Week 3

Walk, Cycle, Eulerian Circuit and Degrees in Graph*

3.1 Walk and Cycle

3.1.1 Every closed odd walk contain an odd cycle.

Theorem 3.1. *Every closed odd walk contain an odd cycle.*

Proof. Base Case ($l = 1$)

It is obvious because it is single loop cycle.

Inductive Step ($l \longrightarrow l + 1$)

Case1: If there is no repetition of vertex in walk, then a closed walk = a closed cycle.

Case2: If there is repetition in vertex in the walk, and let us suppose v is vertex which repeats. Break the walk into two v-v walks (say w1 and w2). Since |w1| + |w2| = odd, that means either w1 or w2 is odd walk. And surely they both are less than L. From the induction one of them (odd walk one) contain odd cycle. □

3.1.2 Propositions

Proposition 3.2. *A simple path is a bipartite graph*

Proposition 3.3. *C_n is bipartite iff n is even.*

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3.1.3 Union of Graphs

The union of graphs $G_1, G_2, \cdots, G_k$ written as $\cup_{i=1}^{k} G_i$ is the graph with vertex set $\cup_{i=1}^{k} E\left(G_i\right)$ and edge set $\cup_{i=1}^{k} V\left(G_i\right)$

Theorem 3.4. *A graph is bipartite iff it has no odd cycle.*

Proof.

If it does not contain a cycle: Take one vertex in X and next vertex in Y.

If it contain even cycle: Partition the graph such that each even length cycle is one subgraph. We know Cn is bipartite for even length cycle. □

3.2 Eulerian Circuit

3.2.1 Eulerian Graph

Properties of Eulerian Graph:

- A graph is Eulerian if it has a closed path containing all the edges.
- We call a closed path a circuit when we do not specify the first vertex but keep the list in cyclic order.
- An Eulerian circuit in a graph is a circuit containing all the edges.

Lemma 3.5. *If every vertex of a graph has degree at least 2, then G has cycle.*

Proof. It can be proof very easily suppose you are traversing the graph through DFS, it can be easily see that we always able to traverse the node other than the parent node. So if we start from a0, a1... we can easily argue that after maximum n iterations we will again come to a node which is already visited, which is proof our assumption that G has a cycle. □

Theorem 3.6. *A graph G is Eulerian iff it has at most one non-trivial component and all its vertices have even degree.*

Proof. Assume a graph G has at most one non-trivial component and all its vertices have even degree.

Base Case ($edges(m) = 0$)

It is obvious G is Eulerian

Inductive Step ($m \longrightarrow m + 1$)

- Each vertex has at least two degree.

- G contains cycle, from Lemma 3.5. Say C.
- Remove E(C) from G to construct G‘.
- We combine these Eulerian cycles with C to construct an Eulerian circuit as follows: Traverse C until a component of G’ appear, then traverse Eulerian cycle of that component, come back to C and repeat this.

□

3.3 Vertex Degrees and Counting

3.3.1 Degree of a vertex

Definition: Degree of a vertex v in a graph G written as $d(v)$ is number of edges incidents to v, except that each loop at v counts twice. The maximum and minimum number of degrees are denoted by $\Delta(G)$ and $\delta(G)$ respectively.

3.3.2 Order and size of a graph

- The order of a graph G, n(G) is number of vertices in G.
- The size of a graph G, e(G) is number of edges in G.

3.3.3 Handshaking Lemma(Degree-sum formula)

If G is a graph then

$$\sum_{v \in V(G)} d(v) = 2e(G)$$

It is very easy to see why this is true because always adding an edge increase the degree of the graph by 2.

3.3.4 Questions

Q1. Which among the following can be Eulerian Graph?

(a) Every node has even degree.

(b) Only one node has odd degree.

(c) Exactly Two nodes have odd degree.

(d) More than two nodes have odd degree.

Ans. a, c.

Q2. What will be $\Delta(G)$ and $\delta(G)$ for k-regular graph?

Ans. k

Q3. In a class with 9 students, each student sends valentine cards to three others. Determine whether it is possible that each student receives cards from the same three students to who he or she sends the card.

Ans. We can let sending and receiving card to other person as an single edge, now the question reduces to that there is any graph with 9 nodes and each node having degree 3, which we know is not possible because sum of degree of all nodes is 27, which is odd that can't be possible in a graph, due to Handshaking Lemma.