Graph Theory Week 3 - Lecture 5 notes scribing
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Walk and Cycle

Walk can be a set of connected or repeating edges. It’s has a limited length which is equal to the number of edges covered.

Types of walk:

Closed Walk – starting from a point and reaching back to the same point.

Open walk – can start from one point and reach a different point.

1. To prove every closed odd walk contain an odd cycle.

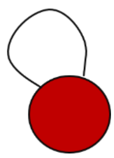
Proved using induction

For l=1, obviously true.

If we consider it is true for l < L, then we have to show that this is also true for L.

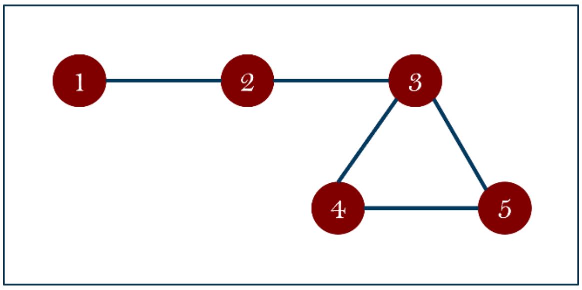
Case1: If there is no repetition of vertex in walk, then a closed walk = a

closed cycle.



If L = 1 its true. Hence proved.

Case2: If there is a repeating vertex in the walk, and let's say v is the repeating vertex. Dividing the walk into two v-v walks is a good idea (say w1 and w2).



Odd close walk w: 1 2 3 4 5 3 2 1
w1 = 1 2 3 3 2 1
w2 = 3 5 4 3
| w1 | + | w2 | = | w | - should be odd
For this either one of | w1 | or | w2 | is odd

Since | w1 | + | w2 | = odd, that means either w1 or w2 is odd walk. And surely, they both are less than L. From the induction one of them (odd walk one) contain odd cycle.

If | w2 | is odd | w2 | is odd closed walk and | w2 | < | w | = L (:: assuming that length of w is L)

Using Induction hypothesis we can say | w2 | contains odd cycle.

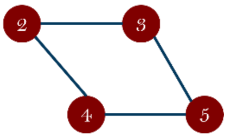
This implies | w | contains an odd cycle

Induction Hypothesis - For any walk of size less than L, the following is true if:
Every closed odd walk contains an odd cycle.

2. A closed even walk need not contain a cycle.

A simple path is a bipartite graph

C_n is bipartite if n is even.



Bipartite Graph

Each edge connects a vertex in U to a vertex in V in a bipartite graph, which includes vertices that can be divided into two different and independent sets U and V.

Union of graphs

The union of graphs G_1, G_2, G_3 ….. G_k written as $\bigcup_{i=1}^{k} Gi$ is the graph with vertex set $\bigcup_{i=1}^{k} E(Gi)$ and edge set $\bigcup_{i=1}^{k} V(Gi)$.

Theorem: A graph is bipartite if it does not have an odd cycle.

Condition: Assume that the graph is bipartite and that X and Y are two independent sets. To complete a cycle, one must travel from X to Y to X OR Y to X to Y at least once. As a result, an odd cycle cannot exist in a bipartite graph.

Sufficient Condition: If G doesn't have an odd cycle, it either doesn't have one or has one.

If there is no cycle: take one vertex in X and the next vertex in Y.

It includes an even cycle => Partition the graph so that each cycle of even length is a separate subgraph.

For even length cycles, we know Cn is bipartite.

If G has no odd cycle => it does not contain a cycle OR it contain even cycle.

It does not contain a cycle => take one vertex in X and next vertex in Y. It includes an even cycle => Partition the graph so that each cycle of even length is a separate subgraph.

For even length cycles, we know Cn is bipartite.

Eulerian Graph

If a graph has a closed path that contains all of the edges, it is said to be Eulerian.

Eulerian Circuit

When we don't define the initial vertex but keep the list in cyclic order, we call it a circuit.

In a graph, an Eulerian circuit is a circuit that contains all of the edges.

Lemma: If every vertex of a graph has degree at least 2, then G has cycle.

Modified version: If every vertex of a graph has degree atleast 2 then G is a cycle.

Proof: Using contrapositive method

$P \rightarrow Q \equiv \sim Q \Rightarrow \sim P$

P ≡ Every vertex of graph has degree >= 2

Q ≡ G has a cycle

~ P: ∃ a vertex of a graph G has degree < 2

~ Q: G has no cycle

G = Collection of trees (Forest)

Let T be complement of G

Case 1: T is trivial graph.

- ⇨ ∃ vertex that has degree = 1

Case 2: T is non trivial

- ⇨ T must have at least 2 end vertices (Property of tree)
- ⇨ End vertices have degree = 1 (two vertices)
- ⇨ ∃ a vertex that has degree = 1 < 2

~ Q => ~ P => P => Q

Theorem: A graph G is Eulerian iff it has at most one non-trivial component and all its vertices have even degree.

Proof:

Part 1: Assume that graph G is Eulerian

- ⇨ G has closed path containing all the edges (closed path no repetition of edges)
- ⇨ There will be incoming and outgoing edges for all vertices.
- ⇨ Every node has even degree

Since, G is a closed path it cannot have more than one non-trivial components.

Part 2: Suppose G has at most one non- trivial component and each node of G has even degree.

If no of edges (m) = 0 then G is Eulerian.

Assume this is true for all graphs having above properties and less than m edges. (Using Induction Hypothesis)

Consider a graph G`, containing one non-trivial component and having m edges and each node has degree = even.

- ⇨ Each vertex of G` has atleast 2 degree
- ⇨ G` contains cycle (Lemma)

Say this cycle is C. Remove all edges corresponding to C i.e. E(C) from G` to construct G``

1. G`` has less than m edges
2. Each vertex of G`` has even degree
3. G` can have multiple components and each of the multiple possible component will have above 2 properties.

- ⇨ Therefore, All of them are eulerian.

Lecture 6

Vertex Degrees and Counting

Degree of a vertex: The number of edges incident to a vertex v in a graph G is expressed as d(v), except that each loop at v counts twice. The maximum and minimum number of degrees are represented by $\Delta(G)$ and δ(G).

Max and min degree for a k regular graph will be k.

- ⇨ Order and size of graph:

The number of vertices in a graph G, denoted by n(G).
The number of edges in a graph G, denoted by e(G).

⇨ Order and size of complete graph K_n.

Order = n
Size = n (n-2)

⇨ Order and size of complete bipartite graph K_{mxn}.

Order = m + n
Size = m x n

Handshaking Lemma

If g is a graph

$\sum_{v \in V(G)} d(v)$ = 2 X e(G)

If e(G) = 1 , $\sum d(v)$ = 1+1 = 2

Proof: We will use Induction Hypothesis

Assume e(G) = k then

$\sum_{v \in V(G)} d(v)$ = 2 X k

We have to prove that if e(G) = k+1 then

$\sum_{v \in V(G)} d(v) = 2\ X\ (k+1)$

Form G we will construct a graph G`

Case1: No increase in vertex.

$\sum_{v \in V(G)} d(v) = 2\ X\ k$

$\sum_{v \in V(G`)} d(v)$ = $\sum_{v \in V(G)} d(v)$ + 2

= 2k + 2
= 2(k+1)

Case 2: Increase in vertex
$\sum_{v \in V(G`)} d(v) = \sum_{v \in V(G)} d(v)$ + 2
= 2k + 2
= 2(k+1)

Hence, we could see the theorem is proved used using induction.

Lemma: A graph cannot have exactly one node with odd degree.

$\sum d(v)$ = (k-1) x even + 1 x odd
= even + odd
= odd
!= 2 x edges