# Week 2

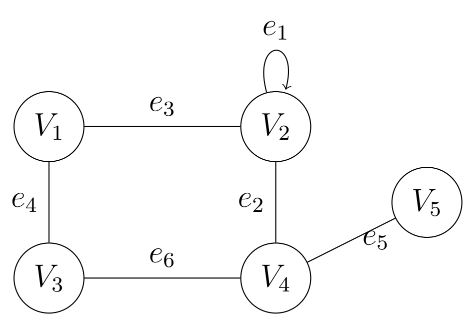
# Fundamental Concepts of Graph Theory*

## 2.1 What is Graph?

### 2.1.1 Definition

A Graph(G) can be defined as G = (V,E) where V(G) is the set of the vertices V and E(G) is the set of edges E. The Graphs are often represented by means of diagram, where the points represents the vertices and the line segment joining the vertices represents the edges.

**Example:**




In the above graph vertices are $\{v_1, v_2, v_3, v_4, v_5\}$ and edges are $\{e_1, e_2, e_3, e_4, e_5, e_6\}$. Edge $e_1$ is a loop because both of its end points are same

### 2.1.2 Subgraph

A subgraph(S) of a graph(G) is a graph such that $V(S) \subseteq V(G)$ and $E(S) \subseteq E(G)$ and the assignment of endpoints to edges in S is the same as in G. We say $S \subseteq G$ and "G contains S".

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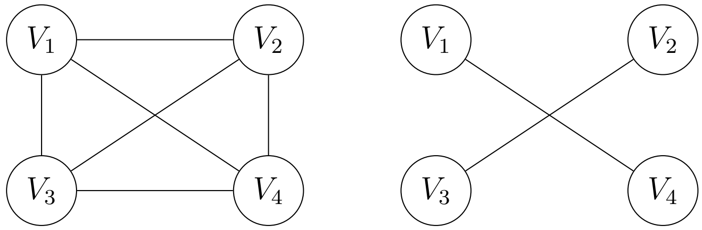
**Example:**

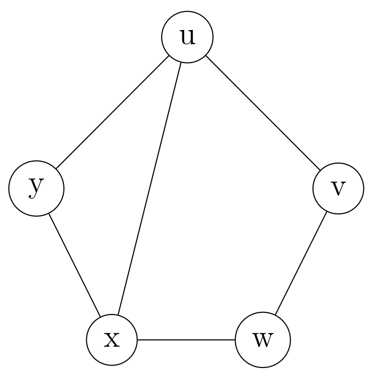



The graph on the right is the subgraph of the graph on the left.

### 2.1.3 Independent Set and Clique

An independent set is a set of pairwise non adjacent vertices. It is also known as stable set. A clique is a set of pairwise adjacent vertices.

**Example:**




In the above graph, $\{u, w\}$ is a independent set of size 2 (It is the largest independent set) and $\{u, x, y\}$ is the only clique of size 3. All edges are clique of size 2.There is no clique of size 4.

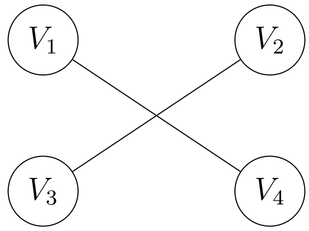
**Note:** If we delete edge ux from G it will be a graph without clique or independent set of size 3.

### 2.1.4 Chromatic Number

The Chromatic Number $\rho(G)$ of a graph G,is the minimum number of colors needed to label the vertices so that the adjacent vertices are coloured differently.

**Example:**

The graph given below has chromatic number = 2.

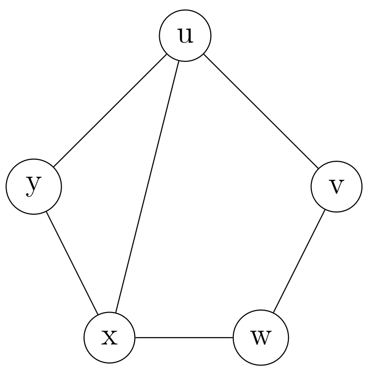


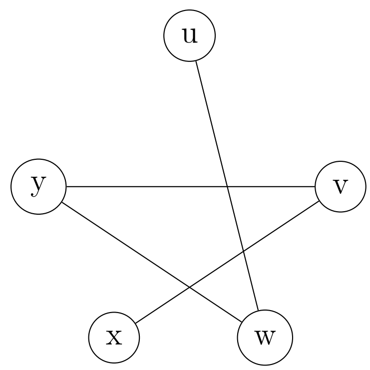

### 2.1.5 Complement of a Graph

The complement of a simple graph G represented as $\bar{G}$ is the simple graph with vertex set V(G) defined by $uv \in E(\bar{G})$ if and only if $uv \notin E(G)$.

**Note:** To find complement of graph
1 Remove all the edges which are there in G.
2 Add all those edges which are not there in G.

**Example:**




Both the above graphs are complements of each other.

**Note:** Independent set of G is same as Clique of $\bar{G}$ and vice-versa.

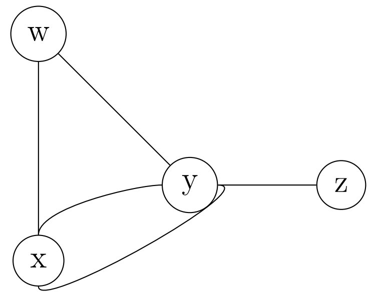
## 2.2 Representation of Graphs

### 2.2.1 Adjacency Matrix

An adjacency matrix of a graph G is a square matrix of dimension $|V| \times |V|$ whose $(i-j)th$ element denote number of edges between ith and jth vertices here $|V|$ denotes the number of vertices. It is denoted as $A(G)$.

**Example:**

1




The graph above has the adjacency matrix as:

$$A(G) = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 2 & 0 \\ 1 & 2 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

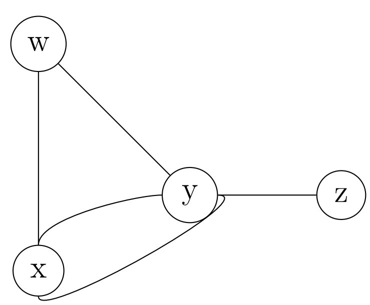
2 If we consider the facebook graph and write the adjacency matrix then each row sum defines number of friends i.e. degree.

**Note:** 1 An adjacency matrix is always symmetric.
2 Weights of the graph can also be used as the entries of the adjacency matrix.
3 It is usually huge sized sparse matrix therefore uses unnecessary storage.

### 2.2.2 Incident Matrix

An Incident matrix of a graph G is matrix of dimension $|V| \times |E|$ whose $(i-j)th$ element is equal to 1 if the ith vertex is the end point of the jth edge here $|V|$ denotes the number of vertices and $|E|$ denotes the number of edges.It is denoted as $M(G)$.

**Example:**



The graph above has the incident matrix as:

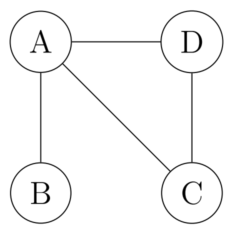
$$M(G) = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

**Note:** 1 Incident matrix is advantageous over adjacency matrix for graph with less connections.
2 For a fully connected graph of order n,the incidence matrix has the order $n \times \binom{n}{2}$.

### 2.2.3 List

Representing graphs as a linked list of neighbours.

**Example:**




The above graph has linked list as:

$\boxed{A} \rightarrow \boxed{B} \rightarrow \boxed{C} \rightarrow \boxed{D}$

$\boxed{B} \rightarrow \boxed{A}$

$\boxed{C} \rightarrow \boxed{A} \rightarrow \boxed{D}$

$\boxed{D} \rightarrow \boxed{A} \rightarrow \boxed{C}$

## 2.3 Isomorphism

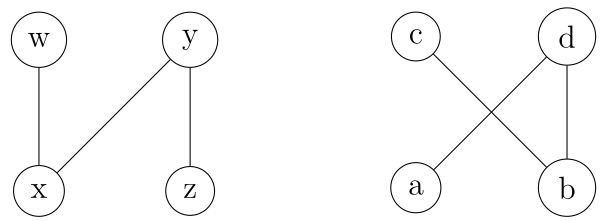
### 2.3.1 Definition:

An isomorphism from a simple graph G to a simple graph A is defined as a bijection $f : V(G) \rightarrow V(A)$ such that $uv \in E(G)$ if and only if $f(u)f(v) \in E(A)$.It is denoted as $G \cong A$.

**Note:** In isomorphism we need to check whether the two graphs have similar structure or not even if they look different. they may have some function connecting them.

**Example:**

Check whether the graphs given below are isomorphic




We can define a function $f : V(G) \to V(S)$ such that $f(w) = c, f(x) = b, f(y) = d, f(z) = a$. Hence,they are isomorphic.
We can also check isomorphism with the help of adjacency matrix.

$$A(G) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} A(S) = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{pmatrix}$$

By arranging the rows order of A(G) as w,y,z,x we get

$$A(G) = A(S) = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{pmatrix}$$

Hence, $G \cong S$

### 2.3.2 Proposition:

**Statement:** The isomorphic relation is an equivalence relation on the set of simple graphs.

**Proof:** Let G , H , I be three graphs such that $G_1 = (V_1, E_1), G_2 = (V_2, E_2), G_3 = (V_3, E_3)$
Reflexive property: To prove: $G_1 \cong G_1$
There exists an identity function $I : V_1 \to V_1$ such that G is isomorphic to itself. Thus $G_1 \cong G_1$.
Symmetric property: Given $G_1 \cong G_2$
To prove: $G_2 \cong G_1$
Since $G_1 \cong G_2$ implies that there exists a bijection $f : V_1 \to V_2$ then there exist bijection $f^{-1} : V_2 \to V_1$ this is because of the fact that "$uv \in E(G_1)$ if and only if $f(u)f(v) \in E(G_2)$" yields "$xy \in E(G_2)$ if and only if $f^{-1}(x)f^{-1}(y) \in E(G_1)$".Thus,$G_2 \cong G_1$.
Transitive property : Given $G_1 \cong G_2$ and $G_2 \cong G_3$

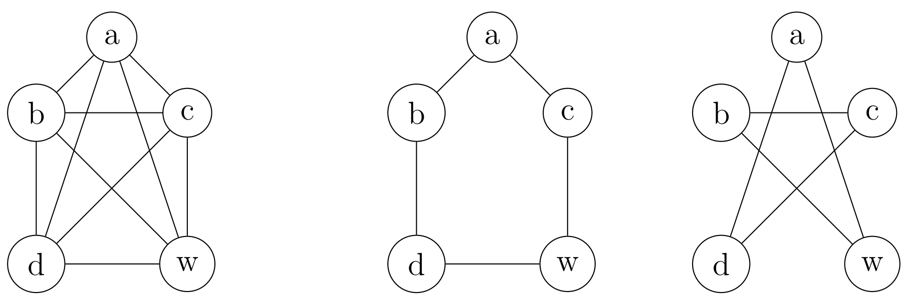
To prove : $G_1 \cong G_3$
Since $G_1 \cong G_2$ then there exist bijections $f_1 : V_1 \to V_2$ and $f_2 : V_2 \to V_3$ such that ” $uv \in E(G_1)$ if and only if $f_1(u)f_1(v) \in E(G_2)$” and ”$xy \in E(G_2)$ if and only if $f_2(u)f_2(v) \in E(G_3)$”.for every $xy \in E(G_2)$ we get $uv \in E(G_1)$ such that $f(u) = xandf(v) = y$ implies there exist $f_2 O f_1 : V_1 \to V_3$ such that ”$uv \in E(G_1)$ if and only if $f_2(f_1(u))f_2(f_1(v)) \in E(G_3)$”. Thus $G_1 \cong G_3$ .
Hence, Isomorphism is an equivalence relation.

### 2.3.3 Decomposition

A decomposition of a graph is a list of subgraphs such that each edge appears in exactly one subgraph in the list.

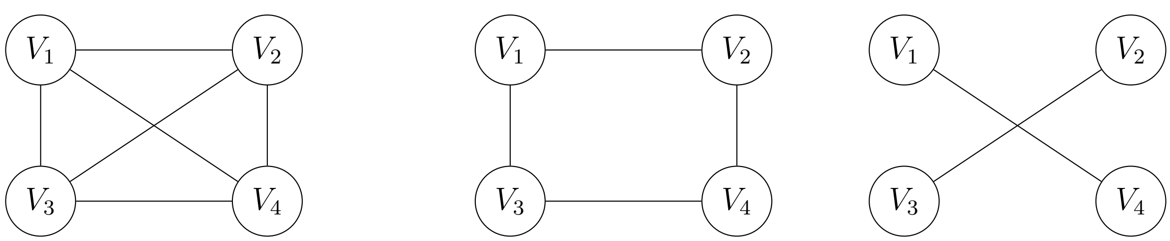
**Example:**

1 Decompose $K_5$ into two $C_5's$.




The two graphs on right hand side represents two $C_5's$ which are formed by the decomposition of $K_5$(Which is on the left side)

2 Can $K_4$ be decomposed into two $C_4's$




No, It cannot be decomposed into two cycles.Here the graph on left hand side is $K_4$ and the other two graphs are formed after the decomposition.

**Questions** 1 What will be the chromatic number for an empty graph having n vertices ?
sol: 1 because n vertices no edge so you can use only one color.
2 What will be the chromatic number for a tree having more than 1 vertex ?
sol: 2
3 The chromatic number of star graph with 3 vertices is greater than that of a tree with same number of vertices.
sol: False, chromatic number of star graph with 3 vertices is 2 and tree with 3 vertices is 2 .i.e. they are same in number.

## 2.4 Connection in Graphs

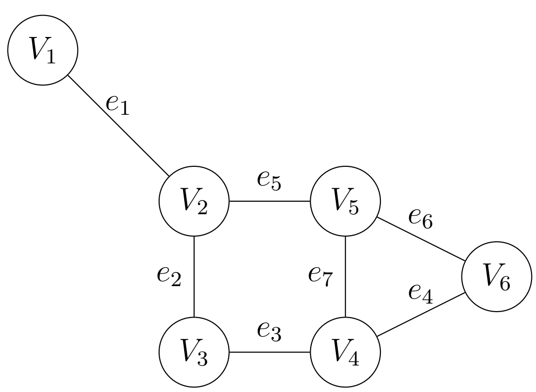
### 2.4.1 Walk, Path and Cycle

A walk in a graph is a sequence of edges such that each edge (except the first one) starts with a vertex where the previous edge ended. The length of a walk is equal to the number of edges in it.
A path is a walk in which all the edges are distinct and its length is equal to the number of edges. Also a path in which all the vertices are distinct is known as simple path.
A cycle is a path in which first and last vertex are distinct.

**Note:** 1 Repeated edges are allowed in a walk.
2 Path is not unique.

**Example:**

Consider the following graph




In the above graph $(e_1, e_2, e_3, e_4, e_6, e_7), (e_1, e_5, e_6, e_4, e_7, e_6, e_4, e_3)$ are the walks of length 6 and 8 respectively.$(e_1, e_5, e_7, e_4, e_6)$ is a path of length 5.$(e_1, e_5, e_7, e_4)$ is a simple path of length 4 and $(e_5, e_7, e_3, e_2)$ is a cycle.

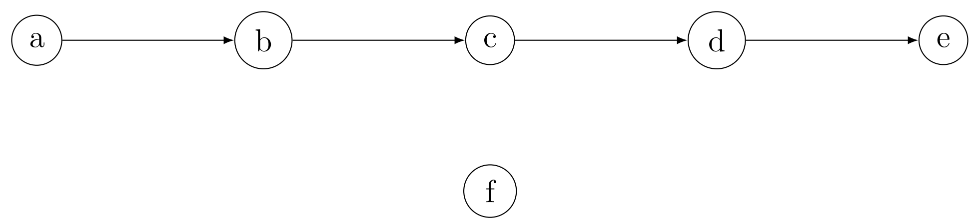
### 2.4.2 Connected and Disconnected graph

A graph is said to be connected if there exist a path between any two pair of vertices in that graph.Otherwise the graph is disconnected.

**Connected Components:** Connected component of graph G is a maximal connected subgraphs of G.Here maximal subgraph means those connected subgraphs which are not contained in any larger connected subgraph of G.

**Example:**

How many components does the following graph has




This graph has two connected components.

### 2.4.3 Proposition

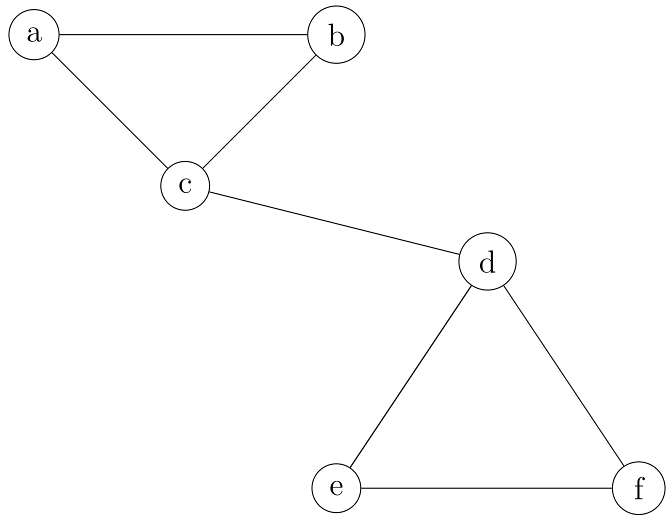
**Statement:** Every graph with n vertices and k edges has atleast n-k components.
Proof: With the help of k edges the maximum number of nodes that we can connect so as to form one connected components = k+1.
The nodes left disconnected = n-k.
Hence, the minimum number of connected components $= n - (k + 1) + 1 = n - k$.

### 2.4.4 Cut-edge and cut-vertex

A cut-edge or cut-vertex of a graph is an edge or vertex whose deletion increases the number of components.

**Example:**

Find out the cut-vertex and cut-edge in the following graph.



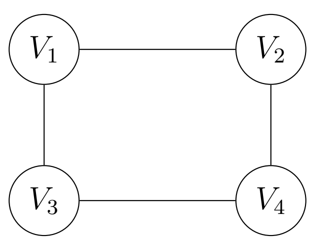

In the above graph cut-edge is cd and the cut-vertices are c and d.

**Theorem:** An edge is a cut-edge if and only if it belongs to no cycle.

### 2.4.5 Bipartition of a Graph

A Bipartition of a graph is a specification of two disjoint independent sets in G whose union is V(G).

**Example:**




This graph has bipartition as $X = \{V_1, V_4\}$ and $Y = \{v_2, v_3\}$

### 2.4.6 Theorem

**Statement:** A graph is bipartite if and only if it has no odd cycle.
Proof:Necessity- Given: G is a bipartite graph.
To prove: It has no cycle.
Since G is bipartite graph.It can be divided into two sets of bipartition such that every walk alternates between them,so every return to the original partite set happens after an even number of steps. Hence G has no odd cycle.
Sufficiency- Given: let G has no cycle. To prove:G is a bipartite graph .i.e. G has a bipartition

of each non-trivial component.
Let x be a vertex in a non-trivial component I and for $y \in V(I)$ define$f(y)$ be the minimum length of a xy-path and we know that I is connected this implies $f(y)$ is defined for each $y \in V(I)$.
Define $X_1$ =$\{y \in V(I) : f(y)$ is even$\}$ and $Y_1 = \{y \in V(I) : f(y)$ is odd$\}$. Now let us consider $X_1$ is not independent then there exist adjacent vertices $y, y' \in X_1$ .With the help of y,y' we can create a closed walk using the shortest xy path and vv' edge and the reverse of the shortest path.We know that a closed odd walk contains an odd cycle which contradicts the fact that G has no odd cycle.Therefore $X_1$ and $Y_1$ are independent sets.
Hence,I is an XY-bigraph.

### 2.4.7 Proposition

1 A simple path is a bipartite graph.
Proof: As we know that simple path contains no cycle. By theorem 2.4.6 a graph is bipartite if it has no odd cycle.
Hence, simple path is bipartite graph.
2 $C_n$ is bipartite if and only if n is even.
Proof: Necessity- Given:$C_n$ is bipartite.
To prove:n is even.
As $C_n$ is bipartite by theorem 2.4.6 a graph is bipartite if it has no odd cycle.Hence,$C_n$ has even cycle.
Sufficiency- Given: n is even.
To prove: $C_n$ is bipartite.
By theorem 2.4.6 a graph is bipartite if it has no odd cycle. Hence,$C_n$ is bipartite.