Week 9

Minimum Cost Flow Network, Maximum Flow, Flow Network, Max cut Min Flow Theorem *

9.1 Minimum Cost flow Network

The minimum-cost flow problem is an optimization and decision problem to find the cheapest possible way of sending a certain amount of flow through a flow network.

PROBLEM: Find out minimum cost flow for transporting all the material from supply nodes to demand nodes.

Solution: Optimization formulation:
we have to minimize $\sum_i \sum_j c_{ij}x_{ij}$

1. $-x_{12} + x_{13} = b1$
2. $-x_{12} + x_{24} + x_{25} = b2$
3. $-x_{13} + x_{34} + x_{35} = b3$
4. $-x_{24} - x_{34} + x_{45} = b4$
5. $-x_{25} - x_{45} + x_{35} = b5$

Multiplying all equations by Wi
$(w_1 - w_2)\, x_{12} + (w_1 - w_3)\, x_{13} + (w_2 - w_4)\, x_{24} + \cdots + \cdots = \sum w_i b_i$

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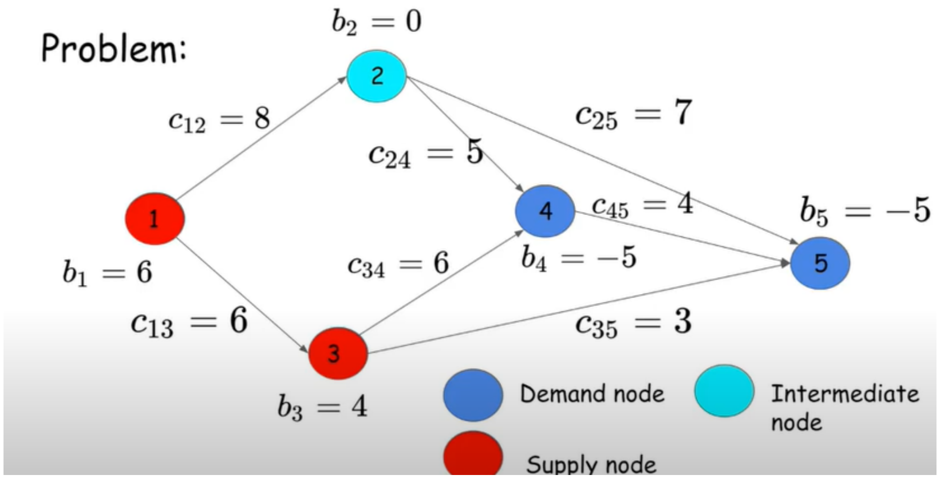

Figure 9.1: Problem 1

$$\sum_i \sum_j^{\prime} (w_i - w_j)\, x_{ij} = \sum_i^{1} w_i b_i$$

If we apply following constraint

$w_i - w_j \leq c_{ij}$

$$\sum_i w_i b_i = \sum_1 \sum_1^{\prime} (w_i - w_j)\, x_{ij} \leq \Sigma' c_{ij} x_{ij}$$

This implies:

w$_i - w_j \leq c_{ij}$

$w_i - w_j - c_{ij} \leq 0$

Now we are going to find the flow of network which satisfies the above condition and that will be the minimum cost flow network. For the given problem the minimum cost will be 81.

9.2 Maximum Flow

MOTIVATION:

Imagine a oil refinery at Mathura producing oil and it has a warehouse in Chennai. There are multiple path from the source (Mathura) to destination(Chennai) with each path having some capacity of fluid flow. Such graph are known as flow network.(fig.9.2)

PROBLEMS WHICH CAN BE MODELLED AS FLOW NETWORK:

1. Liquids through pipes.
2. Current through electrical networks.

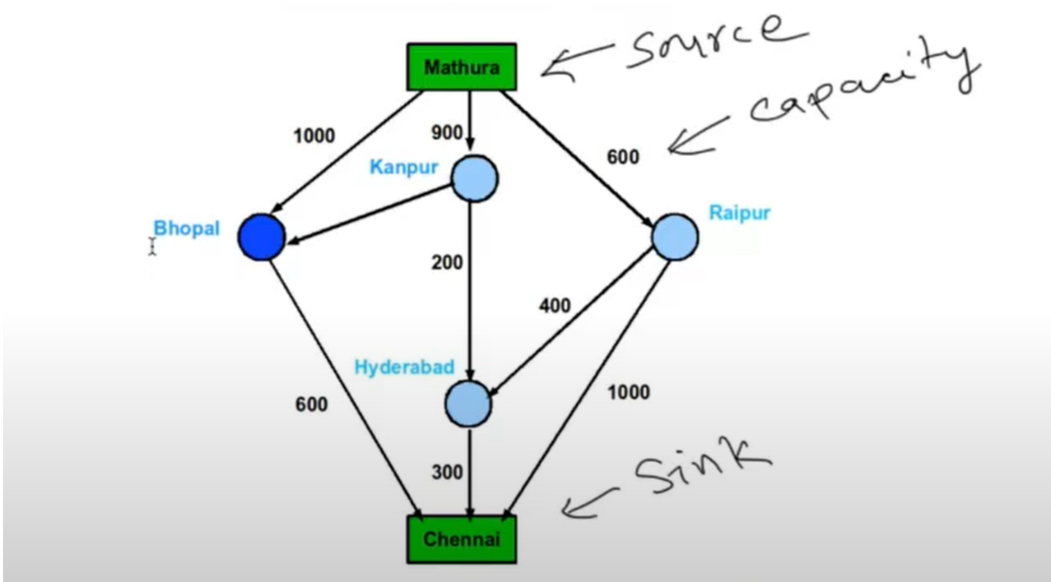

Figure 9.2: max flow

3. Information through communication networks.
4. Vehicles through roads.

In flow network

1. Each vertex other than source and sink is a conduit junction. they do not store any material.
2. Each edge can be though of a conduit for the material with a predefined capacity.

9.3 Flow Network

A flow network $G = (V, E)$ is a directed graph in which each edge $(u, v) \in E$ has a non-negative capacity $c(u, v) \geq 0$ In a flow network we distinguish two vertices source(s) and sink(t).

Note: A flow network is always a connected graph thus in any flow network

$$|E| \geq |V| - 1$$

9.4 FLOW

Let $G = (V, E)$ be a flow network with a capacity function c, let s be the source of the network and t be the sink. Then a flow in G is defined as a real valued function $f : V \times V \longrightarrow \mathcal{R}$

that satisfies following three properties:

1. Capacity constraint:
$$f(u,v) \le c(u,v); \quad \forall u,v \in V$$
2. Skew Symmetry:
$$f(u,v) = -f(v,u); \quad \forall u,v \in V$$
3. Flow Conservation:
$$\sum_{u\in V} f(u,v) = 0; \quad \forall u \in V - \{s,t\}$$

or equivalently
$$\sum_{v\in V} f(v,u) = 0; \quad \forall u \in V - \{s,t\}$$

9.5 Total Net Flow

1.Total positive flow entering a vertex v is defined by: $\sum_{u\in V, f(u,v)>0} f(u,v)$

2. Similarly, We can define total positive flow leaving a vertex v as: $\sum_{v\in V, f(v,u)>0} f(v,u)$

3. Total net flow of vertex v is defined as total positive flow leaving vertex v minus total positive flow entering that vertex.

9.6 Residual Network, Residual capacity and Augmenting Path

Residual Network: Given a flow network $G = (V,E)$ and a flow f the residual network of G induced by flow f is $G_f = (V, E_f)$ where
$$E_f = \{(u,v) \in V \times V : C_f(u,v) > 0\}$$

Residual Capacity: The amount of flow we can push from u to v before exceeding the capacity $c(u,v)$ is the residual capacity of $c(u,v)$.

Augmenting Path: Given a flow network $G = (V,E)$ and a flow f an augmenting path p is a simple from s to t in the residual network. Residual capacity of path: is the minimum residual capacity along the path.

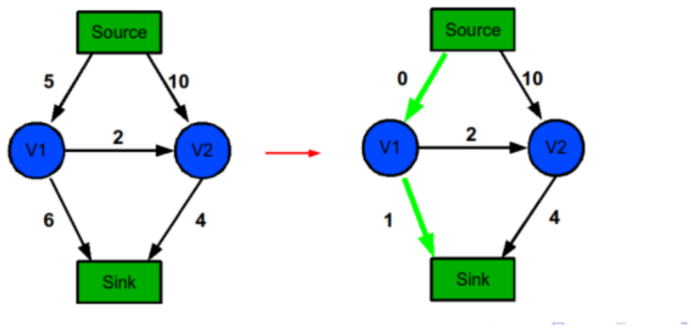

Figure 9.3: Residual Network

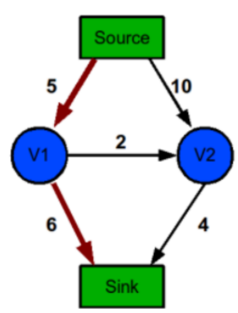

Figure 9.4: Residual capacity of Path : 5

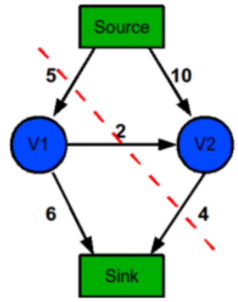

Figure 9.5: capacity of this cut is 9

9.7 Cut

A cut $C(S, T)$ of flow network $G = (V, E)$ is a partition of set of vertices V into two sets disjoint sets S and T. Capacity of a cut is the capacity of edges going from vertices belonging to S to vertices belonging to set T.(fig 9.5)

9.8 Max cut min flow theorem

If f is a flow in flow network $G = (V, E)$ with sources s and sink t, then the following conditions are equivalent:
(1) f is a max flow in G
(2) The residual network G_f contains no augmenting path
(3) There exist a cut $C(S, T)$ with capacity f.

FORD FLUKERSON ALGORITHM

Ford-Fulkerson algorithm is a greedy approach for calculating the maximum possible flow in a network or a graph.

TIME COMPLEXITY:

While finding augmenting path one has to traverse $O(|E|)$ each time. Thus if $max-flow = |f^*|$ then at worst case the complexity of the algorithm based on Ford-Flukerson: $O(|f^*||E|)$

Efficient Ford-Flukerson Method: Edmonds-Karp Algorithm:

Edmonds-Karp Algorithm finds the augmenting path with a breadth first search. and has a complexity of $O\left(|V||E|^2\right)$