Week 3

Eulerian circuit, Vertex degrees and Hand shaking lemma*

The section describes the contents taught in Week-03.

3.1 True/False

1. A closed even walk need not contain a cycle.
   Sol) Let us consider an example,
   The Figure 3.1 is a closed even walk (e1,e1) and it does not contain a cycle. So this

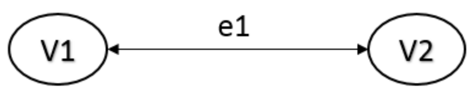

Figure 3.1: Set of 2 vertices

is a contradiction to the above statement. So statement is false.

2. Every closed odd walk contain an odd cycle.
   Sol) Let us consider an example,
   The Figure 3.2 is a closed odd walk (e1,e2,e3,e4,e1) and it also contain a cycle. So the system satisfies this example. Now let us try to prove the above statement using law of induction.

Lemma 3.1: Every closed odd walk contain an odd cycle.

Proof. For a walk of size 1, that is l=1, the statement is true as shown in the figure 3.3,

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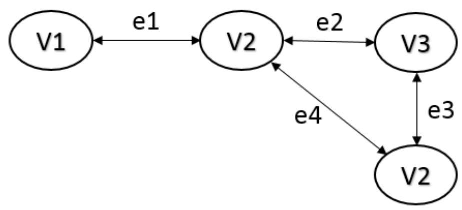

Figure 3.2: Set of 4 nodes with a odd cycle

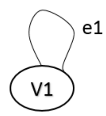

Figure 3.3: For l=1

Induction hypothesis: For any walk of size less than L, the following is true:-
"Every closed walk contains an odd cycle"

Case 1: If there is no repetition of vertex in walk, then a closed walk is a closed walk.
Case 2: If there is repetition in vertex in the walk,and let us suppose V is vertex which repeats. Break the walk into 2 V-V walks (say W1 and W2). Since $|W1|+|W2|$ =odd, that means either W1 or W2 is odd walk and surely both are less than L. From the induction, one of them (odd walk one) contains odd cycle.
From Fig 3.4 we get,

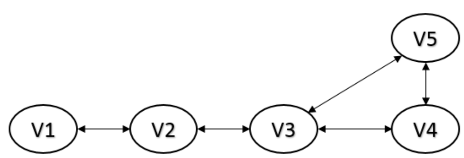

Figure 3.4: Considering 5 vertices with a odd ccycle

Closed odd path (W) : V1 V2 V3 V4 V5 V3 V2 V1.
$|W1| + |W2| = |W|$ =odd

Let W1: V1 V2 V3 V3 V2 V1
and W2: V3 V4 V5 V3
Now, $|W1| \rightarrow$ odd or $|W2| \rightarrow$ odd.
Say $|w2|$ is an odd closed walk and $|w2|$ <L. Therefore by induction hypothesis, W2 contain odd cycle. Hence W contain odd cycle. Hence proved. □

3.2 Eulerian Circuit

- A graph is Eulerian if it has a closed path containing all the edges.
- We call a closed path a circuit when we do not specify the first vertex but keep the list in cyclic order.
- An eulerian circuit in a graph is a circuit containing all the edges.

Lemma 3.2: If every vertex of a graph has degree at least 2 then $\mathcal{G}$ has a cycle.

Proof. Let us try to prove the above lemma by using contrapositive method.
$P \Rightarrow Q \equiv \neg Q \Rightarrow \neg P$
$P \equiv$ Every vertex of graph has degree ≥ 2.
$Q \equiv$ G has a cycle.
$\neg P \equiv \exists$ a vertex of graph $\mathcal{G}$, that has degree < 2.
$\neg Q \equiv \mathcal{G}$ has no cycle.

$\mathcal{G}$ is the collection of trees [Forests]
Let T be a component of $\mathcal{G}$.

Case 1: T is trivial graph.
$\exists$ vertex that has n vertices of degree=1.

Case 2: T is non-trivial.
T must have 2 end vertices.

Any tree which is not trivial, implies that T must have at least 2 end vertices. (Property of a tree)
End vertices have degree = 1.
Therefore $\exists$ 2 vertices such that they have degree $= 1 < 2$.

$\neg Q \Rightarrow \neg P \equiv P = Q$.

□

Determine whether true/false
If every vertex of a graph has degree at least 2, then $\mathcal{G}$ is a cycle.

The above statement is false. The Figure 3.5 shows a counter example of the above statement.

Theorem 3.1. *$\mathcal{G}$ is eulerian iff it has at most 1 non-trivial component and all its vertices have even degree.*

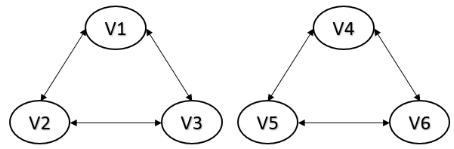

Figure 3.5: A counter example

Proof. Suppose $\mathcal{G}$ is eulerian.
$\Rightarrow \mathcal{G}$ has closed path containing all the edges.
$\Rightarrow$ There will be incoming and outgoing edges for all vertices. They and equal and are in pair.
$\Rightarrow$ Every node has even degree.
Since $\mathcal{G}$ is a closed path, it can not have more than one non-trivial component.
This competes the proof of first part.

Suppose $\mathcal{G}$ has at most one non-trivial component and each node of $\mathcal{G}$ has even degree.
If no. of edges (m)=0 then, $\mathcal{G}$ is eulerian.
Assume this is true for all graphs having above properties and less than M edges (Induction hypothesis).
Consider a graph containing one non-trivial component and having M edges and each node has degree = even.
$\Rightarrow$ Each vertex of $\mathcal{G}$' has at least 2 degree.
$\Rightarrow \mathcal{G}$' contains cycle (Lemma)
Say this cycle is C.
Remove all edges corrosponding to C from $\mathcal{G}$' and construct $\mathcal{G}$".
1) $\mathcal{G}$" has less than M edges.
2) Each vertex of $\mathcal{G}$" has even degree.
Each of the multiple possible component will have above two properties. Therefore, all of them are eulerian.
Hence proved. □

3.3 Vertex degrees and counting

3.3.1 Degree of a vertex:

Degree of a vertex V in a graph $\mathcal{G}$ written as d(v) is the number of edges incident to V, except that each loop at V counts twice. The maximum and minimum number of degrees are denoted by $\Delta(\mathcal{G})$ and $\sigma(\mathcal{G})$ respectively.

What will be $\Delta(\mathcal{G})$ and $\sigma(\mathcal{G})$ for k-regular graph? The maximum ($\Delta(\mathcal{G})$ and minimum ($\sigma(\mathcal{G})$)) number of vertices are both k for a k-regular graph.

3.3.2 Order and size of a graph:

The order of a graph $\mathcal{G}$, $n(\mathcal{G})$ is the number of vertices in $\mathcal{G}$.
The size of a graph $\mathcal{G}$, $e(\mathcal{G})$ is number of edges in $\mathcal{G}$.

What is the order and size of complete graph K_n? The order will be equal to number of vertices (n) and size will be equal to $\binom{n}{x}$.

What is the order and size of complete bipartite graph $K_{m,n}$? The order will be equal to number of vertices (m+n) and size will be equal to $m \times n$.

3.4 Handshaking Lemma (1^{st} theorem of graph theory)

Theorem 3.2. *Degree-sum formula: If $\mathcal{G}$ is a graph then,*

$$\sum_{v \in V(\mathcal{G})} d(v) = 2e(\mathcal{G}) \tag{3.1}$$

Example: An example of the lemma is shown in Fig 3.6,
Here the sum of degree of all the nodes = 1+2+2+3+2+2 = 12.

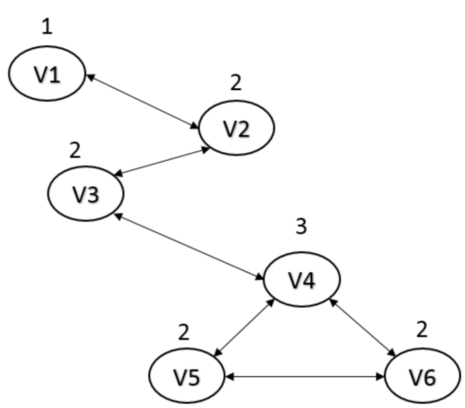

Figure 3.6: Example of handshaking lemma

No. of edges = 6
So sum of degree = No. of edges ×2

Proof. Let us try to prove the above theorem using the method of induction.
If e($\mathcal{G}$) = 1, then $\sum$ d(v) = 1+1 = 2.

Induction hypothesis: Assume e($\mathcal{G}$) = k, then $\sum_{v \in v(\mathcal{G})}$ d(v)=2×k

To prove: If e($\mathcal{G}$)=k+1, then $\sum_{v \in v(\mathcal{G})'}$ d(v)=2×(k+1)
From $\mathcal{G}$ we will construct $\mathcal{G}$'.
Case 1: No increase in vertex as shown in Fig 3.7. For graph $\mathcal{G}$, it is given as,

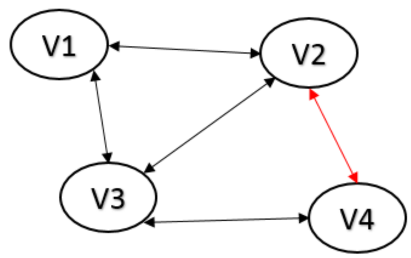

Figure 3.7: Red edge is added to existing graph

$$\sum_{v \in v(\mathcal{G})} d(v) = 2 \times k \tag{3.2}$$

Using (3.2), we get,

$$\sum_{v \in v(\mathcal{G})'} d(v) = \sum_{v \in v(\mathcal{G})} d(v) + 2 \tag{3.3}$$
$$= 2k + 2 \tag{3.4}$$
$$= 2(k+1) \tag{3.5}$$

Case 2: There is increase in vertex as shown in Fig 3.8.
For new graph it is given as,

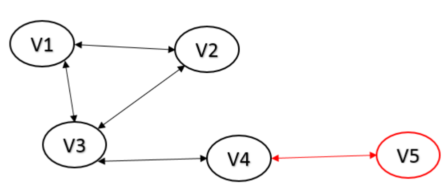

Figure 3.8: Red vertex is added to existing graph

$$\sum_{v \in v(\mathcal{G})'} d(v) = \sum_{v \in v(\mathcal{G})} d(v) + 2 \tag{3.6}$$

$$= 2k + 2 \tag{3.7}$$

$$= 2(k + 1) \tag{3.8}$$

Hence proved.

□

Corollary: A graph can not have exactly one node with odd degree.

Proof.

$$\sum_{v \in v(\mathcal{G})} d(v) = (k - 1) \times even + 1 \times odd \tag{3.9}$$

$$= odd \tag{3.10}$$

This is not possible. Hence proved. □