Week 13

Planer Graph*

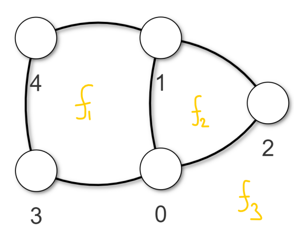
13.1 Introduction

Definition: Planer graph is a graph that has drawing without crossing.

Properties of planer Graph:

- They are sparse graph for large vertex graph
- They are 4-colourable.
- Efficient operation

Face: Also knows as regions in the graph



As in the given graph there are 3 faces form by nodes 0, 1, 4, 3 and 0, 1, 2 and an external region

*Lecturer: Dr. Anand Mishra. Scribe: Vedanshu Goyal.

13.2 Euler Condition

Theorem 13.1. *For any connected Planer Graph*

$$v - e + f = 2$$
$$\nu = \#\ \textit{of vertices}$$
$$e = \#\ \textit{of edges}$$
$$f = \#\ \textit{of faces.}$$

Proof. We can easily check this is always true for tree.
Because,

$$v - e + f = v - (v - 1) + 1 = 2$$

Now the addition of any number of edges in this graph will also increase the number of cycle or number of faces in the graph by the same amount which will cancel each other in the formulae, so this formulae is always true for connected Planer Graph. □

13.3 Dual Graph

Definition The dual graph G^* of a planar graph G is also a planar graph whose vertices correspond to the faces of G. The edges of G corresponds to the edges of G as follows if e is an edge where one side its face-x and another side is face-y then in G^* there is an edge between vertex corresponding.

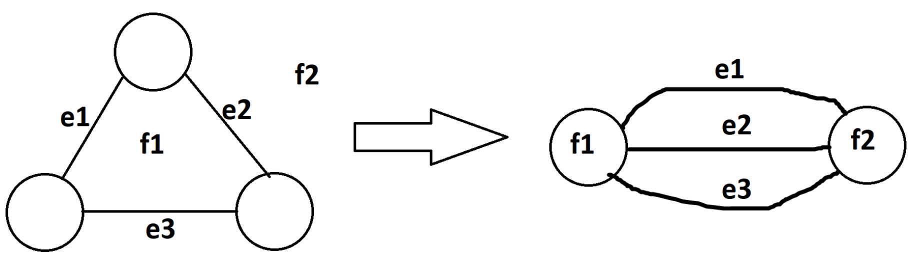


Figure 13.1: Dual Graph

In a planer graph,

$$2e = \sum_i l(fi) \tag{13.1}$$

l(f) is the length of face f.

Theorem 13.2. *Suppose G is a connected graph with v nodes, e edges, f faces and $v \geqslant 3$ then*

$$e \leq 3v - 6$$

Proof. .

$$\begin{aligned}
& v \geqslant 3 \Rightarrow \;\; \text{2e} \;\; = \sum_{i=1}^{i} \ell(fi) \\
& \geqslant 3 + 3 + \ldots \;\; \text{f times} \qquad (13.2) \\
& = 3 \\
& \quad\;\; 2e \geqslant \text{3f}
\end{aligned}$$

$$\begin{aligned}
& v - e + f = 2 \\
& f = e - v + 2 \quad (3f \leq 2e \\
& 3f = 3e - 3v + 6 \leq 2e \qquad (13.3) \\
& \Rightarrow e - 3v + 6 \leq 0 \\
& \Rightarrow e \leq 3v - 6
\end{aligned}$$

□

13.4 Planarity Test Algorithm

Given a graph G to check planarity, do the following steps recursively and get a graph H.

1. Remove all self loops.
2. Remore parallel edges
3. Remove vertex havirg degree = 2 and merge the edges incident on that rertex

$$G \rightarrow H$$
$$G \text{ is planar if } H \text{ is planar}$$

H is planer if

- H has one edge or
- $H = K_4$ or
- $e \leq 3v - 6$

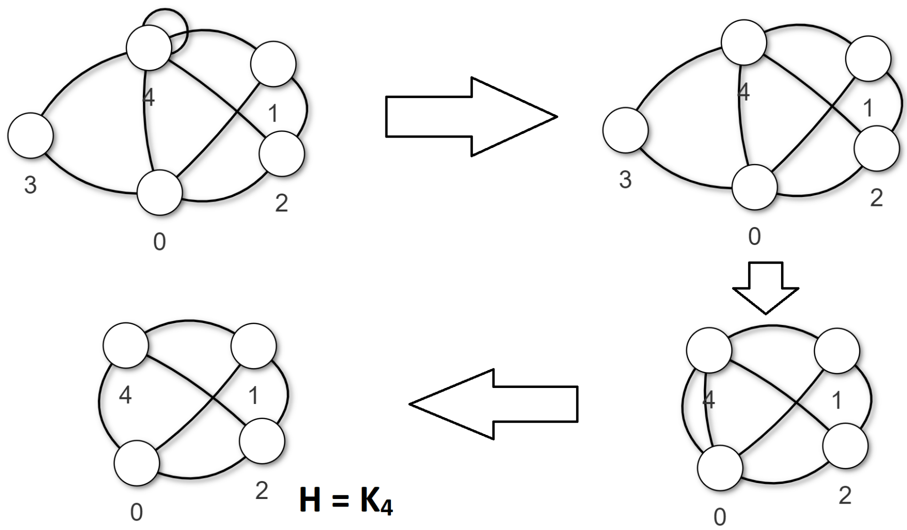


Figure 13.2: Planarity Test Algorithm

Theorem 13.3. *The following statements are equivalent for a planar graph:-*

- *G is a bipartite graph*
- *Every face of G has even length*
- *The dual graph G^* is Eulerian.*

Theorem 13.4. *Every simple planar graph has " a " vertex of degree at most* 5.

Proof. We know that Every simple planar graph with v vertices has at most $3v - 6$ edges for $v \geqslant 3$

Hence the sum of degree is at most $6v - 12$

So, There is surely a vertex whose degree < 6. □