Week 3

Friday: Eulerian Circuits*

3.1 Walk and Cycle in Graphs

Definitions:

Walk:

It is a sequence of edges such that each edge(except the first one) starts with a vertex where the previous edges ended.

When starting and the ending vertices in a walk are same then it is called a **Closed Walk**. When starting and the ending vertices in a walk are not same then it is called a **Open Walk**.

Cycle:

When a graph is traversed such that there is no repetition of either edges or vertices(except the stating and the ending vertex), then the traversal is called a Cycle.

Propositions under consideration:

1. Every closed odd walk contain an odd cycle

Proof by Induction

Base Condition:
It will be a closed walk with length 1, i.e., $l = 1$. This satisfies the given statement as a closed walk of length 1 traverses a cycle of length 1.

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Induction Hypothesis:
Let us assume that $l > 1$ for all closed walks shorter than W.

To Prove:
We want to prove that closed odd walk of W on a graph G is a odd cycle, such that $l < W$.

Proof:
Given graph G, we will have the following cases

Case 1:
We here are considering Closed odd walk. If the Closed odd walk under consideration has no vertices repeating, other than the first and the last last vertex, then the given Closed odd walk in itself is a Closed odd cycle. This becomes a Trivial case.

Case 2:
In this case we consider repeating vertices. According to the definition of cycle, we know that neither the edges nor the vertices should repeat. Therefore we break the given Closed odd walk of length W into 2 $v-v$ walks. Since W is a Closed odd walk, i.e., is of odd length, therefore either one of these $v - v$ walks should be odd (Odd/Even + Even/Odd = Odd).

Also from the Induction Hypothesis, we know that all Closed odd walks of $W > l > 1$ are odd cycles.

Therefore we can conclude that the Closed odd walk that is present in the $v - v$ walks after breaking the Closed odd walk of length W is an Odd cycle and is shorter than W. Therefore W contains an Odd cycle, where W is a Closed odd walk for a graph G

Example:

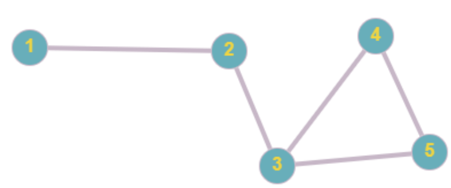

Figure 3.1: Example-1

Let us consider the following Closed odd walk on the above graph G:

$W = v_1, v_2, v_3, v_4, v_5, v_3, v_2, v_1$
$|W_1| = v_1, v_2, v_2, v_1$
$|W_2| = v_3, v_2, v_1$

$|W_2| = 3$, which is odd and also $|W_2| < |W|$
Therefore G that has a Closed odd walk, also has a odd cycle.

2. A closed even walk need not contain a cycle.

To justify the above statement we consider the graph as shown below, with 2 vertices v_1, v_2 and an edge connecting the both e_1.

Figure 3.2: Example-2

Let us consider the following Closed even walk on the above graph G:

$$W = e_1,\ e_1$$

Then we can say that it is not a closed cycle, as edges are repeating.
Therefore G that has a Closed even walk need not contain cycle.

3. A simple path is a bipartite graph.

Path is a traversal in graph in which there is no repetition of both edges and vertices. Path can start anywhere and end anywhere.
Since there is no repetition of edges as well as vertices, we can divide the graph into 2 sets of vertices, where each vertices in one set have edges to vertices in other sets only, but not in its own set.

Figure 3.3: Example-3

4. C_n is bipartite iff n is even.

C_n represents a cyclic graph with n vertices. If n is odd, then there will always be an vertex which joins another vertex to form an edge, i.e., to form a odd cycle. In such cases we will not be able to color all n vertices with 2 colors. When n is even, we will get a even cycle which has no repeating vertex(except the first and last) and edges. Hence it forms a closed path. As we know that a simple path is bipartite, therefore we can conclude that C_n is bipartite iff n is even.

Figure 3.4: Example-4

3.2 Eulerian Circuits

Konigsberg Bridge Problem:

Over the river Preger, there were 7 bridges in the city of Königsberg. The people in Königsberg wanted a closed trail over the bridges meaning all bridges to be traversed in a single trip without repeating the traversal through the same bridge and coming back to the same place, where the traversal started.

Below is the picture of bridges in the city of Königsberg.

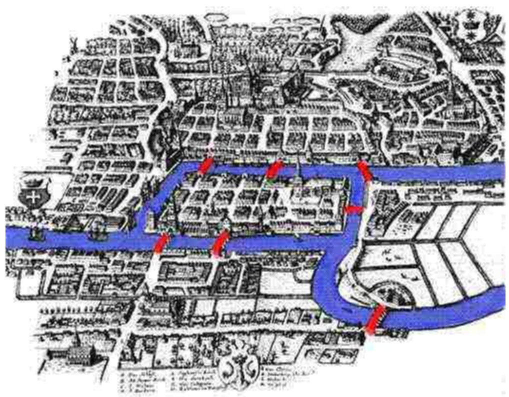

Figure 3.5: Königsberg Bridge Problem

This can be modelled as a graph as shown:

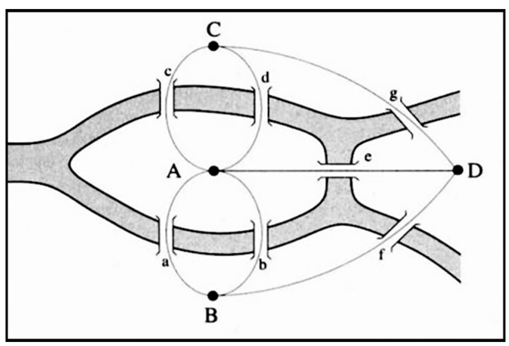

Figure 3.6: graph model

Solution:
It is not possible to have a closed trail covering all bridges.

The Swiss mathematician **Leonhard Euler** stated the sufficient conditions for a graph to have a closed trail. These are as follows:

1. all vertices of graph under consideration should have an even degree

2. all edges of the graph under consideration should belong to the same component of the graph

In honor to his contribution, all such graphs are called Euler Graphs.

Definition:

A graph is **Eulerian** if it has a **closed trail** containing all edges.
Note that we call a closed trail a circuit when we do not specify the first vertex but keep the list in cyclic order.
An **Eulerian circuit** or **Eulerian trail** in a graph is a circuit or trail containing all the edges.

Lemma:

If every vertex of a graph G has degree at least 2, then G contains a cycle.

Proof:
To prove we take Contapositive of the given Lemma
The contapositive statement becomes:

If G contains no cycles, then G has a vertex with degree less than 2.
Let us now suppose that G has no cycles.
Then G must be a forest, as graph G may also have more than one connected components in it.
Let T be a component of G.

Case 1: If T is trivial, then T and hence G, has a vertex of degree 0.
Case 2: If T is nontrivial, then T is a nontrivial tree.
From this we can say that T has at least two end-vertices.
As we know, end vertices have a degree of 1 and this implies that there exists atleast a vertex with degree less than 2.

Therefore from both **Case 1 and Case 2**, we know that if a graph G has no cycles, then G has at least one vertex with degree less than 2.

This proves the above contrapositive statement.
Since implications are equivalent to contrapositive statements, this proves the lemma:
If every vertex of a graph G has degree at least 2, then G contains a cycle.

Examples:

Check whether the graphs below are Eulerian graphs or not.

Let us consider these graphs given below and check whether they are Eulerian Graphs or not.

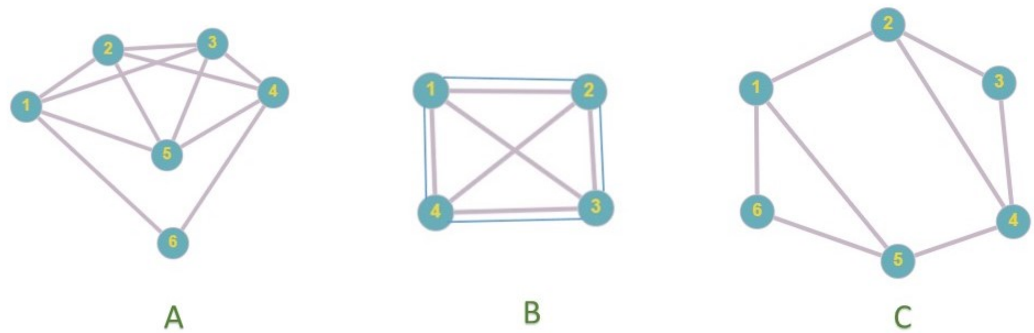

Figure 3.7: Examples

A: The graph A is Eulerian. We see that all vertices have an even degree. The sequence of vertices that is to be followed to cover all the edges with no repetition of edges is: $v_6, v_1, v_5, v_3, v_2, v_1, v_3, v_4, v_2, v_5, v_4, v_6$.

B: All vertices in the graph have an odd degree, which is 5. Hence we can say that the graph is not Eulerian.

C: All vertices in the graph have don't have an even degree. Vertices v_1, v_2, v_4, v_5 have degree 3. Hence we can say that the graph is not Eulerian.

Theorem:

A graph G is Eulerian iff it has atmost one non-trivial component and all its vertices have even degree.

Necessary Condition:
We assume a graph G is Eulerian and we have to prove that G has atmost one non-trivial component and all vertices of G have a even degree.

Proof for Necessary Condition:

Since we have assume graph G is Eulerian, this implies that it has closed trail containing all the edges.
As the graph G has a closed trail, we can say that there will be an incoming and outgoing edge for all the vertices (as when ever we enter a vertex using one edge, we will also be leaving that vertex using another edge).
As each vertex has an outgoing edge corresponding to each incoming edge, we have all vertices of G with even degree.
Now this component should be one and only non-trivial component of the graph G, else having more than one non-trivial components will leave some of the edges uncovered, hence closed trail might not be possible.

Hence, this proves the necessary condition of:
A graph G is Eulerian iff it has atmost one non-trivial component and all its vertices have even degree.

Sufficient Condition:
We assume a graph G has atmost one non-trivial component and all its vertices have even degree, then we have to prove that the graph G under consideration, is Eulerian.

Proof for Sufficient Condition using Induction:

Base Condition:
Let us represent the number of edges with M.
If number of edges $M = 0$, then G is Eulerian, is true, which is a trivial condition.

Induction Hypothesis:
Let us assume that this statement is true for all graphs having the above properties (atmost one non-trivial component and all vertices of G have an even degree) and edges less than M.

To Prove:
A graph G containing one non-trivial component having M edges and all its vertices with even degree, is Eulerian.

Proof:
Let us consider a graph G containing one non-trivial component having M edges. Each vertex of graph G has even degree.
$\Rightarrow$ Each vertex has atleast degree of 2, as we have assumed that G has a non-trivial component.
$\Rightarrow$ G contains a cycle (Using Lemma: If every vertex of a graph G has degree atleast 2, then G contains a cycle). We assume this cycle as C.

We now perform the following operations:
We remove all the edges forming the cycle C from G to form a graph $G^{'}$,
which has edges $= e(C) < M$, where M represents total number of edges in graph G. The graph $G^{'}$ is formed by removing the cycle C from graph G,
$\Rightarrow$ all vertices in $G^{'}$ has even degree.

Here $G^{'}$ has edges less than M, and each vertex of $G^{'}$ has even degree. But $G^{'}$ can have more than one component.
We now apply the Induction hypothesis on each component of $G^{'}$ and therefore we can say that $G^{'}$ contains Eulerian Cycle.

We can combine these Eulerian cycles with C to construct an Eulerian circuit as follows:
We traverse C until a component of $G^{'}$ appear, then traverse Eulerian cycle of that component, come back to C and repeat these steps.

Hence, this proves the sufficient condition:
A graph G containing one non-trivial component having M edges and all its vertices with even degree, is Eulerian.

3.3 Thought of the day Problem

Problem:

Consider the scenario as shown in the graph which represents Baby Euler's House. Baby Euler has just learned to walk. He is curious to know if he can walk through every doorway in his house exactly once, and return to the room he started in.

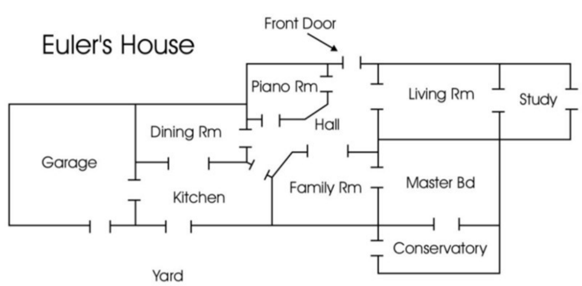

Figure 3.8: Baby Euler's House

Will baby Euler succeed? Can baby Euler walk through every door exactly once and return to a different place than where he started? What if the front door is closed?

Solution:

1. Baby Euler succeeds to walk through all the doorways present in his house starting from the front door. The room from which he starts the traversal is the Hall and end his closed trail covering all the doorways by reaching Hall. This is only possible when front door is open.
The closed trail is as traversed as shown:

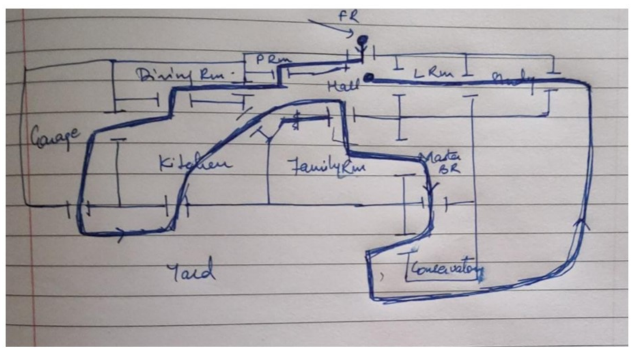

Figure 3.9: Baby Euler's Closed Trail

2. Yes. If the baby Euler starts his trail from the Hall, then there exists a traversal, which will cover all the doorways and this ends at the front door.

3. If the front door is closed, then both closed trail (Q.1 starting from front door to reach hall and covering all the doorways to end the closed trail at hall) and open trail (Q.2 starting hall and covering all the doorways to end the open trail at front door) are not possible.

3.4 Questions

Q: Which among the following can be Eulerian Graph?

1. Every node has even degree.

2. Only one node has odd degree.
3. Exactly Two nodes have odd degree.
4. More than two nodes have odd degree.

Answers:

1. It can be Eulerian.

2. In graph theory we can never have a graph with one odd degree vertex. Hence it is not possible to construct such graphs as an edge is a path between 2 vertices. Therefore it is not Eulerian.

3. It can be Eulerian.

4. It can never be Eulerian.

Week 3

Saturday: Vertex Degree and Counting*

3.1 Degree of a Vertex

Definition:

The **degree** of vertex v in a graph G, written $d_G(v)$ or $d(v)$, is the number of edges incident to v, except that each loop at v counts twice. The maximum degree is of a vertex v in graph G is denoted by $\Delta(G)$ and the minimum degree is denoted by $\delta(G)$.
Let us consider the graphs given below to calculate degree of vertices:

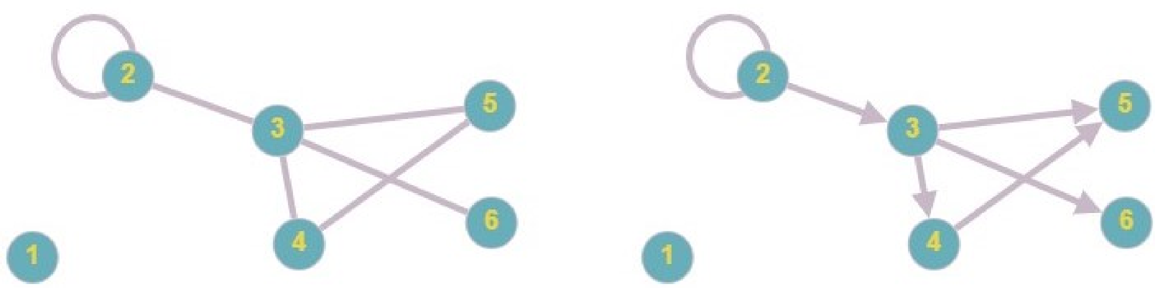

Figure 3.1: Example-1

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The degrees are as follows:

Undirected Graph(Example−1)

	vertex	Degree
1	1	0
2	2	3
3	3	4
4	4	2
5	5	2
6	6	1

Directed Graph(Example−1)

	Vertex	In−Degree	Out−Degree
1	1	0	0
2	2	1	2
3	3	1	3
4	4	1	1
5	5	2	0
6	6	1	0

For the Undirected Graph in the above figure, $\Delta(G) = 4, (v = 3)$ and $\delta(G) = 0, (v = 1)$. Also, for a Regular Graph (K_n) with n vertices, $\Delta(K_n) = \delta(K_n)$.

3.2 Order and Size of Graphs

Definition:

The **order** of a graph G, written $n(G)$, is the number of vertices in G. The **size** of a graph G, written $e(G)$, is the number of edges in G.
An n-vertex graph is a graph of order n.
For $n \in N$, the notation $[n]$ indicates the set $\{1, ..., n\}$.

For a Complete Graph with n vertices (K_n):
Order $= n$
Size $= \{(n^2 - n)/2\}$

For a Complete Bipartite Graph $(K_(m, n))$:
Order $= m + n$
Size $= m * n$

3.3 Handshaking Lemma

Lemma:

If G is a graph, then

$$\sum_{v \in V(G)} d(v) = 2e(G).$$

It is also called Degree-sum formula.

Explanation:

2 vertices make an edge. Summing degrees of all the vertices will lead in considering an edge twice. This is because each edge has two vertices and contributes to the degree at each vertex.

Proof by Induction:

Base Condition:
$e(G) = 1$. $\sum_{v \in V(G)} d(v) = 2e(G) = 2(1) = 2$ is true, as an edge is made by 2 vertices.

Induction Hypothesis:
Assume $e(G) = k$, then $\sum_{v \in V(G)} d(v) = 2e(G) = 2 * k$.

To Prove:
If $e(G^{'}) = k + 1$, then $\sum_{v \in V(G')} d(v) = 2e(G^{'}) = 2 * (k + 1)$.

Proof:
Given graph G with k edges, we will construct graph $G^{'}$ from G.

Case 1:
Adding 1 edge to G, we get $G^{'}$ such that: $\sum_{v \in V(G')} d(v) = \sum_{v \in V(G)} d(v) + 2$ and we kow from induction hypothesis that $\sum_{v \in V(G)} d(v) = 2 * k$.
Therefore $\sum_{v \in V(G')} d(v) = 2 * k + 2 = 2 * (k + 1)$

Case 2:
Adding 1 vertex to G, we get $G^{'}$ such that: $\sum_{v \in V(G')} d(v) = \sum_{v \in V(G)} d(v) + 2$ and we kow from induction hypothesis that $\sum_{v \in V(G)} d(v) = 2 * k$.
Therefore $\sum_{v \in V(G')} d(v) = 2 * k + 2 = 2 * (k + 1)$

Slido Question:

A 49-regular graph with 41 vertices has how many edges ?

- 1004
- Such regular graph can't exist
- 2009
- 8

Such regular graph can't exist is the answer, as the given graph has 41 vertices, each with degree = 49. Therefore sum of degrees of all the edges is equal to $49 * 41$, which is 2009 and according to the Handshaking Lemma, $2009 = 2 * e(G)$. Since 2009 is odd, therefore it is not divisible by 2 and number of edges can't be fractions. Hence, no such regular graph exists.

Problem:

In a class with 9 students, each student sends valentine cards to three others. Determine whether it is possible that each student receives cards from the same three students to who he or she sends the card.

The above question can be modelled into a graph such that, each vertex is connected to 3 other vertices, such that each vertex has degree of 3.

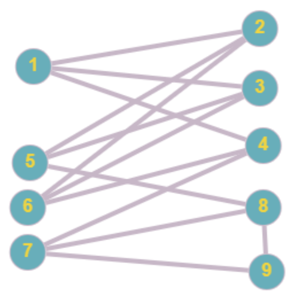

Figure 3.2: Graph-1

Using Handshaking Lemma on the above graph, we get: $3 * 9$, equal to 27(summation of degrees for all the 9 vertices). This should be equal to $2 * e(G)$, which violates the Handshaking lemma. Therefore it is not possible that a student receives cards from the same three students to whom he or she sends the card.