Week 7

Graph Theory*

7.1 Hall's Theorem

Hall's theorem or Hall's Marriage theorem is defined for a bipartite graph. It shows the necessary condition required for a matching to exists such that the set X is saturated.

7.1.1 Matching in a graph

A matching graph is a set of edges of a graph where there are no edges adjacent to each other. In other words, there should not be any common vertex between any two edges.

Consider the following graph G_0

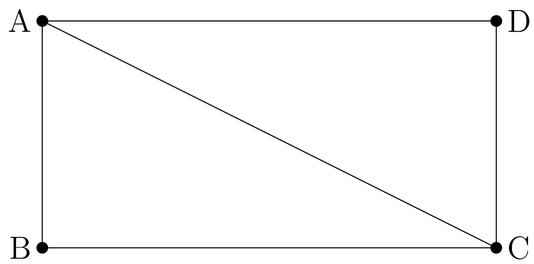

Figure 7.1: Graph G_0

There are multiple ways of choosing a set of edges such that no edge is adjacent to each other. Thus, the possible matchings of the graph can be any of the following:

- ϕ (trivial matching)
- $\{(A, B)\}$
- $\{(C, D)\}$
- $\{(A, C)\}$
- $\{(A, B), (C, D)\}$ (maximum matching)

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The matching can be of the following types:

- **Maximal matching**: A matching in which no more edges of G can be added to it.
- **Maximum matching**: A matching with the maximum possible number of edges.
- **Perfect matching**: A matching in which every vertex of the graph is incident to exactly one edge in the matching.

7.1.2 Theorem statement

The Hall theorem can be described as follows:

Let X, Y be the disjoint sets of vertices of the Graph $G_{m,n}$, where $|X| \leq |Y|$.

Let $S \subseteq X$ be a group of vertices of X, then the set $N(S) \subseteq Y$ is the set of vertices that are adjacent to any of the vertices in S. In other words, $N(S)$ is the neighbourhood of S.

Then, it is possible to have a matching that saturates X if and only if

$$|N(S)| \leq |S| \qquad \forall S \subseteq X \tag{7.1}$$

The statement (7.1) is also known as the Hall's condition.

7.1.3 Applications

This theorem can be used to check if a matching is possible such that all elements of a disjoint set of a graph is saturated.

Example scenario: Given a placement scenario, where 20 companies are to select among 20 candidates, is it possible that every candidate is selected for some company?

The solution to this problem depends on how the favoured candidates for each company. As long as the Hall's condition is satisfied for all subsets of candidates, a perfect matching will exist. To ensure this, all possible subsets must be checked. If any such subset is found that does not satisfy the Hall's condition, it can be concluded that the matching is not possible.

It is to be noted while that such a problem becomes exponentially difficult to compute as the number of elements of the set is increased.

7.1.4 Proof

Since the statement includes 'if and only if', it implies that the proof should cover the following for a XY Graph G

- If the Hall's condition is satisfied by G, there exists a matching such that X is saturated
- If there exists a matching such that X is saturated, then's the Hall's condition is satisfied by G.

It is easy to prove the latter, since if such a matching indeed exists, then all the elements in X have been uniquely matched to elements in Y. This must imply that for all possible $S \subseteq X$, every element in S must have an edge to at least one element in Y. Therefore,

$$\forall S \subseteq X, \qquad |S| \leq |N(S)| \tag{7.2}$$

Proving the former can be done using proof of the contrapositive, which would be as follows:

Contrapositive statement: If there does not exists a matching M in a XY Graph such that X is saturated, then the hall's condition is not satisfied, that is,

$$\exists S \subseteq X, \qquad |S| > |N(S)| \tag{7.3}$$

Now we can also make use Berge's theorem to complete the proof as follows

Consider a maximum matching M in a bipartite XY graph. According to the statement, this matching does not saturate X. Say it leaves a vertex $u \in X$ of the graph unsaturated by the matching.

Now, Consider two sets $P \subseteq X$ and $Q \subseteq Y$ considered as follows:

- P is the set of end points of M-alternating paths starting from u with the last edge belonging to M
- Q is the set of end points of M-alternating paths starting from u with the last edge not belonging to M

Given the way these sets are constructed, P will have at least one vertex more than Q, since P contains u, which does not have a matching, and all the elements in $P \cup Q$ are included in M except for the vertex u. Therefore,

$$|P| = 1 + |Q| \implies |P| > |Q| \implies |P| > |N(Q)| \tag{7.4}$$

Thus, there exists a set of vertices that does not satisfy Hall's condition. Hence we have proven the statement "If there does not exists a matching M in a XY Graph such that X is saturated, then the hall's condition is not satisfied".

7.2 Independent Sets and Covers

7.2.1 Vertex Cover

A vertex cover V_c of a graph G is a set $Q \subset V(G)$ such that it contains at least one end point of every edge.

Consider the following graph G_1:

The possible vertex covers are

- $\{B, E\}$ (Minimum sized vertex cover)
- $\{A, C, E\}$

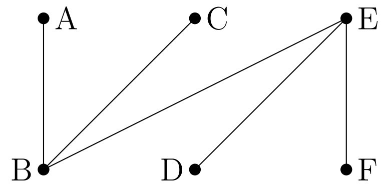

Figure 7.2: Graph G_1

- $\{A, C, D, F\}$
- $\{A, B, C, D, E, F\} = V(G_1)$ (Trivial vertex cover)

Taking another example, consider the following graph G_2:

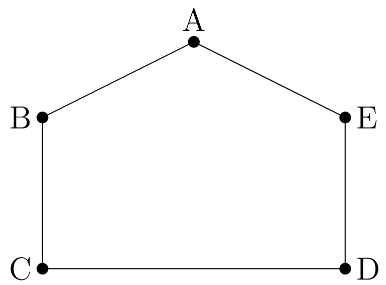

Figure 7.3: Graph G_2

The possible vertex covers are

- $\{A, C, E\}$ (Minimum sized vertex cover)
- $\{B, D, E\}$ (Minimum sized vertex cover)
- $\{A, B, C, D, E\} = V(G_2)$ (Trivial vertex cover)

7.2.2 Edge cover

An edge cover of a graph is a set of edges such that every vertex of the graph is incident to at least one edge in the set.

Thus, for the graph G_1, possible edge cover sets can be

- $\{(A, B), (B, C), (D, E), (E, F)\}$ (Minimum sized vertex cover)
- $\{(A, B), (B, C), (B, E), (D, E), (E, F)\} = E(G_1)$ (Trivial edge cover)

Similarly, for the graph G_2, edge cover sets can be

- $\{(A, B), (C, D), (D, E)\}$ (Minimum sized vertex cover)
- $\{(A, B), (A, E), (C, D)\}$ (Minimum sized vertex cover)
- $\{(A, E), (B, E), (D, E)\}$ (Minimum sized vertex cover)
- $\{(A, B), (B, C), (C, D), (D, E), (E, A)\} = E(G_1)$ (Trivial edge cover)

7.2.3 Independent sets

Independent sets can be defined as a set of vertices that are non-adjacent.

Thus, for the graph G_1, the possible independent sets can be

- ϕ (Trivial independent set)
- $\{A, C, E\}$
- $\{B, D, F\}$
- $\{A, C, D, F\}$ (Maximum sized independent set)

Similarly, for the graph G_2, the possible independent sets can be

- ϕ (Trivial independent set)
- $\{A, C\}$
- $\{B, D\}$
- $\{A, C, D\}$ (Maximum sized independent set)

It is easy to note that the minimum sized vertex cover and the maximum sized independent sets are able to cover the entire vertex set of graph. There is a theorem to support this claim.

7.3 Relation between cover sets and independent sets

Lemma We describe the following functions for a simple graph G:

- $\alpha(G)$: Maximum size of independent set in G
- $\alpha'(G)$: Maximum size of matching in G
- $\beta(G)$: Minimum size of vertex cover in G
- $\beta'(G)$: Minimum size of edge cover in G

Then, these functions satisfy the following for all simple graph G:

$$\alpha(G) + \beta(G) = |V(G)| = \alpha'(G) + \beta'(G) \tag{7.5}$$

7.3.1 Proof

We include proof for only $\alpha(G) + \beta(G) = |V(G)|$ as follows:

Let S be an independent set of maximum size for any graph G. Then

$$|S| = \alpha(G)$$

Now, it is trivial to see that all the vertices in $\bar{S}$ are adjacent to at least one vertex in S. Thus all vertices in $\bar{S}$ is adjacent to every vertex in G. Therefore, $\bar{S}$ is a valid vertex cover.

Also, since S is the maximum sized independent set, the vertex cover $\bar{S}$ is minimum sized. Therefore

$$\beta(G) = |\bar{S}|$$

Thus,

$$|V(G)| = |S| + |\bar{S}| \implies |V(G)| = \alpha(G) + \beta(G)$$

7.3.2 Additional Theorem

With the above function definitions, the following also holds true

$$\alpha'(G) = \beta(G) \tag{7.6}$$

Using (7.5) and (7.6), we can also conclude the following

$$\begin{aligned} \alpha(G) + \beta(G) &= \alpha'(G) + \beta'(G) && \text{from (7.5)} \\ \alpha(G) + \beta(G) &= \beta(G) + \alpha(G) && \text{from (7.6)} \end{aligned}$$

This leads to

$$\alpha'(G) = \beta(G) \tag{7.7}$$