# Week 12

# Graph Coloring*

It is assigning vertices of a graph to certain sets such that each set contains independent vertices with respect to each other

**k-coloring** of a graph is labeling f : V(G) $\rightarrow S = \{C_1, C{-}2, ........, C_n\}$ where S is the set of k-colors and k-coloring is proper if adjacent vertices have different labels

## 12.1 Chromatic Number

The minimum number of colors required for vertex coloring of graph G is called as the chromatic number of G, denoted by X(G)

X(G) = 1 if and only if 'G' is a null graph. If 'G' is not a null graph, then for all other cases X(G) $\geq$ 2.

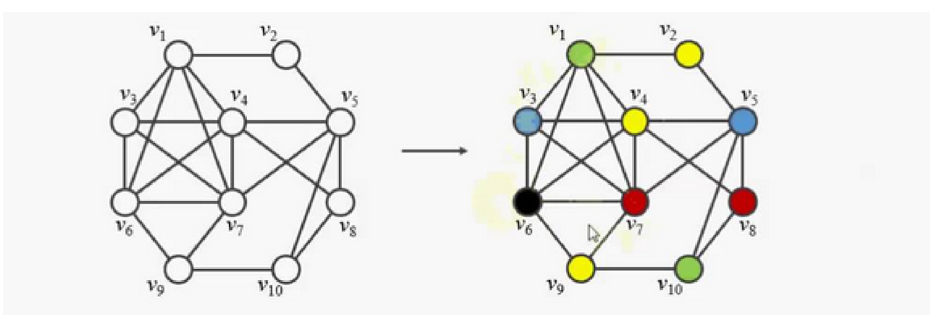


Figure 12.1: {4-coloring of the graph and also 4 is the chromatic number of the graph}

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### 12.1.1 Greedy Algorithm for graph coloring

There is no efficient algorithm available for coloring a graph with minimum number of colors. The problem NP Complete problem. There are approximate algorithms to solve the problem. One such is greedy algorithm to assign label.

**Algorithm**

1. Choose any vertex and color it
2. Do following for remaining V-1 vertices
   (a) Choose any non-colored vertex
   (b) Color it with the lowest color that has not been used in any of the previously colored adjacent vertices to it
   (c) If all of the previously used colors are assigned to the adjacent vertices to it, then assign a new color to it
3. The number of colors used is the approximate chromatic number of the graph

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**Note :- Let W(G) be the clique size in the graph G then for every graph the chromatic number of the graph X(G) $\geq$ W(G)**

**Note :- Let X($G_1$), X($G_2$), X($G_3$),..........,X($G_n$) be the chromatic numbers of the n components of the graph then chromatic number of the whole graph X(G) = max(X($G_1$), X($G_2$), X($G_3$),.....,X($G_n$))**

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### 12.1.2 Bounds on Chromatic Number

**Trivial Bounds**

$0 \leq$ X(G) $\leq \|V(G)\|$

where $\|V(G)\|$ is the number of vertices in the graph G

also X(G) $\leq \|\Delta(G)\|$

where $\|\Delta(G)\ \|$ is the maximum degree in the graph

**Welsh-Powell Bound**

If G has a degree sequence $d_1 \geq d_2 \geq d_3 \geq ...... \geq d_n$
then
$X(G) \leq 1 + max_i\{min(d_i, i-1)\}$

### 12.1.3 Chromatic Numbers of some popular graphs

**Star graph**

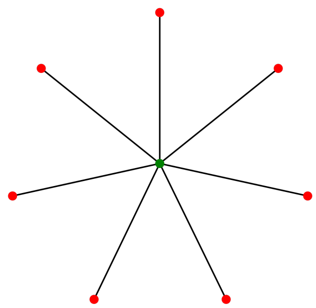

Figure 12.2: {star graph : chromatic number - 2}

$\|\Delta(G)\| = n - 1$

so X(G) $\leq$ n

and X(G) = 2 so it satisfies the bound we have discussed

**Complete Graph**

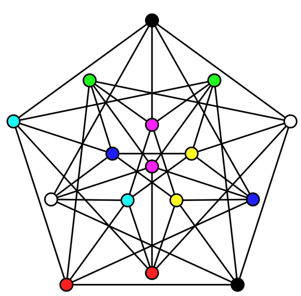

Figure 12.3: {complete graph : chromatic number - n}

$\|\Delta(G)\| = n - 1$

so X(G) $\leq$ n

and X(G) = n so it satisfies the bound we have discussed

**Wheel graph**

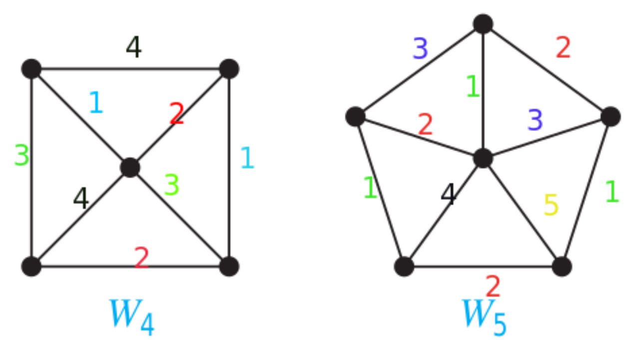


Figure 12.4: {wheel graph}

$\|\Delta(G)\| = n - 1$

so X(G) $\leq$ n

and X(G) $= \begin{cases} 3 & \text{if n is odd,} \\ 4 & \text{if n is even.} \end{cases}$

**Cycle**

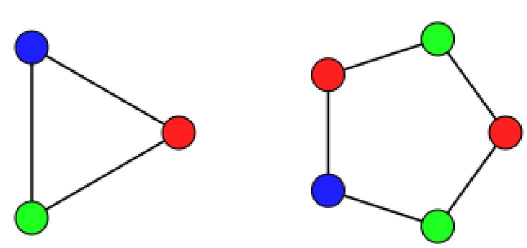

Figure 12.5: {cycle graph}

$\|\Delta(G)\| = 2$

so X(G) $\leq 3$

and X(G) $= \begin{cases} 3 & \text{if n is odd,} \\ 2 & \text{if n is even.} \end{cases}$

## 12.2 Interval Graphs

A graph is an interval graph if it has an intersection model consisting of intervals on a straight line.

in other words

an interval graph is an undirected graph formed from a set of intervals on the real line, with a vertex for each interval and an edge between vertices whose intervals intersect. It is the intersection graph of the intervals.

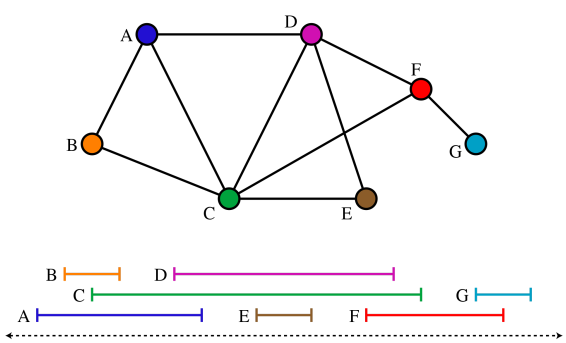


Figure 12.6: {Interval graph}

Consider the following problem, These 4 activities given below in the table are to be scheduled.

| Scheduling Problem | | |
|---|---|---|
| Activity | Start Time | End Time |
| a | 1 | 4 |
| b | 4 | 8 |
| c | 2 | 5 |
| d | 6 | 7 |

Now two questions can arise :-

1. How many minimum rooms are required for scheduling all the activities

2. How many activities can be scheduled in one room

These two questions can be answered if we construct the interval graphs of the following problem.
the corresponding interval graph would look like

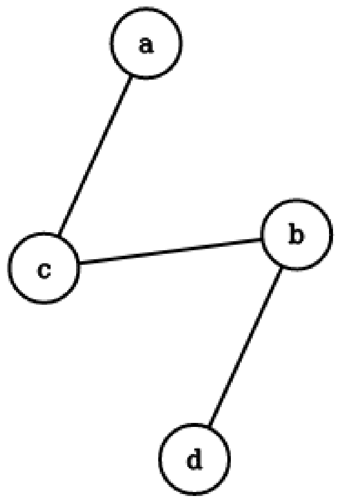


Figure 12.7: {Corresponding interval graph of the above problem}

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**Answer to question 1 would be equal to chromatic number of the graph**

**Answer to question 2 would be the maximum number of alternate vertices that can be colored using one color** ———————————————————————-

**Note :- Not every graph is an interval graph**
**for Example $C_4$**

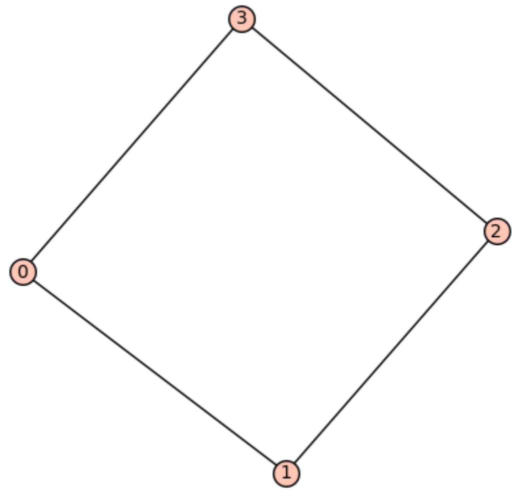


Figure 12.8: {$C_4$ cycle graph}

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### 12.2.1 What Heuristic can be used to solve the scheduling problem

**Heuristic 1 - Shortest event first**

**Note :- This heuristic will not work in the following scenario**

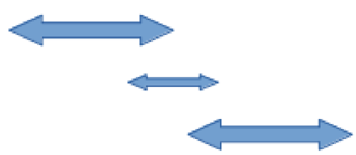

Figure 12.9

according to this heuristic only 1 activity can be scheduled in a room, but the real answer is 2

**Heuristic 2 - Starting with the first event and selecting the next possible events**

**Note :- This heuristic will not work in the following scenario**

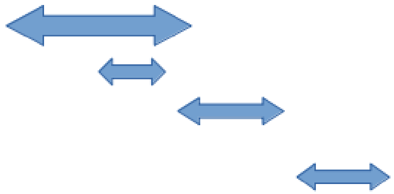

Figure 12.10

according to this heuristic only 2 activity can be scheduled in a room, but the real answer is 3

**Heuristic 3 - Earliest Ending first**

**Note :- This is the correct heuristic for solving the given problem This will work in all scenarios**

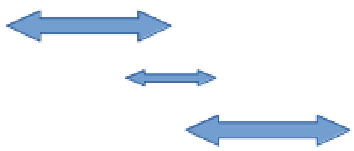

Figure 12.11

Answer for this problem using this heuristic is 2

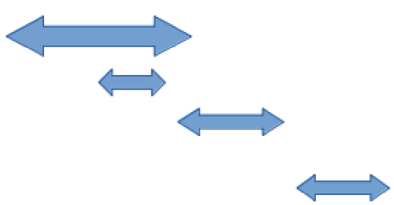

Figure 12.12

Answer for this problem using this heuristic is 3