Week 5

Kernel, Orientations and Distance*

The lecture encompasses the concept of kernels in directed graphs or digraphs, vertex degree, orientations of digraphs, tournaments and the concept of kings in tournaments along with basic introductory definition of trees and properties like distance, diameter, eccentricity and radius in graphs.

5.1 Kernel

The section elaborates on kernel definition along with examples.

5.1.1 Definition

Kernel in a directed graph or digraph D is a set of vertices $S \subseteq V(D)$ $s.t$ S induces no edges (S is an independent set) and every vertex outside S has a successor in S.

5.1.2 Example

Consider an example of a directed graph G with six nodes labeled as A, B, C, D, E, F as shown in the below figure.

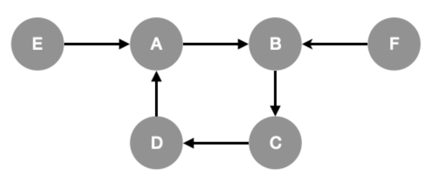

Figure 5.1: Directed graph G

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In the example Figure 5.1, the vertex set *V(G)* is the set, *V(G)* = $\{A,\ B,\ C,\ D,\ E,\ F\}$. We need to find the kernel set S. Let *S1* be the kernel set for digraph *G*. According the kernel definition, *S1* has to be a subset of the vertices set *V(G)*. Consider the set *S1* where, *S1* = $\{A,\ C,\ F\}$. Clearly *S1* $\subseteq$ *V(G)* and here, the successor of all the nodes $v \notin$ *V(G)* lies in *S1*. Also, *S1* is an independent set as it does not induce any edge. Thus *S1* is a valid kernel set of digraph *G*.

Consider another set *S2* where, *S2* = $\{B,\ D,\ E\}$. Here, the successor of all the nodes $v \notin$ *V(G)* lies in *S2* and clearly *S2* $\subseteq$ *V(G)*. Also, *S2* is an independent set as it does not induce any edge. Thus *S2* is also a valid kernel set of digraph *G*. This shows that Kernel set is not unique for any directed graph *G* *i.e.* multiple kernel sets exists.

5.1.3 Proposition

Proposition Directed graph C_n, where n is odd and the successor of v_i is v_{i+1} where, $i \in (1, n-1)$ and for $i = n$, successor $(v_i) = v_1$, do not contain any kernel set.

Proof. Consider a directed cycle C_n with odd number of vertices or nodes as shown below.

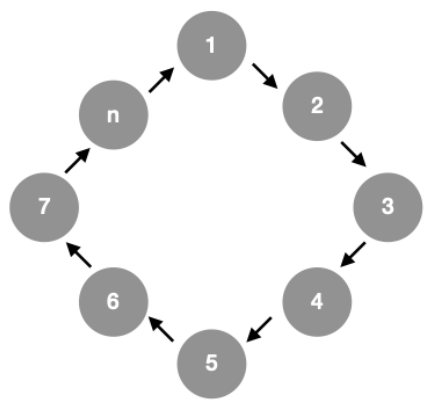

Figure 5.2: Directed cycle C_n

Here, every node has only one successor *i.e.* successor of v_i is v_{i+1} where, $i \in (1, n-1)$ and for $i = n$, successor $(v_i) = v_1$.

The kernel set S induces no edges and also successor of every node vertex $v \notin V(C_n)$ lies in S. This implies that if we are choosing a node vertex v in kernel set S then we have to ignore the successor of v and if we are not choosing a node then we have to choose the successor of that node in kernel set S. Thus we have to choose alternate nodes.

Let say we are choosing vertex v_1, then the next chosen vertex will be v_3 and further next will be v_5 and so on. Since n is odd, thus the last chosen vertex will be v_n. But, the successor of v_n is v_1 and thus introducing this vertex v_n will induce an edge in kernel set S and not including this vertex will lead to not including the successor of the vertex v_{n-1}.

Thus, we cannot create a kernel set S for directed graph C_n, where n is odd and where successor of v_i is v_{i+1} where, $i \in (1, n-1)$ and for $i = n$, successor $(v_i) = v_1$. $\square$

5.2 Vertex Degree

In a directed graph or digraph D, each edge has a direction *i.e.* there are edges which are going out from a vertex and also there are edges which are coming towards a vertex from some other vertex.

5.2.1 Definition

For a vertex v in a digraph S, the **out-degree** of a vertex v is the number of edges going out from that vertex v to other vertex in the digraph. The **in-degree** of a vertex v is the number of edges falling on or coming towards that vertex v from other vertex in the digraph.

The **out-degree** of a vertex v is denoted by $d^+(v)$ and the **in-degree** of a vertex v is denoted by $d^-(v)$.

5.2.2 Example

Consider an example of a directed graph G with four nodes labeled as A, B, C, D as shown in the below figure.

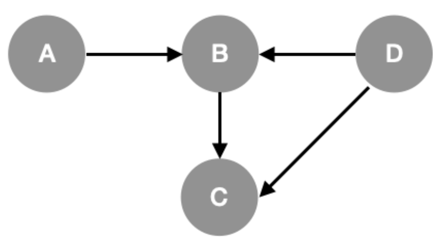

Figure 5.3: Directed graph G

In the example Figure 5.3, the vertex set $V(G)$ is the set, $V(G) = \{A, B, C, D\}$. We need to find the in-degree and out-degree of each vertex in G. For vertex A, $d^+(A) = 1$ and $d^-(A) = 0$. For vertex B, $d^+(B) = 1$ and $d^-(B) = 2$. For vertex C, $d^+(C) = 0$ and $d^-(C) = 2$. For vertex D, $d^+(D) = 2$ and $d^-(A) = 0$.

5.2.3 Proposition

Proposition In a digraph G, $\sum_{v \in V(G)} d^+(v) = \sum_{v \in V(G)} d^-(v) = |e(G)|$.

Proof. For each directed edge uv in digraph G where, $uv \in E(G)$ and u, $v \in V(G)$, If an edge is going out from a vertex then it has to come towards another vertex *i.e.* if an edge is outgoing from a vertex then it has to be incoming at some other vertex in the digraph. It is because each edge starts from a vertex u and ends at another vertex v.

Thus, the summation of out-degrees will be equal to the summation of in-degrees of all vertex in digraph G and this sum will be equal to number of edges in that digraph G *i.e.* size of edge set of G. □

5.3 Orientations and Tournaments

5.3.1 Definition

For any graph G, **Orientation** of that graph G is a digraph D obtained from G by choosing an orientation or direction for each edge $xy \in E(G)$ *i.e.*, $x \to y$ or $y \to x$.

We are given with a graph and we have to make an orientation out of it by choosing some direction for each edge according to requirements.

5.3.2 Example

Consider an example of an undirected graph G with four nodes labeled as A, B, C, D as shown in the below figure.

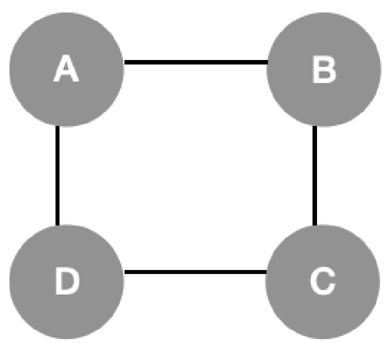

Figure 5.4: Undirected graph G

We have constructed an orientation graph D of an undirected graph G from Figure 5.4 in the manner shown below.

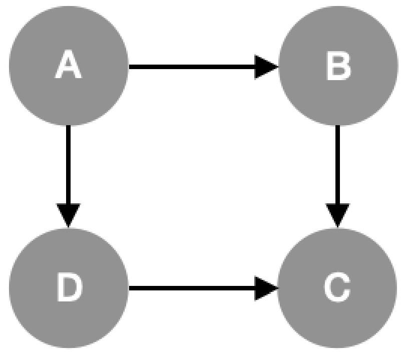

Figure 5.5: Orientation graph D of undirected graph G

5.3.3 Definition

For a complete graph G, **Tournament** is an orientation graph of that complete graph G.

5.3.4 Example

Consider an example of an undirected complete graph G with four nodes labeled as 1, 2, 3, 4, as shown in the below figure.

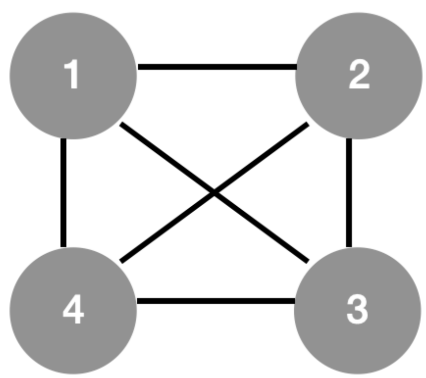

Figure 5.6: Complete graph G

We will construct a tournament graph T from the complete graph G shown in above Figure 5.6. Orientation or direction of an edge $uv \in E(G)$ is chosen in a way $u \rightarrow v$ if $u \leq v$, where $u, v \in V(G)$.

Below is the tournament graph T constructed from the graph G in Figure 5.6.

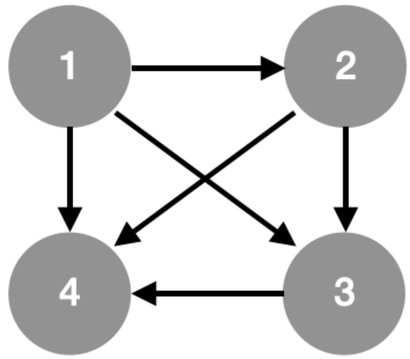

Figure 5.7: Tournament graph T

5.3.5 Definition

For a tournament T, **King** of a tournament is that vertex from where all other vertices could be reached through a path of length at most 2.

It is also possible that multiple kings are present there for a tournament graph T instead of a unique king.

5.3.6 Example

Consider an example of a tournament T with five nodes labeled as A, B, C, D, E, as shown in the below figure.

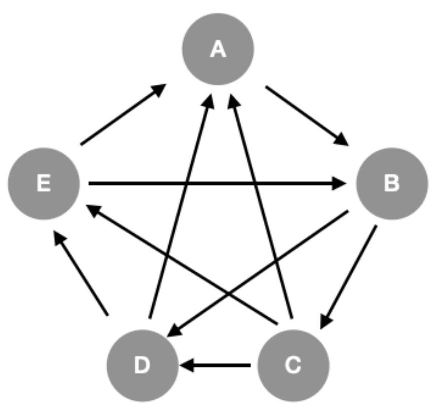

Figure 5.8: Tournament graph T

We will find out all the possible kings in the above shown tournament T Figure 5.8.

From vertex A, We cannot react to vertex E in path of length atmost 2. Thus, A is not a king in T. From vertex B, we can reach to all other vertices in atmost two steps. Thus, B is a king in T. From vertex C, we can reach to all other vertices in atmost two steps. Thus, C is also a king in T. From vertex D, we cannot reach to vertex C in atmost two steps. Thus, D is not a king in T. From vertex E, we can reach to all other vertices in atmost two steps. Thus, E is also king in T.

B, C and E are kings in the above tournament T in Figure 5.8 elucidating that multiple kings are possible for a tournament.

Consider the below example of a tournament $T^{'}$ with three nodes labeled as A, B, C, as shown in the below figure.

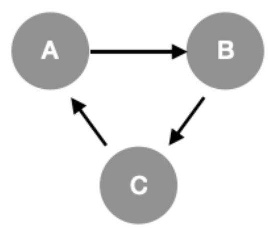

Figure 5.9: Tournament graph T $^{'}$

Here, in Figure 5.9 all the nodes are kings. It is thus possible for all the nodes in a tournament graph to be king.

5.3.7 Proposition

Proposition Every Tournament has a king.

Proof. We will prove this proposition by contradiction.

Consider a Tournament T with vertex set *V(T)* and edge set *E(T)*. Suppose u is a node, where $u \in$ *V(T)*, which has the highest out-degree in tournament T.

Since T is a tournament graph thus, node u will have some successors and some predecessors in T, it will be true that sum of all the successors and predecessors plus the node u will be equal to the total number of nodes in T.

i.e. $|N_o(u)| + |N_i(u)| + 1 = |V(T)|$

Where $N_o(u)$ is set of successor nodes of u or the nodes to which u is connected by an outgoing edge and $N_i(u)$ is the set of predecessor nodes of u or the nodes to which u is connected by an incoming edge in T.

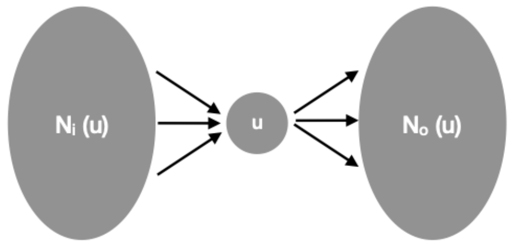

Figure 5.10: Tournament T

Above Figure 5.10 demonstrates the pictorial representation of node u in tournament T where every node other than u will be connected to u in T.

Suppose that the node u is not a king in the above shown tournament. Now, every node is connected to every other node in T. This implies that every node in $N_o(u)$ will be connected to every node in $N_i(u)$ *i.e.* there is a connection where we just don't know the direction of that connection.

Let us assume a direction for the connection between nodes in $N_o(u)$ and nodes in $N_i(u)$.

Case 1 The direction of connection between nodes in $N_o(u)$ and nodes in $N_i(u)$ is $N_o(u) \to N_i(u)$.

In this case, the node u is directly connected to nodes in $N_o(u)$ *i.e.* the nodes in $N_o(u)$ are directly reachable from u through a path of length 1. Whereas the nodes in $N_i(u)$ are not directly reachable from u. However, the direction of connection between nodes in $N_o(u)$ and nodes in $N_i(u)$ is $N_o(u) \to N_i(u)$. This implies that each node in $N_i(u)$ will be reached by a node in $N_o(u)$. Thus, the nodes in $N_i(u)$ are reachable from u through a path of length 2.

It shows that all nodes are reachable from u in a path of length atmost 2. Thus u is a king of the tournament T. This contradicts our supposition that u is not a king.

Case 2 The direction of connection between nodes in $N_o(u)$ and nodes in $N_i(u)$ is $N_i(u) \to N_o(u)$.

Let there be a node vertex v where, $v \in N_i(u)$. Now, v has an outgoing edge to all the nodes in $N_o(u)$ and also to the node u.

$\Rightarrow$ $\mathrm{d}^+(v) = 1 + |N_o(u)|$ $i.e$ $d^+(v) > d^+(u)$

This contradicts that the node u has the highest out-degree in tournament T. This implies that this type of edges cannot exist in tournament T. Thus, there will be some incoming edges to the node v from $N_o(u)$.

Thus, if node v has an incoming edge from $N_o(u)$, it can be reached from node u in a path of length 2. Thus every node $v \in N_i(u)$ is reachable by u in a path of length 2.

It shows that all nodes are reachable from u in a path of length atmost 2. Thus u is a king of the tournament T. This contradicts our supposition that u is not a king.

From both the cases, It shows that every tournament has a king.

$\square$

5.3.8 Problem

Prove or Disprove If D is an orientation of a simple graph with 5 vertices then the vertices of D cannot have distinct out-degree.

Proof. Consider a tournament D with five nodes labeled as 1, 2, 3, 4, 5 with $u \to v$ if $u < v$ where, $u, v \in V(D)$.

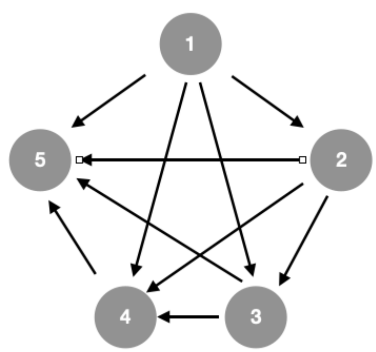

Figure 5.11: Tournament D

In the above Figure 5.11, out-degree of node 1 is 4, node 2 is 3, node 3 is 2, node 4 is 1 and node 5 is 0.

Clearly all nodes have distinct out-degrees.

$\square$

5.4 Trees

Trees are also graphs but they have certain special properties.

5.4.1 Definition

A **tree** is an acyclic connected graph which has a unique path between any two vertices. End point nodes in tree are called leaf nodes and there is a unique path between trees because cycles aren't allowed.

5.4.2 Properties

Following are the properties of trees :

1. Trees are connected and are acyclic.
2. Deleting a leaf node from a n vertex tree results in a n-1 vertex tree.
3. n vertex tree contains n-1 edges.
4. For any two nodes *u, v* there exists only one unique path between them.
5. Every edge of a tree is a cut-edge.
6. Tree is a bipartite graph and the chromatic number of tree is 2.

5.5 Distance in a graph

This section encompasses the properties like distance, diameter of a graph G, eccentricity of a node and radius of a graph along with examples.

5.5.1 Definition

If a graph G has a u-v path, then the **distance** between u to v is written as *d(u, v)* and is the least length of u-v path. The **diameter** is the maximum of distance between any two pair of vertices. The **eccentricity** of a vertex u is the maximum distance it has with any other node in that graph G. The **radius** is the minimum of eccentricities of all the nodes.

5.5.2 Example

Consider a graph G with 12 nodes labeled as A, B, C, D, E, F, G, H, I, J, K, L, as shown in the below figure.

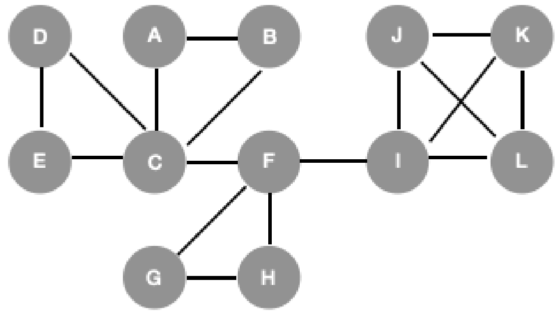

Figure 5.12: Graph G

We need to find the properties - maximum distance between nodes, diameter of graph, eccentricity of nodes and radius of graph.

In the graph G in Figure 5.12, the maximum distance between two nodes is 4, *d(D, K) = 4*. As the diameter of graph is the maximum of distance between any two nodes, thus, diameter of graph G is 4, *diameter(G) = 4*. The eccentricities of all nodes are as, *eccentricity(A) = 4*, *eccentricity(B) = 4*, *eccentricity(C) = 3*, *eccentricity(D) = 4*, *eccentricity(E) = 4*, *eccentricity(F) = 2*, *eccentricity(G) = 3*, *eccentricity(H) = 3*, *eccentricity(I) = 3*, *eccentricity(J) = 4*, *eccentricity(K) = 4*, *eccentricity(L) = 4*. Now, radius of graph is the minimum of eccentricities of all nodes, thus, radius of graph is 2, *radius(G) = 2*.

Theorem 5.1. *If G is a simple graph then **diameter** (diam G) ≥ 3 implies **diameter** (diam G^c) ≤ 3.*

Proof. Given a simple graph G with ***diameter(G)*** ≥ 3.

$\Rightarrow$ $\exists$ *u, v* $\in$ *V(G) s.t. u and v are not adjacent. Also u and v have no common neighbour.*

If u and v are adjacent then in this case the diameter will be 1 and if u and v have common neighbour then the diameter will be 2.

$\Rightarrow$ *uv does not exists and* $\forall x \in V(G) - \{u, v\}$*, ux and vx does not exists simultaneously.*

Thus, G^c will surely have uv edge, ux and vx existing simultaneously and also ux and vx' exist for some $x' \in V(G) - \{u, v\}$.

Case 1 G^c contains uv.

In this case the diameter will be 1.

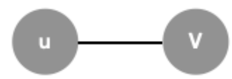

Figure 5.13: uv

Case 2 G^c contains ux and vx simultaneously.

In this case the diameter will be 2.

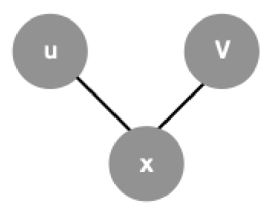

Figure 5.14: ux and vx

Case 3 G^c contains ux and $vx^{'}$ for some $x^{'} \in V(G) - \{u, v\}$.

In this case the diameter will be 3.

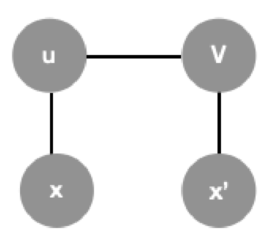

Figure 5.15: ux and ux$^{'}$

Considering all the above three cases, **diameter** (diam G) ≥ 3 implies **diameter** (diam G^c) ≤ 3.

$\square$

5.5.3 Problem

Question Compute diameter and radius of $K_{m,n}$.

In a fully connected bipartite graph, each node in first set is connected to each node in second set. Thus, the maximum distance between any two node if $m > 1$ and $n > 1$ is the distance between two nodes of same set which is 2.

$$diam\ (K_{m,n}) = \begin{cases} 1 & \text{if } m = n = 1, \\ 2 & \text{if } m = 1 \ \& \ n > 1, \\ 2 & \text{if } m > 1 \ \& \ n = 1, \\ 2 & \text{if } m > 1 \ \& \ n > 1. \end{cases}$$

$$radius\ (K_{m,n}) = \begin{cases} 1 & \text{if } m = n = 1, \\ 1 & \text{if } m = 1 \ \& \ n > 1, \\ 1 & \text{if } m > 1 \ \& \ n = 1, \\ 2 & \text{if } m > 1 \ \& \ n > 1. \end{cases}$$