Indian Institute of Technology- Jodhpur

Graph Theory and Applications(GTA)
COURSE CODE: CSL7410

Lecture Scribing Assignment: Week 7

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Week 7

Hall's Theorem*

7.1 Some Theorems and Lemma

In this Section we will learn about the Properties of bipartite graph and some related lemma

7.1.1 Hall's Theorem

Statement : Let G be a Bipartite graph with bipartition (X,Y) . Then G contains a matching that saparates every vertex in X if and only if $\mid N(S) \mid \geq \mid S \mid$ for all S⊆X.

Proof: Let us assume that G contains a matching M which saturate every vertex in x Let s be a subset of x.

$\implies$ The vertices of S are matched under M with distinct vertices in $N(S)$

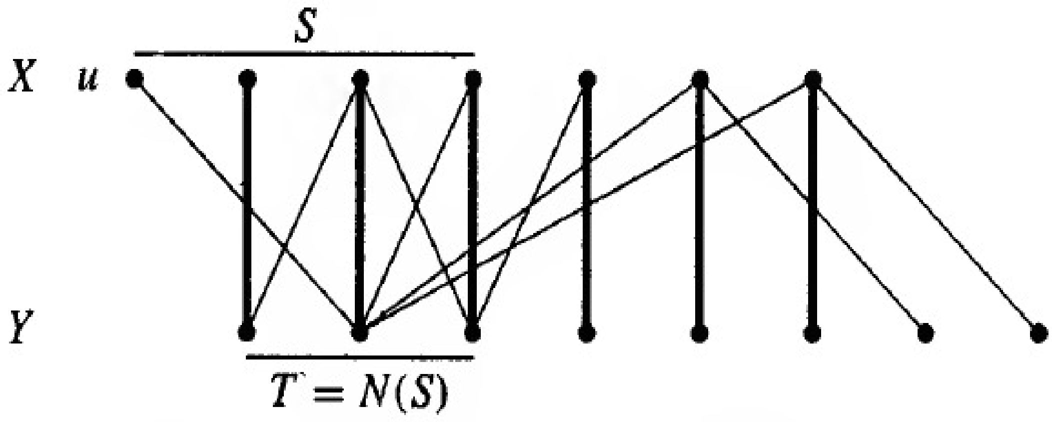
$|N(S)| \geqslant |S|$ for all $S \subseteq X$

Converse:-

Let us consider a bipartite graph G let us assume that $|N(S)| \geqslant |S| \quad \forall S \subseteq X$

To prove:- G contains a matching that saturates every vertex in X.

*Lecturer: Anand Mishra. Scribe: Sachin Kumar (M20MA064).



Let us assume the contradiction that g contains no matching saturating all vertices in X.

Let us assume that M^* is a maximum matching in G.

$\implies M^*$ does not saturates all vertices in X.

Let u be an M^* unsaturated vertex in X let Z denotes the set of vertices connected to u by M^* - alternating Path.

By Berge's theorem it follows that u is the only M^* - unsaturated vertex in Z
let the set S = Z ∩ X and T = Z ∩ Y

clearly, the vertices of S-set(u) are matched under M* with the vertices.

Since,$\mid T \mid = \mid S \mid -1$
N(S)⊆T   N(S)=T

$\mid N(S) \mid = \mid T = \mid S \mid -1$

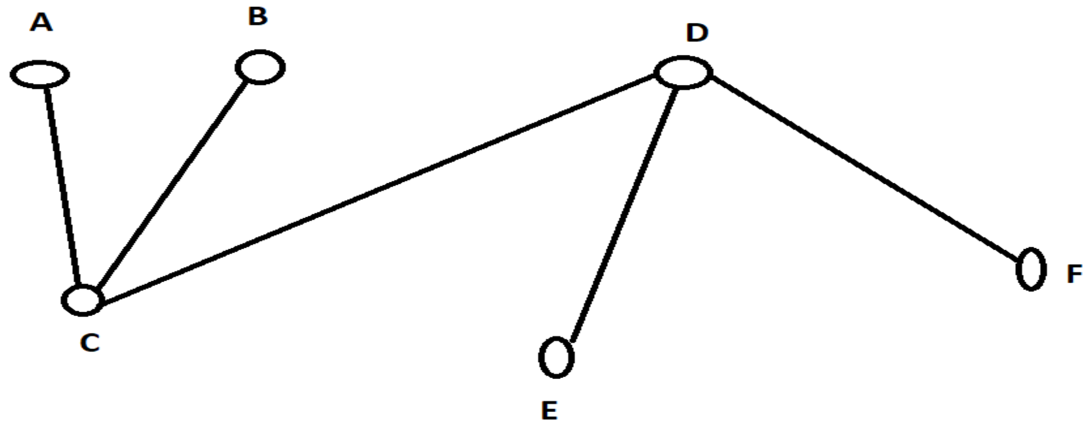
$\implies \mid N(S) \mid < \mid S \mid$

$\mid N(S) \mid \geqslant \mid S \mid$

Some definitions

Vertex cover : A vertex cover of graph C is a set $\theta \subseteq V(G)$ that contains at least one end point of every edge.
example:

$$\theta_1 = \{C, D\}$$
$$\theta_2 = \{A, B, E, F\}$$
$$\theta_3 = \{D, A, B\}$$
$$\theta_4 = \{C, E, F\}$$



Edge Cover: An edge cover of a graph is set of edge such that every vertex of the graph is incident to the at least one edge of the set.
example:

$$EG = \{AC, BC, DE, DF\}$$

Independent set: set of vertices which. is not adjacent.
example: independent set $= \{A, B\}$ or

or $\{E, F\}$ of $\{A, B, E, F\}$

$\alpha(G) =$ maximum size of independent set
$\alpha'(G) =$ maximum size of matching
$\beta(G) =$ minimum size of vertices cover
$\beta'(G) =$ minimum size of edge
$n(G) =$ total number of vertices in graph G

$$\alpha(G) = \{A, B, E, F\}$$
$$\alpha'(G) = \{AC, DE\} \text{ or, } \{BC, DF\}$$
$$\beta(G) = \{C, D\}$$
$$\beta'(G) = \{AB, AC, BC, DE, DF\}$$

7.1.2 Theorem

Statement: if G is a simple graph then diam $(G) \geq 3$ then diam $(G^c) \leq 3$

proof: if diam (G) ≥ 3 that implies
(1) $\exists$ u & v such that $uv \notin E(G)$
(2) u & v does not have common neighbour.

hence diam $(G^c) \leqslant 3$

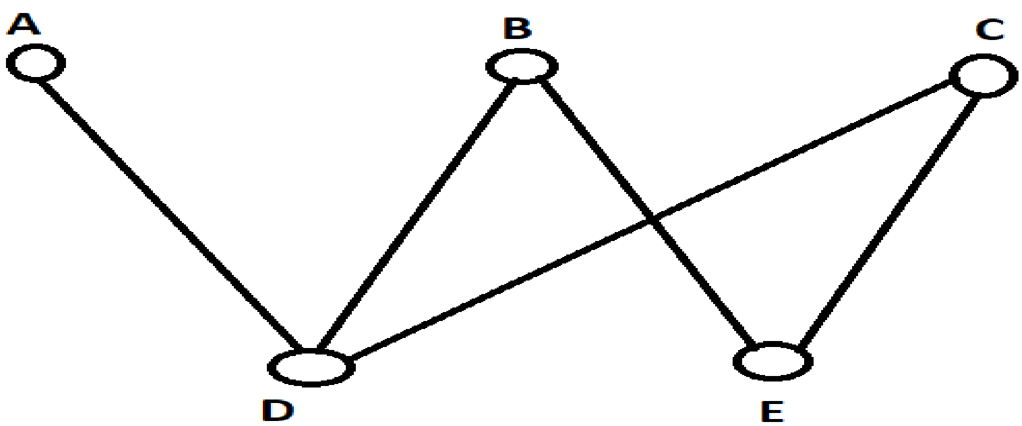
7.1.3 Lemma:

Statement: sum of maximum size of independent set and minimum size of vertex cover is equal to total number of vertices

$$\text{i.e.} \quad \alpha(G) + \beta(G) = n(G)$$

Proof: let s be an independent set of maximum size then every edge is incident to at least one vertex of is

$$S = \{A, B, C\}, \bar{S} = \{D, E\}$$



$SU\bar{S} = V(G)$

$\bar{S}$ covers all the edges

$\bar{S}$ is minimum size vertex cover $\Rightarrow \beta(G) = \bar{S}$ S is maximum size of independent set

$$\Rightarrow S = \alpha(G)$$
$$\therefore \quad \alpha(G) + \beta(G) = |S| + |\bar{S}| = n(G)$$

hence on $\alpha(G) + \beta(G) = n(G)$

7.1.4 theorem

Statement : if G is a graph without isolated vertices, then

$$\alpha'(G) + \beta'(G) = n(G)$$

7.1.5 Theorem

Statement: if G is a bipartite graph with no isolated vertices then $\alpha(G) = \beta'(G)$

Proof: we have already know

$\alpha(G) + \beta(G) = n(G)$ ....(i)

$a'(G) + \beta'(G) = n(G)$........... (ii)

$\alpha'(G) = \beta(G)$ .... (iii)

from (i) and (ii)

$\alpha(G) + \beta(G) = \alpha'(G) + \beta'(G)$

$\implies \alpha(G) + \beta(G) = \beta(G) + \beta'(G)$

$\implies \quad \alpha(G) = \beta'(G)$

7.1.6 theorem

Statement : G be a bipartite graph then $\alpha(G) = \frac{n(G)}{2}$ iff G has perfect matching .

Proof: Since $\alpha(G) + \beta(G) = n(G)$

$$\begin{aligned} \implies \alpha(G) &= n(G) - \beta(G) \\ &= n(G) - \alpha'(G) \end{aligned}$$

if G has perfect matching then maximum size of matching

$$\alpha'(G) = \frac{n(G)}{2}$$

$$\text{So} \quad \alpha(G) = n(G) - \frac{n(G)}{2}$$

$$\alpha(G) = \frac{n(G)}{2}$$