Indian Institute of Technology- Jodhpur

Graph Theory and Applications(GTA)
COURSE CODE: CSL7410

Lecture Scribing Assignment:Week 13

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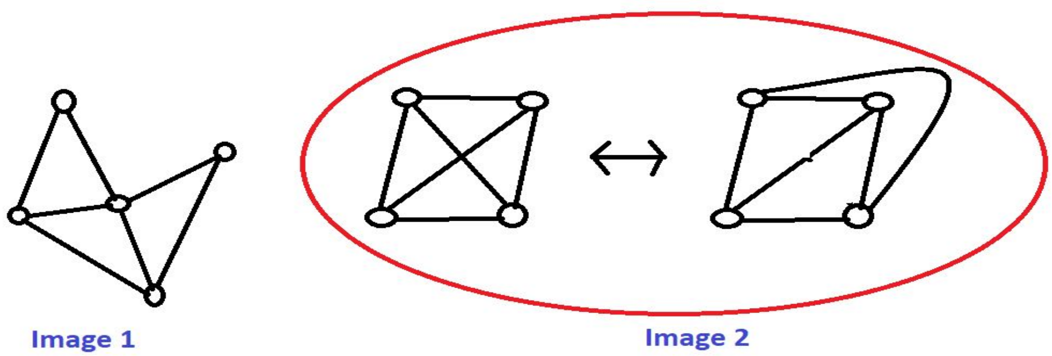
Week 13

Planner Graph and it's Properties*

In this section we will discuss about the Planner graph and it's properties and will know about some conditions ,theorems and some terms used in graph theory.

13.1 Planner Graph

Planner Graph is a graph that has a drawing without crossing.



Above Two images are the example of the Planner Graph.

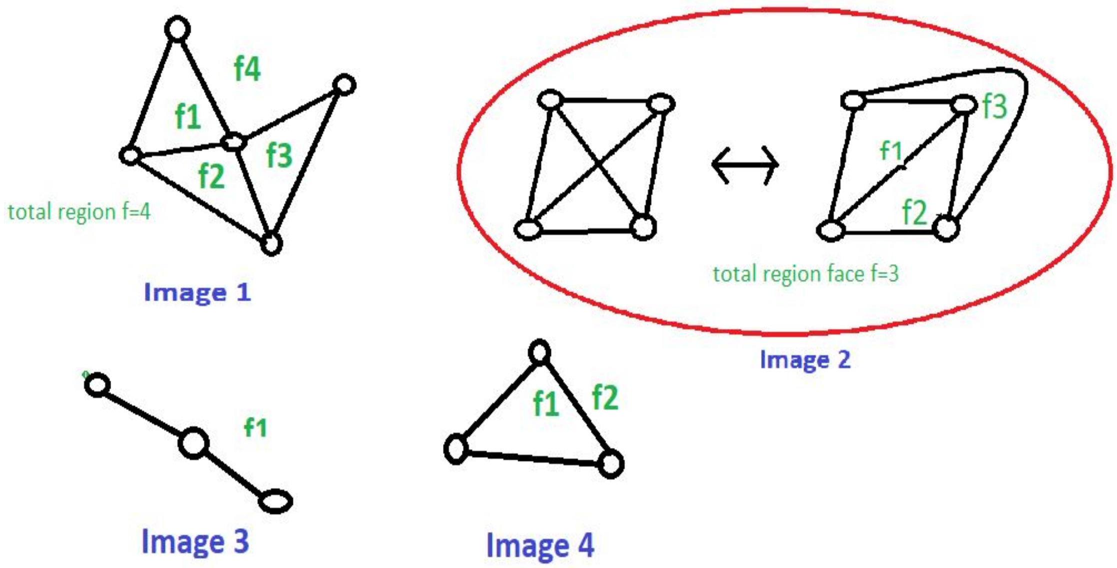
13.1.1 Properties of Planner Graph

- They are Sparse Graph for Large Graph.
- They are 4-colourable.
- Efficient operation.

*Lecturer: Anand Mishra. Scribe: Aditya Mishra (M20MA201).

13.2 Face

Face is the regions in the Graph.



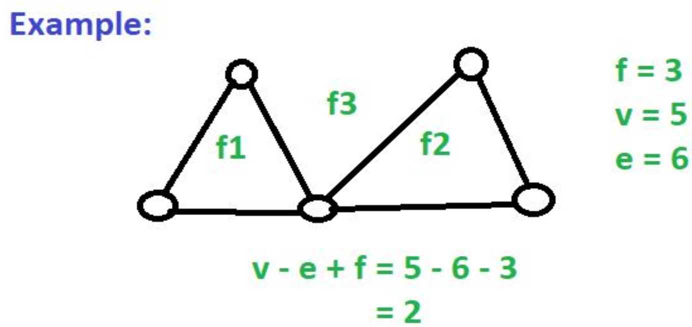
By the above images we can understand the definitions of face of a particular graph more clearly.

13.3 Euiler's Condition

For any connected Planner Graph ,

$$\boxed{\text{v - e + f = 2}}$$

$$v = \text{number of vertices}$$
$$e = \text{number of edges}$$
$$f = \text{number of faces}$$



Proof : By induction on no. of edges,
<u>Base Case :</u>

$$e = 0$$
$$v = 1$$
$$f = 1$$

now here we can clearly see that
$v - e + f = 1 - 0 + 1 = 2$

<u>Induction :</u> Assign upto n edges in a planner connected graph , this is true :
$v - e + f = 2$
Let G be a graph with n+1 edges.

Case 1 : G does not contain a cycle.

number of vertices in G(v) = n+2
number of edges in G(e) = n+1
number of faces in G(v) = 1

n+2-(n+1)+1 = 2

Case 2 : G contains at least one cycle
say we remove P from a cycle and call the result graph G'.
In G'

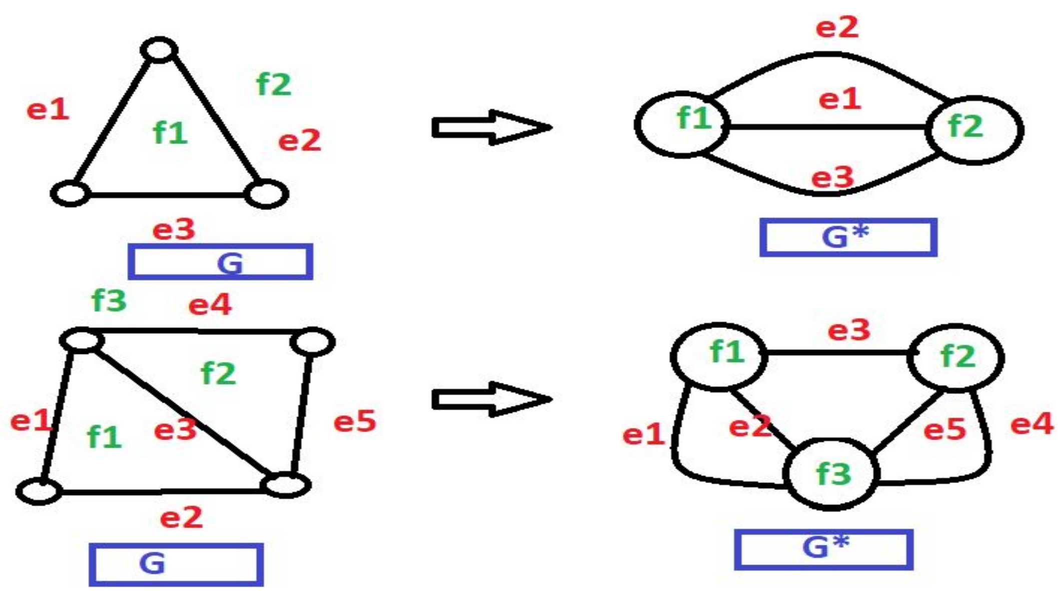
number of vertices in G(v) = v
number of edges in G(e) = e-1
number of faces in G(v) = f-1

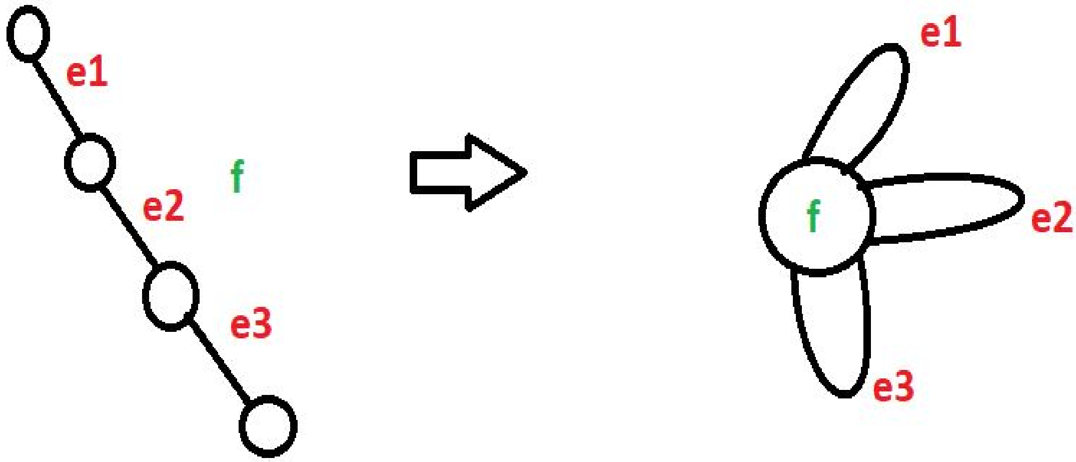
$$v-(e-1)+(f-1) = 2$$
$$\implies v-e+1+f-1=2$$
$$\implies v-e+f=2$$

13.4 Dual Graph :

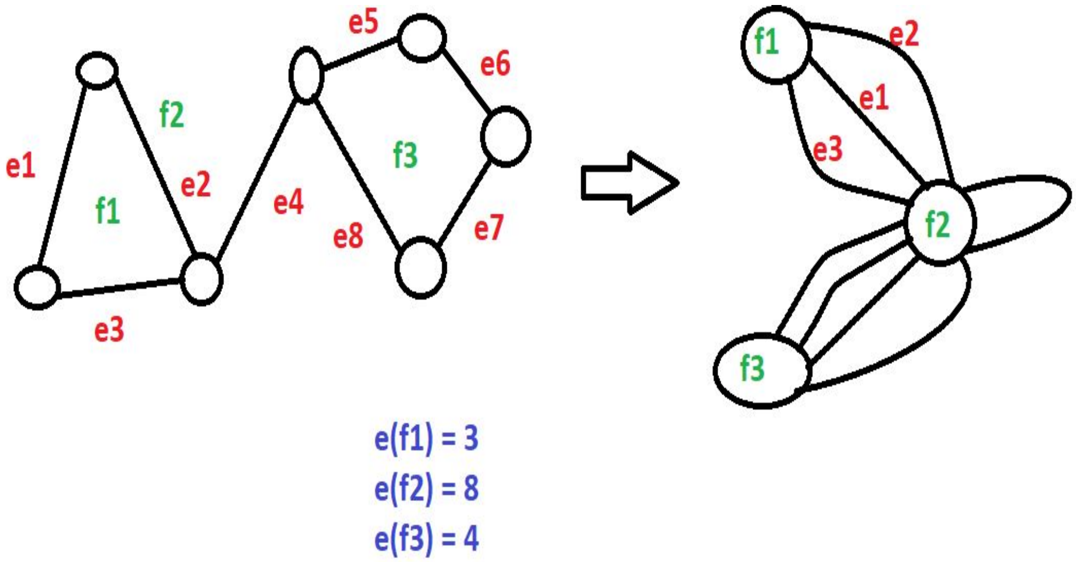
The Dual Graph G* of a planner graph is also a planner graph whose vertices corresponds to the faces of G . the edges of G* corresponds to the edges of G as fallows

- if e is an edges where one side is face-X and another side is face-Y then in G* , there is an edges between vertex corresponding to face-X and face-Y





13.5 Length of a Face



Statement : If $l(f_i)$ denotes length of face f_i in a planner graph.

$$2e = \sum_i l(f_i)$$

in the dual graph

$$l(f_i) = degree(f_i)$$

$$e = e$$

Handshaking Lemma in G*

$$2e = \sum_i degree(f_i)$$

$$\boxed{2e = \sum_i l(f_i)}$$

Statement : Suppose G is a connected planner graph with v nodes , e edged , f faces and v $\geq$ 3 , then

$$\boxed{e \leq 3v - 6}$$

v $\geq$ 3 $\implies$ In G*

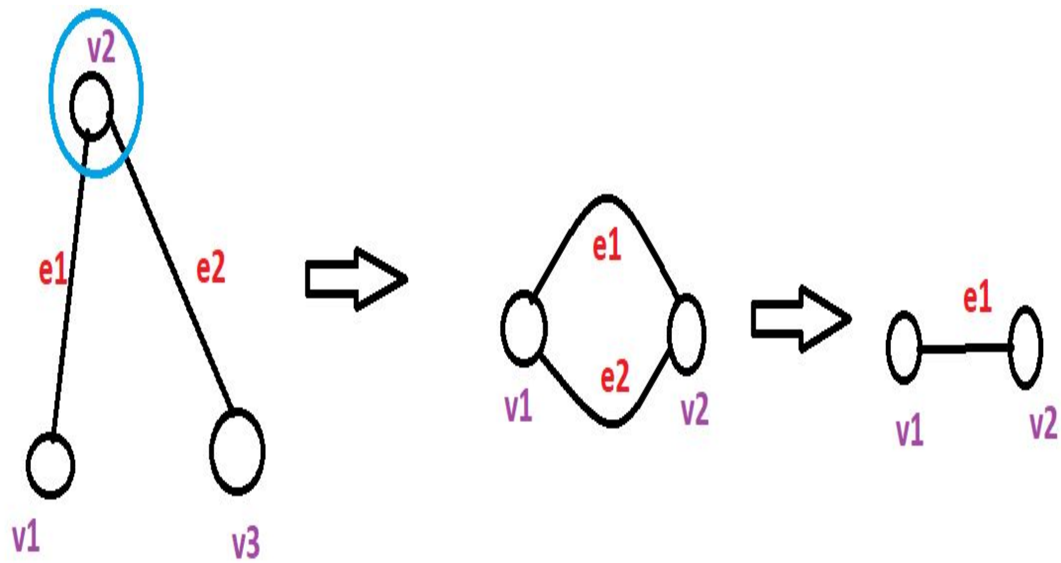
2e = $\sum_i^f l(f_i)$
2e $\geq$ 3 + 3 + 3 +........f times
2e $\geq$ 3f

v - e + f = 2
f = e - v +2 and 3f $\leq$ 2e
3f = 3e - 3v +6 $\leq$ 2e
$\implies$ e - 3v + 6 $\leq$ 0

e $\leq$ 3v - 6

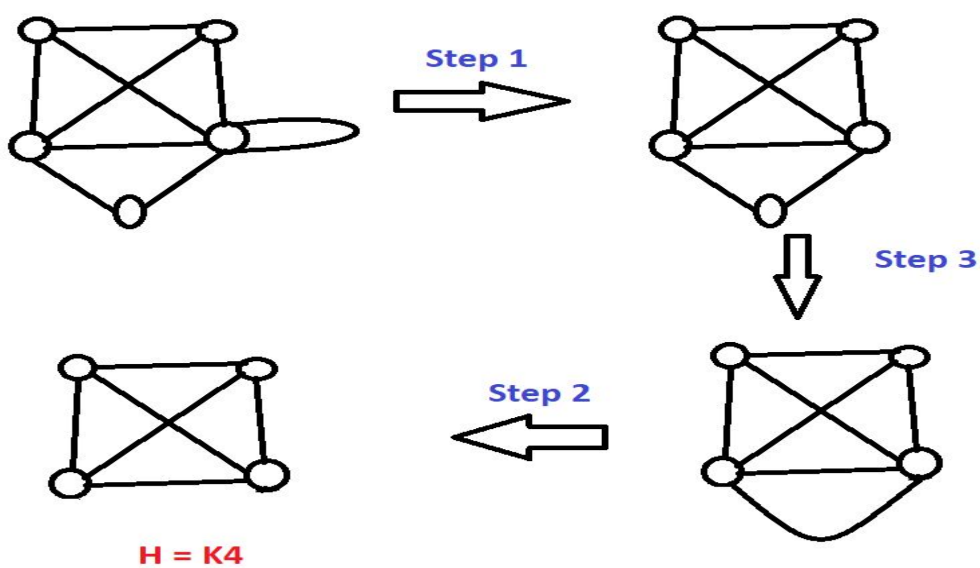
13.6 Planarity Test Algorithm :

- Remove all self loops.
- Remove parallel edges.
- Remove vertex having degree v = 2 and merge the edges incident on that vertex



Statement : G $\rightarrow$ H , G is planner if H is planner if

- H has one edge. Or
- H $\equiv K_4$ Or
- e $\leq$ 3v - 6



13.7 Theorem

Statement : The fallowing statements are equivalent for a planner graph :-

1. G is a Bipartite Graph
2. Every face of G has even Length
3. the dual graph G* is Eulerian

Proof : (1) $\implies$ (2) G contains only even length cycle .
We know that cycles make faces
$\implies$ Every face of G has even length.
Every face has even length in G
$\implies$ degree of each node in G* is even.
$\implies$ G* is Eulerian.

13.8 Theorem

Statement : Every Simple Planner Graph has "a" vertex of degree at most 5.

Proof : We know that every simple planner graph with v vertices has at most 3v - 6 edges , for v $\geq$ 3

Hence the sum of degree is at most 6v - 12

There is surely a vertex whose degree less than 6.