Week 7

Lecture 13 and 14 scribing*

7.1 Quiz 2 Solutions discussion

* Answers are written in bold font

Que 1. By removing how many edges K100 can become bipartite graph?
a. 4951
b. 100
c. 4850
d. 50

Que 2. Which among the following is not a valid graphic sequence for a simple graph:
(A) 4,3,2,1 (B)4,3,3,2,2,1
a. Both A and B
b. Only B
c. Neither A Nor B
d. Only A

Que 3. How many minimum number of edges needs to be removed so that Cn becomes bipartite for sure?
a. n-1
b. 1
c. 0
d. n-2

Que 4. A node v is added to a bipartite graph K1,2. Further, v is connected to all the vertices of K1,2. Then what will be the chromatic number of resultant graph?
a. 4
b. 3
c. 1
d. 2

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Que 5. Swati misunderstood the circuit as cycle, for her which of the following properties of cycle will still hold:

i. Cycle is a closed walk.
ii. By removing last edge of cycle, it becomes a path
iii. Even length cycles are 2-colorable.
iv. Edges do not repeat in a cycle.

a. Only i, iii and iv
b. Only ii and iii
c. Only i, ii and iv
d. Only i and iv
e. All of them

Que 6. If G is an n-vertex tournament with indegree equal to outdegree at every vertex, then what can be said about n?

a. n¿3
b. n¿1
c. n is even
d. n is odd

Que 7. Suppose diameter of simple graph having only one connected component is greater than or equal to 3. then, consider the following two statements:

A. there exist two vertices u and v that are not connected.
B. for all vertex x (except u and v), x is either connected to u or v, but not both.

a. both A and B true
b. both A and B are not necessarily true
c. only A is true
d. only b is true

Que 8. Compute Radius and Diameter of K100,100

a. 50, 100
b. 2, 2
c. 100, 100
d. 1, 2

Que 9. Which among the following is not a property of a tree?

A. Every tree has radius = diameter
B. Every tree has radius = 1
C. Complement of a tree is also a tree.
D. A tree can have two or more perfect matching.

a. all are false
b. only C is true
c. only A and C are true
d. only B is true
e. only A is true

Que 10. what will be the minimum size of maximal matching in C4 and P4
a. 2,2
b. 4,4
c. 2,1
d. 1,1

7.2 Hall's Theorem

Statement: An X-Y Bigraph G has a matching that saturates X iff

$$| N(S) | \geqslant |S| ... \forall S \subseteq X$$

Here, N(S)$\subseteq Y$s a set of neighbours of elements in S.

Proof: Necessary Condition: Suppose X-Y bigraph has matching that saturates X, Then obviously

$$| S | \leqslant | N(S) | ... \forall S \subseteq X$$

Sufficient Condition: if $\forall S \subseteq X, | N(S) | \geqslant |S|$ Then there is a matching that saturates X.

We shall prove the following contrapositive: if there is not such matching M that saturates X,

then $\exists S \subseteq X$, Such that $| S | \leqslant | N(S) |$

Let u $\in X$ be a vertex unsaturated by a matching M.

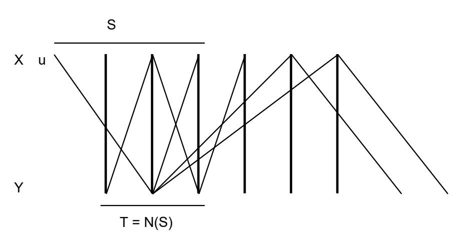

Figure 7.1: Hall's Theorem

Suppose two subsets S $\subseteq X$and T $\subseteq Y$ are considered as follows:

S = End points of m-alternating paths starting from u with the last edge belonging to M

T = End points of M-alternating paths starting from u with the last edge not belonging to M

$|S| = 1 + |T| = 1 + |N(S)|$

$\Rightarrow |S| \leqslant |N(S)|$

7.2.1 Vertex Cover:

A Vertex cover of a graph G is a set $\theta \subseteq V(G)$ that contains at least One end point of every edge.

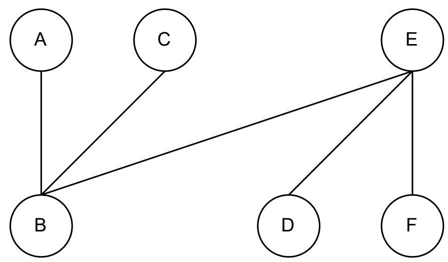

Figure 7.2: Vertex Cover

$\theta_1 = \{B, E\}$

$\theta_2 = \{A, B, C, D, E, F\}$

$\theta_3 = \{A, C, E\}$

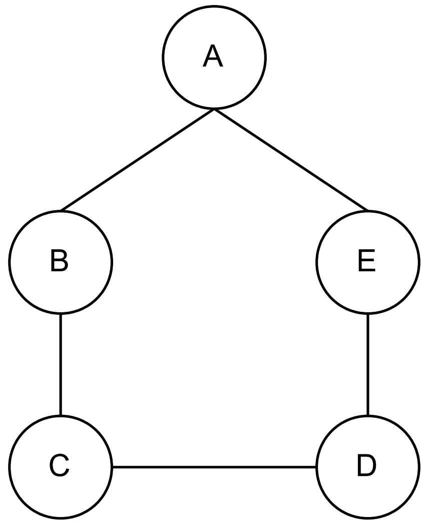

Figure 7.3

min VC = {A, C, D}
maximal matching = {BC, DE}

Independent sets:
The independence number of a graph is the maximum size of an independent set of vertices.
Ind. set = {A, C}

$\alpha(G)$ = maximum size of id set
$\alpha'(G)$ = maximum size of matching
$\beta(G)$ = minimum size of VC
$\beta'(G)$ = minimum size of edge cover

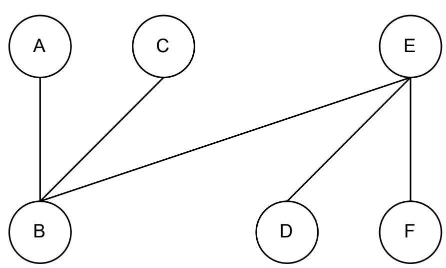

Figure 7.4

For above graph,
$\alpha(G) = 4$

$\alpha'(G) = 2$
$\beta(G) = 2$
$\beta(G) = 1$

also, $\alpha(G) + \beta(G) =\mid V \mid$

Proof for $\alpha(G) + \beta(G) = n(G)$

Proof:
Let S be an Independent set then every edge is incident to at least one vertex of S̄

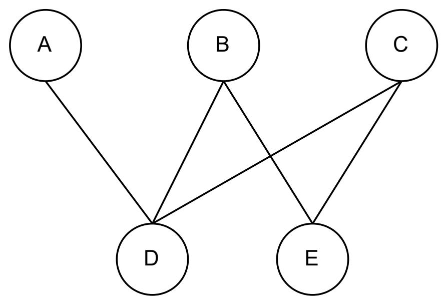

Figure 7.5

S = { A, B, C }
S̄ = { D, E }
S $\cup \bar{S} = V(G)$
S̄ covers all the edges

S̄ is minimum size vertex cover
$\beta(G) =\mid \bar{V} \mid$

S is maximum size of independent set $\alpha(G) =\mid S \mid$

Now $\alpha(G) + \beta(G) =\mid S \mid + \mid \bar{S} \mid=\mid V(G) \mid= n(G)$

Hence, $\alpha(G) + \beta(G) = n(G)$
and, $\alpha'(G) + \beta'(G) = n(G)$

Theorem: If G is a bipartite graph with no isolated vertices then
$\alpha(G) = \beta'(G)$

$\alpha(G) + \beta(G) = n(G)$
$\alpha'(G) + \beta'(G) = n(G)$
$\alpha'(G) = \beta(G)$

$\alpha(G) + \beta(G) = \alpha'(G) + \beta'(G) = \beta(G) + \beta'(G)$
Hence, $\alpha = \beta'$

Theorem: Let G be a bipartite graph prove that $\alpha(G) = n(G)/2$, if and only if G has a perfect matching.

Proof: In a graph G , S $\subseteq V(G)$ is an independent set if and only if S is a vertex cover and hence

$\alpha(G) + \beta(G) = n(G)$.
$\alpha(G) = n(G) - \beta(G) = n(G) - \alpha'(G)$

if G has Perfect matching then n(G)/2 will be the maximum size of matching,
Hence, $\alpha'(G) = n(G)/2$
so, $\alpha(G) = n(G) - n(G)/2 = n(G)/2$

7.3 Diameter problem Theorem:

Theorem: If G is a simple graph, then if diam(G) $\geqslant$ 3 then diam(G^c) $\leqslant$ 3.

Proof: When Diam G ¿ 2, there exist non adjacent vertices u,v $\in V(G)$with no common neighbor.

Hence For all x $\in V(G) - \{u, v\}$ has at least one of {u, v} as non neighbour.
This makes x adjacent in G^c to atleast one of u, v in G^c.
Since also uv $\in E(G^c)$, for every pair x, y-path of length at most 3 in G^c through {u, v}. Hence diam(G^c) $\leqslant$ 3.

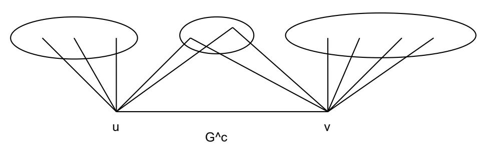

Figure 7.6

References: D.B. West. Introduction to graph theory, 2nd edition, prentice hall of india. 2002.