INDIAN INSTITUTE OF TECHNOLOGY- JODHPUR

GRAPH THEORY AND APPLICATIONS(GTA-2)
COURSE CODE: CSL7410

Lecture Scribing Assignment: Week 10

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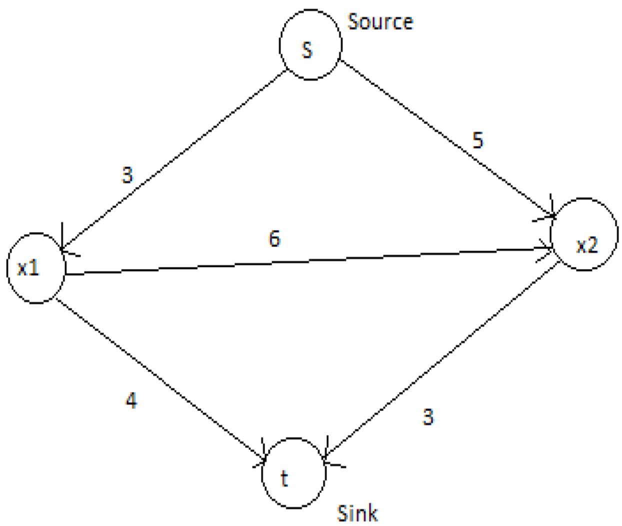
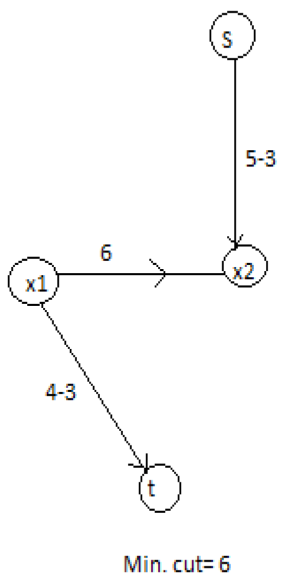
1 Pseudo Boolean function

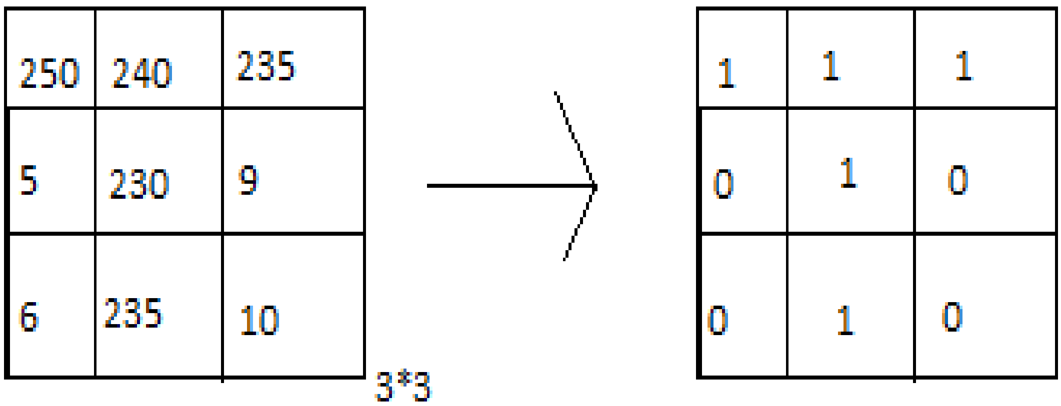
Definition: a pseudo-Boolean function is a function of the form
function f: { 0,1} $\rightarrow R$
f(x)=$3x_1+4\bar{x_2}+5x_2+3\bar{x_2}+6\bar{x_1}x_2$
The degree of the pseudo-Boolean function is simply the degree of the polynomial in this representation
Graph construction as follow-:

1. Every cut of that graph corresponds to some assignment to variables.

2. Min cut=minimum cost of assignment
   f(x)=$3x_1+4\bar{x_2}+5x_2+3\bar{x_2}+6\bar{x_1}x_2$







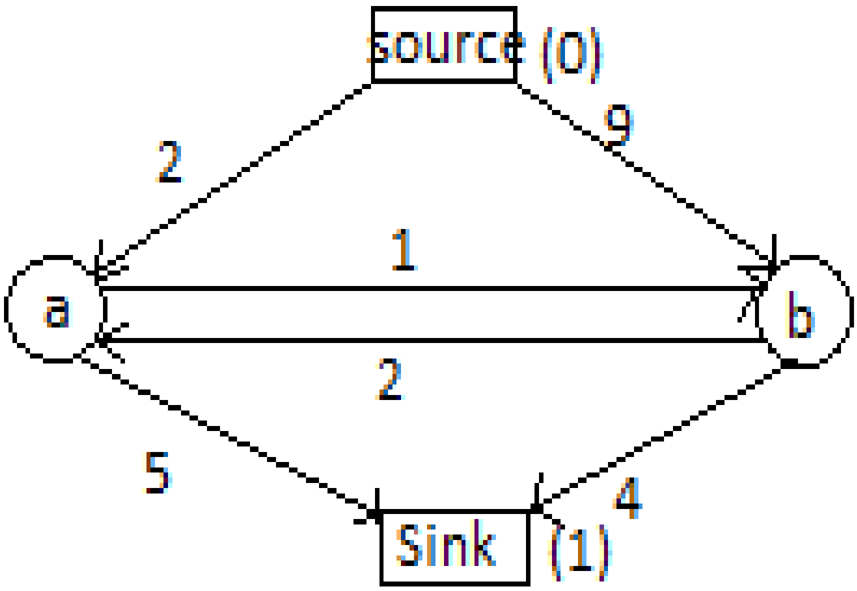
(255-P(x_1))x_1+P(x_1)$\bar{x_1}$

Construct a graph such that-

1. Any cut-corresponds to an assignment of x

2. The cost of the cut is equal to energy of x;

Let us understand how a graph is constructed for a given energy ;for simlicity set us assume there are two pixels $a_1$and a_2
E(a_1,a_2)=$2a_1+5\bar{a}_1+9a_2+4\bar{a}_2+2\bar{a}_2a_1+\bar{a}_1a_2$



implemented issues-

Question-what energy function can be minimized?

1. General energy function:NP had to minimize,only apporimate solutions available.

2. Easy energy functions (Sub modular functions) are graph representable and solvable in polynomial time.

Question-what is sub-modular function.
let f be a function defined over a set of boolean variable X $= (x_1, x_2, ..., x_n)$

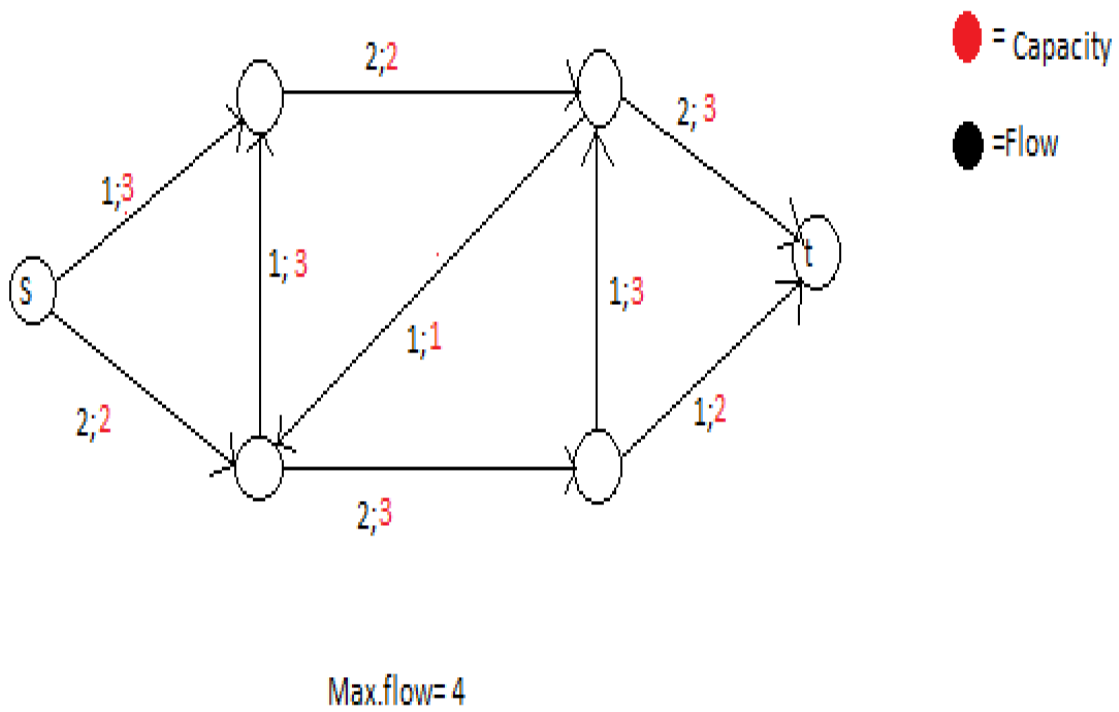
1. All function of one boolean variable are sub-modular.

2. A function of two boolean variable is sub-modular if
f(0,0)+f(1,1)$\leq$ f(0,1) + f(1,0)

3. In general a function is sub-modular if all its projection to two variable are sub-modular.

2 Flow Network

G(V,E) directed graphs two distinguished vertices source(s) and sink(t)
each edge (u,v)$\in$ E ,non-negative Capacity C(u,v)
If (u,v)$\notin$E then C(u,v)=0



Given a Flow Network G,Find a flow with maximum value on G.
Flow- A Flow on G is function f:V*V$\rightarrow$C(u,v)

3 Flow Conservation:

for all u$\in$V-(s,t)
$\sum_{u\in V}$f(u,v)=0

4 Skew-symmetry:

for all u,v$\in$V,f(u,v)=-f(v,u)

The value of a flow f,denoted |f|

|f|= $\sum_{v\in V} f(S,v) = f(S,V)\rightarrow$ Implicit Summation

f(S,V_4)= -f(V_4,S)
= -1
Simple Properties-:

- f(X,X)=0
- f(X,Y)=-f(Y,X)
- f(X $\cup$ Y,Z)=f(X,Z)+f(Y,Z) , if X $\cap$ Y=ϕ

Theorem: |f|=f(V,t)
Proof

$$|f| = f(S,V) = f(V,V) - f(V-S,V)$$

since f(V,V)=0

$$|f| = f(S,V) = f(V,V-S)$$

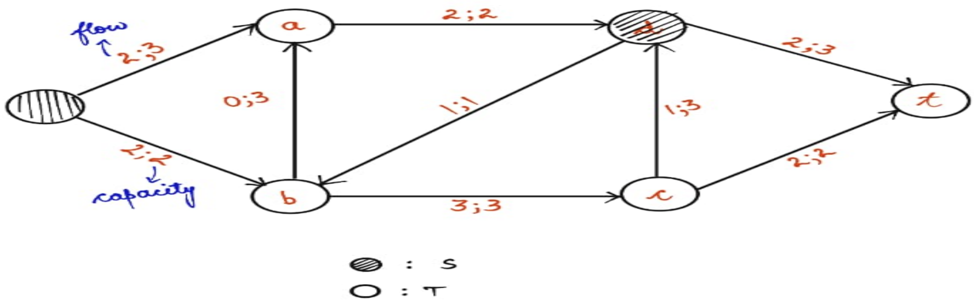
$$|f| = f(S,V) = f(V,t) + f(V,V-S-t)$$

$$|f| = f(S,V) = f(V,t) - f(V-S-t,V)$$

$$|f| = f(S,V) = f(V,t)$$

5 Cuts

A cut(S,T) of a flow network G=(V,E) is a partition of V s.t s$\in$S and t$\in$T
if f is a flow on G then flow across the cut f(S,T)



6 Capacity cut

C(S,T)
= (3+2)+(1+3)
= 9
The value of any flow is bounded by the capacity of any cut
Another characterization of flow value :
Lemma:- For any flow f and any cut (S,T) , we have
|f|= f(S,T)
Proof:-
f(S,T) = f(S,V)- f(S,S)
= f(S,V)
= f(s,V) + f(S-s,V)
= f(s,V) =|f|

7 Residual Network;-

$$G(V, E)$$

$$G_f(V, E_f) : -$$

strictly positive residual capacities

$$C_f(u,v) = C(u,v) - f(u,v) > 0$$

edges in E_f admit more flow if (v,u)$\notin$ E,C(V,u)=0'
but f(V,u)=-f(u,V)