Chapter 7

Hall's Theorem, Vertex Cover*

First, we will talk about the solution of quiz 2 in this course.

7.1 Hall's Marriage Theorem

Theorem 7.1. *An $X - Y$ bipartite graph G has a matching that saturates X if and only if*

$$|N(S)| \geq |S| \quad \forall S \subseteq X$$

Here, $|N(S)| \geq |S|$ is set of neighbours of elements in S.

This theorem was proved by Philip Hall(1935). First we see an example to understand this theorem better. In figure 7.1 four students Ram, Sita, Rai and Mohan got placed in for

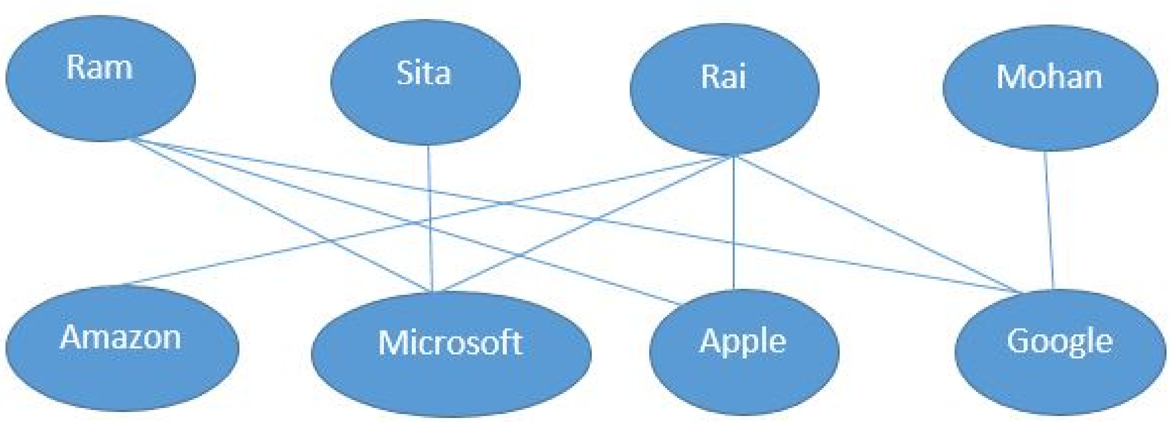

Figure 7.1: Example for Hall's Marriage Theorem

different companies as shown in the graph. Now, if every company has one vacant position then will every one get a job?

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The answer is yes because there will be a perfect matching. Now we will generalize this concept by proving Hall's marriage theorem.

Proof of Hall's marriage theorem : Suppose $X - Y$ bi partite graph haas aa matching that saturates X. Then the following is obvious,

$$|S| \leq |N(S)| \quad \forall S \subseteq X$$

Now if $|S| \leq |N(S)| \quad \forall S \subseteq X$ then as we are working in a bipartite graph then we will match each element of $S \subseteq X$ to exactly one element in $N(S) \subseteq Y$. This way we will get a complete matching that saturates X. So, the condition is sufficient.

Now, we will prove the necessary part by proving contrapositive statement. So, we will prove the contrapositive statement,

If there is no such matching M that saturates X, then $\exists S \subseteq X$ such that $|S| \leq |N(S)|$ Let, $u \in X$ be a vertex unsaturated by matching M. We can visualize this by the below diagram, Suppose two subsets $S \subseteq X$ and $T \subseteq Y$ are considered as follows,

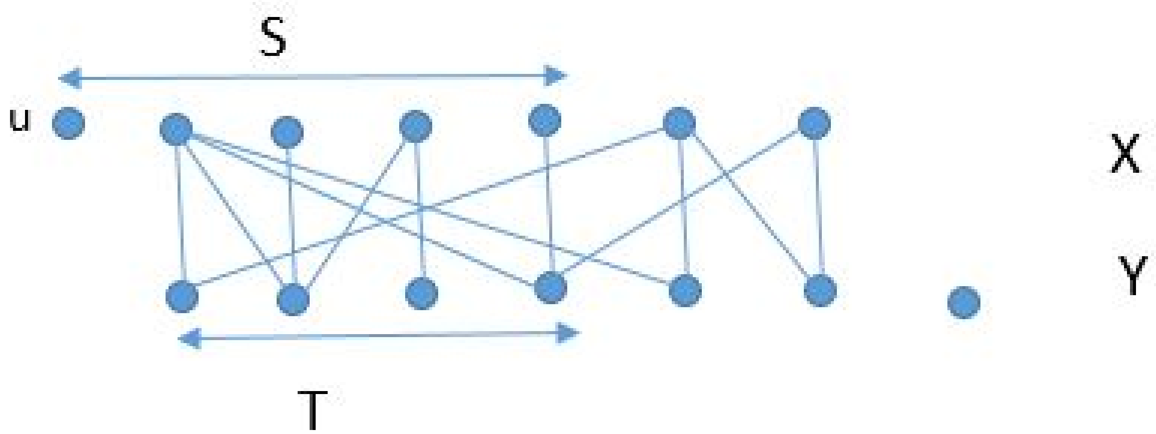

Figure 7.2: Example

S := Ens points of M alternating paths starting from u with the last edge belonging to M.

T := End points of M alternating paths starting from u with the last edge not belonging to M.

We claim that M matches T with $S - \{u\}$. Every vertex of $S - \{u\}$ is reached by an edge in M and which comes from one vertex in T.Since, these dose not exists any M augmenting path then every vertex of T is saturated. Hence there is a bijection from T to $S - \{u\}$.

With $T = N(S)$ clearly,

$|S| = 1 + |T| = 1 + |N(S)|$

$\Rightarrow |S| > |N(S)|$ for this choice of S.

hence, this proves the contrapositive statement. $\square$

Now using an example we will see constrains of the above theorem. Suppose there are 1000 students and 2000 companies. And a graph is given denoting their placement offers denoting by edges. Now to check that if everyone get a job we have to check 2^{1000} times. Which is very time consuming even using computers.

Theorem 7.2 (Degree constrained mapping)**.** *Let, G be a $X - Y$ bipartite graph. If $\forall x \in X$, $deg(x) \leq d$ and $\forall y \in Y$, $deg(y) \geq d$ for some positive integer d then there will be a matching tht saturates X.*

7.2 Vertex cover and edge cover

We will now see definition of vertex cover and edge cover.

Definition 7.3 (Vertex cover)**.** A vertex cover of a graph G is a set $\theta \subseteq V(G)$ that contains at least one end point of every edge in the graph.

Definition 7.4 (Edge Cover)**.** A edge cover of a graph is a set of edges such that every vertex of the graph is incident to at least one edge of the set.

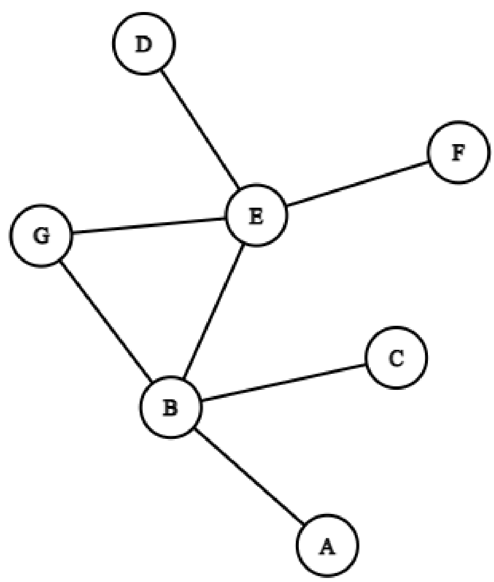

Figure 7.3: Vertex Cover

In figure 7.3 above observe,

- $\theta_1 = \{B, E\}$ is the smallest vertex cover.
- $\theta_2 = \{A, B, C, D, E, F, G\}$ is the trivial vertex cover for all graphs.

- $\theta_3 = \{A, C, D, G, F\}$ is also a vertx cover.

and we also see,

- $\{BE\}$ is the smallest edge cover here.
- $\{EF,\ BC\}$ is also an edge cover.

We also observe in the above graph that $\{DE, GB\}$, $\{E, B\}$ are both maximal matching also $\{D, F, A\}$ an independent set.

Now, we will introduce some notations for better understanding,

- $\alpha(G)$:= maximum size of independent set
- $\alpha'(G)$:= maximum size of matching
- $\beta(G)$:= minimum size of vertex cover
- $\beta'(G)$:= minimum size of edge cover

7.3 Review of previous theorem on diameter of graph

We will discuss proof of a previous theorem discussed previously again.

Theorem 7.5. *If G is a simple graph then if $diam(G) \geq 3$ then $diam(G^c) \leq 3$.*

Now we will see a graph and its complement to understand this statement. Seeing their

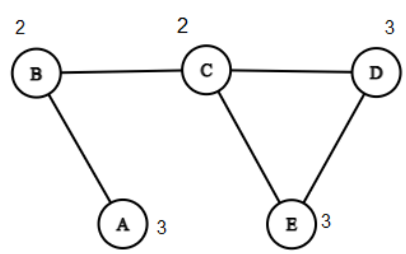

Figure 7.4: G

eccentricities we can verify this above theorem easily.

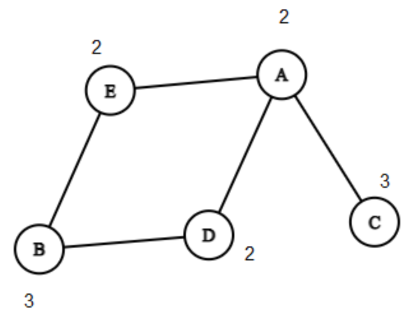

Figure 7.5: G^c

Proof of the above theorem : Given, $diam(G) \geq 3$
$\Rightarrow \exists\, u$ and $v \in V(G)$ such that;

1. $uv \notin E(G)$
2. u and v do not have common neighbour.

$\exists x \in V(G) - \{u, v\}$ and $y \in V(G) - \{u, v\}$ such that $uv \in E(G)$ and $vy \in E(G)$

Also, $\forall x \in V(G) - \{u, v\}$ has at least one of $\{u, v\}$ is non-neighbour. Now, we try to visualize like below,

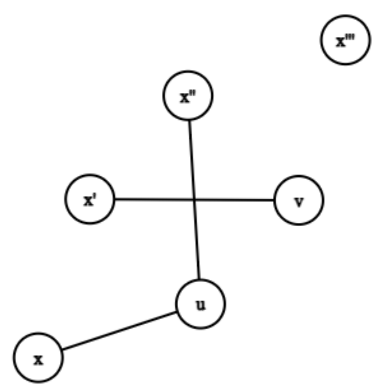

Figure 7.6

$uv \notin E(G)$

$\Rightarrow uv \in E(G^c)$
$\Rightarrow ux''' \in E(G^c)$, $vx''' \in E(G^c)$
This implies that $diam(G^c) \leq 3$. $\square$

Now, we will see another important theorem.

Theorem 7.6. *Let, G be a group. Then, $\alpha(G) + \beta(G) = n(G)$.*

Proof. Let, S be an independent set. Then every edge is incident to at least one vertex of $\overline{S}$.

$$S = \{A,\ B,\ C\}$$
$$\overline{S} = \{D,\ E\}$$
$$S \cup \overline{S} = V(G)$$

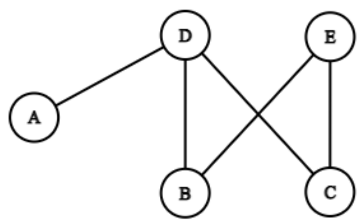

Figure 7.7: G

So, $\overline{S}$ cover all the edges. Hence, $\overline{S}$ is minimum size vertex cover.

$$\beta(G) = |\overline{S}|$$

S is the max size independent set. So,

$$\alpha(G) = |S|$$

So, $\alpha(G)\ +\ \beta(G)\ =\ |S| + |\overline{S}|\ =\ |V(G)|\ =\ n(G)$

$\square$

Theorem 7.7. *Let, G be a group. Then, $\alpha'(G) + \beta'(G) = n(G)$.*

Theorem 7.8. *If G is a bipartite graph with no isolated vertices then $\alpha(G) = \beta'(G)$.*

Proof. we already know,

$$\alpha(G) + \beta(G) = n(G)$$

$$\alpha'(G) + \beta'(G) = n(G)$$

We also know that, $\alpha'(G) = \beta(G)$

From the above three we get, $\alpha(G) + \beta(G) = \alpha'(G) + \beta'(G) = \beta(G) + \beta'(G)$
$\Rightarrow \alpha(G) = \beta'(G)$ $\square$

Proposition 7.9. *Let G be a bipartite graph. Prove that $\alpha(G) = \frac{n(G)}{2}$ if and only if G has perfect matching.*

Proof.

$$\alpha(G) + \beta(G) = n(G)$$

$$\Rightarrow \alpha(G) = n(G) - \beta(G)$$

$$\Rightarrow \alpha(G) = n(G) - \alpha'(G)$$

If G has a perfect matching then the maximum size of matching is $= n(G) - \frac{n(G)}{2} = \frac{n(G)}{2}$. $\square$