Week 12

Lecture 23 & 24*

12.1 Graph Coloring

Motivation

Consider we have to create a timetable for a class. It implies we have some constraints while creating a timetable, such as if a student registered for two subjects, then those two subjects can not be scheduled in the same time slot. If the lectures have to schedule in the same classroom, no two classes can overlap. For example, consider the following table.

	physics	**calculus**	**electronics**	**math**	**OS**
physics	0	1	1	1	0
calculus	1	0	0	1	1
electronics	1	1	0	0	0
math	1	1	1	0	0
OS	0	1	0	1	0

Table 12.1: Row and column names are subject names. 1 in each row represents having some common students with a corresponding column, and 0 means no common students.

There are five subjects in the above table. Here 1 represents that there are common students in both of the subjects. Now the question is, can we find a schedule such that no class overlap in time domain? This question can be answered by a graph coloring problem. To this end 12.1 can be formulated as a graph. For constructing a graph, each subject is considered as a node, and there will be edges between nodes or subjects if they have common students.

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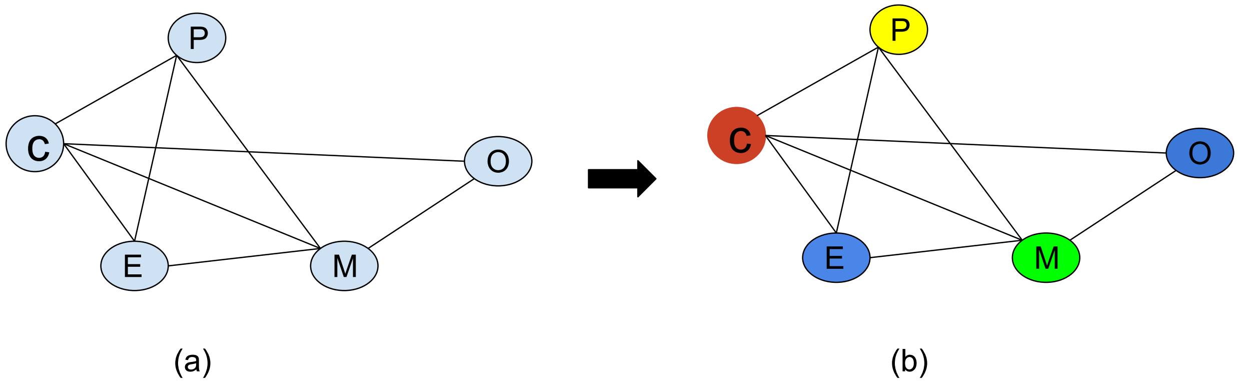

Figure 12.1: (a) Represent the graph corresponding to Table 12.1 with nodes name as first letter of subjects, (b) represent the minimum coloring of the graph.

Figure 12.1 (a) Represent the graph corresponding to the Table 12.1 and Figure 12.1 (b) represents the minimum coloring of the graph. We require a minimum of four colors to color the graph such that adjacent nodes do not have the same color. Hence we can have four different time slots for scheduling all five lectures.

Definition 12.1. A **k-coloring** of a graph G is a labelling $f : V(G) \rightarrow S$, where $S = \{c_1, c_2, ..., c_k\}$ is a subset of k colors (total possible labels are n^k). A k-coloring is proper if adjacent vertices have different labels.

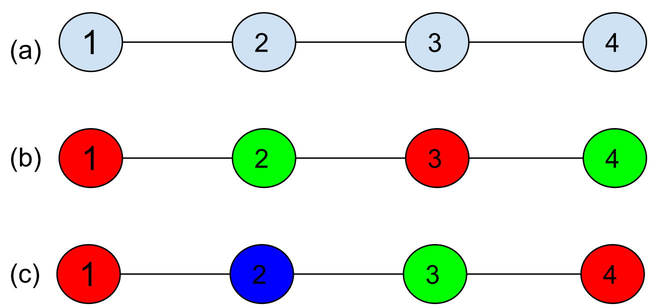

Figure 12.2: (a) A graph with four nodes, (b) and (c) are proper coloring of the graph where.

Example: Consider Figure 12.2 (a) with $S = \{R, G, B, Y\}$ we can color it with (b), (R, G, R, G) or (c), (R, B, G, R) i.e with two or three color respectively. Both (a) and (b) represent proper coloring of (a).

Chromatic Number (χ): The least k such that graph G is proper k-colorable.

A greedy algorithm for Graph Coloring

1. Choose any vertex and color it.
2. Do the following for remaining $|V| - 1$ vertices:
    2.1 Choose any non color vertices.
    2.2 Color it with the lowest numbered color that has not been used on any previously colored vertex adjacent to it.
    2.3 If all previously used colors appears on the adjacent vertices then assign a new color to it.

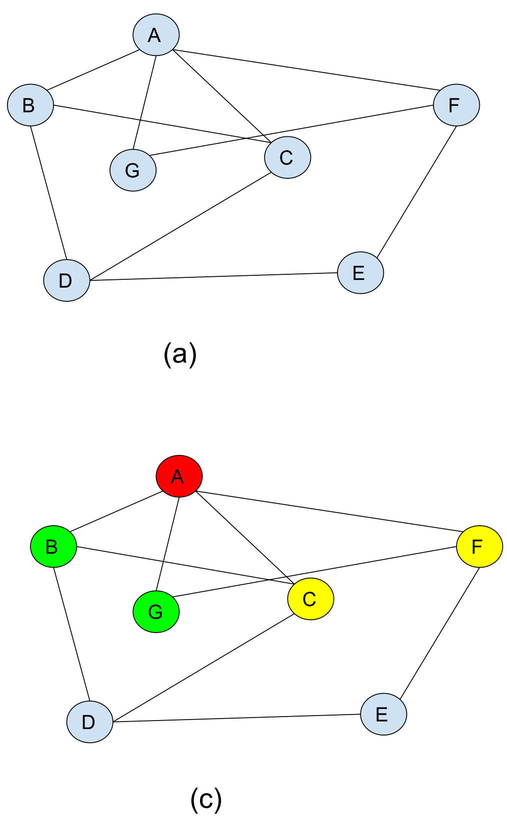

Figure 12.3: (a) A graph with seven nodes, (b) graph after coloring node A (step 1), (c) graph after coloring B,G,C and F respectively, (d) final graph after coloring E then G.

Example: Consider a graph with seven nodes given in figure 12.3 (a) and $S = \{R, G, Y, C, B, W\}$. Figure 12.3 (b) shows the coloring after choosing node A. Figure 12.3 (c) shows coloring after selecting nodes B,G,C and F respectively. Figure 12.3 shows coloring after selecting nodes E and G respectively.

Proposition 1. For every graph G, $\chi(G) \geq$ clique size.
Proof: Vertices of the clique are connected to each other, which means every vertex of the clique requires a distinct color. It implies we need at least clique size colors. Hence the chromatic number of a graph is at least its clique size.

Statement: Given a graph G, $\chi(G) = max(\chi(G_1), \chi(G_2), ..., \chi(G_k))$, where G_i is the i^{th} blocks of graph G.
Proof:[1] Proof by mathematical induction:
If G has only one block, then the claim is immediate. Otherwise G is disconnected or has a cut-vertex v. In either case, we have subgraphs $H1, H2$ whose union is G, such that H_1, H_2 are disjoint (if G is disconnected) or share only the vertex v (if v is a cut-vertex). The blocks of G are precisely the blocks of H_1 and H_2. Each has fewer blocks than G. Thus the induction hypothesis implies that $\chi(G) = max(\chi(H_1), \chi(H_2))$. The lower bound holds because both $H1$ and $H2$ are subgraphs of G. For the upper bound, assume by symmetry that $\chi(H_1) \geq \chi(H_2)$. Starting with an optimal coloring of H_1, we can incorporate an optimal coloring of H_2 by switching a pair of color names to make the coloring agree at v (if G is connected). This produces a proper coloring of G.

Bounding on chromatic number

Trivial upper bound: $\chi(G) \leq |V(G)|$
Trivial lower bound: $\chi(G) > 0$

Proposition 2. $\chi(G) \leq \Delta(G) + 1$, where $\Delta(G)$ is max degree of graph G.
Proof:[2] If we order the vertex in descending order, each vertex has at most earlier $\Delta(G)$ neighbors, so the greedy coloring cannot be forced to use more than $\Delta(G) + 1$ colors. This proves that $\chi(G) \leq \Delta(G) + 1$.

Bounds on the chromatic number on some known graphs

Star graph: As $\Delta(G) = n - 1$ so $\chi(G) \leq n$
Wheel graph: As $\Delta(G) = n - 1$ so $\chi(G) \leq n$
Complete graph: As $\Delta(G) = n$ so $\chi(G) \leq n + 1$
Cycle graph: As $\Delta(G) = 2$ so $\chi(G) \leq 3$

Proposition 3. (Welsh-Powell) If a graph G has degree sequence $d_1 \geq d_2, ..., \geq d_n$, then $\chi(G) \leq 1 + max\{d_i, i - 1\}$.
Proof[2]**:** Let apply greedy coloring to the vertices in nonincreasing order of degree. So color the i^{th} vertex v, , it has at most $min\{d+i, , i|1\}$ earlier neighbors, so at most this many colors appear on its earlier neighbors. Hence the color we assign to v_i, is at most $1 + min\{d_i, i-\}$.

[1] Proof is taken from here: https://bayanbox.ir/view/2051120523710881534/Introduction-to-graph-theory-solution-manual.pdf
[2] https://bayanbox.ir/view/2051120523710881534/Introduction-to-graph-theory-solution-manual.pdf

This holds for each vertex, so we maximize over i to obtain the upper bound on the maximum color used.

12.2 Interval Graphs

Motivation

Consider there is fest in your college. You are asked to organize different events, and every event organizer has given you the duration along with a start and finish time. Now the questions are, (a) In one Hall, how many maximum events can be organized? (b) If all activities have to be performed, then how many rooms are required?
The first question can be answered using formulating it as **Interval Scheduling** problem. The second question can be answered using **Graph Interval coloring**.

activity	**start**	**end**
a1	1	4
a2	4	8
a3	2	5
a4	6	7

Table 12.2: a1 to a4 are activities with given start and end time.

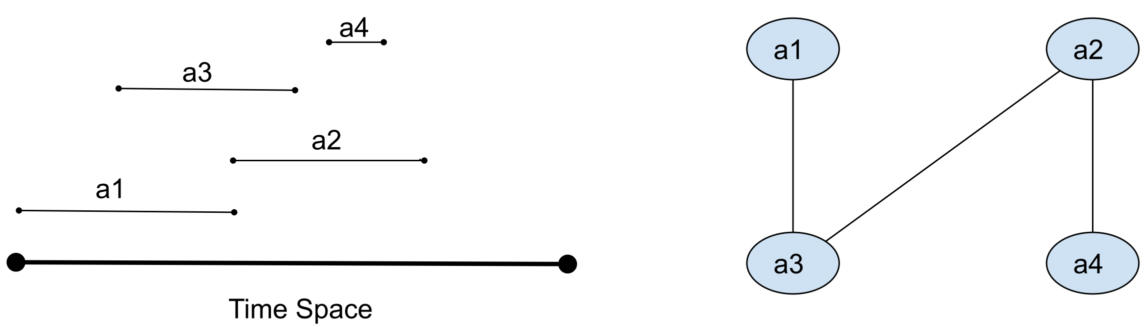

Figure 12.4: Left part of the image represent the activities in time domain and rigth part is corresponding interval graph of Table 12.2.

For example, consider Table 12.2 which represent four activity along with their start and end time. The first part of the figure 12.4 represents the same activities in time-space, and the second part is the corresponding interval graph. The nodes in the interval graph represent the activity, and there is an edge between activities if they overlap in time-space. Now answer of second question is two as the corresponding graph has chromatic number as 2.

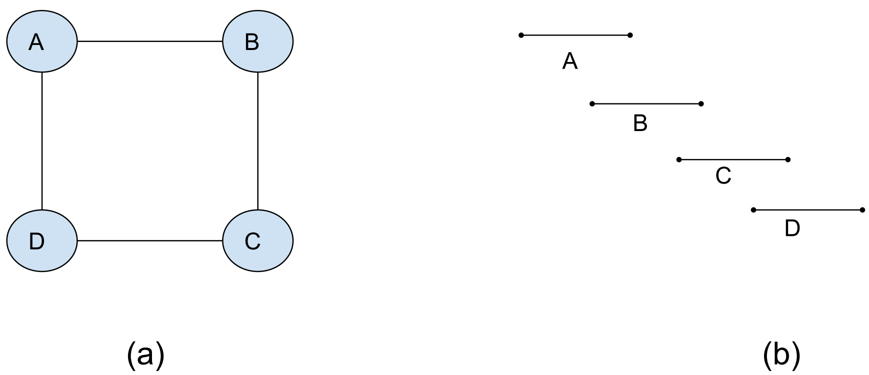

Figure 12.5: (a) represent C_4 and (b) is overlapping activities or nodes of C_4 where A and D activities not able to overlap in corresponding time-space.

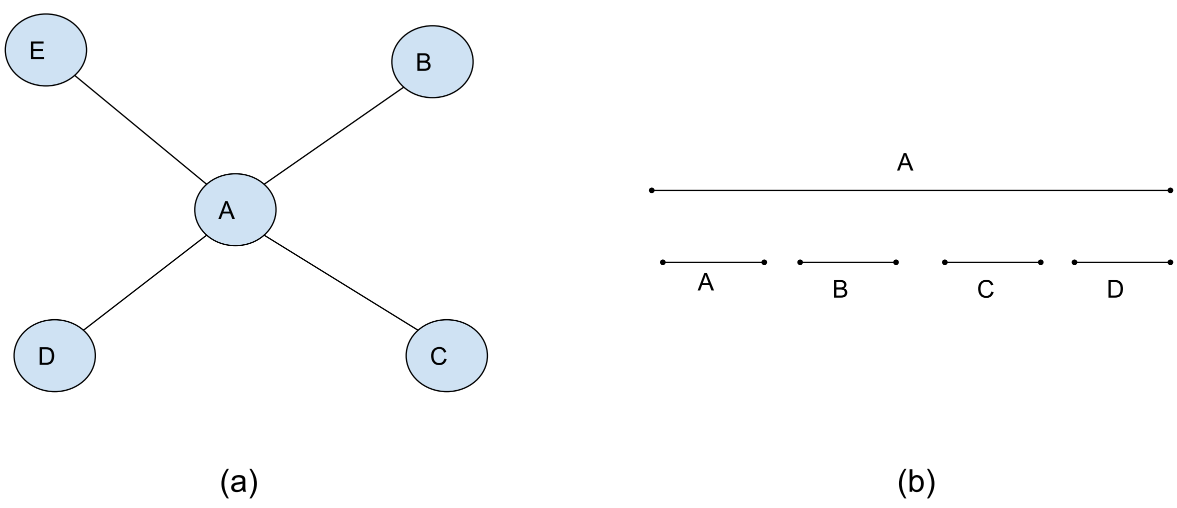

Figure 12.6: (a) represent a star graph and (b) is overlapping activities or nodes in corresponding time-space.

Note: (i) Not every graph is an interval graph. For example consider figure 12.5 (a) where graph is C_4 and figure 12.5 (b) is corresponding time-space representation of C_4. We can see that activities or nodes A and D are not overlapping in time-space, and it is not possible

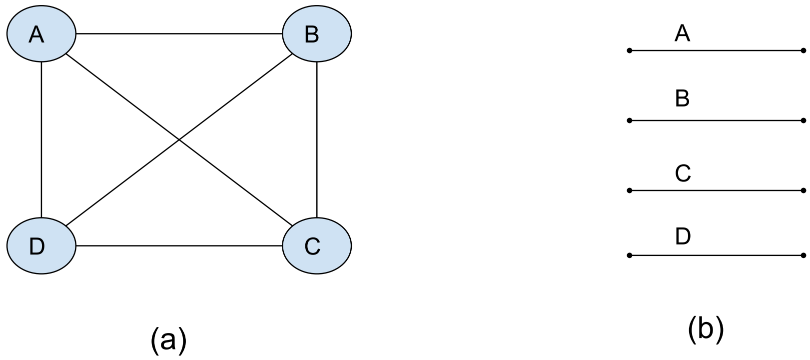

Figure 12.7: (a) represent a complete graph, K_4 and (b) is overlapping activities or nodes in corresponding time-space.

to make the overlap while maintaining other overlapping constraints. Hence C_4 is not an interval graph.
(ii) Start graph is an interval graph. For example consider fig 12.6 (b) where start graph represented in time-space.
(iii) Complete graph is an interval graph. For example consider fig 12.7 (b) where complete graph, K_4 represented in time-space.

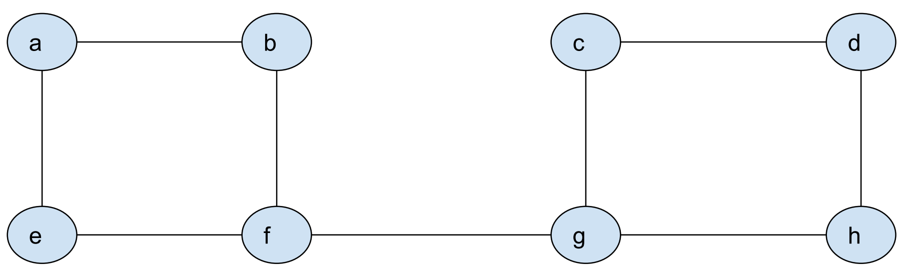

Figure 12.8: A graph with 8 nodes. It is not a interval graph.

Property of Interval Graphs

A graph that has no induced subgraph as C_4 and it's complement graph has transitive orientation is an interval graph.
For example in figure 12.8 we can have an induced graph as C_4 when we remove nodes g, c, d and e. So the graph is not an interval graph.

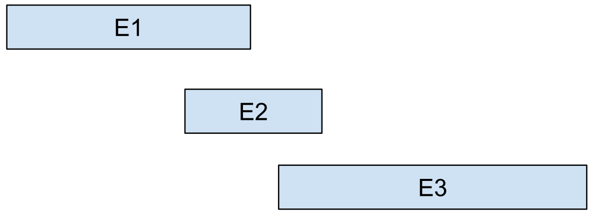

Figure 12.9: Three activities E1, E2 and E3. According to heuristic 1 answer is E2 i.e one which is not optimal.

Greedy Approaches for Room/Interval Scheduling Problem

Heuristic 1: Select shortest event first.
Not always give optimal solution. For example in figure 12.9 this heuristic will select E2 so answer will be one. So this heuristic fails to give optimal answer as optimal answer will we two (E1 and E3).

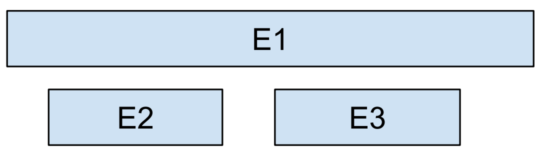

Figure 12.10: Three activities E1, E2 and E3. According to heuristic 2 answer is E1 i.e one which is not optimal.

Heuristic 2: Start with first event then select next possible event that begin as early as possible.
Not always give optimal solution. For example in figure 12.10 it will give E1 i.e one which is not optimal as optimal answer is E2 and E3 i.e two.

Heuristic 3: Select earliest finish time first.
This heuristic always give optimal solution. For example, consider figures 12.9 and 12.10 it will give (E1, E3) and (E2, E3) i.e two and two and these are corresponding optimal solutions.

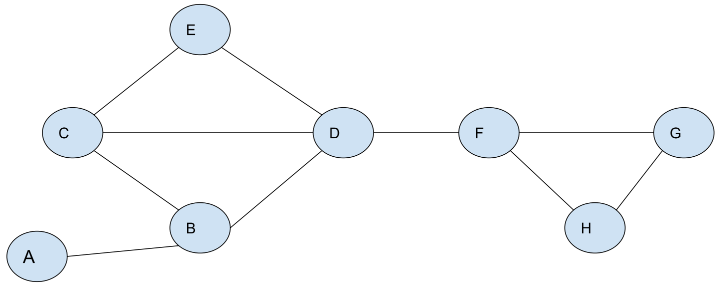

Figure 12.11: Graph with eight nodes.

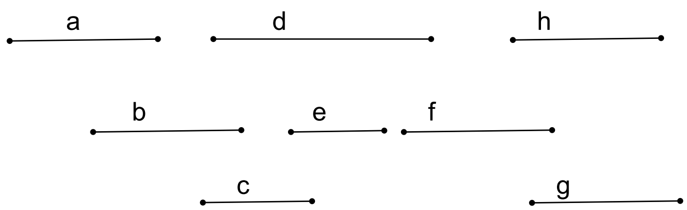

Figure 12.12: Time-space representation of graph given in figure 12.12

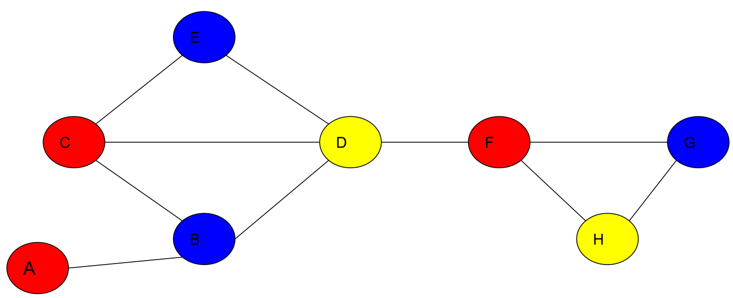

Figure 12.13: Coloring of the graph given in figure 12.11

Question: Consider graph in figure 12.11. Find (a), How many events can be conducted in a single room? (b) How minimum rooms are required so that all events can be organized?
Ans (a): Consider corresponding activities in time-space in figure 12.12. After applying earliest time first we get a,c and f i.e three events.
Ans (b): After applying Earliest start time greedy coloring on graph i.e following the greedy order (a,b,c,d,e,f,h,g) we get 3-colorable graph as represented in figure 12.13. So answer is three.