Week 7

Hall's Theorem, Vertex Cover, Edge Cover*

7.1 Hall's Marriage Theorem

Statement: An x-y Bigraph G has a matching that Saturates x iff $|N(S)| \geqslant |S| \forall S \in x$
Here $N(s) \subseteq Y$ is a set of neighbours of elements in S.

S = End points of M-a7ternating paths starting from u with the last edge belonging to M.
T = End points of M-atternating path starting from u with the 1 ast edge not belonging to M.

$|S| = 1 + |T| = 1 + |N(S)|$
$|S| > |N(S)|$ |
Hence Proved.
]Proof: An x-y Bigraph G has a matching that Saturates x iff $|N(S)| \geqslant |S| \forall S \in x$
We shall prove the following contrapositive: if there is not such matching m that saturates x, then $\exists S \subseteq x, s.t|S| > |N(S)|$
Letu $\in X$ be a vertex unsaturated by a matching M.
Suppose two subsets $s \subseteq X$ and $T \subseteq Y$ are considered as follows:-

S = End points of M-a7ternating paths starting from u with the last edge belonging to M.
T = End points of M-atternating path starting from u with the 1 ast edge not belonging to M.

$|S| = 1 + |T| = 1 + |N(S)|$
$|S| > |N(S)|$ |

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Hence Proved.

7.2 Vertex Cover

A Vc of a graph G is a set $\theta \subseteq V(G)$ that contains at least one end point of every edge.
For example:
In fig. 7.1, set of vertex cover are:
i. B,E
ii. A,B,C,D,E,F
iii. A,C,E

7.3 Edge Cover

An edge cover of a graph is a set of edges such that every vertex of the graph is incident to at least one edge of the set.
In fig. 7.1, set of Edge cover are:
i. AB,ED
ii. BC,EF

$\alpha(G)$ = maximum size of independent set.
$\alpha'(G)$ = maximum size of matching.
$\beta(G)$ = minimum size of vertex cover
$\beta'(G)$ = minimum size of edge cover

for fig 7.1,
$\alpha(G) = 4$
$\alpha'(G) = 2$
$\beta(G) = 2$
$\beta'(G) = 1$

7.4 Theorem 1

Sum of maximum size of independent set and maximum size of vertex cover is equal to the number of vertices i.e
$\alpha(G) + \beta(G) = n(G)$
Proof: - Let S be an independent set of max size then every edge is incident to at least one vertex of $\bar{s}$.
In fig. 7.1,
S=A,C,D

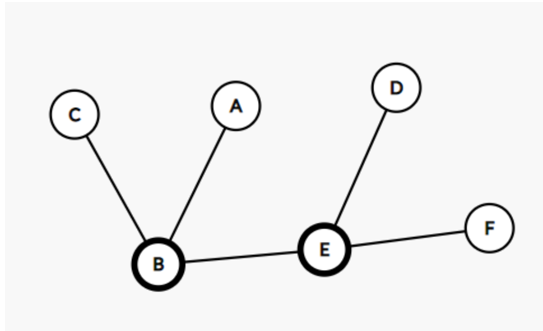

Figure 7.1: Graph of vertex cover

$\bar{S}$=B,E
therefore, SU$\bar{S}$=V(G)

$\bar{S}$ is minimum size vertex cover.

$$\beta(G) = |\bar{s}|$$

S is max size independent set

$$\alpha(G) = |s|$$

$\therefore \alpha(G) + \beta(G) = |s| + |\bar{s}| = |v(G)| = n(G)$
$Also,$
$\alpha'(G) + \beta'(G) = n(G)$

7.5 Theorem 2

Statement :If G In a bipartite graph with no isolated vertices then
$\alpha(b) = \beta'(a)$
Proof:
i) $\alpha(G) + \beta(G) = n(G)$
$ii)\alpha'(G) + \beta'(G) = n(G)$
$iii)\alpha'(G) = \beta(G)$

$from(i), (ii)(iii), weget$

$\alpha(b) = \beta'(a)$

Hence proved, In a bipartite graph with no isolated vertices then
$\alpha(b) = \beta'(a)$

7.6 Problem

Let G be a bipartite graph. prove that $\alpha(G) = \frac{n(G)}{2}$; iff G has perfect matching.
solution:

$$\begin{aligned} \alpha(G) + \beta(G) &= n(a) \\ \alpha(u) &= n(G) - \beta(a) \\ &= n(G) - \alpha'(n) \end{aligned}$$

G has perfect matching, So it has maximum size of matching $= \frac{n(G)}{2}$
hence, $\alpha(G) = \frac{n(G)}{2}$