Week 13

Planer Graphs*

The section describes the contents taught in Week-13.

13.1 Planer Graphs

Planer graph is a graph that can be drawn without crossing of any edge.

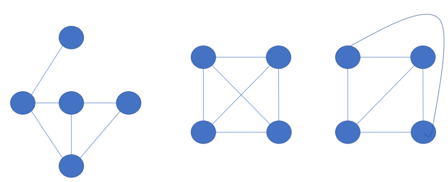

Figure 13.1: Planer graphs

Fig 13.1 shows an example of planer graph.

1. They are sparse graph for large vertex graph (graph having less number of vertices).

2. They are 4-colourable. item Efficient operation.

13.2 Face

Face is defined as the regions in the graph.

The Figure 13.2 shows the number of faces in different graphs.

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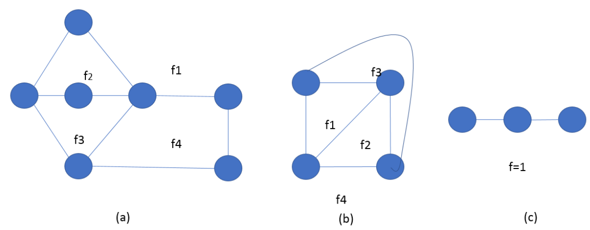

Figure 13.2: Number of faces=4 for (a),(b) and 1 for (c)

13.3 Euler's condition

For any connected planer graph: $v - e + f = 2$
Where, v = number of vertices,
e = number of edges,
f = number of faces.

For Example

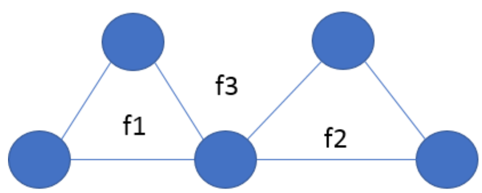

Figure 13.3: f=3, v=5 and e=6

So we can verify that, v-e+f=5-6+3=2.

Proof. Using Induction on number of edges.
Base case:-
e=0, v=1, f=1.
This implies $v - e + f = 1 - 0 + 1 = 2$. So true for base case.
Induction:- Let us assume that upto nedges in a planer conneccted graph, this is true:
$v - e + f = 2$
Let us try to prove that it is true for n+1 edges.
Let G be a graph with n+1 edges.
Case 1: G does not contain a cycle.
number of vertices in $G = n + 2$
number of edges in $G = n + 1$
number of faces in $G = 1$

Table 13.1: Chord and conflicting chords

Chord	Conflicting chord
AD	CE, BE
BD	AC, CE
....	....

Therefore, $v - e + f = n + 2 - n - 1 + 1 = 2$.

Case2: G contain atleast a cycle.
Say we remove edge "p" from a cycle and call the resultant graph G'.
In G'
no. of vertices =v
no. of edges =e-1=n
no. of faces = f-1
Therefore, $v - e + f = v - e + 1 + f - 1 = 2$.
Hence proved. □

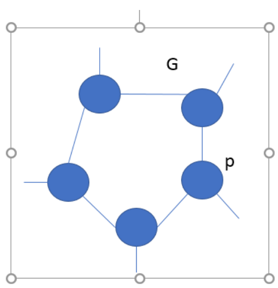

Figure 13.4: p^{th} edge is removed

13.4 Chords and conflicting chords:

- Chords: connection of any 2 vertices.
- Conflicting chords: Two chords conflict if their end points occur in alternate order on the cycle.

Table 13.1 gives the chord and conflicting chord for K_5 shown in fig 13.5. $K_{3,3}$ The graph is a non-planer graph as shown in fig 13.6.

13.5 Dual graph:

The dual graph G^* of a planer graph G is also a planer graph whose vertices corrospondes to the faces of G. The edges of G^* corrospondes to the edges of G as follows:

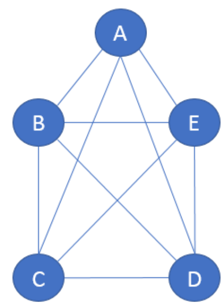

Figure 13.5: K_5

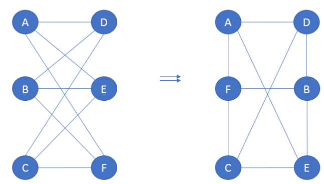

Figure 13.6: Non-planer graph

If e is an edge where one side is face-x and other side is face-y, then in G^* there is an edge between vertex corrosponding to face x and face y.
The examples are shown in the figure below:

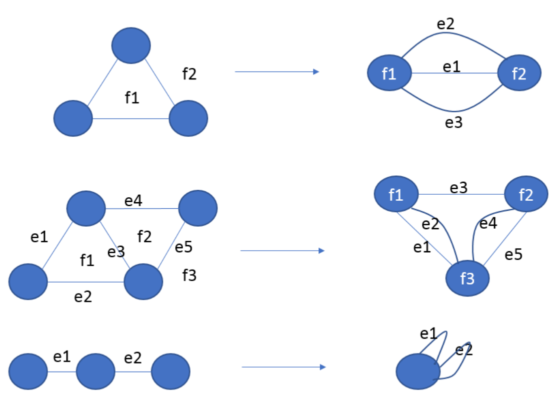

Figure 13.7: Dual graph

1. Every subgraph of a planer graph is also planer : True
2. Every subgraph of a non-planer graph is a non-planer graph :False (By example of $K_{3,3}$)

13.6 Length of a face:

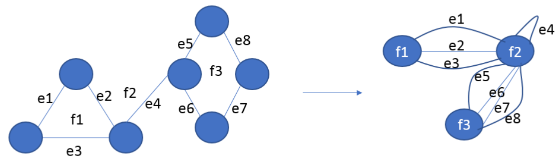

Figure 13.8: Example

As shown in the figure 13.8, length (f_1)=3,
length (f_2)=9,
length(f_3)=4

Statement: If l(f_i) denote length of face f_i in a planer graph, we have

$$2e = \sum_i l(f_i) \tag{13.1}$$

Proof. In the dual graph, $l(f_i)$=degree(f_i) in G^*
Handshaking lemma in G^*

$$2e = \sum_i degree(f_i) \tag{13.2}$$

$$2e = \sum_i l(f_i) \tag{13.3}$$

Hence proved □

Statement: If G is a connected planer graph with v nodes, e edges, f faces and v≥3 then,

$$e \leq 3v - 6 \tag{13.4}$$

Proof. If $v \geq 3$ then in G^*, $3f \leq 2e$
Since, $2e = \sum_i l(f_i)$
$\geq 3 + 3.....3$ (f-times)
=3f
So, v-e+f=2
f=e-v+2
3f=3e-3v+6
We can say,
3e-3v+6≤2e
Therefore, $e \leq 3v - 6$ Hence proved. □

13.7 Planarity test algorithm:

The algorithm is given as:

1. Remove all self loops.

2. Remove parallel edges.

3. Remove vertex having degree 2 and merge the edges incident on that vertex.

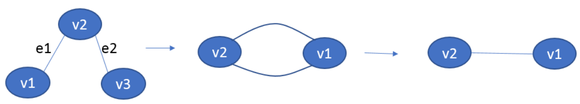

Figure 13.9: Steps to check planerity

$G \rightarrow H$ means, G is planer if His planer.
H is planer if,

1. H has one edge

2. $H = K_4$ or

3. $e \leq 3v - 6$

Theorem 13.1. *The following statements are equivalent for a planer graph:*

- *G is a biparatite graph.*

- *Every face og G has even length.*

- *The dual graph G^* is Eulerian.*

Proof. $(a) \rightarrow (b)$ G contains only even length cycle. We know that cycles make faces.
Every face of G has even length.
Degree of each node in G^* is even. So, G^* is eulerian. Hence proved. ☐

Theorem 13.2. *Every simple planer graph has a vertex of degree at most 5.*

Proof. We know that every simple planer graph with vertices has at most 3v-6 edges, for $v \geq 3$.
Hence the sum of degree is at most 6v-12. Ther is surely a vertex whose degree < 6.
Hence proved. ☐