-(u)

.

The matching between T and S - (u) yields T≤ N(S). In fact, T=N(S). Suppose that y∈Y-T has a neighbor v∈ S. The edge vy cannot be in M, since u is unsaturated and the rest of S is matched to T by M. Thus adding vy to an M-alternating path reaching v yields an M-alternating path to y. This contradicts y does not belongs toT, and hence vy cannot exist.

With T = N(S), we have proved that IN(S)| = |T| = |S| -1 <[S] for this choice of S. This completes the proof of the contrapositive.