Indian Institute of Technology Jodhpur
Real Time Autonomous Systems
Major Exam

Date: May 1, 2021, Max Marks: 20

Max Time: 120 min (extra 15 min for submission)

`Attempt all the questions. Best of luck` ☺

1. Consider a function $f : \mathbb{R}^2 \rightarrow \mathbb{R}^3$ defined as $f(\mathbf{x}) = \begin{bmatrix} 2x_1^2 + 2x_1x_2 \\ x_1x_2 \\ 2x_1x_2 + x_2^2 \end{bmatrix}$. Find the a linear approximation of this function at the point $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$. Here, $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ is a vector in $\mathbb{R}^2$. **[1.5 Marks]**

2. Let X, Y, and Z be three random variables. Assume that X is independent of Z given the value of Y. Also, assume that Y is independent of Z. Show that $p(X|Z) = p(X)$. **[2 Marks]**

3. Let $\mathbf{x} \in \mathbb{R}^n$ be a zero mean random vector with covariance matrix $\mathbf{C_x}$. Let $\mathbf{y} \in \mathbb{R}^n$ be another random vector defined as $\mathbf{y} = \mathbf{Ax}$, where $\mathbf{A} \in \mathbb{R}^{n \times n}$ is a known matrix. Show that the random vector $\mathbf{y}$ will have zero mean vector and covariance matrix $\mathbf{C_y}$ equal to the matrix $\mathbf{AC_xA}^\top$. **[1.5 Marks]**

4. Consider a Linear dynamical system with motion model $\mathbf{x}_t = \mathbf{A}_t\mathbf{x}_{t-1} + \mathbf{B}_t\mathbf{u}_t + \boldsymbol{\epsilon}_t$. Here, $\mathbf{x}_t \in \mathbb{R}^n$ is the state at time t, $\mathbf{u}_t \in \mathbb{R}^p$ is the control vector, $\mathbf{A}_t$ and $\mathbf{B}_t$ are known matrices, and $\boldsymbol{\epsilon}_t \sim \mathcal{N}(\mathbf{0}, \mathbf{R}_t)$ is an independent Gaussian noise random vector. Show that $p(\mathbf{x}_t|\mathbf{u}_t, \mathbf{x}_{t-1})$ is a multivariate Gaussian distribution with mean vector $\mathbf{A}_t\mathbf{x}_{t-1} + \mathbf{B}_t\mathbf{u}_t$ and covariance matrix $\mathbf{R}_t$. **[2 Marks]**

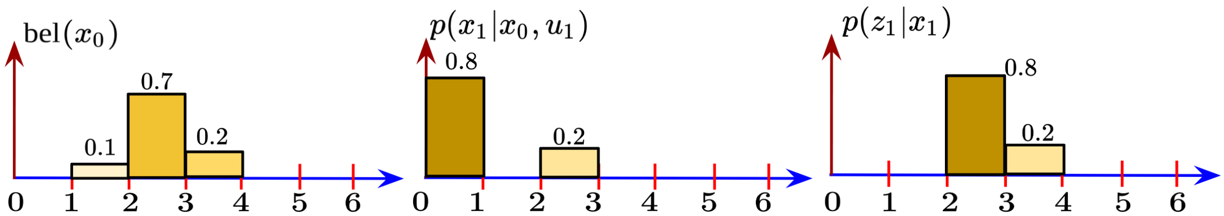
5. Consider a set of given correspondences $\{(\mathbf{P}_i, \mathbf{p}_i)\}_{i=1}^n$ between the known 3D points $\mathbf{P}_i \in \mathbb{R}^3$ and their respective ground-truth 2D projections $\mathbf{p}_i \in \mathbb{R}^2$, design an algorithm to find the extrinsic camera matrices $(\mathbf{R}, \mathbf{t})$ that represent the camera transformation with respect to the world coordinate system. Assume that the intrinsic camera matrix $\mathbf{K}$ is known to us. **[2 Marks]**

6. Consider a non-linear dynamical autonomous system with the state-transition model defined by below system of equations.

$$
\begin{aligned}
\begin{bmatrix} x_t \\ y_t \\ \theta_t \end{bmatrix} &= \begin{bmatrix} x_{t-1} \\ y_{t-1} \\ \theta_{t-1} \end{bmatrix} + \begin{bmatrix} -\frac{u_t}{\omega_t}\sin\theta_{t-1} + \frac{u_t}{\omega_t}\sin(\theta_{t-1} + \omega_t\Delta t) \\ \frac{u_t}{\omega_t}\cos\theta_{t-1} - \frac{u_t}{\omega_t}\cos(\theta_{t-1} + \omega_t\Delta t) \\ \omega_t\Delta t \end{bmatrix} + \boldsymbol{\epsilon}_t. \\
\mathbf{x}_t &= g(\mathbf{u}_t, \mathbf{x}_{t-1}) + \boldsymbol{\epsilon}_t
\end{aligned}
$$

Here, Δt is the time duration between two states, $\boldsymbol{\epsilon}_t \sim \mathcal{N}(\mathbf{0}, \mathbf{R}_t)$ is the noise vector, $\mathbf{x}_t = \begin{bmatrix} x_t \\ y_t \\ \theta_t \end{bmatrix}$ is the robot pose and $\mathbf{u}_t = \begin{bmatrix} u_t \\ \omega_t \end{bmatrix}$ is the control data at time t. Linearize this model at $\mathbf{x}_{t-1} = \boldsymbol{\mu}_{t-1}$ as $\mathbf{x}_t = g(\mathbf{u}_t, \boldsymbol{\mu}_{t-1}) + \mathbf{G}_t(\mathbf{x}_{t-1} - \boldsymbol{\mu}_{t-1}) + \boldsymbol{\epsilon}_t$ for some matrix $\mathbf{G}_t$. Find the matrix $\mathbf{G}_t$. **[3 Marks]**

7. In this question, apply one iteration of the discrete Bayes filter algorithm for updating the belief of the robot about its state. Assume that the environment is one-dimensional and tessellated into six equally spaced cells. The robot's initial belief $\mathrm{bel}(x_0)$ is represented in Figure 1(a). We can interpret it as follows: at the start ($t = 0$) the robot is in the 2nd cell with a probability of 0.1, in the 3rd cell with a probability of 0.7, and in the 4th cell with a probability of 0.2. The probabilistic motion model (state-transition model) is represented in Figure 1(b). This model must be interpreted in this way: between time $t = 0$ and time $t = 1$, the robot may have moved by zero, or two units to the right with probabilities 0.8 and 0.2, respectively. What will be the predicted robot belief distribution $\overline{\mathrm{bel}}(x_1)$ after this movement? Let us now assume that the robot uses its onboard range-finder and measures the distance from the origin. Assume that the measurement model $p(z_1|x_1)$ is given in above Figure 1(c). This plot tells us that the distance of the robot from the origin can be 3 or 4 units with probabilities 0.8 and 0.2, respectively. Find the belief distribution $\mathrm{bel}(x_1)$ after the measurement update step. **[4 Marks]**



(a) Initial Belief (b) State Transition Distribution (b) Measurement Model

Figure 1:

8. Consider the below graph representing a grid map having seven nodes. The cost of going from one to another adjacent node is labeled on each edge of the graph. Consider that the 1st vertex represents the initial location of the robot and the 9th vertex represents the goal location for the robot. Find the optimal path and the corresponding cost using the Dijkstra algorithm. Now swap the goal position and initial position of the robot and find the optimal path and the corresponding cost using the Dijkstra algorithm. Are both the path the same? Also, are these paths optimal? **[4 Marks]**

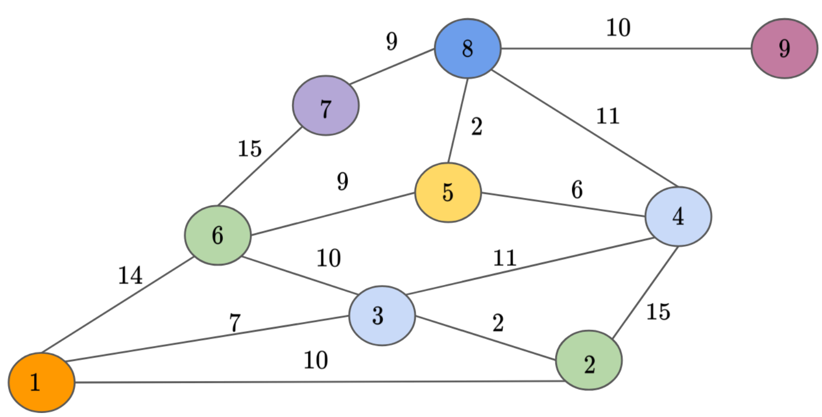


Figure 2: