Week 10

Pseudo Booleon function, Image segmentation Using Graph cut, Sub Modular Function, flow Networks, flow, Cuts, Residual Network and Residual Capacity*

10.1 Pseudo Booleon function

$f : \{0, 1\} \longrightarrow R$
$f(x) = 3x_1 + 4\bar{x}_1 + 5x_2 + 3\bar{x}_2 + 6\bar{x}_1 x_2$
$x_1, x_2 \in \{0, 1\}$
Goal is to minimize $f(x)$

We will do it using Graph cut

Graph construction as follows:
(1). Every cut of that graph corresponds to some assignment to variables.
(2). Min cut = minimum cost assignment.

Graph is shown in figure 1

Now we will find min cut of this graph.
min cut = 6
So, x1 = 1 and x2 = 0
f(x) = 3+ 0 + 0 + 3 + 0 = 6 (It is the minimum possible value of f(x)).

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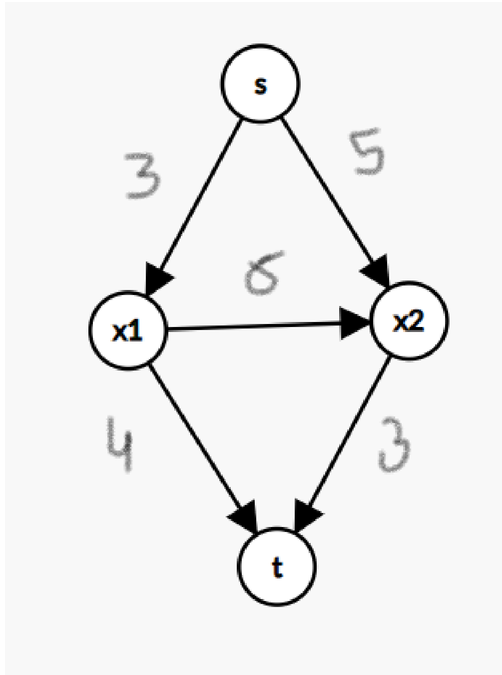

Figure 10.1: figue 1

10.2 Image segmentation Using Graph cut

Let us consider the problem of image segmentation in energy minimization framework. The energy we need to minimize is of following form:

$$E(x) = \sum_i c_i x_i + \sum_{i,j} c_{i,j} x_i (1 - x_j)$$

or we wish to find out the global minima of the above energy i.e.

$$x^* = \text{argmin}_x E(x)$$

Construct a graph such that
- Any cut corresponds to an assignment of x
- The cost of the cut is equal to energy of x : $E(x)$

Let us understand how a graph is constructed for a given energy: (for simplicity let us assume there are two pixels a_1 and a_2
$E(a_1, a_2) = 2a_1 + 5\bar{a}_1 + 9a_2 + 4\bar{a}_2 + 2a_1\bar{a}_2 + \bar{a}_1 a_2$
The graph for this function is shown in fig. 2.

Now,we can can calculate the minimum value of this function with the help the graph that we have made (as shown in fig. 2.) using graph cut.

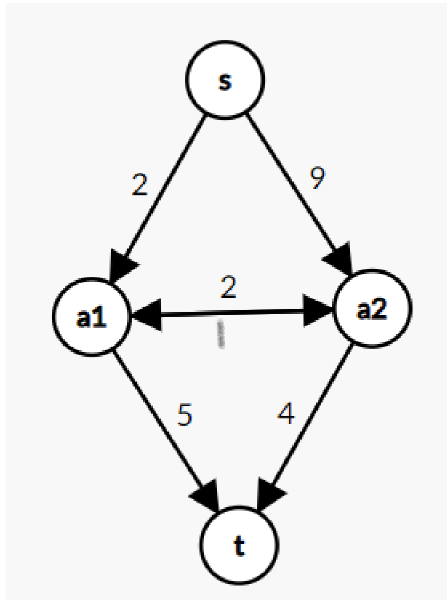

Figure 10.2: figue 2

10.3 Sub Modular Function

Let f be a function defined over set of boolean variables $x = \{x_1, x_2, \ldots, x_n\}$, then
(1) All functions of one boolean variables are sub modular.
(2) A function f of two boolean variable is sub-modular if $f(0,0)+f(1,1) \leq f(0,1)+f(1,0)$
(7) In general a function is sub-modular if all its projection to two variables are sub-modular.

10.4 flow Networks

A flow network G = (V, E) is a directed graph in which each edge $(u,v) \in E$ has a non-negative capacity $c(u,v) \geq 0$
In a flow network we distinguish two vertices source(s) and sink(t).

10.5 flow

Let $G = (V,E)$ be a flow network with a capacity function c, let s be the source of the network and t be the sink. Then a flow in G is defined as a real valued function $f : V \times V \longrightarrow \mathcal{R}$ that satisfies following three properties:

(1) Capacity constraint:

$$f(u,v) \leq c(u,v); \quad \forall u,v \in V$$

(2) Skew Symmetry:

$$f(u,v) = -f(v,u); \quad \forall u, v \in V$$

(3) Flow Conservation:

$$\sum_{u \in V} f(u,v) = 0; \quad \forall u \in V - \{s,t\}$$

- Total positive flow entering a vertex v is defined by: $\sum_{u \in V, f(u,v)>0} f(u,v)$
- Similarly, We can define total positive flow leaving a vertex v as: $\sum_{v \in V, f(v,u)>0} f(v,u)$
- Total net flow of vertex v is defined as total positive flow leaving vertex v minus total positive flow entering that vertex.

Some Simple Properties of Flow Networks:

$f(u,u) = 0$
$f(u,v) = -f(v,u)$
$f(u \cup v, w) = f(u,w) + f(v,w) + u, v = \varnothing$

10.6 Cuts

A cut (S,T) of a flow net $G = (V,E)$ is a partition of V such that $s \in S$ and $t \in T$

If f is a flow on G then the flow across the cut is f(S, T)

Capacity of a cut is the capacity of edges going from vertices belonging to S to vertices belonging to set T.

Value of any flow is bounded by the capacity of any cut.

10.7 Residual Network and Residual Capacity

Residual Network: Given a flow network $G = (V,E)$ and a flow f the residual network of G induced by flow f is $G_f = (V, E_f)$ where

$$E_f = \{(u,v) \in V \times V : C_f(u,v) > 0\}$$

Residual Capacity: The amount of flow we can push from u to v before exceeding the capacity $c(u,v)$ is the residual capacity of $c(u,v)$.