Week 3

Walk and Cycle; Eulerian Circuit; Vertex Degrees and Counting*

In maths, a graph is what we might normally call a network. It consists of a collection of nodes, called vertices, connected by links, called edges. The degree of a vertex is the number of edges that are attached to it. The degree sum formula says that if you add up the degree of all the vertices in a (finite) graph, the result is twice the number of the edges in the graph.

Some basics of the topic:
1. Walk – A walk is a sequence of vertices and edges of a graph i.e. if we traverse a graph then we get a walk. Vertices and Edges can be repeated.

2. Trail – Trail is an open walk in which no edge is repeated. Vertex can be repeated.

3. Circuit – Traversing a graph such that not an edge is repeated but vertex can be repeated and it is closed also i.e. it is a closed trail. Vertex can be repeated. Edge can not be repeated.

4. Path – It is a trail in which neither vertices nor edges are repeated i.e. if we traverse a graph such that we do not repeat a vertex and nor we repeat an edge. As path is also a trail, thus it is also an open walk. Vertex not repeated. Edge not repeated.

5. Cycle – Traversing a graph such that we do not repeat a vertex nor we repeat a edge but the starting and ending vertex must be same i.e. we can repeat starting and ending vertex only then we get a cycle. Vertex not repeated. Edge not repeated.

3.1 Propositions of Walk and Cycle

1. Every closed odd walk contain an odd cycle.

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2. A simple path is a bipartite graph.

3. C_n is bipartite iff n is even.

3.2 Theorem

Theorem 3.1. *A graph is bipartite iff it has no odd cycle.*

Proof. First, suppose that G is bipartite. Then since every subgraph of G is also bipartite, and since odd cycles are not bipartite, G cannot contain an odd cycle.

Now suppose that G is a non-trivial graph that has no odd cycles. We must show that G is bipartite. So we must determine a partition of the vertices of G into independent sets. It is enough to prove our result for connected graphs since if G is bipartite, so is every component of G (and vice versa).

So, now consider any vertex a of G. Let A = v : d(v,a) is even. Similarly, define, B = v : d(v,a) is odd. Clearly then V(G) = A B . We will be finished if we can show that A and B are independent sets.

So we assume that A is not independent and show that this leads to a contradiction. Suppose that x and y are adjacent vertices of A. We may assume that for some integers k, m that d(a, x) = 2k, and d(a, y) = 2m.

Now let P be a shortest a-x path, and Q a shortest a-y path.

Say P is a = $v_0 v_1 v_2 \ldots v_{2k}$ = x and Q is a =$u_0 u_1 u_2 \ldots u_{2m}$ = y.

We might notice here that y cannot be on P and x cannot be on Q .

Let w be the vertex in $V(P) \cap V(Q)$ that is closest to x. So, w = v_j = u_j where d(a,w) = j . So now consider P‘ , the w-x subpath of P‘ and Q‘ , the w-y subpath of Q. Then $V(P‘) \cap V(Q‘) = w)$.

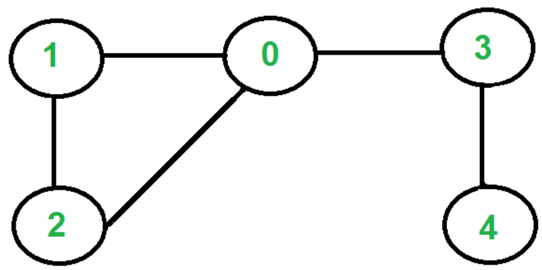
But then the cycle formed by following P‘ from w to x, then the edge xy, and then following Q‘ in reverse from y to w is an odd cycle; more precisely, the cycle

w = $v_j v_{j+1} v_{j+2} \ldots v_{2k-1} x y u_{2m-1} u_{2m-2} \ldots w$ has length (2k - j)+)(2m - j) + 1 = 2(k + m - j) + 1, which is odd.

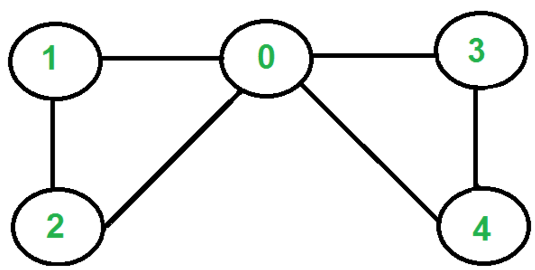
But this contradicts the assumption that G has no odd cycles. Thus it must be that A is independent. A similar argument shows that B is independent. So our result is proven. □

3.3 Eulerian Circuit

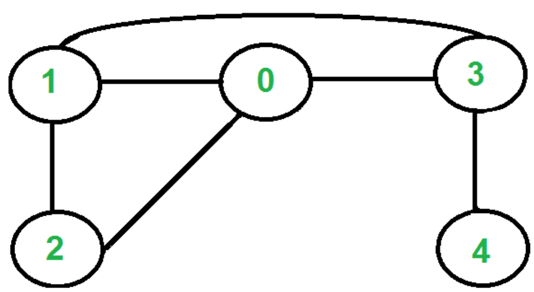
Eulerian Path is a path in graph that visits every edge exactly once. Eulerian Circuit is an Eulerian Path which starts and ends on the same vertex.



The graph has Eulerian Paths, for example "4 3 0 1 2 0", but no Eulerian Cycle. Note that there are two vertices with odd degree (4 and 0)



The graph has Eulerian Cycles, for example "2 1 0 3 4 0 2"
Note that all vertices have even degree



The graph is not Eulerian. Note that there are four vertices with odd degree (0, 1, 3 and 4)

Figure 3.1

A graph is called Eulerian if it has an Eulerian Cycle and called Semi-Eulerian if it has an Eulerian Path. Following are some interesting properties of undirected graphs with an Eulerian path and cycle.

3.3.1 Eulerian Cycle

An undirected graph has Eulerian cycle if following two conditions are true.

a) All vertices with non-zero degree are connected. We don't care about vertices with zero degree because they don't belong to Eulerian Cycle or Path (we only consider all edges).

b) All vertices have even degree.

3.3.2 Eulerian Path

An undirected graph has Eulerian Path if following two conditions are true.

a) All vertices with non-zero degree are connected. We don't care about vertices with zero degree because they don't belong to Eulerian Cycle or Path (we only consider all edges).

b) If zero or two vertices have odd degree and all other vertices have even degree. Note that only one vertex with odd degree is not possible in an undirected graph (sum of all degrees is always even in an undirected graph)

3.3.3 Theorem

Theorem 3.2. *If every vertex of a graph has degree at least 2, then G has cycle.*

Proof. We shall prove the contrapositive, i.e., if G contains no cycles, then G has a vertex with degree less than 2. To this end, suppose that G has no cycles. Then G must be a forest. Let T be a component of G. If T is trivial, then T, and thus G, has a vertex of degree 0. If T is nontrivial, then T is a nontrivial tree and Theorem implies that T has at least two end-vertices. These are of degree 1. Consequently, if G has no cycles, then G has at least one vertex with degree less than 2. □

3.3.4 Theorem

Theorem 3.3. *A graph G is Eulerian if and only if G has at most one nontrivial component and its vertices all have even degrees.*

Sufficient condition: G' has less than M edges and Each vertex of G' has even degree. It can have more than one component. By the induction hypo. each component of G' contain Eulerian Cycle.

Proof. Necessity: Suppose G is Eulerian. All edges are on a Eulerian cycle. Therefore, all edges are in one component. Other components have no edge. Thus, they are isolated vertices. Let us fix an orientation of the Eulerian circuit. For any vertex v in the nontrivial component, the number of edges leaving v is equal to the number of edges entering v. The degree d_v is the sum of edges which are either leaving or entering v. Thus,d_v is even.

Sufficiency: We will prove it by induction on the number m of edges.
If m = 0, the Eulerian cycle is empty. It holds.
Suppose that the statement holds for any graph with at most m edges. In another words, if a graph G with at most m edges has at most one nontrivial component and its vertices all have even degrees, then G is Eulerian.

Now we consider a graph with m+1 edges, which has at most one nontrivial component H and whose vertices all have even degrees. By above Lemma, it contains a cycle C. Deleting all edges on C from G, H might be breaking into several components, say H_1, H_2, . . . , H_r. The degree of a vertex v either decreases by 2 if $v \epsilon C$, or remains the same if v in C. All degrees remain even after deleting the edges of C.

Each component H_i has at most m edges. By inductive hypothesis, There is an Eulerian circuit C_i for each component H_i. Since G has only one non-trivial component, the cycle C must intersect with every component H_i. Pick one vertex $v_i \epsilon V(C) \cap$V (H_i). The vertices v_1, v_2, . . . , v_r break the cycle C into r paths, say $v_1 P_1 v_2, v_2 P_2 v_3, ..., v_r P_r v_1$. Arrange Eulerian circuit C_i so that the starting vertex and end vertex is v_i. Now we construct an Eulerian circuit as follows:
$C_1 P_1 C_2 P_2 ... C_r P_r v_1$
It contains all edges of G. □

We combine these Eulerian cycles with C to construct an Eulerian circuit as follows: Traverse C until a component of G' appear, then traverse Eulerian cycle of that component, come back to C and repeat this.

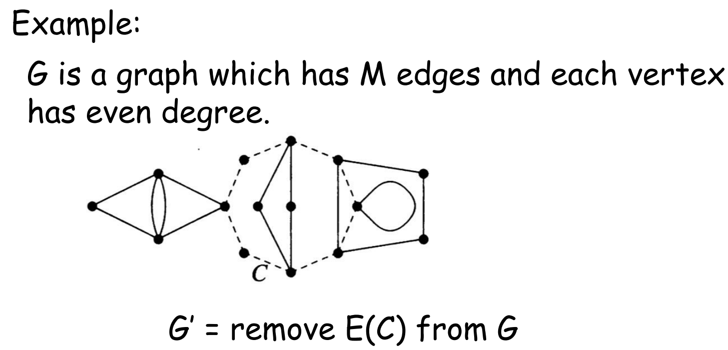


Figure 3.2

3.4 Vertex Degrees and Counting

3.4.1 Degree of a vertex

Degree of a vertex v in a graph G written as d(v) is number of edges incidents to v, except that each loop at v counts twice. The maximum and minimum number of degrees are denoted by $\Delta(G) and \delta(G)$ respectively.

3.4.2 Order and size of a graph

The order of a graph G, n(G) is number of vertices in G.

The size of a graph G, e(G) is number of edges in G.

3.4.3 Handshaking Lemma

Theorem 3.4. *1st Theorem of Graph Theory (Degree-sum formula): If G is a graph then*

$\sum_{v \epsilon V(G)} d(v) = 2e(G)$

Proof. If G is a graph then

$\sum_{v \epsilon V(G)} d(v) = 2e(G)$

if e(G)=1 , $\sum d(v0 = 1 + 1 = 2$

Ind hypo: Assume e(G)=k, then

$\sum_{v \epsilon V(G)} d(v) = 2 * k$

We have to prove that if e(G)=k+1 then $\sum_{v \epsilon V(G‘)} d(v) = 2 * (k + 1)$

From G we will construct G‘

Case 1: no increase in vertex

$\sum_{v \epsilon V(G)} d(v) = 2 * k$

$\sum_{v \epsilon V(G‘)} d(v) = \sum_{v \epsilon V(G)} d(v) + 2$

=2 k + 2

=2 (k+1)

Case 2: There is increase in vertex.

$\sum_{v \epsilon V(G‘)} d(v) = \sum_{v \epsilon V(G)} d(v) + 2$

=2 k + 2

2(k+1)

□

3.4.4 Lemma

Theorem 3.5. *A graph can not have exactly one node with odd degree.*

Proof. $\sum d(v) = (k - 1) * evenedges + 1 * oddedges$

= even + odd

=odd

$\neq 2 * edges$

Hence proved $\square$

3.5 Extra

1. A graph is said to be regular if $\Delta(G) = \delta(G)$.

2. The order of a graph G=(V,E) is $|V|$, as the size of G is $|E|$.

3. K_n is a regular graph. Each vertex has degree n-1.

4. $K_{m,n}$ is regular if and only if m=n. Then, the degree is always n.

5. A connected regular graph that has the same order and size is a cycle.

6. Hypercubes are regular graphs.

7. Every graph has an even number of vertices of odd degree.

8. A k-regular graph (i.e. a regular graph in which the degree of each vertex is k) has $\frac{k|V|}{2}$ edges.

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