Week 7

Lecture 17 and 18 scribing*

7.1 Lecture 17

7.1.1 Assignment problem in Optimization

Cost Matrix:

$C_{n,n} = [C_{ij}]$
$x_{ij} = 1$ means person i is assigned task j
Constraints:

$$\sum_i x_{ij} = 1$$

$$\sum_j x_{ij} = 1$$

That means one task would be done by one person.
minimize

$$\sum_i \sum_j c_{ij} x_{ij}$$

Matrix C
$C'_{nxn} = []$
In above matrix C' rows goes as $v_1, v_2, ..., v_j, ..., v_n$
And column goes as $u_1, u_2, ..., u_i, ..., u_n$
$C'_{ij} = C_{ij} - (u_i + v_j)$
$C'_{ij} \geqslant 0$
Now previous optimization function
minimize

$$\sum_i \sum_j c_{ij} x_{ij}$$

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Subject to

$$\sum_i x_{ij} = 1$$

$$\sum_j x_{ij} = 1$$

Introduce following variables,
u_i for all i = 1,... n — (1)
v_j for all j = 1,... n — (2)

Now multiplying these variables with constraints,

$(u_1 + v_1)x_{11} + (u_2 + v_2)x_{22} + (u_1 + v_{12})x_{12} + (u_2 + v_1)x_{21} + ... + (u_n + v_n)x_{nn} = u_1 + u_2 + ... + u_n + v_1 + v_2 + ... + v_n$

$\sum_i \sum_j (u_i + v_j)x_{ij} = \sum_i u_i + \sum_j v_j$

$\Rightarrow \sum_i u_i + \sum_j v_j = \sum_i \sum_j (u_i + v_j)x_{ij} \leqslant \sum_i \sum_j c_{ij}x_{ij}$

Hence we want to minimize term on right side in below equation,
$\sum_i \sum_j (u_i + v_j)x_{ij} \leqslant \sum_i \sum_j c_{ij}x_{ij}$

Dual Optimization
max $\sum_i u_i + \sum_j v_j$
Subject to $C_{ij} \geqslant u_i + v_j$

Problem:

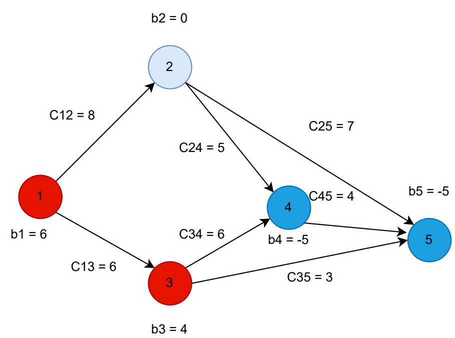

Figure 7.1

Problem: Find out minimum cost flow for transporting all the material from supply nodes to demand nodes.

Optimization formulation:

$\min \sum_i \sum_j c_{ij}x_{ij}$
Subject to: $x_{12} + x_{13} = b1$
$-x_{12} + x_{24} + x_{25} = b2$
$-x_{13} + x_{34} + x_{35} = b3$
$-x_{24} - x_{34} + x_{45} = b4$
$-x_{25} - x_{45} + x_{35} = b5$

minimum coast flow network problem: $\min \sum_i \sum_j c_{ij}x_{ij}$
Now multiply above each equations by w1, w2, w3, w4, w5 respectively and add all these equations.

$(w1 - w2)x_{12} + (w1 - w3)x_{13} + (w2 - w4)x_{24} + ... + ... = \sum \sum w_i b_i$

$\sum_i \sum_j (w_i - w_j)x_{ij} = \sum_i w_i b_i$

if we apply following constraint $w_i - w_j \leqslant c_{ij}$

$\sum_i w_i b_i = \sum_i \sum_j (w_i - w_j)x_{ij} \leqslant \sum_j c_{ij}x_{ij}$

$\max \sum_i w_i b_i$
s.t. $w_i - w_j \leqslant c_{ij}$
$w_i - w_j - c_{ij} \leqslant 0$

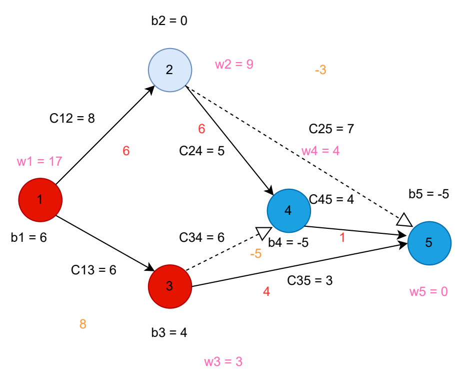

Figure 7.2

Cost = 6 * 8 + 6 * 5 + 4 * 5 + 6 * 4 = 122

For non-basic edges verify if $w_i - w_j - c_{ij} isnegativeornot$.

now Cost = 6 * 8 + 6 * 5 + 4 * 3 + 1 * 4 = 94

By repeating these step until all non basic edges satisfy w$_i - w_j - c_{ij} isnegative$

Hence final Optimal cost we are getting is 81.

7.2 Lecture 18

7.2.1 Maximum flow Motivation:

Consider a oil refinery at Mathura producing oil with warehouse in Chennai there would be multiple paths from source to destination with each path having different capacity of fluid flow, such graph is known as flow network.

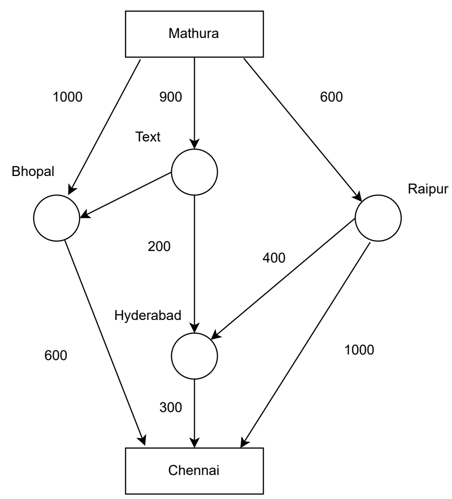

Figure 7.3

Que. Which problems can be modelled as flow network?

1. Liquids flowing through the pipes
2. Current through electrical networks.
3. Information through Communication networks.
4. vehicles movement on roads.

In a flow network,

1. Vertices other than sink and source do not stores any material and they are conduit junction.

2. Each edge can have predefined capacity.

following questions can be answered through such flow networks.

Que. What is maximum amount of flow possible in a given flow network?

following terms would be important to answer this question, like Flow network, Flow, Max-Flow problem.

Flow networks:
A flow network is a directed graph in which each edge (u,v) $\in$as non-negative capacity c(u,v) ≥ 0. we distinguish two vertices as Source(s) and sink(t).
In any flow network, $\mid E \mid \geq$ —V— - 1 (always a connected graph).

Flow:
Let G = (V,E) be a flow network with a capacity function c, let s be the source of the network and t be the sink. the a flow in G is defined as a real valued function f: V X V $\longrightarrow$ R that satisfies following three properties:
1. Capacity Constraint.
2. skew symmetry.
3. flow conservation.

Total net flow of a vertex v is defined as total positive flow leaving vertex v minus total positive flow entering that vertex.

7.2.2 Maximum flow Problem

with source s and sink t we want to find a flow of maximum value.

Residual Network: Given a flow network G = (V,E)and a flow f the residula network of G induced by flow f is G$_f = (V, E_f) where,$
E$_f = (u, v) \in VXV : C_f(u, v) > 0$

Residual Capacity: The amount of flow we can push from u to v before exceeding the capacity c(u,v) is the residual capacity of c(u,v).

Augmenting path: Given a flow network G=(V,E) and a flow f, an augmenting path p is a simple path from s o t in the residual network.
Residual capacity of path is the minimum residual capacity along the path.

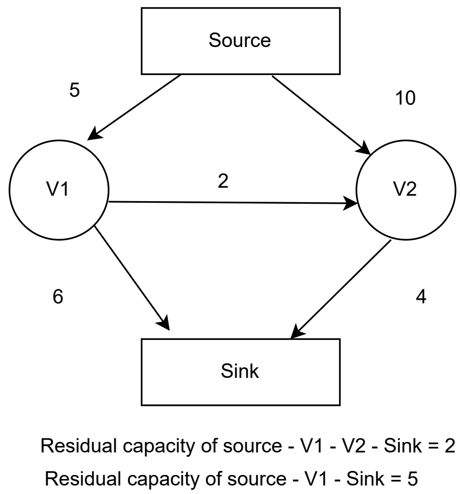

Figure 7.4

CUT: A cut C(S, T) of flow network is a partition of set of vertices into 2 disjoint sets S and T.

Capacity of edges going from vertices belonging to S to vertices belonging to set T, is the capacity of cut.

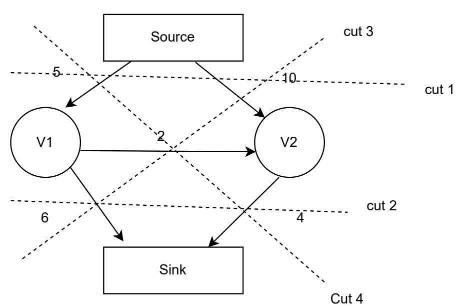

Figure 7.5

cut 1: S = $\varnothing$, T = V_1, V_2
cut 2: S = $\mathrm{v}_1, v_2, T = \varnothing$
cut 3: S = $\mathrm{v}_1, T = V_2$
cut 4: s = $\mathrm{v}_2, T = v_1$

Capacity:
cut 1: 15
cut 2: 10
cut 3: 18
cut 4: 9

7.2.3 Max flow - min cut theorem:

Following conditions are equivalent If f is a flow in G = (V, E) with sources s and sink t:

1. f is a max flow in G.
2. The residual network G_f contains no augmenting path.
3. There exist a cut C(S, T) with capacity f.

Ford-Fulkerson Method

Find an augmenting path p and augment flow f against flow f against p.

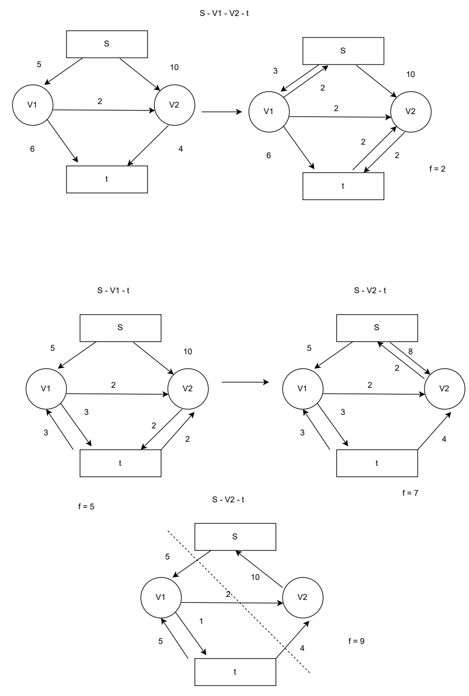

Figure 7.6

While finding augmenting path one has to traverse O(| E |) each time. Thus if max - flow = | $f*$ | then at worst case the complexity of algorithm based on Ford - Fulkerson is O(| $f*$ || E |).

References: D.B. West. Introduction to graph theory, 2nd edition, prentice hall of india. 2002.