# Week 7

# Hall's Theorem, Vertex Cover, Edge Cover, Matching, Independent Set and Relations between them*

## 7.1 Hall's Theorem

In an X,Y-bipartite graph, the objective is to find a matching that saturates X. For that Hall gave a necessary and sufficient condition to saturate X,given below:

**Theorem 7.1.** ***Hall's Theorem***
*An $X,Y$-bigraph $G$ has a matching that saturates $X$ if and only if $|N(S)| \geq |S|$ for all $S \subseteq X$.*

*Proof.* Necessary Condition:
Suppose $X,Y$-bigraph $G$ has a matching that saturates $X$. Then clearly

$$|S| \leq |N(S)|, \qquad \text{for all } S \subseteq X$$

Sufficient Condition:
*To Prove* : If $\forall\ S \subseteq X$, $|N(S)| \geq |S|$, then there is a matching that saturates $X$.
We will prove its contrapositive statement given below.
*Contrapositive* : If there is not such matching $M$ that saturates $X$, then $\exists\ S \subseteq X$, such that

$$|S| > |N(S)|$$

Let $u \in X$ be a vertex unsaturated by a matching $M$, where $M$ is a matching consisting of dark lines in figure (7.1).
Suppose two subsets $S \subseteq X$ and $T \subseteq Y$ are considered as follows:
$S$ = End points of $M$-alternating path with the last edge belonging to $M$

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$T$ = End points of $M$-alternating path starting from $'u'$ with the last edge not belonging to $M$

From the fig (7.1),
$S$ contains four vertices including $'u'$ and $T$ contains three vertices.
$\Rightarrow |S| = 1 + |T| = 1 + |N(S)|$
$\Rightarrow |S| > |N(S)|$
Hence Proved.

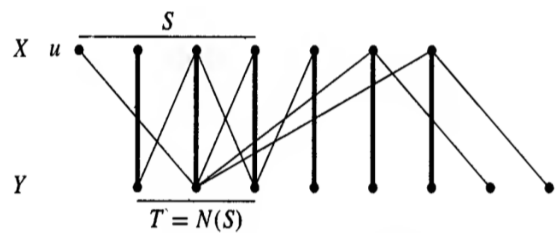


Figure 7.1: Graph showing Hall's Theorem with two partition sets $X$ and $Y$

□

Let's take an example given below to understand Hall's Theorem.

**Example 7.2.** Consider a bigraph given below showing persons in one partition set and rooms in which they can stay in other partition set. Each person is eligible to stay in their respective adjacent rooms. Will everyone get a room?
Ans: By Hall's Theorem, we can say that everyone will get a room, since for every subset $S \subseteq X$ (X=Persons), $|N(S)| \geq |S|$.

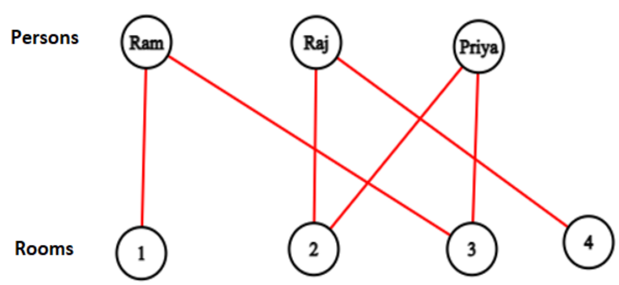


Figure 7.2: Graph with partition sets, Persons and Rooms

## 7.2 Vertex Cover, Edge Cover, Matching, Independent Set

**Definition 7.3.** A **vertex cover** of a graph $G$ is a set $Q \subseteq V(G)$ that contains at least one endpoint of every edge. The vertices in $Q$ **cover** $E(G)$.

**Example 7.4.** Given below are two examples of simple graphs showing vertex cover with orange color vertices.

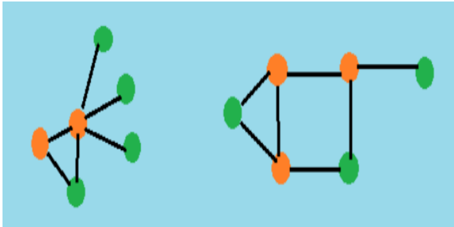

Figure 7.3: Size of vertex cover is (i)2 (ii)3

**Definition 7.5.** An **edge cover** of a graph $G$ is a set $L$ of edges such that every vertex of $G$ is incident to some edge of $L$. We say that the vertices of $G$ are **covered** by the edges of $L$.

**Example 7.6.** In the figure given below, $L = \{AE, AF, BC, BD\}$ is the edge cover of the given graph.

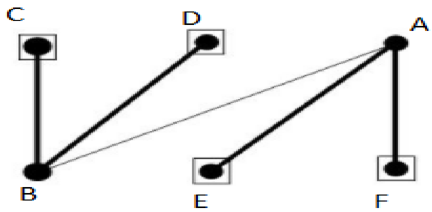


Figure 7.4: Size of edge cover is 4

**Definition 7.7.** An **independent set** of a graph $G$ is a set of pairwise non-adjacent vertices.

**Example 7.8.** In the figure (7.5), $I = \{C, D, E, F\}$ is an independent set.

**Some Notations:** $G$ is a graph

- $\alpha(G)$ = maximum size of independent set
- $\alpha'(G)$ = maximum size of matching
- $\beta(G)$ = minimum size of vertex cover
- $\beta'(G)$ = minimum size of edge cover
- $n(G)$ = total number of vertices in $G$

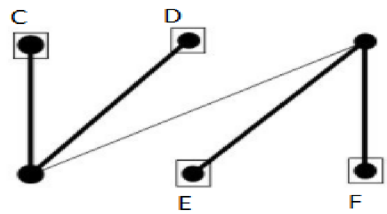


Figure 7.5: Size of independent set is 4

## 7.3 Relations between Vertex Cover, Edge Cover, Matching, Independent Set of a Graph $G$

**Theorem 7.9.** *If $G$ is a bipartite graph,then the maximum size of a matching in $G$ equals the minimum size of a vertex cover of $G$, i.e. $\alpha'(G) = \beta(G)$.*

We will give an example to check this theorem.

**Example 7.10.** Consider the graph given below,

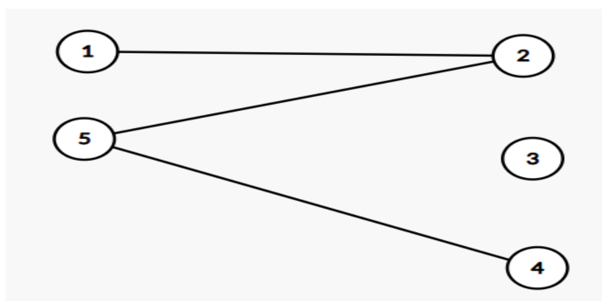


Figure 7.6

Here, maximum matching $= \{12, 54\} \Rightarrow \alpha'(G) = 2$
minimum vertex cover $= \{2, 4\} \Rightarrow \beta(G) = 2$
Hence, $\alpha'(G) = \beta(G)$.

**Lemma 7.11.** *In a graph $G$, $\alpha(G) + \beta(G) = n(G)$.*

*Proof.* Let $S$ be an independent set of maximum size of a graph $G$. And $\overline{S}$ be the complement of $S$. Then every edge is incident to at least one vertex of $\overline{S}$.
$\Rightarrow \overline{S}$ covers all the edges.
$\Rightarrow \overline{S}$ is minimum size vertex cover.
i.e. $\beta(G) = |\overline{S}|$
And $S$ is maximum size independent set.
i.e. $\alpha(G) = |S|$
$\therefore \alpha(G) + \beta(G) = |S| + |\overline{S}| = |V(G)|$, where $V(G)$ is the set of total vertices in $G$.

$\because |V(G)| = n(G)$
$\Rightarrow \alpha(G) + \beta(G) = n(G)$
Hence Proved. $\square$

Let's take an example to apply this lemma.

**Example 7.12.** Consider the graph given below,

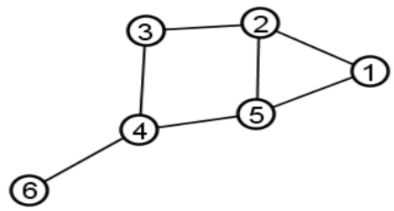


Figure 7.7

Here, maximum independent set $= \{3, 5, 6\} \Rightarrow \alpha(G) = 3$
minimum vertex cover $= \{1, 2, 4\} \Rightarrow \beta(G) = 3$
total number of vertices $= 6 = n(G)$
Hence, $\alpha(G) + \beta(G) = n(G)$.

### 7.3.1 Theorems related to Graphs without Isolated Vertices

**Theorem 7.13.** *If $G$ is a graph without isolated vertices, then $\alpha'(G) + \beta'(G) = n(G)$.*

*Proof.* Let $M$ be a maximum matching in $G$. We construct an edge cover of $G$ by adding to $M$ one edge incident to each unsaturated vertex. We have used one edge for each vertex, except that each edge of M takes care of two vertices, so the total size of this edge cover is $n(G) - |M|$, as desired.
Since a smallest edge cover is no bigger than this cover, this will imply that

$$\beta'(G) \leq n(G) - \alpha'(G) \tag{7.1}$$

Now let $L$ be a minimum edge cover. If both endpoints of an edge $e$ belong to edges in $L$ other than $e$, then $e \notin L$, since $L - \{e\}$ is also an edge cover. Hence each component formed by edges of $L$ has at most one vertex of degree exceeding 1 and is a star (a tree with at most one non-leaf). Let $k$ be the number of these components. Since $L$ has one edge for each non-central vertex in each star, we have $|L| = n(G) - k$. We form a matching $M$ of size $k = n(G) - |L|$ by choosing one edge from each star in $L$.
Since a largest matching is no smaller than this matching, this will imply that

$$\alpha'(G) \geq n(G) - \beta'(G) \tag{7.2}$$

Then from equations (7.1) and (7.2), we get

$$\alpha'(G) + \beta'(G) = n(G)$$

$\square$

To show the theorem above, we will take an example.

**Example 7.14.** Consider the same graph again,

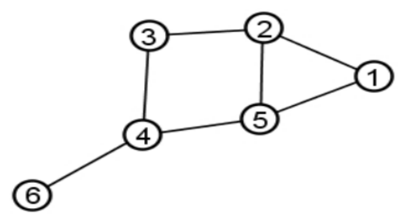


Figure 7.8

Here, maximum matching = $\{15, 23, 46\} \Rightarrow \alpha'(G) = 3$
minimum edge cover = $\{15, 23, 46\} \Rightarrow \beta'(G) = 3$
total number of vertices = $6 = n(G)$
Hence, $\alpha'(G) + \beta'(G) = n(G)$.

**Corollary 7.15.** *If $G$ is a bipartite graph without isolated vertices, then $\alpha(G) = \beta'(G)$.*

*Proof.* We will use Lemma 7.11 and Theorem 7.13 to prove this corollary.
From Lemma 7.11,

$$\alpha(G) + \beta(G) = n(G) \tag{7.3}$$

And from Theorem 7.13,

$$\alpha'(G) + \beta'(G) = n(G) \tag{7.4}$$

Then from equations (7.3) and (7.4), we have

$$\alpha(G) + \beta(G) = \alpha'(G) + \beta'(G) \tag{7.5}$$

Since $G$ is a bipartite graph, so from Theorem 7.9, $\alpha'(G) = \beta(G)$
Applying this to (7.5), we get

$$\alpha'(G) = \beta(G)$$

Hence Proved.

□

We will give an example to show this corollary.

**Example 7.16.** Consider the bipartite graph (7.9), given below,
Here, maximum independent set = $\{1, 2, 3\} \Rightarrow \alpha(G) = 3$
minimum edge cover = $\{14, 25, 35\} \Rightarrow \beta'(G) = 3$
Hence, $\alpha(G) = \beta'(G)$.

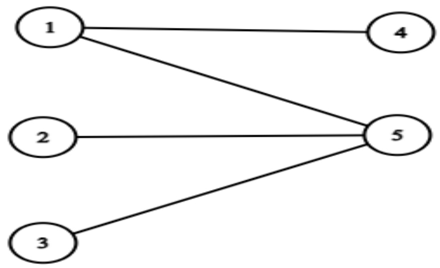


Figure 7.9

**Problem:** Let $G$ be a bipartite graph. Prove that $\alpha(G) = \frac{n(G)}{2}$ iff G has perfect matching.

**Solution** From Lemma (7.11),

$$\alpha(G) + \beta(G) = n(G)$$
$$\alpha(G) = n(G) - \beta(G)$$

Since $G$ is a bipartite graph, then from theorem (7.9), $\alpha(G) = n(G) - \alpha'(G)$ , $[\because \alpha'(G) = \beta(G)]$ when $G$ is a bipartite graph

If $G$ has perfect matching then the maximum size of matching of $G = \frac{n(G)}{2}$
i.e. $\alpha'(G) = \frac{n(G)}{2}$
$\Rightarrow \alpha(G) = n(G) - \frac{n(G)}{2} = \frac{n(G)}{2}$

Hence Proved.

## 7.4 Review of one theorem of Week 5

**Theorem 7.17.** *If $G$ is a simple graph, then if $diam(G) \geq 3$ then $diam(G^c) \leq 3$.*

*Proof.* Given, $diam(G) \geq 3$
$\Rightarrow \exists$ $u$ and $v$ such that :

1. $uv \notin E(G)$ where $E(G)$ is the set of edges
2. $u$ and $v$ do not have common neighbour

$\Rightarrow \forall x \in V(G) - \{u, v\}$ has at least one of $\{u, v\}$ is non-neighbour, where $V(G)$ is the set of vertices.

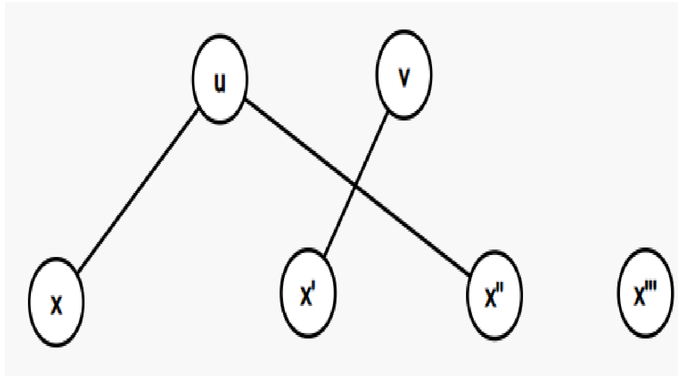


Figure 7.10: Graph of $G$

Let $V(G) = \{u, v, x, x', x'', x'''\}$
$\because uv \notin E(G) \Rightarrow uv \in E(G^c)$
similarly, $ux, ux''', vx, vx'', vx''', xx', x'x'', x''x''' \in E(G^c)$
We get the following graph of $G^c$.

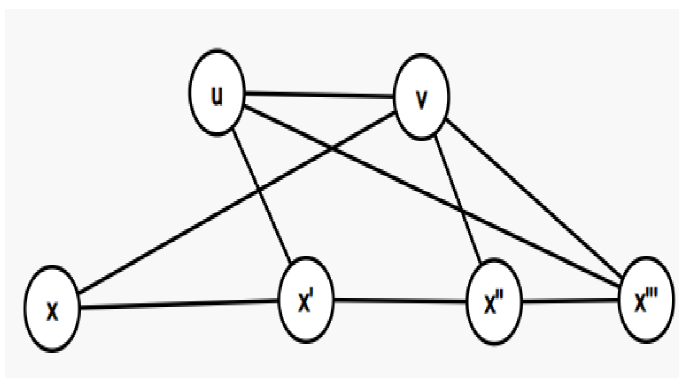


Figure 7.11: Graph of $G^c$

From the graph above, $diam(G^c) \leq 3$
Hence Proved. □