INDIAN INSTITUTE OF TECHNOLOGY- JODHPUR

GRAPH THEORY AND APPLICATIONS(GTA)
COURSE CODE: CSL7410

Lecture Scribing Assignment: Week 12

Done by:
Deepanshi jindal
M20MA202

April 3, 2022

॥ त्वं ज्ञानमयो विज्ञानमयोऽसि ॥

Week 12

Graph Coloring and Interval Graphs *

12.1 Graph Coloring

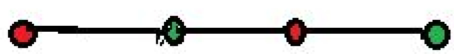
In this Section we will learn about graph coloring methods , some properties of chromatic numbers and interval graphs .

12.1.1 Definition

A k-coloring of graph G is a labeling f:V(G)$\mapsto S, whereS =$C$_1, C_2, ...c_k$ is a set of k colors .A k-coloring is proper if adjacent vertices have different lables.A graph is k-colorable if it has a proper k-coloring.The chromatic number $\chi(G)$ is the least k such that G is k-colorable.

Note: We can not color the graphs with loops because we can not assign different color to a vertex itself.So in this study all the graphs are loopless.Multiple edges are also pointless because extra copies do not affect coloring.In this study, we will usually think about simple graphs and we will name edges by their endpoints.The statements made without restricting to simple graph remain valid when multiple edges are allowed.

Example 12.1. $S = \{R, G, B, Y\}$



coloring 1(2 colors)

coloring 2(3 colors)

*Lecturer: Anand Mishra. Scribe: Deepanshi jindal(M20MA202).

We know that the least k such that the grpah is k-colorable is the chromatic number. So in this example the least $k = 2$ and hence the chromatic number of the graph is 2.

12.1.2 Definition:

A graph G is k-chromatic if $\chi(G) = k$. A proper k-coloring of a k-chromatic graph is an optimal coloring . If $\chi(H) < \chi(G) = k$ for every proper subgraph H of G, then G is color-critical or k-critical.

12.1.3 Greedy Algorithm:

The greeedy coloring relative to a vertex ordering $v_1, v_2, ...vn$ of V(G) is obtained by coloring vertices in the order $v_1, v_2, ...v_n$, assigning to v_i the smallest indexed color not already used on its lower indexed neighbours.

12.1.4 Definition:

The clique number of a graph G is the maximum size of a set of pairwise adjacent vertices(clique) in G.it is denoted by $\omega(G)$.

Proposition: For every graph G, $\chi(G) \geq \omega(G)$
Proof: The bound holds because vertices of a clique require distinct colors.

Example 12.2. : forC_4 graph and C_3 graphs $\chi(G) = \omega(G)$

Problem: Prove that $\chi(G)$ of a graph $G = max\{\chi(G_1), \chi(G_2), ...\chi(G_k)\}$. where $G_1, G_2, ...G_k$ are k-components of the graph.

Answer: The chromatic number of a graph equals the maximum of the chromatic numbers of its components. Since there are no edges between components, giving each component a proper coloring produces a proper coloring of the full graph. On the other hand, every proper coloring of the full graph must restrict to a proper coloring on each component.

12.1.5 Bounds on Chromatic number:

The trivial upper bound on the chromatic number is $\chi(G) \leq |V(G)|$. This holds the equality when the graph is the complete graph.

The trivial lower bound is $\chi(G) > 0$.

Propostion:

$\chi(G) \leq \Delta G + 1$, where ΔG is the maximum degree of G.

Proof: In a vertex ordering, each vertex has at most ΔG earlier neighbors,so the greedy coloring cannot be forced to use more than $\Delta G + 1$ colors. This proves constructively that $\chi(G) \leq \Delta G + 1$.

Example 12.3. For star graph $\Delta G = n - 1$ and $\chi(G) = 2$. So, $\chi(G) \leq n$.

also for complete graph $\Delta G = n$ and $\chi(G) = n$, So $\chi(G) \leq n + 1$.

For wheel graph ΔG= n-1 and $\chi(G) = 3 or 4$, so $\chi(G) \leq n$ and for cycle $\Delta G = 2$ and $\chi(G) = 2 or 3$. so $\chi(G) \leq 3$

Proposition:

(Welsh Powel): If a graph G has degree sequence $d_1 \geq d_2 \geq ... \geq d_k$, then $\chi(G) \leq 1 + max_i min\{d_i, i - 1\}$

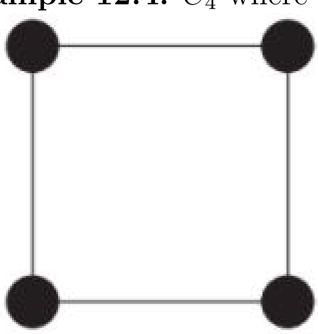
proof: We apply greedy coloring to the vertices in nonincreasing order of degree. When we color the ith vertex v_i, it has at most $min\{d_i, i - 1\}$ earlier neighbors, so at most this many colors appear on its earlier neighbors. Hence the color we assign to v_i is at most $1 + min\{d_i, i - 1\}$. This holds for each vertex, so we maximize over i to obtain the upper bound on the maximum color used.

12.2 Interval Graphs:

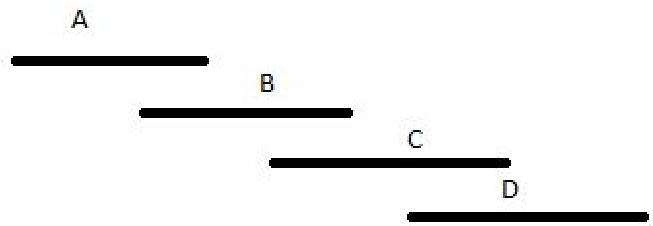
An interval representation of a graph is a family of intervals assigned to the vertices so that vertices are adjacent if and only if the corresponding interval intersect. A graph having such a representation is an interval graph.

Note: Not every graph is an interval graph.

Example 12.4. C_4 where the vertices are named as A,B,C,D.



It's interval representation is defined as shown below which is not correct for the graph so this graph is not an interval graph.



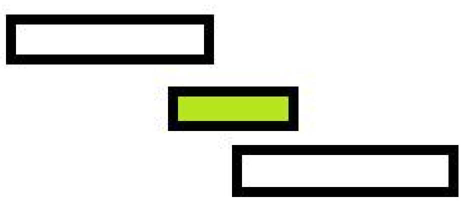
Example 12.5. Star graph is also not an interval graph because the interval representation does not exist for this.

The complete graph K_n is an interval graph, and can be represented by any family of intervals that contain a common point.

Definition: A graph that has no induced subgraph as C_4 and it's complement graph has transitive orientation is an **interval graph**

Example 12.6. If we have a list of events that will occur in different time intervals and we are intersted to find the maximum number of events that can occur in a single room.then we will take help of these heuristics to handle these kind of problems.

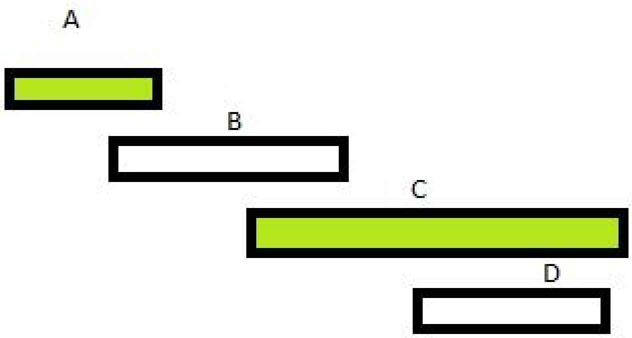
Heuristic 1: Take Shortest events first.
But this Heuristic does not work for all the problems. for counter example- There are some events with their timings in interval representation.



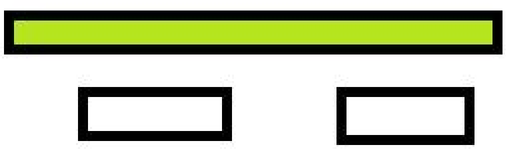
If we take the shortest length event first then our answer is only one which is wrong as there can be occur two events in a room.

Heuristic 2: Start with first event then select next possible events that begins as early as possible.

Example 12.7. : There are four events A,B,C,D with their timings in interval representation.



But this heuristic can also fail for example if we take the timing of these events as :

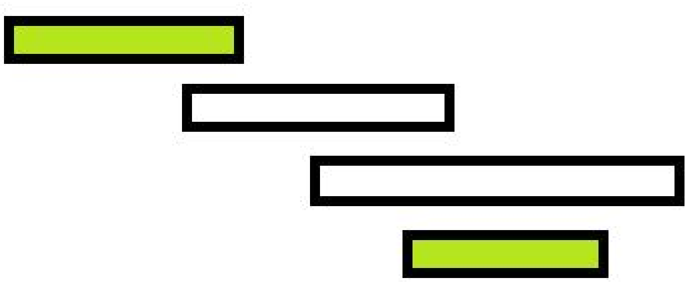


then the answer will be 1 which is wrong as their can occur 2 maximum number of events in a room.

Now there is a final heuristic for the correct answers.

Heuristic-3: Select the earliest finish time interval first . this heuristic will work for all the problems.

Example 12.8. :



Proposition: If G is an interval graph than $\chi(G) = \omega(G)$

Proof: In an interval representation, order the vertices according to the left endpoints of the intervals.Apply greedy coloring, and suppose that x receives k, the maximum color assigned.Since x does not receive a smaller color, the left endpoint a of its interval belongs also to intervals that already have colors 1 through $k - 1$ These intervals all share the point a, so we have a k-clique consisting of x and neighbors of x with colors 1 through k — 1.Hence $\omega(G) \geq k \geq \chi(G)$. since $\chi(G) \geq \omega(G)$, this coloring is optimal.