Duality : Let us introduce the concept of duality using an example :

Suppose there is a dealer who sells two vitamins V_1 and V_2. The two vitamins are also available in foods f_1 and f_2 and their distribution and costs are as per the table below

	f_1	f_2	Daily Requirement
V_1	5	12	60
V_2	7	9	80
Per unit cost :	8	11	

If a consumer wishes to complete his/her daily requirements through foods f_1 and f_2 then the corresponding LPP to minimize the expenditure of the consumer is

$$\min \quad 8x_1 + 11x_2$$

$$\text{s.t.} \quad 5x_1 + 12x_2 \geq 60$$

$$7x_1 + 9x_2 \geq 80$$

$$x_1, x_2 \geq 0$$

where x_1 and x_2 are number of units of f_1 and f_2 purchased by the cusumer.

However, the vitamins are available with the dealer. To maximize the dealer's sale, the corresponding LPP comes out to be

$$\max \quad 60y_1 + 80y_2$$

$$\text{s.t.} \quad 5y_1 + 7y_2 \leq 8$$

$$12y_1 + 9y_2 \leq 11$$

$$y_1, y_2 \geq 0$$

where y_1 and y_2 are prices of vitamins V_1 and V_2 respectively.

The above two LPPs constructed represent the same problem but with a different perspective (and is reffered as dual of the other). While the original problem is reffered as primal, the second one in general is refferred as the dual

If primal is the problem of the form

$$\max \quad c^T x$$

$$\text{s.t.} \quad AX \leq b$$

$$X \geq 0$$

Then the dual is given as

$$\min \quad b^T w$$

$$\text{s.t} \quad A^T w \geq c$$

$$w \geq 0$$

One may observe that the dual of dual is primal itself. Hence duality is a symmetric relation

To write the dual of

$$\begin{aligned} &\min\ b^T w \\ &\text{s.t. } A^T w \geq c \\ &\qquad w \geq 0 \end{aligned}$$

Represent the problem as

$$\begin{aligned} &\max\ -b^T w \\ &\text{s.t. } (-A^T) w \leq -c \\ &\qquad w \geq 0 \end{aligned}$$

and thus the dual of dual is

$$\begin{aligned} &\min\ -c^T y \\ &\text{s.t. } (-A^T)^T y \geq -b \\ &\qquad y \geq 0 \end{aligned}$$

which is same as

$$\begin{aligned} &\max\ c^T y \\ &\text{s.t. } Ay \leq b \\ &\qquad y \geq 0 \end{aligned}$$

which is same as the primal and hence dual of dual is primal.

In general, the dual of an LPP can be computed using following table

Primal	Dual
① is of maximization type	① is of minimization type
② has i^{th} variable "≥ 0"	② has i^{th} equation of "$\geq$" type
③ has i^{th} variable "≤ 0"	③ has i^{th} equation of "$\leq$" type
④ has i^{th} variable unrestricted	④ has i^{th} equation of "$=$" type
⑤ has i^{th} equation of "$\geq$" type	⑤ has i^{th} variable "≤ 0"
⑥ has i^{th} equation of "$\leq$" type	⑥ has i^{th} variable "≥ 0"
⑦ has i^{th} equation of "$=$" type	⑦ has i^{th} variable unrestricted
⑧ Coefficient matrix is A	⑧ Coefficient matrix is A^T

It is worth noting that while nature of equations in primal affects the nature of variables in the dual, nature of equations in dual is determined by the nature of variables in the primal (and relation is given by table above)

Thus if primal is given as

$$\max \; x_1 - 2x_2 + 3x_3 + 2x_4$$

$$\text{s.t.} \quad x_1 + x_2 \geq -1$$

$$x_1 - 3x_2 - x_3 \leq 7$$

$$x_1 + x_3 - 3x_4 = -2$$

$$x_1, x_4 \geq 0, \; x_2, x_3 \text{ unrestricted.}$$

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Then the dual is given by

$$\min \; -w_1 + 7w_2 - 2w_3$$

$$\text{s.t.} \quad \begin{aligned} w_1 + w_2 + w_3 &\geq 1 \\ w_1 - 3w_2 &= -2 \\ -w_2 + w_3 &= 3 \\ -3w_3 &\geq 2 \end{aligned}$$

$w_1 \leq 0$, $w_2 \geq 0$, w_3 is unrestricted.

One may try writing the duals of following LPPs

① $\max \; 5x_1 + 12x_2 + x_3$

$$\text{s.t.} \quad \begin{aligned} x_1 + 2x_2 + x_3 &\leq 5 \\ 2x_1 - x_2 + 3x_3 &= 3 \end{aligned}$$

$x_1, x_2 \geq 0$, x_3 unrestricted

② $\max \; 3x_1 - 2x_2 + 7x_3$

$$\text{st} \quad \begin{aligned} x_1 + x_2 - x_3 &\geq 5 \\ 3x_1 - x_2 + 2x_3 &= 12 \\ 8x_1 + 2x_2 + 5x_3 &\leq 8 \end{aligned}$$

$x_1 \geq 0$, $x_2 \leq 0$, x_3 unrestricted.

From now on, for sake of convenience, our primal shall refer to a problem of maximization type.

Some observations

① If Ⓟ denotes the primal (max. type), Ⓓ denotes the dual and x_0 and w_0 are feasible for Ⓟ & Ⓓ respectively then,

$$c^T x_0 \leq b^T w_0.$$

[Follows from the fact that if $A x_0 \leq b$ and $A^T w_0 \geq c$ then,

$$c^T x_0 \leq (w_0^T A) x_0 = w_0^T (A x_0) \leq w_0^T b$$ and hence result follows

② Primal has an optimal solution if and only if dual has an optimal solution

③ If Primal has unbounded solution then dual is infeasible (and if primal is infeasible then dual has an unbounded solution).

④ If x_0 and w_0 are optimal for Ⓟ & Ⓓ respectively then $c^T x_0 = b^T w_0$

⑤ If x_0 and w_0 are feasible for Ⓟ & Ⓓ respectively and $c^T x_0 = b^T w_0$ then x_0 and w_0 are optimal for their respective problems

⑥ If x_0 and w_0 are feasible for Ⓟ & Ⓓ respectively, then x_0 and w_0 are optimal for their respective problems if and only if

$$w_0^T (b - A x_0) = 0 \quad \text{and} \quad x_0^T (A^T w_0 - c) = 0$$

As already observed, the primal has an optimal solution if and only if the dual has an optimal solution. Infact, the optimal solution to the dual can be determined from the optimal table of the primal. To be precise z_j (and not $z_j - c_j$) of the basic variables of the first simplex table is the optimal solution to the dual, i.e. z_j's (in the final table) of the basic variables of the first simplex table is the optimal soln of the dual.

For example, while solving the LPP

$$\max \; 3x_1 + 2x_2$$
$$\text{s.t. } x_1 + x_2 \leq 4$$
$$2x_1 + x_2 \leq 6$$
$$x_1, x_2 \geq 0.$$

the optimal table computed was

			3	2	0	0
C_B	B	b	a_1	a_2	a_3	a_4
2	x_2	2	0	1	2	−1
3	x_1	2	1	0	−1	1
	$z_j - c_j$:		0	0	1	1

and thus the optimal solution to the dual of the above problem is

$$(w_1^* = 1,\; w_2^* = 1) \quad [\text{i.e. } z_j\text{'s of } x_3 \;\&\; x_4]$$

Consider the LPP

$$\max\ 3x_1+2x_2$$
$$\text{s.t. } x_1+x_2 \geq 1$$
$$x_1+x_2 \leq 7$$
$$x_1+2x_2 \leq 10$$
$$x_2 \leq 3$$
$$x_1, x_2 \geq 0$$

The dual of the above LPP is

$$\min\ -w_1+7w_2+10w_3+3w_4$$
$$\text{s.t. } -w_1+w_2+w_3 \geq 3$$
$$-w_1+w_2+2w_3+w_4 \geq 2$$
$$w_1, w_2, w_3, w_4 \geq 0$$

The above problem (after introducing artificial variables) can be written as

$$\max\ w_1-7w_2-10w_3-3w_4-Mw_7-Mw_8$$
$$\text{s.t. } -w_1+w_2+w_3-w_5+w_7=3$$
$$-w_1+w_2+2w_3+w_4-w_6+w_8=2$$
$$w_i \geq 0 \ \forall i$$

First simplex table

			1	−7	−10	−3	0	0	−M	−M
C_B	B	b	a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8
−M	w_7	3	−1	1	1	0	−1	0	1	0
← −M	w_8	2	−1	1	2	1	0	−1	0	1
			2M−1	−2M+7	−3M+10	−M+3	M	M	0	0
					↑					

			1	−7	−10	−3	0	0	−M	−M
C_B	B	b	a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8
−M	w_7	2	−1/2	1/2	0	−1/2	−1	1/2	1	−1/2
← −10	w_3	1	−1/2	1/2	1	1/2	0	−1/2	0	1/2
			$\frac{M}{2}+4$	$-M/2+2$	0	$M/2-2$	M	$-\frac{M}{2}+5$	0	$\frac{3M}{2}-5$
				↑						

			1	−7	−10	−3	0	0	−M	−M
C_B	B	b	a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8
← −M	w_7	1	0	0	−1	−1	−1	1	1	−1
−7	w_2	2	−1	1	2	1	0	−1	0	1
			6	0	M−4	M−4	M	−M+7	0	2M−7
								↑		

			1	−7	−10	−3	0	0	−M	−M
C_B	B	b	a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8
0	w_6	1	0	0	−1	−1	−1	1	1	−1
−7	w_2	3	−1	1	1	0	−1	0	1	0
			6	0	3	0	7	3	M−7	M

$z_j - c_j \geq 0 \ \forall j \Rightarrow$ above table corresponds to optimal soln of the dual.

Optimal Solution is $w_1 = 0,\ w_2 = 3,\ w_3 = 0,\ w_4 = 0$ & optimal value $= 21$

A z_j for max. problem (and thus $-z_j$ for min problem) provides optimal solution of the dual of the problem solved, the optimal solution to the given LPP is $x_1 = 7,\ x_2 = 0$. (& optimal value is 21).

Dual Simplex method

Dual simplex method is used to solve a LPP while allowing the coloum "b" of simplex table to have negative entries. While simplex method keeps "b" non-negative and tries to make $z_j - c_j$ non-negative ($\geq 0\ \forall j$), dual simplex method keeps $z_j - c_j$ non-negative ($\forall j$) and tries to make "b" non-negative. Note that if both "b" and "$z_j - c_j$" ($\forall j$) are non-negative then the table corresponds to optimal solution of the original problem. Although the method has same time complexity as simplex method; the method has applications to solve specific problems (like integer programming).

The detailed algorithm for Dual Simplex algorithm is given below:

1. Represent the problem as a maximization problem
2. Write all eqns as "$\leq$" type, introduce slack variables and compute the first simplex table
3. If some $z_j - c_j$ is negative, then the method is not applicable
4. If all $z_j - c_j \geq 0$ and coloum b is non-negative then the table corresponds to optimal solution to given LPP.
5. If some b_i's are negative then update the simplex table using the following strategy:

(a) Choose most negative b_i, x_{B_i} leaves.

(b) Compute the ratios $\left\{ \dfrac{z_j - c_j}{a_{ij}} : a_{ij} < 0 \right\}$ i.e. ratios of $z_j - c_j$ with negative entries of ~~exiting~~ i^{th} row. If $\dfrac{z_{j_0} - c_{j_0}}{a_{ij_0}}$ is largest (i.e. least modulus) then x_{j_0} enters the simplex table.

⑥ Update the simplex table untill the coloum "b" is non-negative.

Example :-

$$\min \; 3x_1 + x_2$$
$$\text{s.t.} \quad x_1 + x_2 \geq 1$$
$$2x_1 + 3x_2 \geq 2$$
$$x_1, x_2 \geq 0$$

Writing the problem in desired form, the problem ~~becomes~~ can be written as,

$$\max \; -3x_1 - x_2$$
$$\text{s.t.} \quad -x_1 - x_2 + x_3 \leq -1$$
$$-2x_1 - 3x_2 \quad + x_4 \leq -2$$
$$x_1, x_2, x_3, x_4 \geq 0$$

The first simplex table is

			-3	-1	0	0
C_B	B	b	a_1	a_2	a_3	a_4
0	x_3	-1	-1	-1	1	0
0	x_4	-2	-2	-3	0	1
$z_j - c_j$:			3	1	0	0

$z_j - c_j \geq 0 \ \forall j \Rightarrow$ Dual simplex method is applicable

b_2 is most negative $\Rightarrow x_4$ leaves the simplex table

$\dfrac{z_2 - c_2}{a_{22}}$ is largest (least modulus) & thus x_2 enters. Thus updated simplex table is:

			-3	-1	0	0
C_B	B	b	a_1	a_2	a_3	a_4
0	x_3	$-1/3$	$-1/3$	0	1	$-1/3$
-1	x_2	$2/3$	$2/3$	1	0	$-1/3$
$z_j - c_j$:			$7/3$	0	0	$1/3$

b_1 is most negative $\Rightarrow x_3$ leaves. Further $\dfrac{z_4 - c_4}{a_{14}}$ is largest $\Rightarrow x_4$ enters

Thus updated table is

			-3	-1	0	0
C_B	B	b	a_1	a_2	a_3	a_4
0	x_4	1	1	0	-3	1
-1	x_2	1	1	1	-1	0
$z_j - c_j$:			2	0	1	0