Week 13

Planar Graphs*

In the lectures of week 13, the lecturer has taught the topic of Planar Graphs under the subject Graph Theory. First going by definition, Planar Graphs are defined as "A graph is said to be planar if it can be drawn in a plane so that no edge cross."

13.1 Some basics definitions of the topic

1. Region of a Graph (also known as face): Consider a planar graph G=(V,E).A region is defined to be an area of the plane that is bounded by edges and cannot be further subdivided. A planar graph divides the plans into one or more regions. One of these regions will be infinite.

2. Finite Region: If the area of the region is finite, then that region is called a finite region.

3. Infinite Region: If the area of the region is infinite, that region is called a infinite region. A planar graph has only one infinite region.

The properties of Planar Graph are:

a. They are sparse graph for large vertex graph.

b. They are 4-colourable.

c. They have efficient operation.

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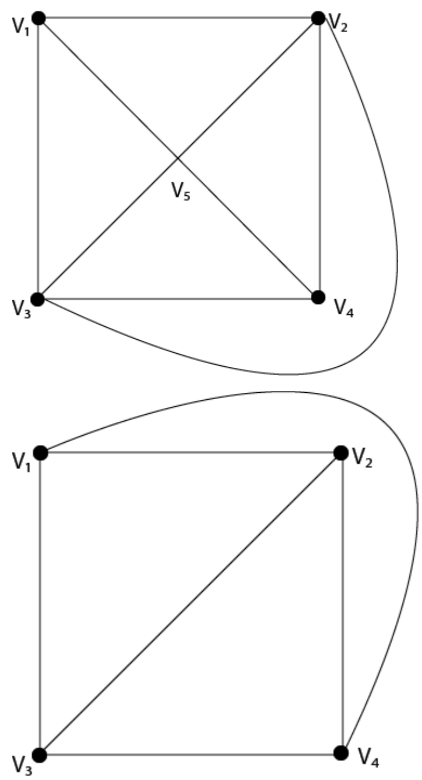
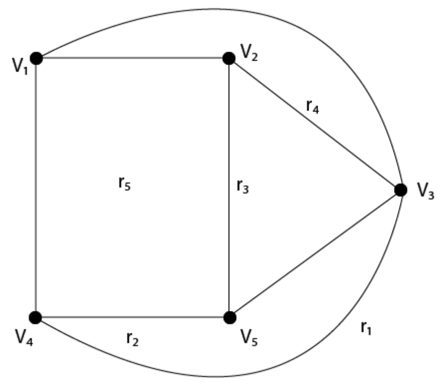


Figure 13.1

Example: Consider the graph shown in Fig. Determine the number of regions, finite regions and an infinite region.



Solution: There are five regions in the above graph, i.e. r_1, r_2, r_3, r_4, r_5.

There are four finite regions in the graph, i.e., r_2, r_3, r_4, r_5.

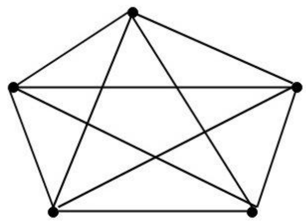
There is only one finite region, i.e., r_1

Figure 13.2

Non-Planar Graph:

A graph is said to be non planar if it cannot be drawn in a plane so that no edge cross.

Example: The graphs shown in fig are non planar graphs.



K_5

These graphs cannot be drawn in a plane so that no edges cross hence they are non-planar graphs.

Figure 13.3

13.2 Theorem

For any planar graph with v vertices, e edges, and f faces, we have $v-e+f=2$ The equation $v-e+f=2$ is called Euler's formula for planar graphs.

Proof: By induction (on the number of edges in the graph)

We can now prove Euler's formula ($v-e+f=2$) works in general, for any connected simple planar graph. Base: If e = 0, the graph consists of a single vertex with a single region surrounding it. So we have 1 - 0 + 1 = 2 which is clearly right. Induction: Suppose the formula works for all graphs with no more than n edges. Let G be a graph with n + 1 edges. Case 1: G doesn't contain a cycle. So G is a tree and we already know the formula works for trees. Case 2: G contains at least one cycle. Pick an edge p that's on a cycle. Remove p to create a new graph G'. Since the cycle separates the plane into two regions, the regions to either side of p must be distinct. When we remove the edge p, we merge these two regions. So G' has one fewer regions than G. Since G' has n edges, the formula works for G' by the induction hypothesis. That is v' - e' + f' = 2. But v' = v, e' = e - 1, and f' = f - 1. Substituting, we find that v - (e - 1) + (f - 1) = 2. So v - e + f = 2

13.3 Chords

A cycle has a chord if there are a pair of vertices that are adjacent, but not along the cycle.

13.4 Dual Graph

The dual graph G* of a planar graph G is also a planar graph whose vertices corresponds to the faces of G. The edges of G* corresponds to the edges of G as follows: if e is an edge where one side is face-x and another side is face-y then in G* there is an edge between vertex corresponding to face-x and face-y.

13.5 True/False

Every subgraph of a planar graph is also plannar. TRUE

Every subgraph of a non-plannar graph is a non plannar graph. FALSE

13.6 Length of a face

When a planar graph is drawn with no crossing edges, it divides the plane into a set of regions, called faces. Each face is bounded by a closed walk called the boundary of the face. By convention, we also count the unbounded area outside the whole graph as one face. The degree of the face is the length of its boundary.

Statement: If $l(f_i)$ denotes length of face f_i in a planar graph then

$2e = \sum_i l(f_i)$ In the dual graph

$l(f_i) = degree(f_i)$

$e = e$

Handshaking lemma in G*

$2e = \sum_i deg(f_i)$

which leads to the equation

$2e = \sum_i l(f_i)$

13.7 Statement

Suppose G is a connected planar graph with v nodes, e edges, f faces an $v \geq 3$ then;

$e \leq 3v - 6$

$v \geq 3 \Rightarrow$ In G*

$2e = \sum_{i=1}^{f} l(f_i)$

$\geq 3 + 3 + 3 + ...ftimes$

$= 3f$

$2e \geqslant 3f$

$v - e + f = 2$

$f = e - v + 2$

$3f \leq 2e$

$3f = 3e - 3v + 6$

$e \leq 3v - 6$

Hence Proved.

13.8 Planarity Test Algorithm

1. Remove all self loops.

2. Remove parallel edges.

3. Remove vertex having degree =2 and merge the edges incident on that vertex.

13.9 Statement

Every face has even length in G.

$\Rightarrow$ degree of each node in G* is even.

$\Rightarrow$ G* is Eulerian.

13.10 Statement

The following statements are equivalent for a planar graph:

(a) G is a bipartite graph.

(b) Every face of G has even length.

(c) The dual graph G* is Eulerian.

(a)$\Rightarrow$(b) G contains only even length cycle. We know that cycles make faces.

$\Rightarrow$ Every face of G has even length.

13.11 Theorem

Every simple planar graph has "a" vertex of degree at most 5.

Proof: We know that every simple planar graph with v vertices has at most 3v-6 edges, for $v \geq 3$

Hence the sum of egree is atmost 6v - 12.

There is surely a vertex whose degree is < 6

13.12 EXTRA: Kuratowski's Theorem

Claim 1: Kuratowski's Theorem: A graph is nonplanar if and only if it contains a subgraph that is a subdivision of $K_{3,3}$ or K_5.

It turns out that any non-planar graph must contain a subgraph closely related to one of these two graphs. Specifically, we'll say that a graph G is a subdivision of another graph F if the two graphs are isomorphic or if the only difference between the graphs is that G divides up some of F's edges by adding extra degree 2 vertices in the middle of the edges.

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