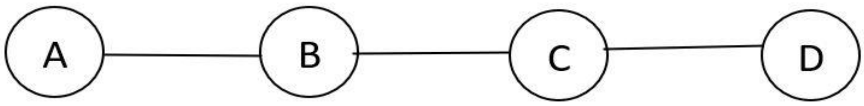
# Week 07

## Graph Theory & Its Application

## Previous Class Discussion

### 7.1 Matching:

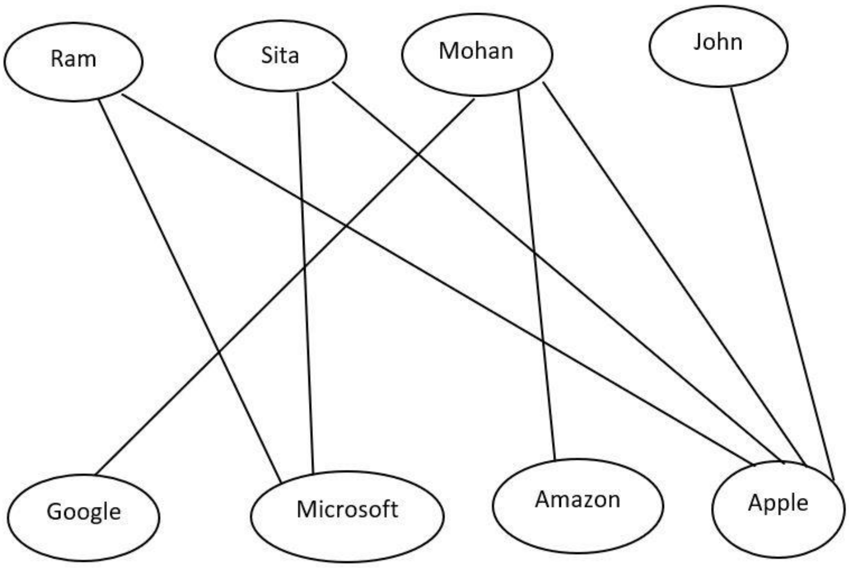
Matching is set of non-loop edges with no-shared end points




Matching = {AB, CD} = M1 ➔ Perfect Matching

= {BC} = M2 ➔ Not Perfect Matching

### 7.2 Hall's Marriage theorem




Scenario: there are four companies visiting for campus placement and there are 4 students. Each companies have its own eligibilities criteria and each company has only one vacancy. Edges denote the person meets the eligibility criteria of that companies.

Question: Will everyone get the Job given every company has only one job?

Sol: This may or may not happen but in current scenario it will not happen.

For this Statement can be written as:

An x-y Bigraph (Bipartite Graph) G has a matching that saturates X iff

$$| N(s) | \geq | s | \qquad \forall s \subseteq X$$

Here, **N(s) ⊆ Y** is a set of neighbors of elements in S.

## Previous Class Discussion Ends

### 7.3 Hall's Theorem

Proof:

Necessary Condition: Suppose X-Y Bigraph has a matching that saturates (means: every element of X is matched with exactly one element of Y) then obviously,

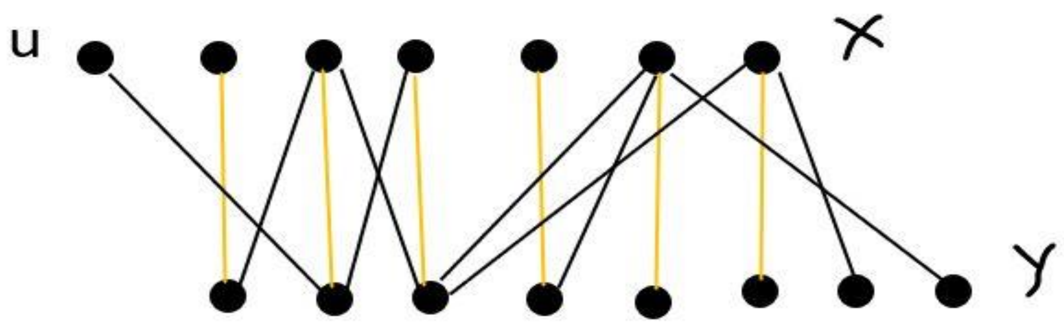
$$|s| \leq | N(s) | \quad \forall s \in X$$

Sufficient condition: If $\forall s \in X$, $| N(s) | \geq |s|$, then there is a matching that saturates X.

We will proof using contrapositive method.

We will proof- If there is not such matching M that saturates X, then $\exists\, s \subseteq X$, such that $|s| > | N(s) |$

Let $u \in X$, be a vertex unsaturated by matching means this u become single, it cannot find its pair.

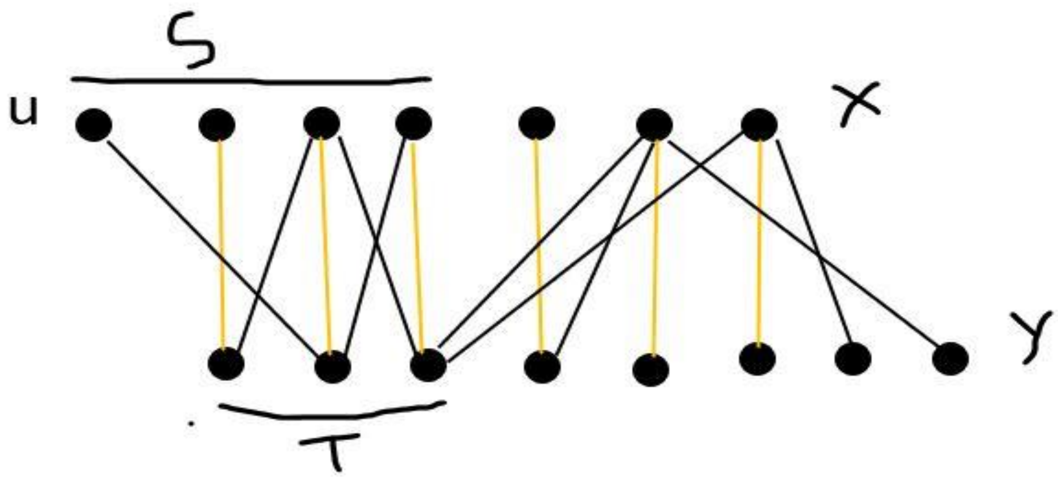


Yellow color shows matching and other are original edges

suppose two subset **$S \subseteq X$** and **$T \subseteq Y$** are considered as follows:

S = End points of M-alternating paths starting from u with the last edge $\in$ M

T = End points of M-alternating paths starting from u with the last edge $\notin$ M




**$|S| = 1 + |T| = 1 + |N(s)|$**

=> **$|S| > |N(s)|$**

Q. There are thousand vertices in x and nearly 5 thousand vertices in Y and there is some edges connection is given, we have to find whether Matching is possible or not? Can we use Hall's Theorem?

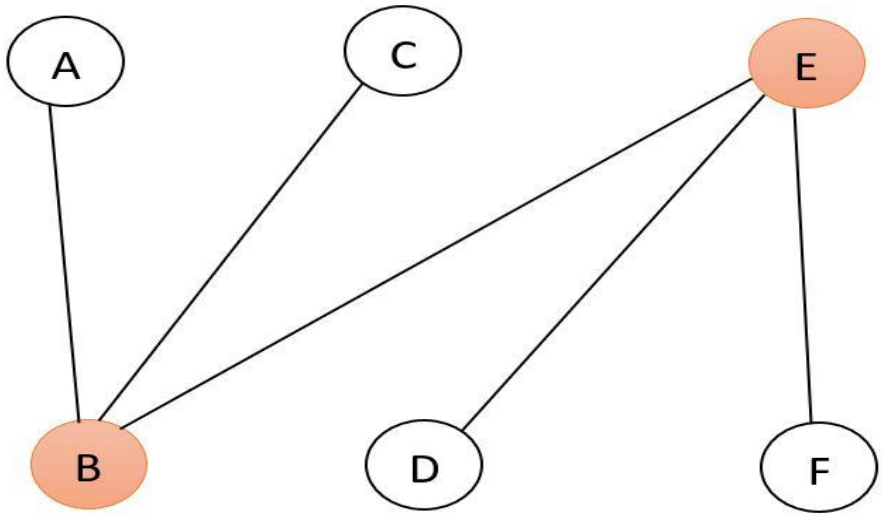
Sol - In theory, Hall's theorem we can apply for fewer subset. But in above condition No of subset possible in 2 power 1000 which is not possible to evaluate.

if $\forall$ **$x \in X$ degree(x) $\leq$ d** and $\forall$ **$y \in Y$ degree(y) $\geq$ d**

if we can find d which satisfy the above condition then there will be matching.

## 7.4 Vertex Cover:

A vertex cover of graph G is a set $Q \in V$ of G that contains at least one end point of every edge.




Vertex Cover will be

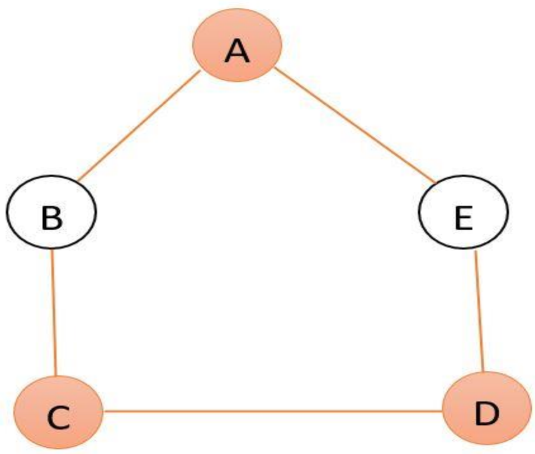
Q1 = **{B, E}** → Minimum Size Vertex Cover

Q2 = {A, B, C, D, E, F}

Q3 = {A, C, E}

There can be may vertex cover but minimal will be Q1.

Another Example



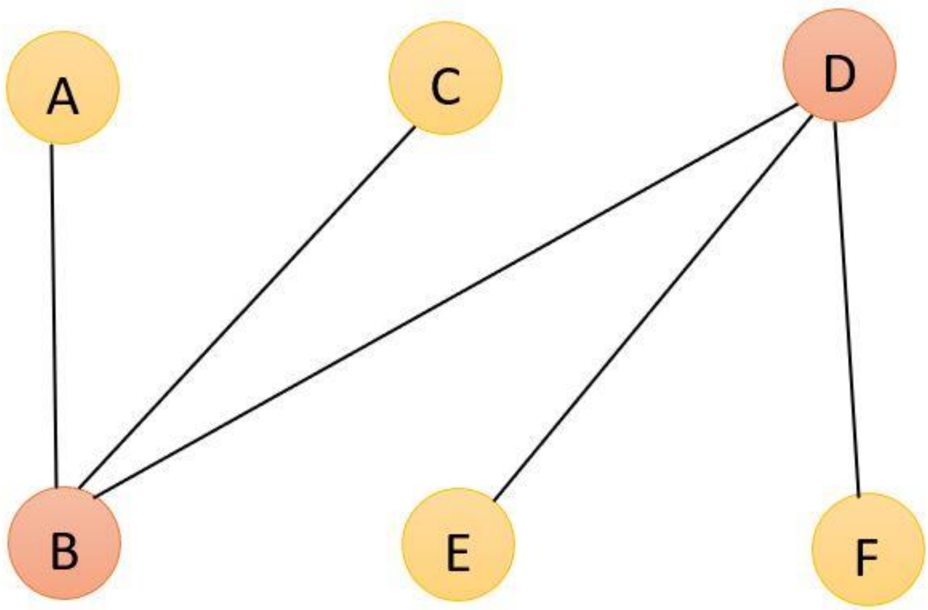

VC = **{A, C, D}** → Minimum Size vertex Cover

Maximal Matching = **{BC, DE}**

## 7.5 Independent Sets:

Independent Sets are those set of vertices which are not adjacent to each other.

Independent sets of above graph = **{A, C}, {A, D}**




Independent Set = {A, C, E, F}

Vertex Cover = {B, D}

There is a relation occurs, that is :

Let **α (G)** = maximum size of independent sets = 4 (For Above Graph)

**α' (G)** = Maximum size of matching = 2 (For Above Graph)

**β (G)** = Minimum size of vertex cover = 2 (For Above Graph)

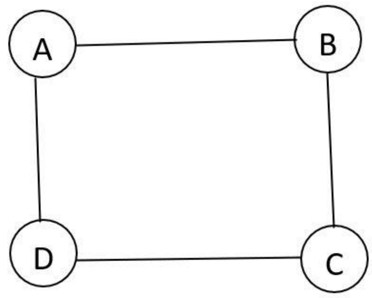
**β' (G)** = Minimum size of **Edge Cover** = 1 {For Above Graph}

Note: **Edge cover** is very similar to vertex cover, Instead of Vertex we consider the edges, so edge cover is defined as set L of E of G such that every vertex of G incident to some edge of L

Edge Cover of Above Graph = **{BD}** → Minimal size edge Cover

**α (G) + β (G) = |V|**

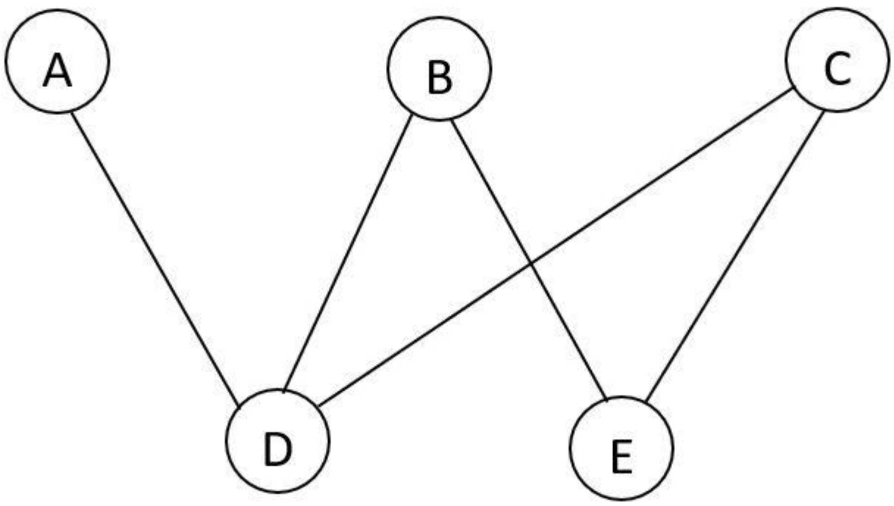
Another Example of Edge Graph




Edge Cover = **{AB, BC}, {AB, AD}** and many

**$\alpha$ (G) + $\beta$ (G) = n(G)**

Proof: Let S be an independent set (non-Adjacent) of max size then every edge is incident to at least one vertex of S'




S = {A, B, C}

S' = {D, E}

S U S' = V(G)

Means, Every Edge is incident to at least one vertex of S'. In other Words, S' coves all the edges.

In other words, we can say, S' is minimum size vertex cover, So, $\boldsymbol{\beta(G) = |S'|}$

And, $\boldsymbol{\alpha(G) = |S|}$

Hence, $\boldsymbol{\alpha(G) + \beta(G) = |S'| + |S| = n(G)}$

And $\boldsymbol{\alpha'(G) + \beta'(G) = n(G)}$

**Corollary:** If G is bipartite Graph with no isolated vertex, then $\boldsymbol{\alpha(G) = \beta'(G)}$

From the Above Proof, we have

$$\alpha(G) + \beta(G) = n(G)$$

$$\alpha'(G) + \beta'(G) = n(G)$$

$$\alpha'(G) = \beta(G)$$

Using Above 3 Equation

$$\alpha(G) + \beta(G) = \alpha'(G) + \beta'(G)$$

$$= \beta(G) + \beta'(G)$$

$\boldsymbol{\alpha(G) = \beta'(G)}$ Hence Proved.

Problem: Let G be a Bipartite Graph. Prove that $\boldsymbol{\alpha(G) = n(G)/2}$, iff G has perfect matching.

Solution: we have already seen, for any graph,

$$\boldsymbol{\alpha(G) + \beta(G) = n(G)}$$

$$\alpha(G) = n(G) - \beta(G)$$

$$= n(G) - \alpha'(G) \qquad [\ \textbf{Since } \boldsymbol{\alpha'(G) = \beta(G)}\ ]$$

Since it is given Graph G has perfect matching, then

Maximum size of Matching = $n(G)/2 = \alpha'(G)$

Hence, $\alpha(G) = n(G) – n(G)/2$ [ **Since** $\boldsymbol{\alpha'(G) = n(G)/2}$]

$\boldsymbol{\alpha(G) = n(G)/2.}$

## 7.6 Theorem for diameter ≥ 3

Theorem: **If G is Simple Graph, then if diameter(G) ≥ 3 then diameter(G') ≤ 3**

Proof: Diameter of any graph is defined as maximum of eccentricity of vertex.

Since diameter(G) ≥ 3 that means, not every node is connected to everything. Therefore, we can say,

$\exists$ u & v such that i) u, v $\notin$ E(G)

ii) u & v do not have common neighbor

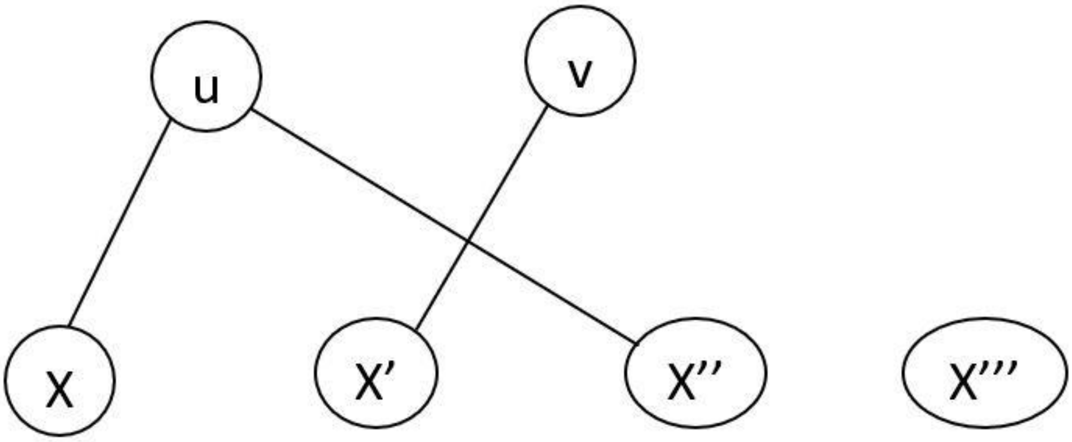
Note: If u & v will have the common neighbor then to diameter more than 3, we have to add more vertex and edges.

Now, $\exists$ x & y vertex $\in$ V(G)-{u, v}

such that ux $\in$ E(G) and vy $\in$ E(G).

In Simpler way, we can write this,

$\forall$ x $\in$ V(G) – {u, v} gas at least one of {u, v} is non neighbor.




Since uv $\notin$ E(G) then uv definitely $\in$ E(G')

And also, ux''' $\in$ E(G') and vx''' $\in$ E(G').



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