Week 3

Bipartite Graph, Eulerian Circuit and Handshaking Lemma*

3.1 Introduction

▶ First of all, we have learnt about the birth of Graph Theory and practical problems where graph theory involves.

▶ Then we define a formal definition of graph, vertex & edge.

▶ Then we have seen the 20 Avatars of Graphs.

▶ Then we have seen Matrices and Isomorphism.

▶ Then we introduces the definitions of walk, trail, path & cycle.

3.2 Bipartite Graph

Now, our aim is to do characterization of Bipartite Graph using cycles. Let's quickly recall the definition of Bipartite Graph.

Definition: A graph G is said to be a Bipartite Graph if its vertex set can be partitioned into two disjoint independent sets.

• Before doing the characterization of Bipartite Graph using cycles, we first see some observations on cycles with the help of the following question:

Question 1: True or False:
(a) A closed even walk contain a cycle.
(b) A closed odd walk contain a cycle.

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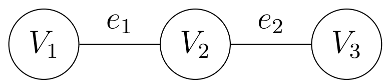
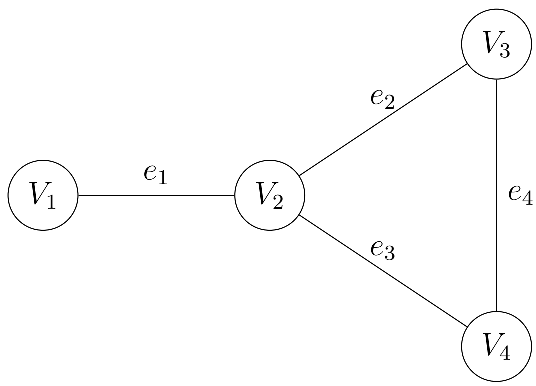
Answer 1: (a) False.



Clearly, here the closed walk is: e_1, e_2, e_2, e_1 and it is an even walk.
But there is no cycle.

(b) True.



Clearly, here the closed walk is:e_1, e_2, e_4, e_3, e_1 and it is an odd walk and the cycle is: e_2, e_4, e_3.

• Since, the second statement is true. Then, we must have to prove it. And in order to prove it we want to take the help of Strong Principle of Induction. So, we have to introduce it first.

Strong Principle of Mathematical Induction: Let Q(n) be a statement with n as positive integer parameter and if
1) Q(1) is true
2) For all n $>$ 1, "Q(k) is true for 1$\leqslant$k $<$ n" $\Rightarrow$ "Q(n) is true".
these condition holds. Then, Q(n) is true for all n.

3.2.1 Lemma:

Statement: Every closed odd walk contains an odd cycle.

Proof: Let W be the closed odd walk and length of the closed odd walk is n.
$\Rightarrow$ n is odd
Since, an odd walk means its length is odd.
Our aim is to apply Strong Principle of Mathematical Induction on the length of the closed

odd walk.
To Show: W contains an odd cycle.
For n=1: we have only one vertex and one edge that means clearly we have W itself a cycle of length one.
Therefore, W contains an odd cycle. The statement is true for n=1.
Let the statement is true for $1 \leqslant n < N$
$\Rightarrow W$ has an odd cycle for n < N.
For n=N: in order to prove for n=N, there are two cases:
Case 1: Suppose there is no repetition of vertex in W and we know that W is a closed odd walk.
$\Rightarrow W$ is itself an odd cycle.
Case 2: Suppose W has repetition of vertex.
Let u be the vertex that repeats. Now, break the walk W into two u,u- closed walks (say w_1 and w_2).
Since, W is of odd length.

$$\Rightarrow |w_1| + |w_2| = |W| = odd$$

$$\Rightarrow either\ |w_1| = odd\ or\ |w_2| = odd$$

without loss of generality, let

$$|w_1| = odd$$

and w_1 is a closed walk
and $|w_1| < |W|$
$\Rightarrow |w_1| < N$.
And we know that W has an odd cycle for n < N.
$\Rightarrow w_1$ has an odd cycle.
Therefore, W has an odd cycle.
now, by combining both the cases, we got that the statement is true for n = N.
$\Rightarrow W$ has an odd cycle for all n.
Hence, Every closed odd walk contains an odd cycle.

• Now, we are ready to learn the characterization of Bipartite Graph using cycles.

3.2.2 Theorem:

Statement: A graph is Bipartite $\Longleftrightarrow$ it has no odd cycle.

Proof: Forward Part: Let G be a Bipartite Graph and X,Y are two independent sets.
To Show: G has no odd cycle. i.e. either G has no cycle or if it has a cycle then it must be of even length.
Suppose G has no cycle.
Then, we are done.
Suppose G has a cycle.

Now, our aim is to show that the length of the cycle must be even.
Since G has a cycle.
In order to make a cycle, we have to travel either from X to Y to X or Y to X to Y.
$\Rightarrow$ the length of the cycle is even.
Hence G has no odd cycle.

Converse Part: Let G has no odd cycle.
$\Rightarrow$ either G has no cycle or it has a cycle of even length only.
To Show: G is a Bipartite Graph.
i.e. G has a bipartition of each nontrivial component.
Let G has no cycle.
$\Rightarrow$ take a vertex in X and a vertex in Y in a trivial manner.
$\Rightarrow$ X and Y are independent.
Let G has a cycle and J be a nontrivial component of G and c be a vertex of J.
For each d $\in$ V(J),define g(d) be the minimum length of c,d-path.
since J is connected.
$\Rightarrow$ g(d) is well defined for each d $\in$V(J).
let X={ d$\in$V(J): g(d) is even} and Y={d $\in$V(J): g(d) is odd}.
now, we just need to show that X and Y are independent.
Without loss of generality, let Y is not independent.
$\Rightarrow \exists$ adjacent vertices d,d' in Y.
now, create a closed odd walk using a shortest c,d-path and an edge dd' and a reverse shortest d',x-path.
and by the Lemma 3.2.1, J has an odd cycle.
$\Rightarrow$ G has an odd cycle.
which is a contradiction.
$\Rightarrow$ X and Y are independent set.
J is an X,Y-bigraph.
Hence, G is Bipartite.

3.2.3 Corollary:

Statement: A path is a Bipartite Graph.

Proof: We know that path is a trail in which there is no repetition of vertices.
$\Rightarrow$ Simple path can not have a cycle.
Therefore by Theorem 3.2.2, A simple path is a Bipartite Graph.

3.2.4 Corollary:

Statement: C_n is Bipartite $\Longleftrightarrow$ n is even.

Proof: We know that C_n has a cycle. (by definition of C_n)
By Theorem 3.2.2, C_n is Bipartite Graph $\Longleftrightarrow$ n is not odd.
Hence, C_n is Bipartite Graph $\Longleftrightarrow$ n is even.

Definition: Let $G_1, G_2, ..., G_n$ be the graphs. The union of graphs $G_1, G_2, ..., G_n$ is the graph having vertex set

$$\bigcup_{j=1}^{n} V(G_j)$$

and edge set

$$\bigcup_{j=1}^{n} E(G_j)$$

The union of graph is denoted by

$$\bigcup_{j=1}^{n} G_j$$

3.3 Eulerian Circuit

• Before introducing the definition of Eulerian Graph and Eulerian Circuit, we should learn that what is the reason behind defining this definition.The reason behind it is Könisberg Bridge Problem. In this problem, we try to find out a walk containing all the edges exactly once and we give this walk a specific name by the following definitions:

Definition: If a graph has a closed trail containing all the edges.Then the graph is called as Eulerian Graph.

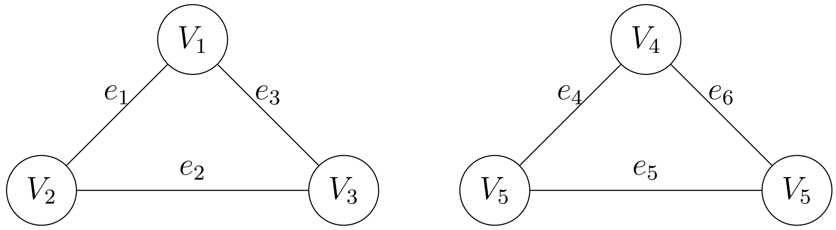
• Before defining the Eulerian circuit, first of all let's define the circuit

Definition: Circuit is a closed trail in which we will not mention the first vertex but keep list in cyclic order.

Definition: A circuit having all the edges is called Eulerian Circuit.

Question 2: True or False:
If every vertex of graph G has at least degree 2.Then G is a cycle.

Answer 2: False.



Here is the graph for countering the statement.Clearly all the vertices are having degree at least 2. But G is not a cycle.

• Now, let us try to modify the above question so that the modified statement will always be true. So, here is the following Lemma:

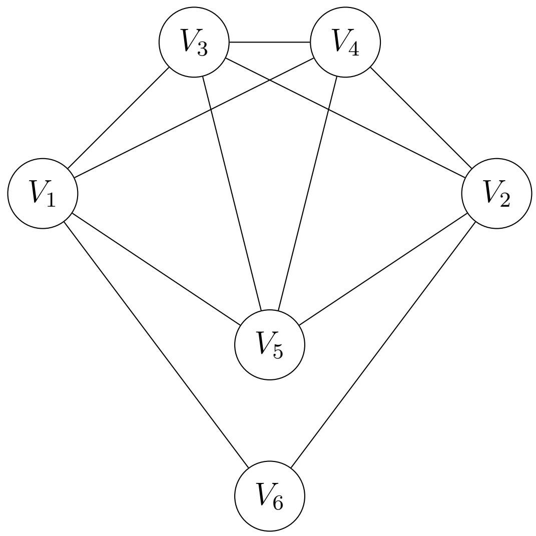
3.3.1 Lemma:

Statement: If every vertex of a graph G has degree at least 2. Then, G has a cycle.

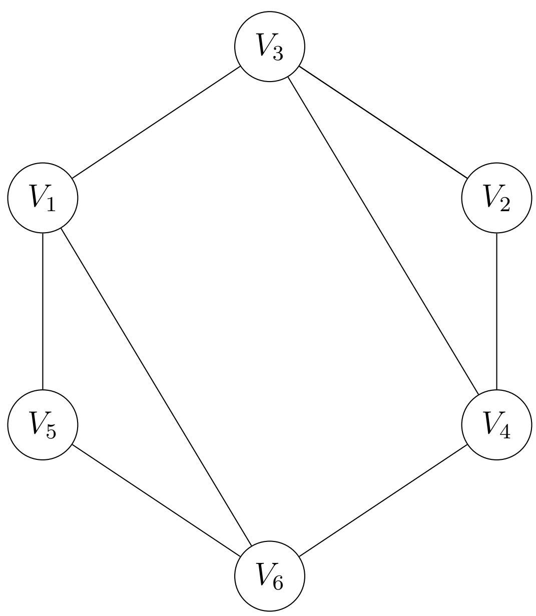
Proof: We will prove it by contrapositive way which means P $\Rightarrow$ Q $\equiv \sim$ Q $\Rightarrow \sim$ P.
And here P $\equiv$ Every vertex of G has degree $\geqslant 2$
$\Rightarrow \sim$ P $\equiv \exists$ a vertex of G having degree < 1.
Similarly, here Q $\equiv$ G has a cycle.
$\Rightarrow \sim$ Q $\equiv$ G has no cycle.
now, let G has no cycle.
$\Rightarrow$ G is a forest (collection of trees) .
now, let U be the component of G.
Case 1: let U is a trivial graph.
$\Rightarrow$ U has only one vertex.
$\Rightarrow$ degree of that vertex $= 0$.
$\Rightarrow \exists$ a vertex of graph G having degree < 1.
Case 2: let U is a non-trivial graph.
$\Rightarrow$ U has atleast two vertices and as U is a component of G.
$\Rightarrow$ They must be connected.
$\Rightarrow$ U has at least two end vertices of degree $\geqslant 1$.
If possible let the end vertex has degree $\geqslant 2$.
$\Rightarrow$ U has a cycle.
which contradicts the property of tree.
$\Rightarrow$ U has at least two end vertices of degree $\geqslant 1$.
$\Rightarrow \exists$ at least two vertices of degree $= 1 < 2$.
Hence, If every vertex of a graph G has degree at least 2.
Then, G has a cycle.

Question 3: Which of the following graph is Eulerian?

(a)



(b)



Answer 3: We want to costruct a path having all the edges that means we have to check each vertex from where we should start so that we can cover all the edges.
For (a) : we can follow the following trail which is closed and covering all the edges:

V_1,V_3,V_4,V_5,V_3,V_2,V_4,V_1,V_6,V_2,V_5,V_1.
For (b) : it will not be any easy task to find out a closed trail covering all the edges.

• Since it is very difficult to check each vertex to get a path having all the edges. Therefore, the following theorem comes into act:

3.3.2 Theorem:

Statement: A graph G is Eulerian

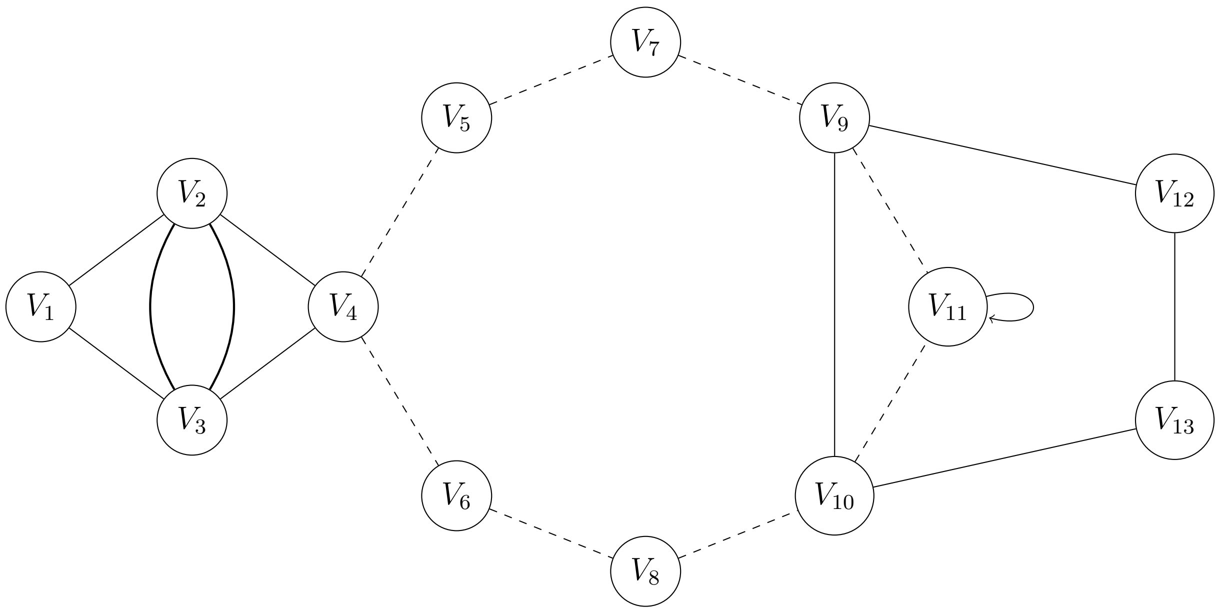
$$\Longleftrightarrow$$

it has at most one nontrivial component and its vertices all have even degree.

Proof: Forward Part: Let G be an Eulerian Graph.
$\Rightarrow$ G must have an Eulerian Circuit, say C_1.
and we know that Eulerian circuit is closed trail containing all the edges.
$\Rightarrow$ at each vertex, there is same number incoming and outgoing of edges.
$\Rightarrow$ every vertex of C_1 has even degree.
and we know that two edges lies in same trail only when they belongs to the same component.
Hence, G has at most one nontrivial component and its vertices all have even degree.

Converse Part: Let G has at most one nontrivial component and its vertices all have even degree.
To Show: G is Eulerian Graph.
let the number of edges in G is M.
now, we will try to apply Strong Principle of Mathematical Induction on M.
let M=0 $\Rightarrow$ there is no edge in G.
$\Rightarrow$ G has only one vertex and our closed trail will consist one vertex.
$\Rightarrow$ G is an Eulerian Graph for M=0.
let the induction hypothesis holds for 0 < n < M.
$\Rightarrow$ G has Eulerian Circuit for 0 < m < M.
now, let G has M edges with at most one nontrivial component and its vertices all have even degree.
therefore, by Lemma 3.3.1,the nontrivial component of G must have a cycle C_2.
let G_1 be the graph obtained by E(C_2).
and we know that any cycle has 0 or 2 edges at each vertex.
$\Rightarrow$ each component of G_1 is also an even graph.
and we know that components are maximal connected subgraphs and number of edges in each component of G_1 is < M.
therefore, by the induction hypothesis each component of G_1 is an Eulerian Circuit.
To get an Eulerian Circuit for G, we start traversing along C_2 and when we enter into a

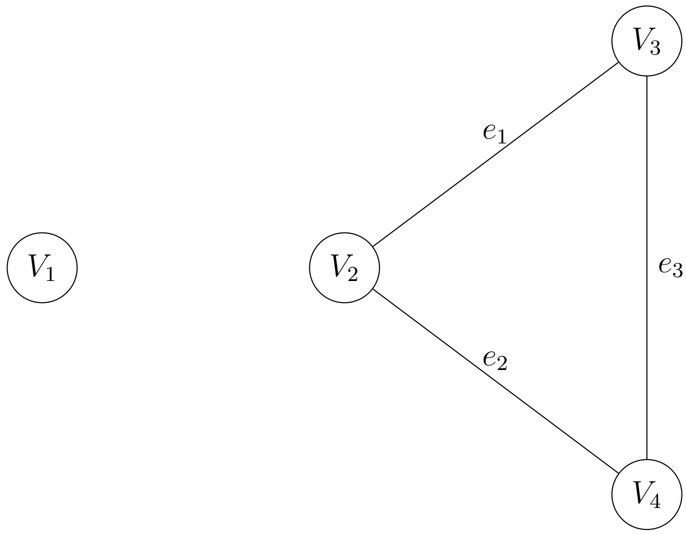
component of G_1 then we traverse along an Eulerian Circuit of that component and this circuit will ends the vertex where we start.
When we complete the traversal of C_2, we will be able to complete an Eulerian Circuit of G.
$\Rightarrow$ G is an Eulerian Graph for n=M.
Therefore, by Strong Principle of Mathematical Induction, G is an Eulerian Graph.



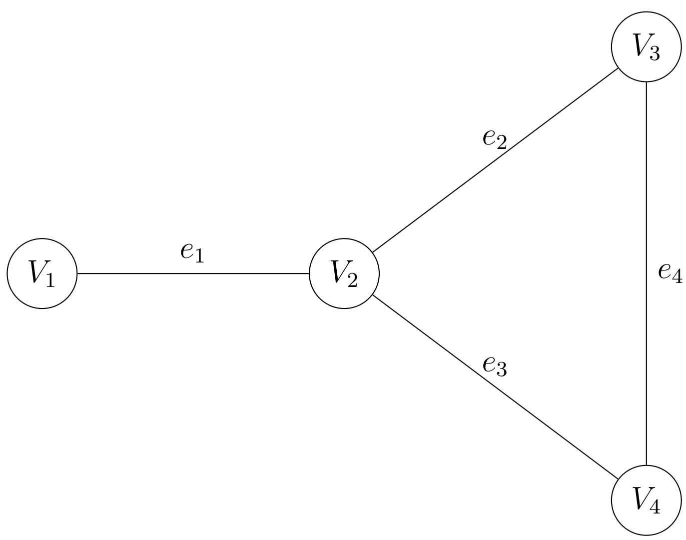
Note: In the Question 3, we are not able to directly about that the graph is Eulerian or not. Now, we are ready to decide directly(means without finding a closed trail) weather the given graph is Eulerian or not.
(a) Clearly, the given graph is a nontrivial component and its vertices V_1,V_2,V_3,V_4,V_5 has degree 4 and the vertex has degree 2 means all vertices have even degree.
Therefore by above theorem, the given graph is Eulerian.
(b) Clearly, the given graph is a nontrivial component and its vertices V_1,V_3,V_4,V_6 has degree 3 which is an odd number.
Therefore by above theorem, the given graph is not Eulerian.

Question 4: Which of the following can be an Eulerian Graph:
(a) Every node has even degree.
(b) Only one node has odd degree.
(c) Exactly two nodes have odd degree.
(d) More than two nodes have odd degree.

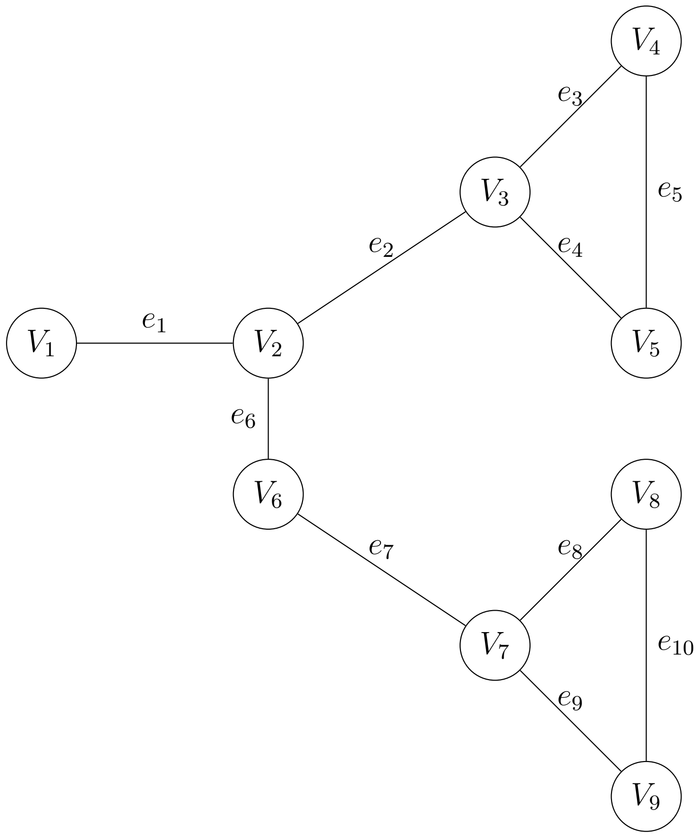
Answer 4: Only (a) can be the Eulerian Graph.
For (a) : we can take G as:



Since each vertex V_1,V_2,V_3,V_4 of G has degree 0,2,2,2 respectively.
Therefore by the theorem 3.3.2, G is Eulerian.
For (b) :
$\nexists$ any graph having only one node as odd degree.(We will prove it by Handshaking Lemma in the next section).
For (c) : we can take G as:



Clearly, the nontrivial component of G is G itself and each vertex V_1,V_2,V_3,V_4 of G has degree 1,3,2,2 respectively and it has two nodes of odd degree.
Therefore, by the theorem 3.3.2, G is not an Eulerian.
For (d) : we can take G as:



Clearly, the nontrivial component of G is G itself and vertex V_1,V_2,V_3,V_7 of G has degree 1,3,3,3 respectively and all the rest vertices have degree 2 and it has four nodes of odd degree. Therefore, by the theorem 3.3.2, G is not an Eulerian.

3.4 Handshaking Lemma

Definition: Let G be a graph. Then, the number of edges incidents to v is called the degree of a vertex v in G and denoted by $d(v)$.

• The maximum number of degree is denoted by $\Delta(G)$ and minimum number of degree is denoted by $\delta(G)$.

Question 5: What is $\Delta(G)$ & $\delta(G)$ in k-regular graph?

Answer 5: k-regular graph means a graph in which every node has degree k.
Therefore, $\Delta(G) = \text{k} = \delta(G)$.

Definition: The number of vertices in G is called the order of a graph G and denoted by $n(G)$.

Definition: The number of edges in G is called the size of a graph G and denoted by $e(G)$.

Question 6: What is $n(G)$ & $e(G)$ in a complete graph K_n?

Answer 6: Since the complete graph K_n has n vertices such that every pair of vertices joined by exactly one edge.
Therefore, $n(G) =$ n.
By definition of complete graph first vertex has to be joined with n-1 vertices and second vertex has to be joined with n-2 vertices and proceeding in the same manner nth vertex has to be joined with only 1 vertex.
$\Rightarrow e(G) =$ (n-1)+(n-2)+...+1
$\Rightarrow e(G) =$ n(n-1)/2.

Question 7: What is $n(G)$ & $e(G)$ in a complete bipartite graph K_m, n?

Answer 7: A complete bipartite graph K_m, n means there are two independent sets having m,n vertices such that each vertex in the first set is joined to each vertex in the second set by exactly one edge.
$\Rightarrow n(G) =$ m+n.
Since, each vertex of the set having m vertices is joined to each vertex in the second set by exactly one edge.
$\Rightarrow e(G) =$ m(1+1+...+1) = mn.

3.4.1 Handshaking Lemma/1st Theorem of Graph Theory/Degree-Sum Formula:

Statement: If G is a graph.Then,

$$\sum_{v \in V(G)} d(v) = 2e(G).$$

Proof: We will try to prove it by applying the Principle of Mathematical Induction on e(G).
let $e(G) = 0 \Rightarrow$ there are only vertices with no edges between them.

$$\Rightarrow \qquad d(v) = 0 \qquad\qquad \forall \qquad v \in G$$

The result is true for $e(G) = 0$.
$\Rightarrow$ the result is true for $e(G) = k$

$$\Rightarrow \qquad\qquad \sum_{v \in V(G)} d(v) = 2k.$$

let $e(G) = k + 1$.
we will construct G' from G.
Case 1: If there is no increase in vertex.
and

$$\sum_{v \in V(G)} d(v) = 2k.$$

$$\Rightarrow \sum_{v \in V(G')} d(v) = 2k + 2.$$

as increase of edge by one will cause increase of degree of vertex by 2.
Case 2: If there is an increase of a vertex.
and

$$\Rightarrow \sum_{v \in V(G)} d(v) = 2k.$$

$$\Rightarrow \sum_{v \in V(G')} d(v) = 2k + 1 + 1.$$

first 1 is for increase an edge in a vertex of G and second one is for the increased vertex in G'.
by combining both the cases,

$$\Rightarrow \qquad \sum_{v \in V(G)} d(v) = 2(k + 1).$$

Therefore, by the Principle of Mathematical Induction,

$$\sum_{v \in V(G)} d(v) = 2e(G).$$

Question 8: How many edges are there in 39-regular graph with 31 vertices?

Answer 8: Since, the each vertex is of 39 degree.
Therefore by Handshaking Lemma,

$$\sum_{v \in V(G)} d(v) = ((39)(31))$$

which is an odd number
but it is not possible because right hand side always gives an even number.
Hence, $\nexists$ any 49-regular graph with 41 vertices.

Question 9: In a class of 7 students, each student sends card to 3 others. Determine weather it is possible that each student receives card from the same 3 students to whom the individual sends the card?

Answer 9: Take the students as a vertices of graph G and sending & receiving is the same thing, therefore consider the edges at each vertex to be 3 means $d(v) = 3$ for any $v \in G$. Therefore by Handshaking Lemma,

$$\sum_{v \in V(G)} d(v) = ((7)(3))$$

which is an odd number
but it is not possible because right hand side always gives an even number.
Hence, $\nexists$ any such situation asked in the question.

Question 10: A graph can not have exactly one node of odd degree.

Answer 10: If possible let $\exists$ a graph G having k-1 vertices with even degree and 1 vertex with odd degree.
Therefore by Handshaking Lemma,

$$\sum_{v \in V(G)} d(v) = ((k-1)even) + ((1)odd) = odd$$

but it is not possible because right hand side always gives an even number.
Hence, A graph can not have exactly one node of odd degree.