Week 13

Planar Graphs*

13.1 Introduction :

▶ In this week, we will learn about "Planar Graphs". In the real world, Planar Graphs have various applications such as cheap designs, high speed highways and railroad designs.

▶ Initially we will start by introducing the formal definition of planar graphs and then we will try to understand the reason why they have many real world applications. Then we will learn about properties of planar graphs. We will introduce some new terms like face, length of a face, Euler's Condition, chords, conflicting chords, dual graphs. At the end, we will learn how to check the planarity of a graph.

13.2 Planar Graph :

Definition : A planar graph is a graph that has a drawing without crossing.

Reason of many real world applications : In order make any design better we want the less crossing in it and planar graphs helps in that. This is main reason why planar graphs have many real world applications application.

Properties of Planar graphs : Planar graphs have the following special properties (that is why we read them) :
(1) Planar graphs are sparse Graphs for large vertex graphs (adjacent matrix involves more number of zeros/less number of edges are present in the graph)
(2) Planar graphs are four colorable
(3) Efficient Operation

*Lecturer: Dr. Anand Mishra. Scribe: Gurubachan.

Example 1 : Decide weather the given graphs are planar or not:
(a)

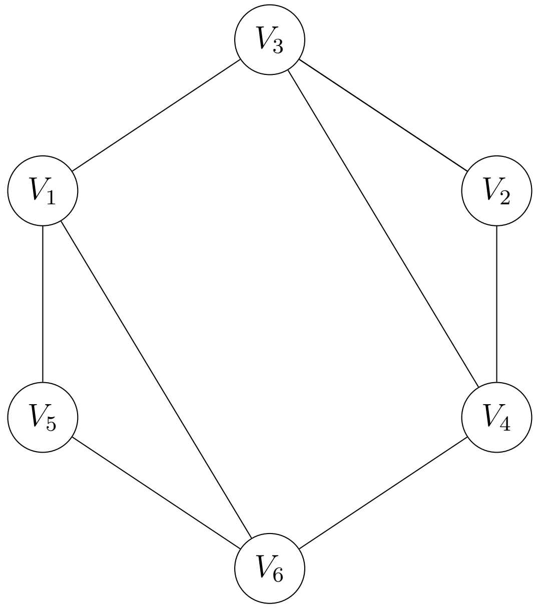


figure 1

(b)

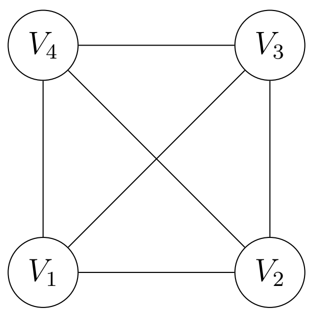


figure 2

Answer 1: We need to just check weather the drawings have any crossing or not:
(a) Clearly there is no crossing in *figure* 1, therefore the given graph in *figure* 1 is planar.
(b)

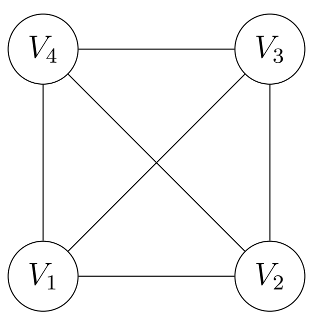


figure 2

$$\Updownarrow$$

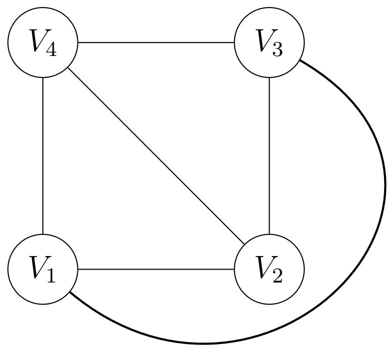


figure 3

Clearly there is no crossing in *figure* 3, therefore the given graph in *figure* 2 is planar.

Faces : It is the number of regions in the graph and is denoted by f .

Example 2 : Calculate the number of faces (f) for the following graphs :
(a)

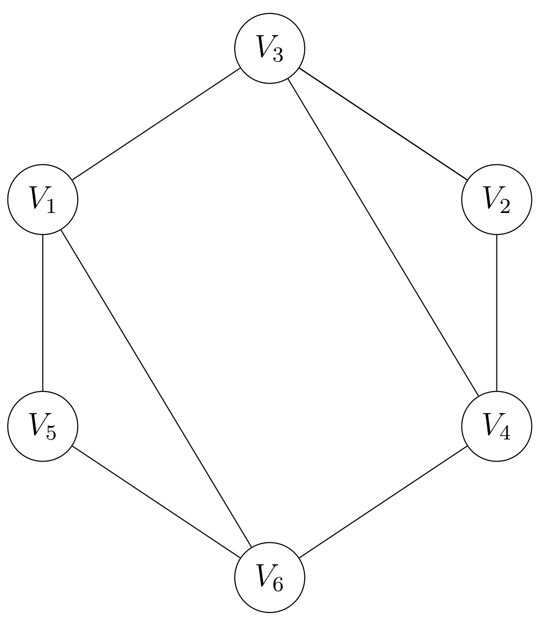


figure 4

(b)

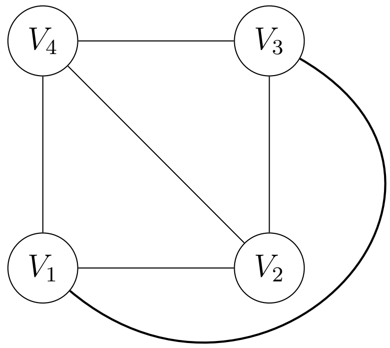


figure 5

(c)

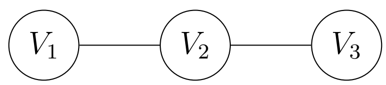


figure 6

Answer 2: (a) After giving the name to the faces, the *figure* 4 looks like *figure* 7 :

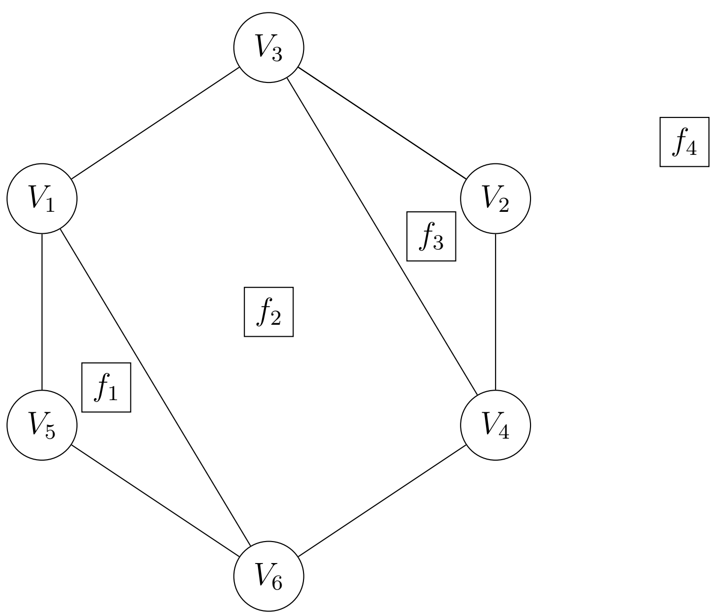


figure 7

Here, $f = 4$ as there are 4 regions.
(b) After giving the name to the faces, the *figure* 5 looks like *figure* 8 :

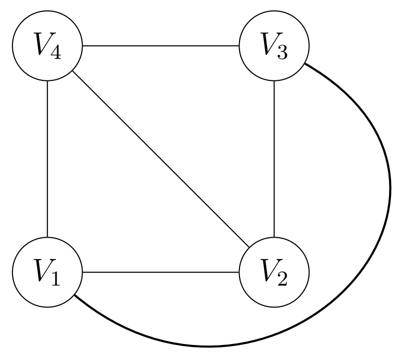


figure 8

Here, $f = 4$ as there are 4 regions.
(c) After giving the name to the faces, the *figure* 6 looks like *figure* 9 :

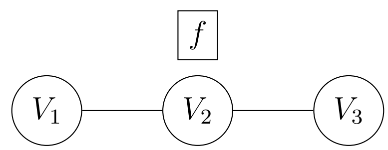


figure 9

Here, $f = 1$ as there is only 1 region.
From the above examples, we can write the following observation:

Observation : Let G be any graph and
(a) If G has no cycle, then $f = 1$
(b)If G has a cycle, then $f > 1$

13.2.1 Euler's Condition :

Statement : Let G be a connected planar graph and v be the number of vertices in G, e be the number of edges in G, f be the no. of faces in G. Then,

$$v - e + f = 2$$

Proof: We will prove this by applying Strong Principle of Mathematical Induction on the number of edges,

For $e = 0$: Since $e = 0$ and G be a connected planar graph

$\Rightarrow$ there is only one vertex in G

$\Rightarrow e = 1, f = 1$

therefore, $v - e + f = 1 - 0 + 1 = 2$

Hence, the result holds for $e = 0$

For $e = n$: Let the result is true for $e = n$

$$v - e + f = 2$$

For $e = n + 1$:

Case 1: If G does not has any cycle

$\Rightarrow$ G is a tree

$\Rightarrow v = n + 2, f = 1$

therefore, $v - e + f = (n + 2) - (n + 1) + 1 = 2$

Case 2: If G has at least one cycle

let us remove edge e_1 from a cycle and call the resultant graph G_1

$\Rightarrow$ number of edges in $G_1 \leqslant n$

$$v_1 - e_1 + f_1 = 2 \qquad \text{...(1)}$$

where, $v_1 =$ the number of vertices in $G_1 = v$

(since removable of an edge does not change the number of vertices in G)

$e_1 =$ the number of edges in $G_1 = n = e - 1$

$f_1 =$ the number of faces in $G_1 = n = f + 1$

therefore by (1)

$$v - (e - 1) + (f + 1) = 2$$

$$v - e + f = 2$$

Hence, the result holds for $e = n + 1$ Therefore By Strong Principle of Mathematical Induction,

$$v - e + f = 2$$

Chords : The edges different from cycle are called chords of the graph.

Conflicting Chords : Two chords are said to be conflicting if their end points occurs in alternate order of the cycle.

eg: From the $figure$ 10, e_6,e_7,e_8,e_9,e_{10} are chords of K_5 as they are different from the cycle e_1,e_2,e_3,e_4,e_5. In this, we can not check the Euler's condition. So, we have to go with the method by checking the conflicting chords.

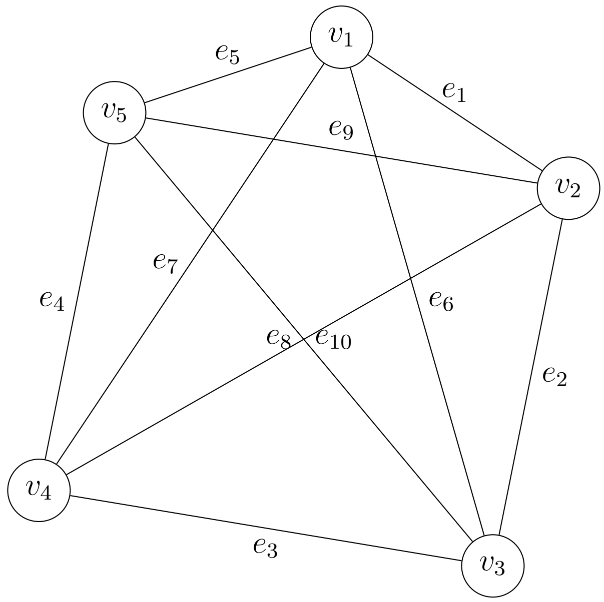

v_1
v_2
v_3
v_4
v_5
e_1
e_2
e_3
e_4
e_5
e_6
e_7
e_8
e_9
e_{10}

figure 10

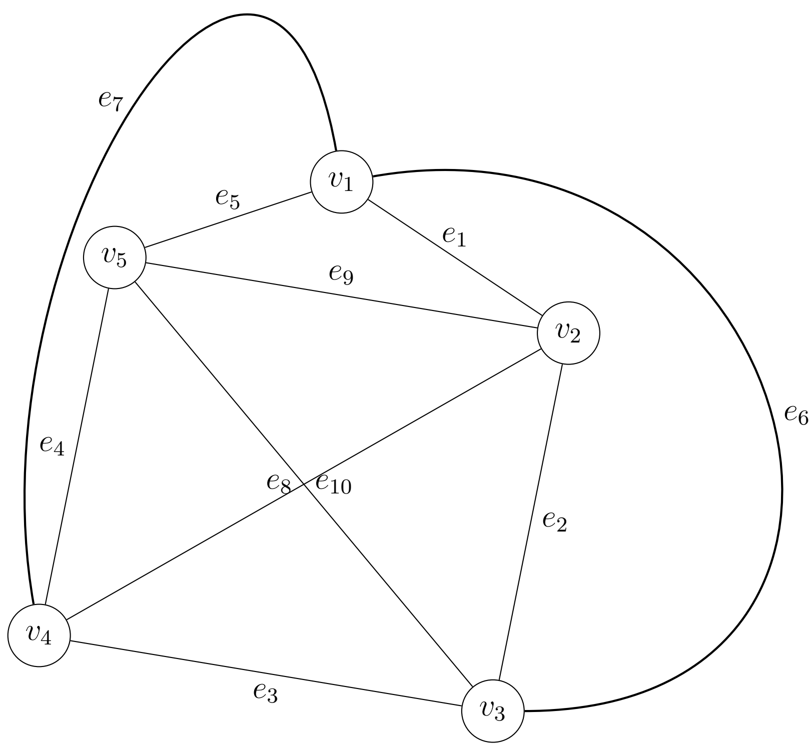

v_1
v_2
v_3
v_4
v_5
e_1
e_2
e_3
e_4
e_5
e_6
e_7
e_8
e_9
e_{10}

figure 11

In the *figure* 10, e_9,e_{10} are conflicting chords with respect to e_6.
Clearly by *figure* 11, K_5 is non-planar.

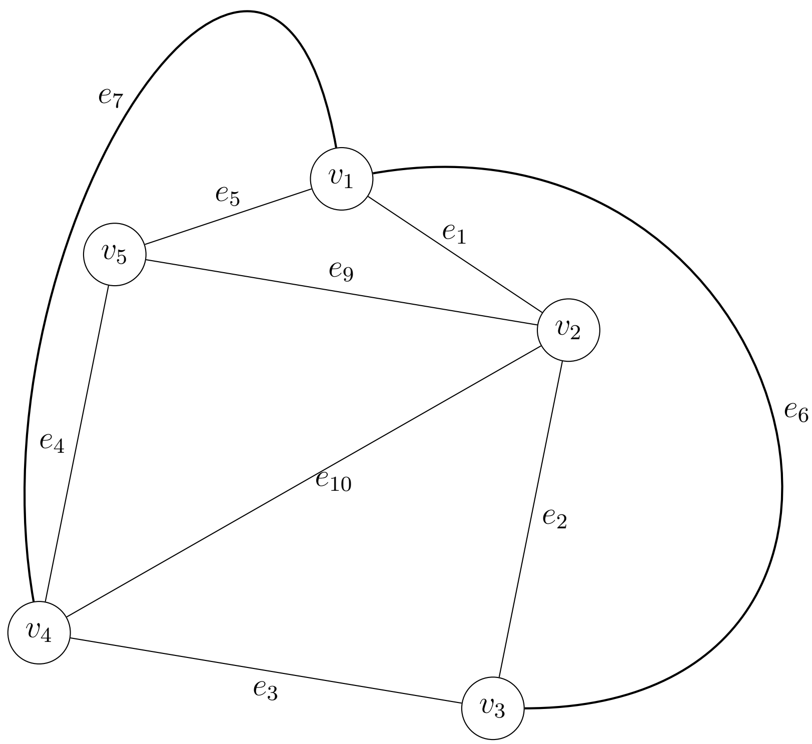
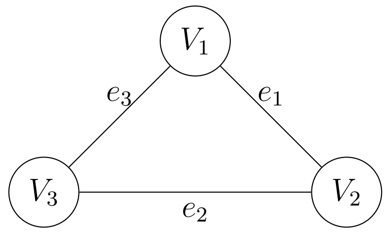


figure 12

But K_5 -e_8 is a planar graph (by *figure* 12).

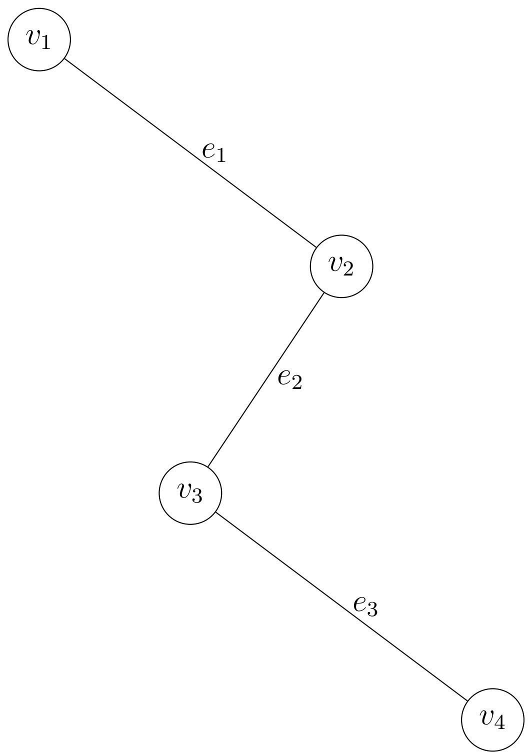
Dual Graph : The dual graph G^* of a planar graph G is also a planar graph whose vertices corresponds to the faces of G and the edges of G^* corresponds to the edges of G as follows if e is an edge where one side is face X and another side is face Y, then in G^* there is an edge between vertex corresponding to face X and face Y.

Example 3 : Draw the dual graphs of the following graphs :
(a)



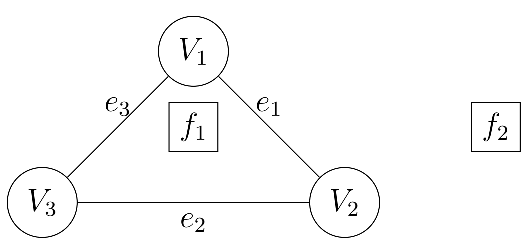
$figure$ 13

(b)



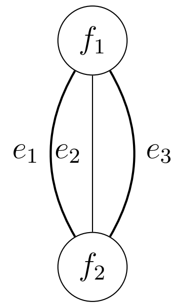
$figure$ 14

Solution 3: (a)After giving the name to the faces the $figure$ 13 looks like $figure$ 15 :



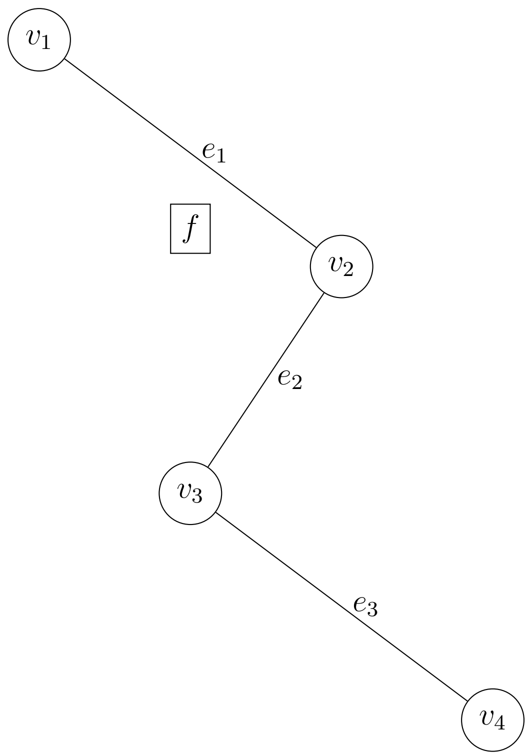
$figure$ 15

and the dual graph of the above graph is:



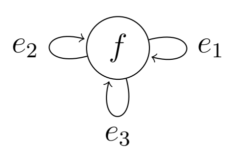
$figure$ 16

(b) After giving the name to the faces the $figure$ 14 looks like $figure$ 17 :



$figure$ 17

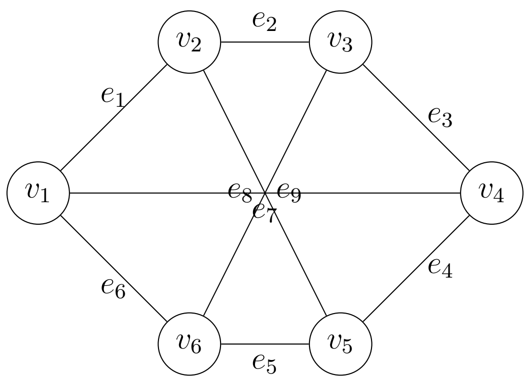
and the dual graph of the above graph is:



$figure$ 18

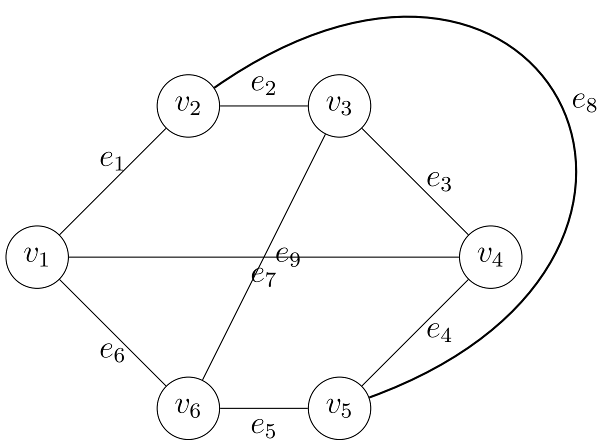
Question : True or False:
(a) Every subgraph of a planar graph is also planar.
(b) Every subgraph of a non-planar graph is a non-planar graph.

Answer : (a) True
Since the planar graph has the drawing without crossing
$\Rightarrow$ the subgraph of the planar graph has also the drawing without crossing
Hence, Every subgraph of a planar graph is also planar.
(b) False
Since $K_{3,3}$ is non-planar



$figure$ 19

$\Updownarrow$



$figure$ 20

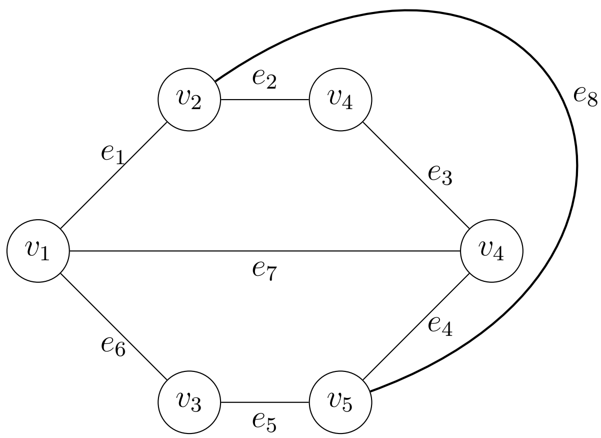


figure 21

but $K_{3,3} - \ e_9$ is a planar graph as it has no crossing from the *figure* 21.

Length of a Face : The total length of a closed walk in a planar graph G bounding the face is called the length of that face in G.

Caution : If e is an cut edge in the boundary of one face f in the planar graph G. Then e contribute the length twice to the face f and denoted by $l(f)$.

Example 4 : Calculate the length of the faces of the following graph:

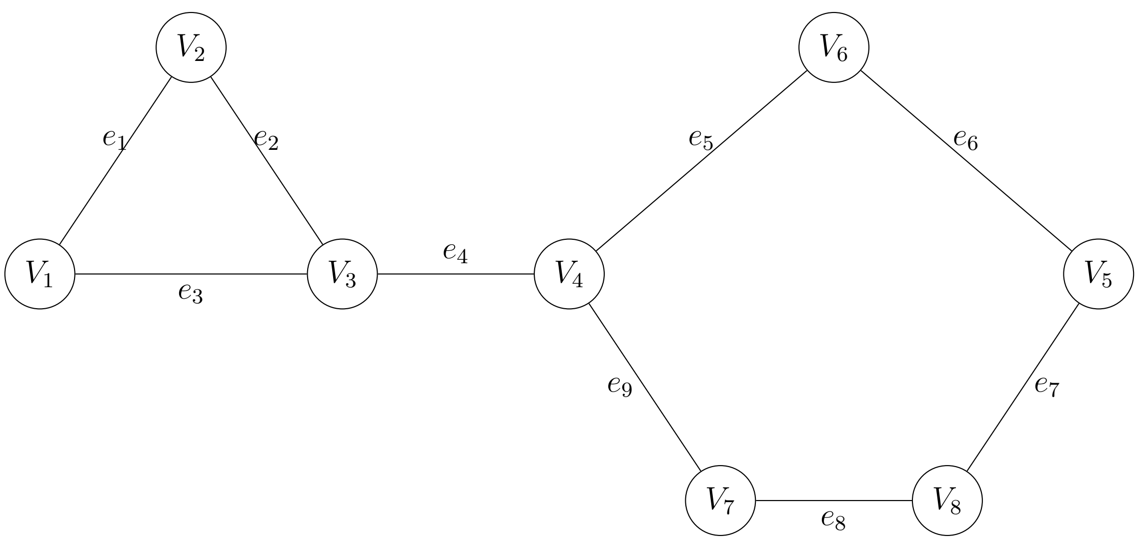


figure 22

Answer 4 : From the below graph $l(f_1) = 3$ (since the closed walk $e_1e_2e_3e_1$ bounding the face f_1 has length is 3),
$l(f_2) = 9$ (since the face f_2 has an cut edge e_4 on its boundary, therefore e_4 contributes 2 in the length of the face f_2 or the closed walk $e_1e_2e_4e_5e_6e_7e_8e_9e_4e_2e_1$ $l(f_2) = 9$,
$l(f_3) = 4$ (since the closed walk $e_5e_6e_7e_8e_9$ bounding the face f_3 has length is 5)

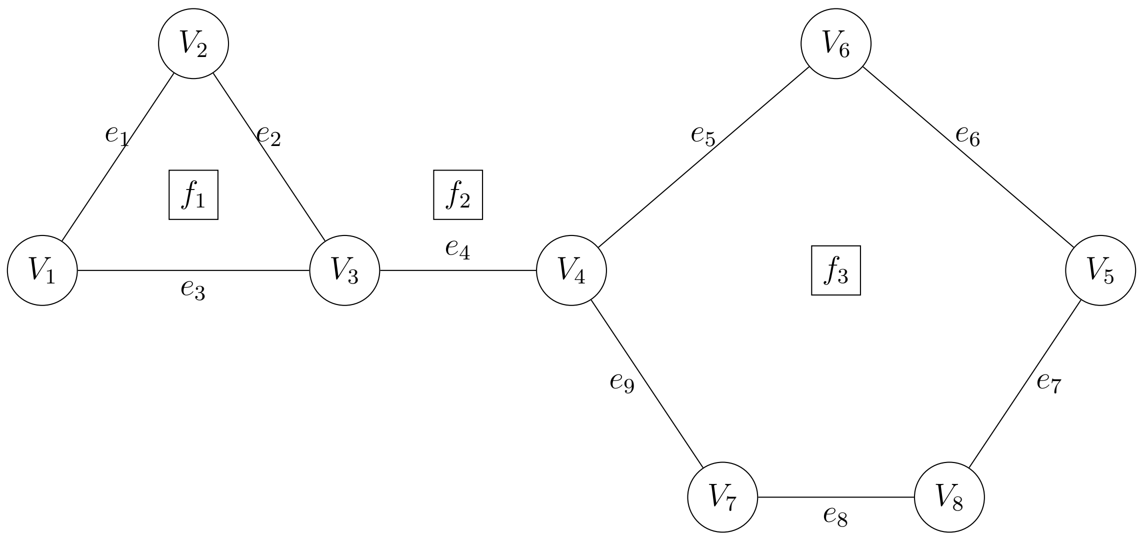


figure 23

13.2.2 Lemma :

Statement : Let G be a planar graph and $l(f_i)$ denotes the length of face f_i. Then,

$$2e = \sum_i l(f_i)$$

Proof : We know that in the dual graph,

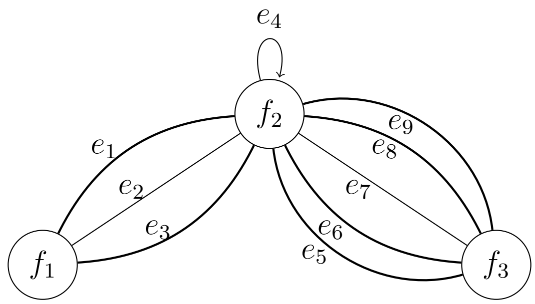
$$l(f_i) = deg(f_i)$$

now by applying Handshaking lemma on G^*,

$$2e = \sum_i deg(f_i) = \sum_i l(f_i)$$

Example 5 : Verify Lemma 13.2.2 from example 4.

Solution 5 : Dual graph of $figure$ 23 is:



$figure$ 24

clearly $\deg(f_1) = 3$, $\deg(f_2) = 10$, $\deg(f_3) = 5$

$$\sum_i deg(f_i) = \sum_i l(f_i) = 18 = 2 * 9 = 2e$$

13.2.3 Proposition :

Statement : Let G be a connected planar graph with v-nodes, f-faces and $v \geqslant 3$. Then,

$$e \leqslant 3v - 6$$

Proof : Since $v \geqslant 3$
by applying Lemma 13.2.2 on G^*,

$$2e = \sum_i l(f_i)$$

$\Rightarrow$ $\quad 2e \geqslant 3 + 3 + 3 + ...3 \; (f - times)$
$\Rightarrow$ $\quad 2e \geqslant 3f \qquad ...(1)$
now by Euler's Condition, $\quad v - e + f = 2$
$\Rightarrow$ $\quad f = 2 + e - v \qquad ...(2)$
by (1) and (2), $\quad 3(e - v + 2) \leqslant 2e$
$\Rightarrow$ $\quad 3e - 3v + 6 \leqslant 2e$
$\Rightarrow$ $\quad e \leqslant 3v - 6$

13.3 Testing of Planarity :

• In this section, we are going to learn that how do we check the given graph is planar or not.
• The following are some simple graphs that are planar:
(a) K_1 :

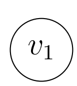


figure 25

(b) K_2 :

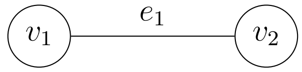


figure 26

(c) K_3 :

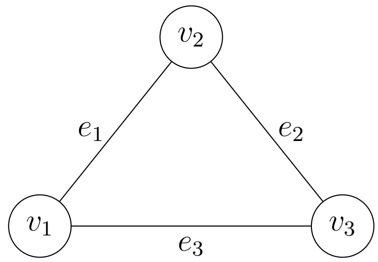


figure 27

(d) K_4 :

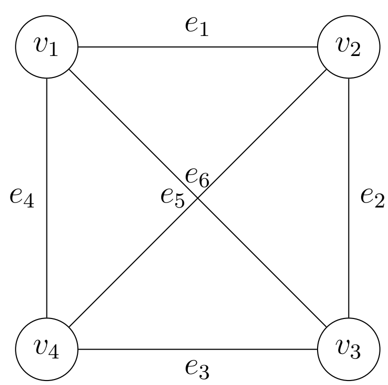


figure 28

$\Updownarrow$

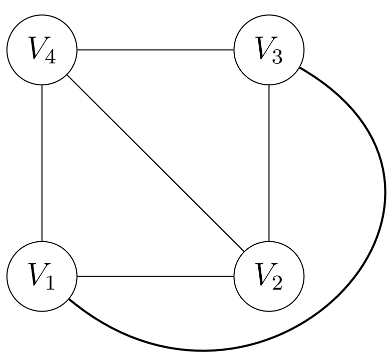


figure 29

Now, we will move to learn planarity test algorithm. In this algorithm our target is to convert very large graph into small graph by applying some elementary reduction techniques:

13.3.1 Planarity Test Algorithm :

Let G be any large graph and we will reduce the large graph G into some small graph H by applying the following elementary reduction techniques:
(I) Remove self loops
(II) Remove parallel edges
(III) Remove vertex of degree 2 and merge the edges incident on that vertex
Then, G is a planar graph $\iff$ H is a planar graph.

Note : We can apply the first elementary reduction techniques as it doesn't include any crossing so removing them does not effect the planarity of the graph and this technique is used only one time while can apply the second and third elementary reduction techniques mutiple times.

• Now the question arises that how to check the planarity of small graph H.

▶ The small graph H is planar if:
(i) H has only one edge Or (ii) $H = K_4$ Or (iii) $e \leqslant 3v - 6$
where, e = number of edges in H, v = number of vertices in H.

Example 6 : Decide weather the given graph is planar or not:

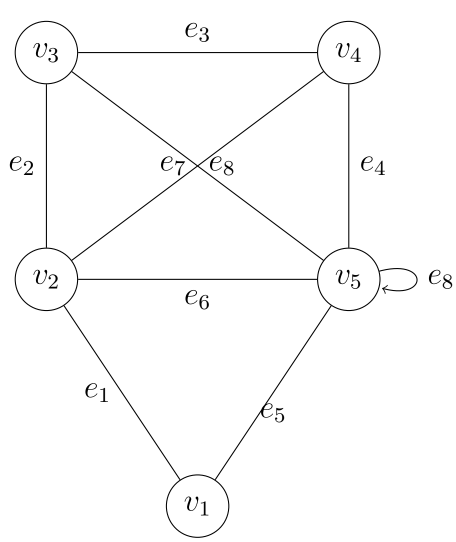


figure 30

Solution 6 : Since the given graph has a self loop, therefore according to elementary reduction technique (I) we should first remove it and the given graph will reduce from *figure* 30 into *figure* 31

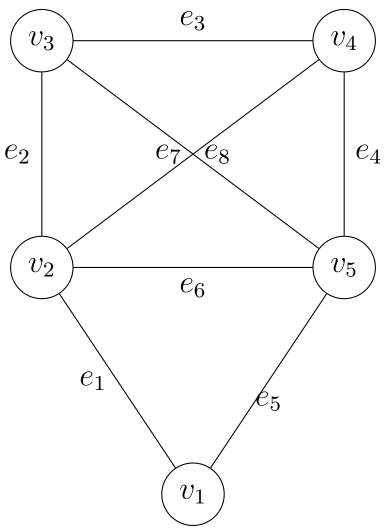


figure 31

now as $\deg(v_1) = 2$, therefore by elementary reduction technique (III) we should first remove the vertex v_1 and merge the edges e_1,e_5 the given graph will reduce from *figure* 31 into *figure* 32

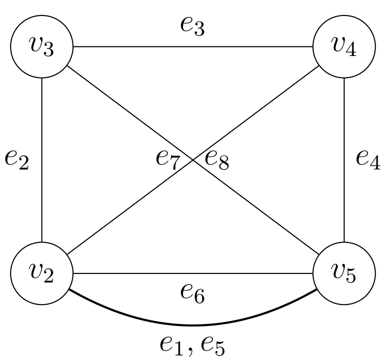


figure 32

now as there are parallel edges, therefore by elementary reduction technique (II) we should remove the e_1, e_5 and merge the edges e_1,e_5 the given graph will reduce from *figure* 32 into *figure* 33

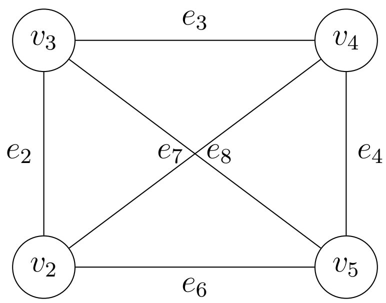


figure 33

Now we can not apply the planar test algorithm again
therefore, *figure* 33 is the required smaller graph H
and as $H = K_4$
$\Rightarrow H$ is a planar graph
Hence, G is also a planar graph (by Planarity Test algorithm).

13.3.2 Theorem :

Statement : For any planar graph G, the following statements are equivalent :
(a) G is a Bipartite Graph
(b) Every face of G has an even length
(c) The dual graph G^* is Eulerian.

Proof : $\underline{(a) \Rightarrow (b)}$:-
As G is a Bipartite Graph
$\Rightarrow$ G has no odd cycle i.e. either G has no cycle or cycle of even length
and we know that cycle make faces Therefore, every face of G has an even length.
$\underline{(b) \Rightarrow (c)}$:- Since every face of G has an even length and when we draw dual graph of G,then faces of G becomes nodes of G^*
and in the dual graph deg(each node in G^*) = length(each face in G) = even
and we know G' is Eulerian $\iff$ it has at most one nontrivial component and its vertices all have even degree.
Hence, G^* is Eulerian.
$\underline{(c) \Rightarrow (a)}$:- Since dual graph G^* is Eulerian
$\Rightarrow$ degree of every vertex of G^* is even.
$\Rightarrow$ degree of each face is even
$\Rightarrow$ length of each face is even(since deg(each node in G^*) = length(each face in G))
and face is bounded by cycles
$\Rightarrow$ every cycle of G is of even length Hence, G is a Bipartite Graph.

13.3.3 Theorem :

Statement : Every simple planar graph has a vertex of degree at most 5.

Proof : Let G be any simple planar graph and v,e be the number of vertices and edges in G respectively

Case 1: If $v < 3$

and G is a simple graph

$\Rightarrow$ deg(every vertex of G) < 3 then we are done

Case 2: If $v \geqslant 3$ As simple graphs are connected graphs

therefore by proposition 13.2.3,

$$e \leqslant 3v - 6 \qquad for \qquad v \geqslant 3$$

by Handshaking Lemma, sum of degrees of nodes in $G \leqslant 2(3v - 6)$

$\Rightarrow$ sum of degrees of nodes in $G \leqslant 6v - 12$ ...(*)

claim: There exists a vertex in G of degree < 6

If possible let G have all the vertices of degree $\geqslant 6$

$\Rightarrow$ sum of degrees of nodes in $G \geqslant 6v$

which contradicts the statement (*)

$\Rightarrow$ there exists a vertex in G of degree < 6

Hence, Every simple planar graph has a vertex of degree at most 5.