Week 5

Kernel, tournaments, king of digraph and Distance, Diameter, Eccentricity*

5.1 Kernel of digraph

A kernel in the digraph D is a subset of vertices S such that S induces no edges (i.e. S is independent set) and every vertex outside S has a successor in S.

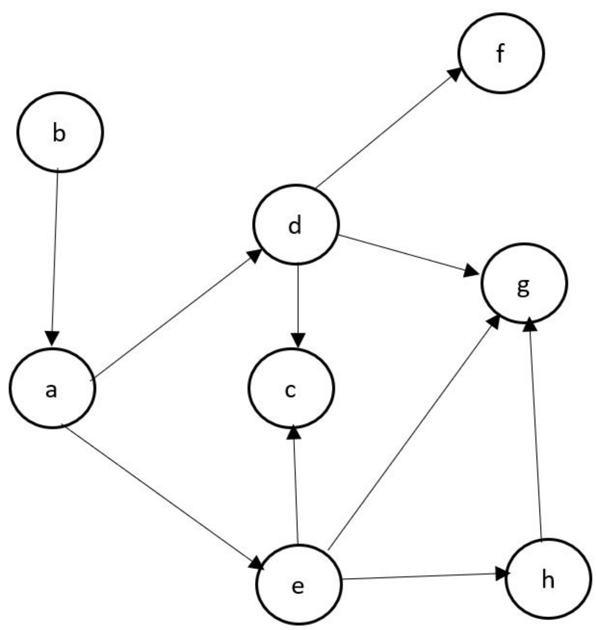

Figure 5.1: Example 1

*Lecturer: Dr. Anand Mishra. Scribe: Shruti Sureshan (M21CS015).

In the above graph the kernel is S={a, f, g, c}

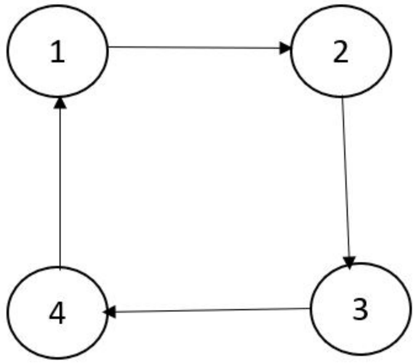

Figure 5.2: Example 2

In the above graph the kernels are S1={1, 3} and S2={2, 4}

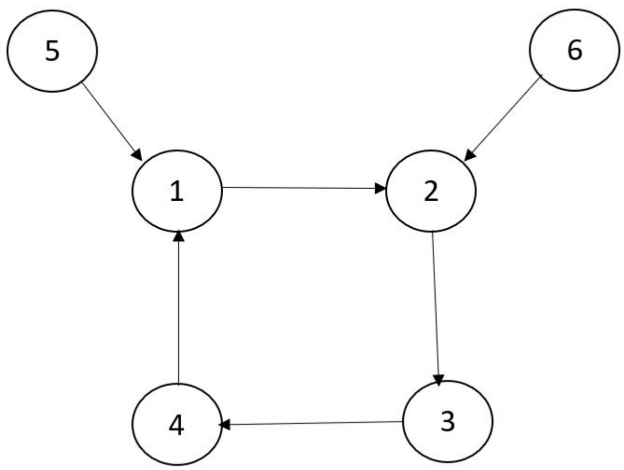

Figure 5.3: Example 3

In the above graph the kernels are S1={1, 3, 6} and S2={2, 4, 5}

Q. Prove that a graph with odd cycles has no kernel

Proof. Let k be the kernel of cycle graph C_n that any v ϵ k should have the following properties:

1. They should be independent

2. w $\notin$ k then w must have a successor in k

Here, every node has only one successor (i.e. alternate ones). In the case of a graph with odd cycles, the issue will come in the end as we have to choose the alternate nodes.
Thus, A graph with odd cycles has no kernel. $\square$

5.2 Outdegree and Indegree

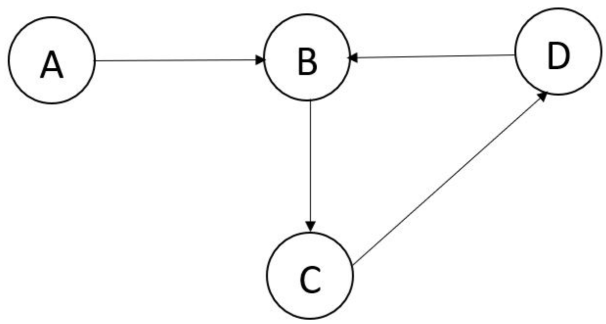

Figure 5.4: Example 1

In the above example, the indegrees and outdegrees are:
d$^+(A) = 1$ $\quad d^+(C) = 1$
$d^-(A) = 0$ $\quad d^-(C) = 1$
$d^+(B) = 1$ $\quad d^+(D) = 1$
$d^-(B) = 2$ $\quad d^-(D) = 1$

In a digraph, the sum of all indegrees is equal to the sum of all outdegrees and it is equal to the number of edges. For v ϵ V(G),

$$\sum d^+(v) = \sum d^-(v)$$

5.3 Orientation

An orientation of graph G is a digraph D obtained from F by choosing an orientation for each edge.

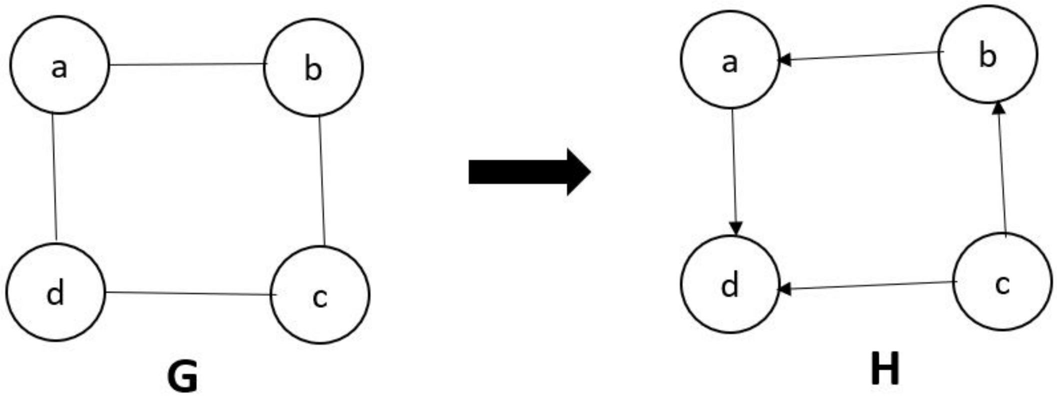

Figure 5.5: Example 1

Tournament is an orientation graph of complete graph. King of a tournament is a vertex from where all other vertices are reachable by a path of length atmost 2.

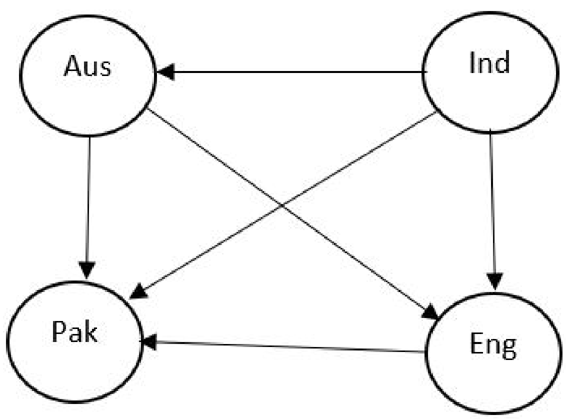

Figure 5.6: Example 1

In the above graph, King is India.

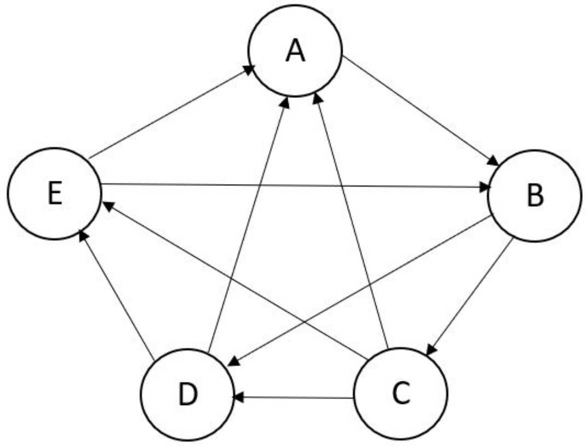

Figure 5.7: Example 2

In the above graph, King={B, C, E}

Theorem 5.1. *Every tournament has a king*

Proof. Suppose u is a node with the highest outgoing degree.

Case 1:

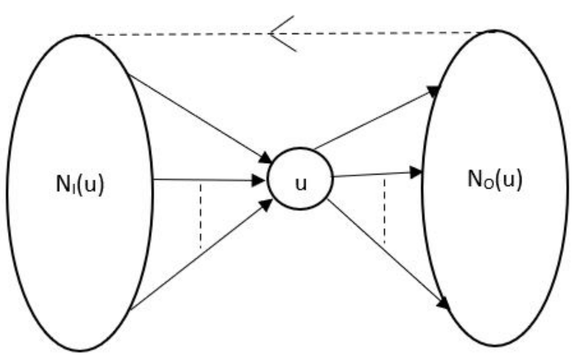

Figure 5.8: Example 1

$|N_o(u)|$ + $|N_I(u)|$ + 1 =$|V(G)|$

Thus, u is king

Case 2:

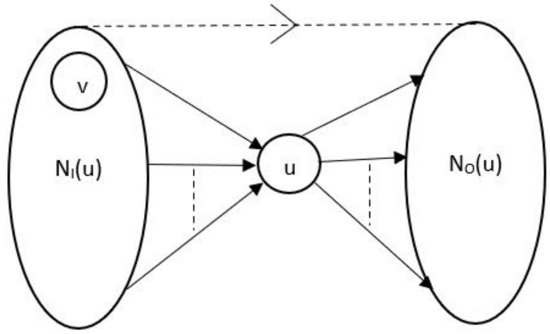

Figure 5.9: Example 2

Number of outgoing edge of node v=1 + $|N_o(u)|$

Number of outgoing edge of node v is greater than number of outgoing edges of u.
This contradicts our assumption of you having highest outgoing degree.
Thus, u is king. □

Q. Prove or disprove: If D is an orientation of a simple graph with 5 vertices then the vertices of D cannot have distinct outdegree.

Proof. Consider the following example:

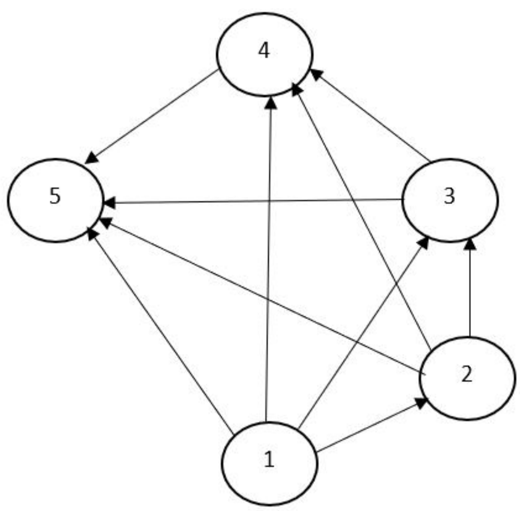

Figure 5.10: Example 1

In the above graph, if x is greater than y then there will be an edge from y to x.
$d^+(1) = 4$
$d^+(2) = 3$
$d^+(3) = 2$
$d^+(4) = 1$
$d^+(5) = 0$

Thus, the statement is incorrect. □

5.4 Trees

Properties:

1. *Deleting a leaf from an n-vertex tree produces an n-1 vertex tree.*
2. *Tree is connected and it has no cycle*
3. *An n-vertex tree contains n-1 edges*
4. *For any pair of vertex (u,v) there exists one and only one path*

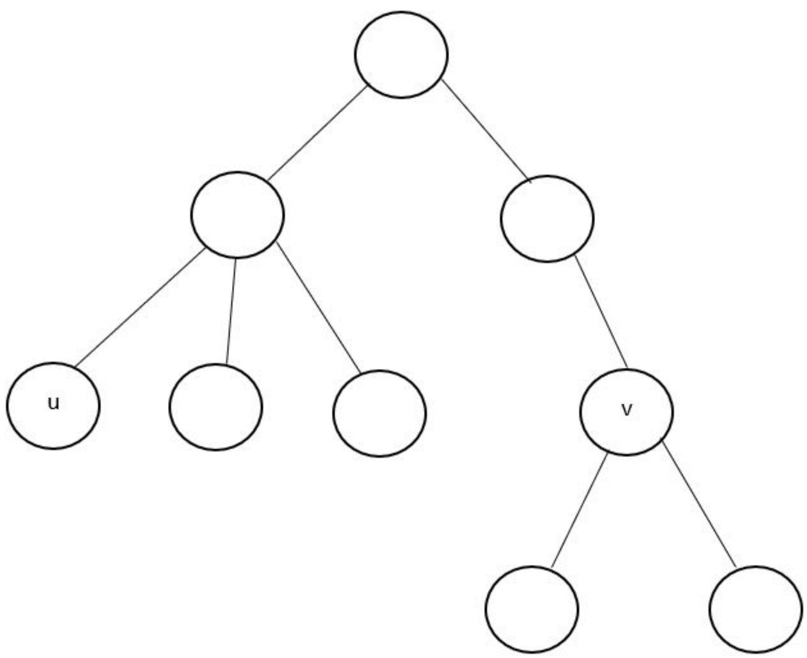

Figure 5.11: Example 1

5. *Every edge of a tree is cut edge*

6. *Tree is bipartite graph*

5.5 Distance, Diameter, radius, eccentricity of a graph

If graph G has a u-v path, then the distance between u to v is written as (u, v) is the least length of u-v path.

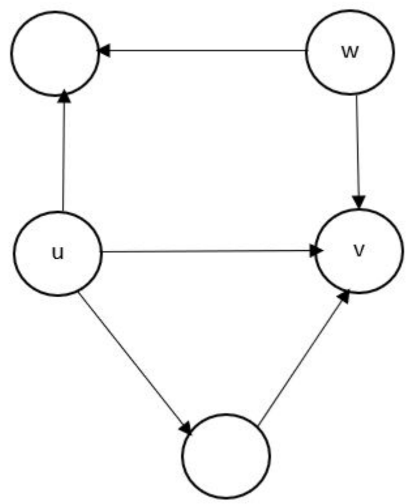

Figure 5.12: Example 1

In the above example, $d(u, v)$=1 $\quad$ *$d(u, w)$=∞*

Diameter: *It is maximum of distances between any two pairs of vertices.*

Eccentricity: *Eccentricity of a vertex u is the maximum of distances it has with any node in the graph.*

Radius: *Radius is the minimum of eccentricity of all nodes.*

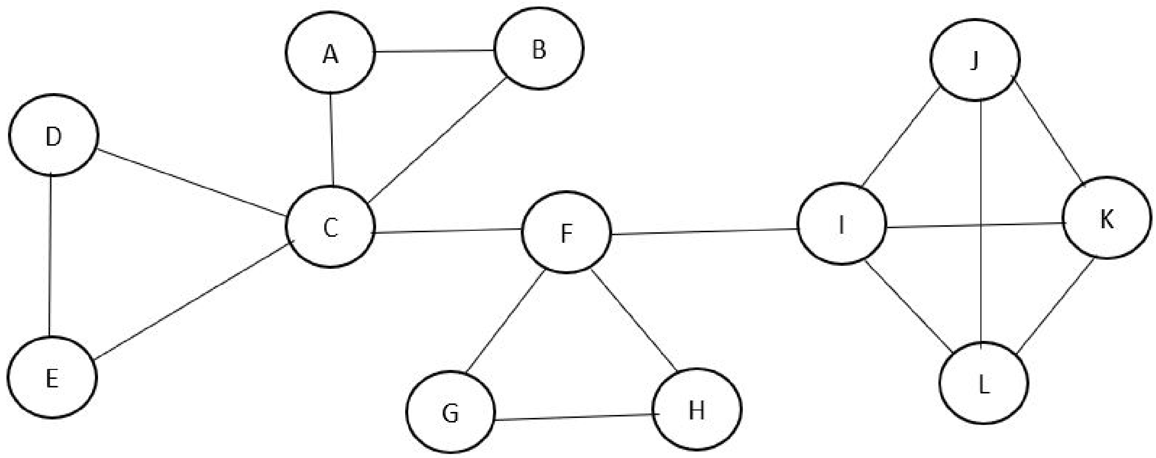

Figure 5.13: Example 2

In the above graph, d(D, K)=4
Eccentricity of the nodes are as follows:

A - 4	*E - 4*	*I - 3*
B - 4	*F - 2*	*J - 4*
C - 3	*G - 3*	*K - 4*
D - 4	*H - 3*	*L - 4*

Diameter is 4
Radius is 2

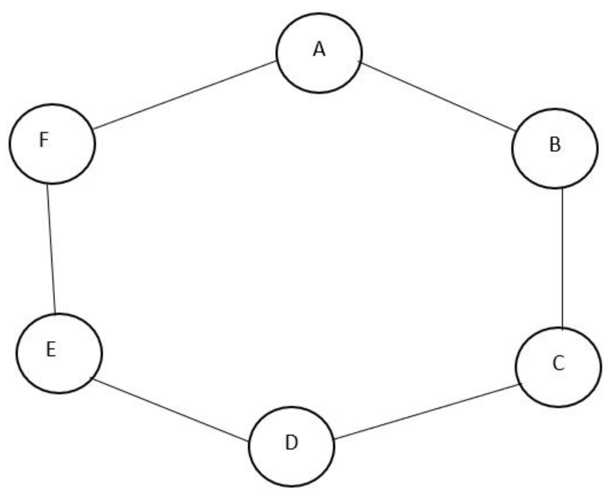

Figure 5.14: Example 3

In the above graph, Eccentricity of the nodes are as follows:
A - 3 D - 3
B - 3 E - 3
C - 3 F - 3
Diameter is 3
Radius is 3

Q. Compute Diameter and Radius of $K_{m,n}$

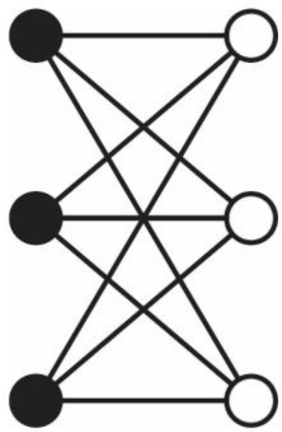

Figure 5.15: Example 4

In the above graph, Eccentricity of each node=2
Diameter of the graph=2
Radius of the graph=2