Indian Institute of Technology- Jodhpur

Graph Theory and Applications(GTA)
COURSE CODE: CSL7410

Lecture Scribing Assignment: Week 4

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Week 4

Vertex, Degree and Counting*

4.1 Graphic Sequences

In this Section we will learn about the degree sequence and graphic sequences. We will learn about the valid and invalid graphic sequences.

4.1.1 Definition

The degree sequence of a graph is the list of vertex degrees, usuallly written in nonincreasing order as , $\mathrm{d}_1 \leq d_2 \leq ... \leq d_n$.

Every graph has a degree sequence, but which sequence occur? That is, given nonnegative integers $d_1, d_2...d_n$. is there a graph with these as the vertex degrees?

As we know, The degree sum formula implies d_i must be even.when we allow multiple loops and edges, TONCAS. So, the nonnegative integers $d_1, d_2, ..., d_n$are the vertex degrees of some graph iff $\sum d_i$ is even.

4.1.2 Defintion

A graphic sequence is a list of nonnegative numbers that is the degree sequence of some simple graph. A simple graph with degree sequence d "realizes" d.

4.1.3 Theorm

(Havel-Hakimi Algorithm) For $n > 1$, an integer list d of size n is graphic iff d' is graphic, where d' is obtained from d by deleting its largest element and subtracting 1 from its next largest elements . The only 1-element graphic sequence is $d_1 = 0$.

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So, With the help of this algorithm , we can say wheather the given sequence of numbers is valid graphic sequence or not?

Example:

5 , 5, 5, 4, 2, 1, 1, 1
4, 4, 3, 1, 0, 1, 1
4, 4, 3, 1, 1, 1, 0
3, 2, 0, 0, 1, 0
3, 2, 1, 0, 0, 0

This is a invalid sequence, we can not find a graph with vertices having 3,2,1,0,0,0 degrees.

Example:

5 5 4 3 2 2 2 1
4 3 2 1 1 2 1
4 3 2 2 1 1 1
2 1 1 0 1 1
2 1 1 1 1 0
0 0 1 1 0
1 1 0 0 0
0 0 0 0

This is a valid graphic sequence.

Question: Prove or disprove that if u and v are only vertices of odd degree in a graph G, then G contains u-v path.

Ans: The degree of a vertex in a component of G is the same as its degree in G. If the vertices of odd degree are in different components, then those components are graphs with odd degree sum, which is the contradiction of Hand-shaking lemma. So, if u and v are only vertices of odd degree in a graph G, then G contains u-v path.

Question: Determine maximum number of edges in a Biparatite subgraph of P_n, C_n and k_n.

Ans: P_n itself is a maximal Biparatite subgraph .So, the maximum number of edges in a Biparatite subgraph of $P_n = n - 1$.

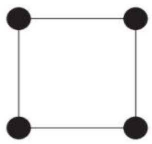
Example:

Here, $P_4 \subseteq p_4$

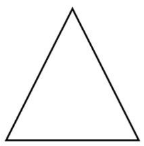
number of edges = 3 (4-1)

In the case of C_n graph , if n is even then C_n it self is a maximal Biparatite subghraph , so when n is even the number of edges in maximal Biparatite subgraph of $C_n = n$ but if n is odd then the graph of C_n has to lose one edge for its maximal Biparatite subgraph . So when n is even, the number of edges in maximal Biparatite subgraph of $C_n = n - 1$.

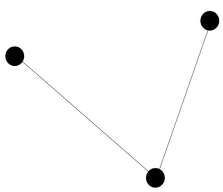
Example: C_4 and C_3



Number of edges in maximal Biparatite subgraph of $C_4 = 4$.



Maximal biparatite subgraph of C_3 will be obtained by removing one edge



So, the number of edges in maximal Biparatite subgraph of $C_3 = 2$.

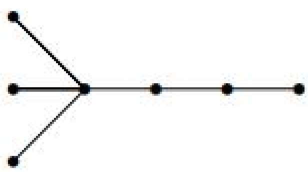
The largest Biparatite Subgraph of K_n is $K_{\lfloor \frac{n}{2} \rfloor, \lceil \frac{n}{2} \rceil}$ which has $\lfloor \frac{n^2}{4} \rfloor$ edges.

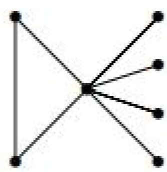
Question: Let l,m,n be nonnegative integers with $l + m = n$. Find necessary and sufficient condition on l,m,n such that there exist a connected simple n-vertex graph with l vertices of even degree and m vertices of odd degree.

Ans: if l,m ,n are nonnegative integers with $l+m = n \geq 1$, then there exists a connected simple n-vertex graph with l vertices of even degree and m vertices of odd degree iff m is even, except for $(l, m, n) = (2, 0, 2)$ .Since every graph has an even number of vertices of odd degree, and the only simple connected graph with two vertices has both degree odd , the condition is necessary .

To prove sufficiency , we construct such a graph G. If $m = 0$, let $G = C_1$(except $g = K_1$ if $l = 1$). For $m > 0$, we can begin with $K_{1,m-1}$, which has m vertices of odd degree , and then add a path of length l beyond one of the leaves.

Alternatively start with a cycle of length l , and add m vertices of degree one with a common neighbour on the cycle. That vertex of the cycle has even degree because m is even. Many other construction also work.It is also possible to prove sufficiency by induction on n for $n \geq 3$, but this approach is longer and harder to get right than an explicit general construction.





4.1.4 Directed Graphs

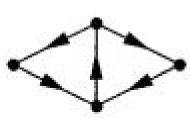
Definition:

A Directed graph or Diagraph G is a triple consisting of a vertex set V(G) , an edge set E(G) , and a function assigning each edge an ordered pair of vertices . The first vertex of the ordered pair is the tail of the edge , and the second is the head; together , they are the endpoints . We say that an edge from its tail to its head.

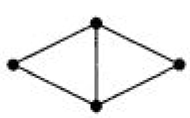
4.1.5 Underlying graph

Definition:

The Underlying graph of a diagraph D is the graph G obtained by treating the edges of D as unordered pairs; the vertex set and edge set remain the same and the endpoints of an edge are the same in G as in D, but in G they become an unordered pair.



D

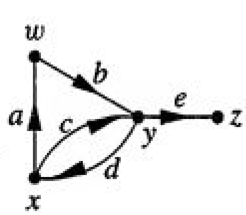


G

4.1.6 representing Directed Graphs:

$$\begin{array}{c} \\ w \\ x \\ y \\ z \end{array} \begin{array}{c} \begin{array}{cccc} w & x & y & z \end{array} \\ \begin{pmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \end{array}$$

A(G)



G

$$\begin{array}{c} \\ w \\ x \\ y \\ z \end{array} \begin{array}{c} \begin{array}{ccccc} a & b & c & d & e \end{array} \\ \begin{pmatrix} -1 & +1 & 0 & 0 & 0 \\ +1 & 0 & +1 & -1 & 0 \\ 0 & -1 & -1 & +1 & +1 \\ 0 & 0 & 0 & 0 & -1 \end{pmatrix} \end{array}$$

M(G)

4.1.7 Weakly and Strongly connected diagraphs:

A diagraph is weakly connected if its underlying graph is connected . A diagraph is strongly connected or strong if for each ordered pair u,v of vertices , there is a path from u to v . The strong components of a diagraph are its maximal strong subgraphs .

4.1.8 Kernal of diagraph :

A kernal in the diagraph D is a set $S \subseteq V(D)$ such that S induces no edges and every vertex outside S has a successor in S.