Week 7

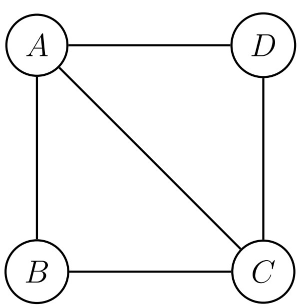
Lecture 13 and 14: Graph Theory *

7.1 Hall's Theorem

According to Hall's marriage theorem, a graph has an X-perfect matching if and only if it does not contain any Hall violators. The following algorithm establishes the theorem's hard direction: it locates either an X-perfect match or a Hall violator.

7.1.1 Matching in a graph

A matching graph is a collection of edges in a graph that are not adjacent to one another. In other words, no two edges should share a vertex. Consider the graph below G_0.
Example :



There are numerous ways to choose a set of edges such that no two are adjacent. Thus, the graph's possible matchings include the following:

1. $\varnothing$ (trivial matching)
2. $\{(A, B)\}$
3. $\{(C, D)\}$

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4. $\{(A, C)\}$
5. $\{(A, B)\}, \{(C, D)\}$ (Maximum matching)

The matching can be of the following types:

- Maximal matching: A matching in which no more edges of G can be added to it.
- Maximum matching: A matching with the maximum possible number of edges.
- A matching in which every vertex of the graph is incident to exactly one edge in the matching.

Algorithm :
The Hall theorem can be described as follows:

Let X, Y be the disjoint sets of vertices of the Graph $G_{m,n}$ where $|X| \leq |Y|$.
Let $S \subseteq X$ be a group of vertices of X, then the set $N(S) \subseteq Y$ is the set of vertices that are adjacent to any of the vertices in S. In other words, $N(S)$ is the neighbourhood of S.

Then, it is possible to have a matching that saturates X if and only if

$$N(S)| \leq |S| \forall S \subseteq X \tag{7.1}$$

Example Application :
This theorem can be used to check if a matching is possible such that all elements of a disjoint set of a graph is saturated.

Example scenario: Given a placement scenario, where 20 companies are to select among 20 candidates, is it possible that every candidate is selected for some company? The solution to this problem depends on how the favoured candidates for each company. As long as the Hall's condition is satisfied for all subsets of candidates, a perfect matching will exist. To ensure this, all possible subsets must be checked. If any such subset is found that does not satisfy the Hall's condition, it can be concluded that the matching is not possible. It is to be noted while that such a problem becomes exponentially difficult to compute as the number of elements of the set is increased.

Solutions :
Since the statement includes 'if and only if', it implies that the proof should cover the following for a XY Graph G

- If the Hall's condition is satisfied by G there exists a matching such that X is saturated
- If there exists a matching such that X is saturated, then's the Hall's condition is satisfied by G.

It is straightforward to demonstrate the latter, as if such a match exists, then all elements in X have been uniquely matched to elements in Y. This implies that for all possible $S \subseteq X$, each element in S must have at least one edge to an element in Y.

Therefore,

$$\forall S \subseteq X, |S| \leq |N(S)| \tag{7.2}$$

The former can be established through the use of

Contrapositive Proof, which is as follows:If there does not exists a matching M in a XY Graph such that X is saturated, then the hall's condition is not satisfied,

$$\exists S \subseteq X, |S| > |N(S)| \tag{7.3}$$

Now, we can use Berge's theorem to complete the proof in the following manner. Consider a bipartite XY graph with a maximum matching M. This matching, according to the statement, does not saturate X. Assume it leaves an unsaturated vertex $u \in X$ of the graph after the matching.

Now, Consider two sets $P \subseteq X$ and $Q \subseteq Y$ considered as follows:

- P is the set of end points of M-alternating paths starting from u with the last edge belonging to M
- Q is the set of end points of M-alternating paths starting from u with the last edge not belonging to M

Given the way these sets are constructed, P will have at least one vertex more than Q, since P contains u, which does not have a matching, and all the elements in $P \cup Q$ are included in M except for the vertex u. Therefore,

$$|P| = 1 + |Q| \Longrightarrow |P| > |Q| \Longrightarrow |P| > |N(Q)| \tag{7.4}$$

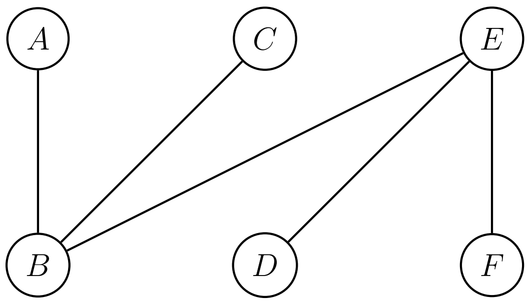
Thus, there exists a set of vertices that does not satisfy Hall's condition. Hence we have proven the statement "If there does not exists a matching M in a XY Graph such that X is saturated, then the hall's condition is not satisfied".

7.2 Independent Sets and Covers

7.2.1 Vertex Cover

A vertex cover V_c of a graph G is a set $Q \subset V$ (G) such that it contains at least one end point of every edge.

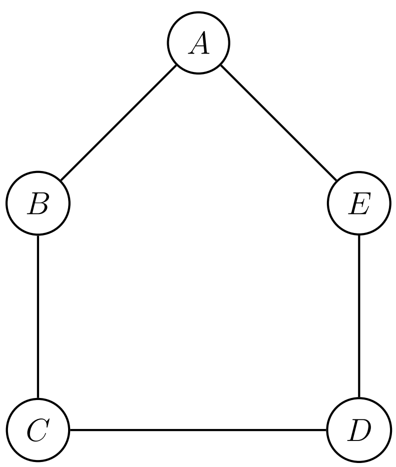
Consider the following graph G_1 below:



The possible vertex covers are

1. $\{A, C, D, F\}$
2. $\{A, B, C, D, E, F\} = V(G1)$(Trivial vertex cover)

Taking another example, consider the following graph G_2 :



The possible vertex covers are

1. $\{A, C, E\}$ (Minimum sized vertex cover)
2. $\{B, D, E\}$ (Minimum sized vertex cover)
3. $\{A, B, C, D, E\} = V(G_2)$ (Trivial vertex cover)

7.2.2 Edge Cover

An edge cover of a graph is a set of edges such that every vertex of the graph is incident to at least one edge in the set. for the graph G_1, possible edge cover sets can be

- $\{(A, B), (B, C), (D, E), (E, F)\}$ (Minimum sized vertex cover)

- $\{(A, B), (B, C), (B, E), (D, E), (E, F)\} = E(G1)$ (Trivial edge cover)

Similarly, for the graph G_2, edge cover sets can be

- $\{(A, B), (C, D), (D, E)\}$ (Minimum sized vertex cover)
- $\{(A, B), (A, E), (C, D)\} = E(G1)$ (Trivial edge cover)
- $\{(A, E), (B, E), (D, E)\}$ (Minimum sized vertex cover)
- $\{(A, B), (B, C), (B, E), (D, E), (E, F)\} = E(G1)$ (Trivial edge cover)

7.2.3 Independent sets

Independent sets can be defined as a set of vertices that are non-adjacent.

Thus, for the graph G_1, the possible independent sets can be

- ϕ(Trivial independent set)
- $\{A, C, E\}$
- $\{(B, D, F)\}$
- $\{A, C, D, F\}$(Maximum sized independent set)

Similarly, for the graph G_2, the possible independent sets can be

- ϕ (Trivial independent set)
- $\{A, C\}$
- $\{(B, D)\}$
- $\{A, C, D\}$(Maximum sized independent set)

It is self-evident that both the smallest vertex cover and the largest independent sets are capable of covering the entire vertex set of the graph. This assertion is backed up by a theorem.

7.3 Independent Sets and Covers

Algorithm : We describe the following functions for a simple graph G

- $\alpha(G)$: Maximum size of independent set in G
- $\alpha'(G)$: Maximum size of matching in G
- $\beta(G)$: Minimum size of vertex cover in G

- $\beta(G)$: Minimum size of edge cover in G

Then, these functions satisfy the following for all simple graph G:

$$\alpha(G) + \beta(G) = |V(G)| = \alpha(G) + \beta(G) \tag{7.5}$$

Solution : We include proof for only $\alpha(G) + \beta(G) = |V(G)|$ as follows:Let S be an independent set of maximum size for any graph G.

$$|S| = \alpha(G) \tag{7.6}$$

Now, it is trivial to see that all the vertices in S are adjacent to at least one vertex in S. Thus all vertices in S is adjacent to every vertex in G. Therefore, S is a valid vertex cover.

Also, since S is the maximum sized independent set, the vertex cover S is minimum sized.

$$\beta(G) = |S| \tag{7.7}$$

So,

$$|V(G)| = |S| + |S| \implies |V(G)| = \alpha(G) + \beta(G) \tag{7.8}$$

Additional Theorem
With the above function definitions, the following also holds true

$$\alpha(G) = \beta(G) \tag{7.9}$$

$$\alpha(G) + \beta(G) = \alpha(G) + \beta(G) \tag{7.10}$$

$$\alpha(G) + \beta(G) = \beta(G) + \alpha(G) \tag{7.11}$$