Optimization
Some Solved Problems

Q1 Find all the BFS for the feasible region given by:

$$2x_1+6x_2+x_3+x_4=6$$
$$6x_1+4x_2+2x_3+4x_4=4$$
$$x_i \geq 0 \ \forall i$$

as $n=4$, $m=2$ substitute two variables equal to zero and solve for other two

$x_1=0,\ x_2=0 \Rightarrow x_3=10,\ x_4=-4$ (not BFS as $x_4<0$)

$x_1=0,\ x_3=0 \Rightarrow x_2=1,\ x_4=0$ (BFS)

$x_1=0,\ x_4=0 \Rightarrow x_2=1,\ x_3=0$ (BFS, same as above)

$x_2=0,\ x_3=0 \Rightarrow x_1=10,\ x_4=-14$ (not BFS as $x_4<0$)

$x_2=0,\ x_4=0 \Rightarrow x_1=-4,\ x_3=14$ (not BFS as $x_1<0$)

$x_3=0,\ x_4=0 \Rightarrow x_2=1,\ x_1=0$ (BFS, same as other two).

Thus the above system has only one BFS, namely $(0,1,0,0)$.

Q2 Using simplex method find the optimal solution to

$$\max\ 2x_1+x_2$$
$$\text{s.t}\ \ 3x_1+5x_2 \leq 15$$
$$6x_1+2x_2 \leq 24$$
$$x_1, x_2 \geq 0$$

The above problem can also be represented as

$$\max\ 2x_1 + x_2$$

$$\text{s.t. } 3x_1 + 5x_2 + x_3 = 15$$

$$6x_1 + 2x_2 + x_4 = 24$$

$$x_i \geq 0 \ \forall i$$

First table:

			2	1	0	0
C_B	B	b	a_1	a_2	a_3	a_4
0	x_3	15	3	5	1	0
← 0	x_4	24	6	2	0	1
	$z_j - c_j$:		-2 ↑	-1	0	0

[as most -ve $z_j - c_j$ enters and least ratio leaves. Thus we get the following table:

Second table

			2	1	0	0
C_B	B	b	a_1	a_2	a_3	a_4
← 0	x_3	3	0	4	1	$-1/2$
2	x_1	4	1	$1/3$	0	$1/6$
	$z_j - c_j$:		0	$-1/3$ ↑	0	$1/3$

x_2 enters, x_3 leaves

Third table

			2	1	0	0
C_B	B	b	a_1	a_2	a_3	a_4
1	x_2	$3/4$	0	1	$1/4$	$-1/8$
2	x_1	$15/4$	1	0	$-1/12$	$5/24$
		$z_j - c_j$:	0	0	$1/12$	$7/24$

as $z_j - c_j \geq 0 \; \forall j$, optimality reached.

opt. solution: $(x_1 = 15/4, \; x_2 = 3/4)$ opt. value: $33/4$.

Q Using Big-M method solve the following LPP

$$\max \; x_1 + 5x_2$$
$$\text{s.t. } 3x_1 + 4x_2 \leq 6$$
$$x_1 + 3x_2 \geq 3$$
$$x_1, x_2 \geq 0$$

The above problem can be written as

$$\max \; x_1 + 5x_2$$
$$\text{s.t. } 3x_1 + 4x_2 + x_3 = 6$$
$$x_1 + 3x_2 - x_4 = 3$$
$$x_i \geq 0 \; \forall i$$

Introducing artificial variables (required), we get the following:

$$\max \; x_1 + 5x_2 - Mx_5$$
$$\text{s.t } 3x_1 + 4x_2 + x_3 = 6$$
$$x_1 + 3x_2 - x_4 + x_5 = 3$$
$$x_i \geq 0 \; \forall i$$

First table

			1	5	0	0	$-M$
C_B	B	b	a_1	a_2	a_3	a_4	a_5
0	x_3	6	3	4	1	0	0
$-M$	x_5	3	1	3	0	-1	1
	$z_j - c_j$:		$-M-1$	$-3M-5$	0	M	0

x_2 enters, x_5 leaves

Second table

			1	5	0	0	$-M$
C_B	B	b	a_1	a_2	a_3	a_4	a_5
0	x_3	2	$5/3$	0	1	$4/3$	$-4/3$
5	x_2	1	$1/3$	1	0	$-1/3$	$1/3$
	$z_j - c_j$:		$2/3$	0	0	$-5/3$	$M-5/3$

x_4 enters, x_3 leaves

Third table:

			1	5	0	0	$-M$
C_B	B	b	a_1	a_2	a_3	a_4	a_5
0	x_4	$3/2$	$5/4$	0	$3/4$	1	-1
5	x_2	$3/2$	$3/4$	1	$1/4$	0	0
	$z_j - c_j$:		$11/4$	0	$5/4$	0	M

$z_j - c_j \geq 0 \ \forall j \Rightarrow$ optimality achieved

optimal solution $(0, 3/2)$ opt value : $15/2$

Q Write the dual of the following LPP

$$\min\ 5x_1 + 2x_2$$

$$\text{s.t. } x_1 + 2x_2 \geq 5$$

$$2x_1 - x_2 \geq 12$$

$$x_1 + 3x_2 \geq 4$$

$$x_1 \geq 0,\ x_2 \text{ unrestricted}$$

Solve the dual. From the optimal table of the dual, find the optimal solution to the above problem.

The dual of the given problem is

$$\max\ 5x_1 + 12x_2 + 4x_3$$

$$\text{s.t } x_1 + 2x_2 + x_3 \leq 5$$

$$2x_1 - x_2 + 3x_3 = 2$$

$$x_1, x_2, x_3 \geq 0$$

The problem can also be written as

$$\max\ 5x_1 + 12x_2 + 4x_3$$

$$\text{s.t } x_1 + 2x_2 + x_3 + x_4 = 5$$

$$2x_1 + x_2 + 3x_3 = 2$$

$$x_i \geq 0\ \forall i$$

Introducing artificial variables, we get the following problem:

$$\max\ 5x_1+12x_2+4x_3-Mx_5$$

$$\text{s.t. } x_1+2x_2+x_3+x_4=5$$

$$2x_1-x_2+3x_3+x_5=2$$

$$x_i \geq 0 \ \forall i$$

First table

		c_j	5	12	4	0	$-M$
C_B	B	b	a_1	a_2	a_3	a_4	a_5
0	x_4	5	1	2	1	1	0
$-M$	x_5	2	2	-1	3	0	1
	z_j-c_j:		$-2M-5$	$M-12$	$-3M-4$	0	0

x_3 enters, x_5 leaves

Second table

		c_j	5	12	4	0	$-M$
C_B	B	b	a_1	a_2	a_3	a_4	a_5
0	x_4	$13/3$	$1/3$	$7/3$	0	1	$-1/3$
4	x_3	$2/3$	$2/3$	$-1/3$	1	0	$1/3$
			$-7/3$	$-\frac{40}{3}$	0	0	$M+4/3$

x_2 enters, x_4 leaves

Third table:

		c_j	5	12	4	0	$-M$
C_B	B	b	a_1	a_2	a_3	a_4	a_5
12	x_2	$13/7$	$1/7$	1	0	$3/7$	$-1/7$
4	x_3	$9/7$	$5/7$	0	1	$1/7$	$2/7$
	z_j-c_j:		$-\frac{11}{7}$	0	0	+ve	+ve

x_1 enters, x_3 leaves

Fourth table

			5	12	4	0	−M
C_B	B	b	a_1	a_2	a_3	a_4	a_5
12	x_2	8/5	0	1	−1/5	2/5	−1/5
5	x_1	9/5	1	0	7/5	1/5	2/5
$z_j - c_j$:			0	0	3/5	29/5	$M-\frac{2}{5}$

$z_j - c_j \geq 0 \; \forall j \Rightarrow$ optimal table reached

optimal soln to problem solved: $(x_1 = 9/5, x_2 = 8/5)$ and optimal value is $\frac{141}{5}$.

Optimal solution to original problem (dual of above)

As (a_4, a_5) constituted the identity matrix in the first iteration, (z_4, z_5), i.e. $(w_1 = z_4, w_2 = z_5)$ ~~is the optimal~~ in the optimal table is the solution to the original problem

optimal solution to original problem is $\left(w_1 = \frac{29}{5}, w_2 = -\frac{2}{5}\right)$

and optimal value is $\frac{141}{5}$.

Q Using Dual Simplex method solve the following LPP

$$\max -3x_1 - x_2$$
$$\text{s.t. } x_1 + x_2 \geq 1$$
$$2x_1 + 3x_2 \geq 2$$
$$x_1, x_2 \geq 0$$

The above problem can be written as

$$\max -3x_1 - x_2$$
$$\text{s.t. } -x_1 - x_2 \leq -1$$
$$-2x_1 - 3x_2 \leq -2$$
$$x_1, x_2 \geq 0$$

which further can be written as

$$\max -3x_1 - x_2$$
$$\text{s.t. } -x_1 - x_2 + x_3 = -1$$
$$-2x_1 - 3x_2 + x_4 = -2$$
$$x_i \geq 0 \ \forall i$$

First table

			-3	-1	0	0
C_B	B	b	a_1	a_2	a_3	a_4
0	x_3	-1	-1	-1	1	0
0	x_4	-2	-2	-3	0	1
	$z_j - c_j$:		3	1	0	0

most negative variable leaves, i.e. x_4 leaves

For entering, least modulus enters (ratio of $z_j - c_j$ with negative enteries of leaving variable's row).

$\Rightarrow x_2$ enters

Second table

C_B	B	b	a_1 (−3)	a_2 (−1)	a_3 (0)	a_4 (0)
← 0	x_3	$-1/3$	$-1/3$	0	1	$-1/3$
−1	x_2	$2/3$	$2/3$	1	0	$-1/3$
	$z_j - c_j$:		$7/3$	0	0	$1/3$ ↑

x_3 leaves,
x_4 enters,

Third table

C_B	B	b	a_1 (−3)	a_2 (−1)	a_3 (0)	a_4 (0)
0	x_4	1	1	0	−3	1
−1	x_2	1	1	1	−1	0
	$z_j - c_j$:		2	0	1	0

as $b_i \geq 0 \ \forall i$ and $z_j - c_j \geq 0 \ \forall j$ optimality achieved
optimal solution is $(x_1 = 0, x_2 = 1)$ optimal value: -1

Note: The solution can be verified by plotting the feasible region, finding corner points (which are BFS) and then choosing the optimal solution