Solving rational expectations models at first order: what Dynare does

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Dynare Model Framework

$$E\left[f_\theta\left(y_{t-1}, y_t, y_{t+1}, u_t \,|\, \Omega_t\right)\right] = 0$$
$$u_s \sim WN(0, \Sigma_u)$$

$t, s \in \mathbb{T}$: discrete time set, typically $\mathbb{N}$ or $\mathbb{Z}$

y_t: n endogenous variables (declared in *var* block)

u_t: n_u exogenous variables (declared in *varexo* block)

Σ_u: covariance matrix of invariant distribution of exogenous variables (declared in *shocks* block)

θ: n_θ model parameters (declared in *parameters* block)

f: n model equations (declared in *model* block)

f_θ is a continuous non-linear function indexed by a vector of parameters θ

$$E\left[f_\theta\left(y_{t-1}, y_t, y_{t+1}, u_t \,|\, \Omega_t\right)\right] = 0$$
$$u_s \sim WN(0,\Sigma_u)$$

Ω_t: information set (*filtration,* i.e. $\Omega_t \subseteq \Omega_{t+s} \forall s \geq 0$)

$E[\,\cdot\,|\Omega_t]$: conditional expectation operator, typically we use shorthand E_t

Rational expectations:

- information set includes model equations *f,* value of parameters θ*,* value of current state y_{t-1}, value of current exogenous variables u_t, invariant distribution (but not values!) of future exogenous variables u_{t+s}
- $\Omega_t = \{f, \theta, y_{t-1}, u_t, u_{t+s} \sim N(0,\Sigma)\}$ for all $t \in \mathbb{T}$, $s > 0$

$$E_t\left[f\left(y_{t-1}, y_t, y_{t+1}, u_t\right)\right] = 0$$

$$E_t\left[f\left(y^{*}_{t-1}, y_t, y^{**}_{t+1}, u_t\right)\right] = 0$$

nk_var_typology.mod

Typology and ordering of variables

$$n = n^{static} + n^{pred} + n^{fwrd} + n^{both}$$

static: appear only at t, not at $t-1$, not at $t+1$

predetermined: appear at $t-1$, not at $t+1$, *possibly* at t

forward: appear at $t+1$, not at $t-1$, *possibly* at t

mixed: appear at $t-1$ and $t+1$, *possibly* at t

Typology and ordering of variables

y_t^* are the *state variables*: predetermined and mixed variables (n^{spred})

y_t^{**} are the *jumper variables"*: mixed and forward variables (n^{sfwrd})

Typology and ordering of variables

Declaration order: as you declare in *var* block

DR (decision-rule) order: used for perturbation

$$y_t = \begin{pmatrix} \textit{static} \\ \textit{predetermind} \\ \textit{mixed} \\ \textit{forward} \end{pmatrix} \quad y_t^* = \begin{pmatrix} \textit{predetermind} \\ \textit{mixed} \end{pmatrix} \quad y_t^{**} = \begin{pmatrix} \textit{mixed} \\ \textit{forward} \end{pmatrix}$$

Perturbation approach

General idea

Step 1: Introduce perturbation parameter

- scale u_t by a parameter $\sigma \geq 0$: $u_t = \sigma \ \varepsilon_t$ with $\varepsilon_t \sim WN(0,\Sigma_\varepsilon)$
- note that this implies $\Sigma_u = \sigma^2 \Sigma_\varepsilon$
- σ is called the *perturbation parameter*
    - non-stochastic, i.e. static model: $\sigma = 0$
    - stochastic, i.e. dynamic model: $\sigma > 0$

General idea

Step 2: define dynamic solution

- find an invariant mapping between y_t and (y^*_{t-1}, u_t):

$$y_t = g(y^*_{t-1}, u_t, \sigma)$$

- $g(\cdot)$ is called the *policy-function* or *decision rule*
- $g(\cdot)$ is unknown, i.e. we need to solve a *functional equation*

General idea

Idea: Maybe we can get g from $E_t\left[f\left(y^*_{t-1}, y_t, y^{**}_{t+1}, u_t\right)\right] = 0$?

Define

- $y_t = g(y^*_{t-1}, u_t, \sigma)$ $\quad$ $y^*_t = g^*(y^*_{t-1}, u_t, \sigma)$ $\quad$ $y^{**}_t = g^{**}(y^*_{t-1}, u_t, \sigma)$
- $y^{**}_{t+1} = g^{**}(y^*_t, u_{t+1}, \sigma) = g^{**}(g^*(y^*_{t-1}, u_t, \sigma), u_{t+1}, \sigma)$

General idea

Rewrite dynamic model:

$$f\left(y^*_{t-1}, y_t, y^{**}_{t+1}, u_t\right)$$

$$= f\left(y^*_{t-1}, g(y^*_{t-1}, u_t, \sigma), g^{**}(g^*(y^*_{t-1}, u_t, \sigma), u_{t+1}, \sigma), u_t\right)$$

$$\equiv F(y^*_{t-1}, u_t, u_{t+1}, \sigma)$$

General idea

Perturbation is based on the *implicit function theorem:*

$$E_t F(y^*_{t-1}, u_t, u_{t+1}, \sigma) = 0 \quad \text{[known]}$$

<u>*implicitly*</u> defines

$$g(y^*_{t-1}, u_t, \sigma) \text{ [unknown]}$$

General idea

We know how to solve for the non-stochastic ($\sigma = 0$) steady-state $\bar{y}$ by solving the *static* model:

$$\bar{f}(\bar{y}) \equiv f(\bar{y}^*, \bar{y}, \bar{y}^{**}, 0) = F(\bar{y}^*, 0, 0, 0) = 0$$

which provides us with the non-stochastic steady-state for $\bar{y}$, $\bar{y}^*$ and $\bar{y}^{**}$

Even though we do not know $g(\,\cdot\,)$ explicitly, we do know its value at $\bar{y}$, $\bar{y}^*$ and $\bar{y}^{**}$:

$$\bar{y}^* = g^*(\bar{y}^*, 0, 0) \quad \text{and} \quad \bar{y} = g(\bar{y}^*, 0, 0)$$

Taylor approximation of g

Let's approximate $g(\cdot)$ around $\bar{y}$ with a 1st order Taylor expansion:

$$y_t = g(y^*_{t-1}, u_t, \sigma)$$

$$y_t \approx \bar{y} + \left[\frac{\partial g(\bar{y}^*,0,0)}{\partial y^*_{t-1}}\right](y^*_{t-1} - \bar{y}^*) + \left[\frac{\partial g(\bar{y}^*,0,0)}{\partial u_t}\right](u_t - 0) + \left[\frac{\partial g(\bar{y}^*,0,0)}{\partial \sigma}\right](\sigma - 0)$$

Some progress: instead of an infinite unknown number of parameters for g, we have now only *three* unknown matrices

Taylor approximation of g

But: how do we obtain these?

➡ Let's approximate $F(\,\cdot\,)$ around $\bar{y}$ with a 1st order Taylor expansion!

Notation Variable Vectors

$$y_0 := y_t\,,\, y_0^* := y_t^*,\, y_0^{**} := y_t^{**},\, u := u_t\,,\, u_+ := u_{t+1}$$

$$y_- := y_{t-1},\, y_-^* := y_{t-1}^*,\, y_-^{**} := y_{t-1}^{**},\, y_+ := y_{t+1},\, y_+^* := y_{t+1}^*\,,\, y_+^{**} := y_{t+1}^{**}$$

$x := y_{t-1}^*$ denotes *previous states*

$$r := \begin{pmatrix} x \\ u \\ u_+ \\ \sigma \end{pmatrix} \qquad z := \begin{pmatrix} y_-^* \\ y \\ y_+^{**} \\ u \end{pmatrix} = \begin{pmatrix} x \\ g(x, u, \sigma) \\ G(x, u, u_+, \sigma) \\ u \end{pmatrix}$$

Notation Jacobian Matrices

$$g_x := \left[\frac{\partial g(\bar{y}^*,0,0)}{\partial y^*_{t-1}}\right],\ g_u := \left[\frac{\partial g(\bar{y}^*,0,0)}{\partial u_t}\right],\ g_\sigma := \left[\frac{\partial g(\bar{y}^*,0,0)}{\partial \sigma}\right]\ \text{[unknown]}$$

$$f_{y^*_-} := \left[\frac{\partial f(\bar{z})}{\partial y^*_{t-1}}\right],\ f_{y_0} := \left[\frac{\partial f(\bar{z})}{\partial y_t}\right],\ f_{y^{**}_+} := \left[\frac{\partial f(\bar{z})}{\partial y^{**}_{t+1}}\right],\ f_u := \left[\frac{\partial f(\bar{z})}{\partial u_t}\right]\ \text{[known]}$$

$$F_x := \left[\frac{\partial F(\bar{r})}{\partial y^*_{t-1}}\right],\ F_u := \left[\frac{\partial F(\bar{r})}{\partial u_t}\right],\ F_{u_+} := \left[\frac{\partial F(\bar{r})}{\partial u_{t+1}}\right],\ F_\sigma := \left[\frac{\partial F(\bar{r})}{\partial \sigma}\right]\ \text{[implicit]}$$

All derivatives are evaluated at the non-stochastic steady-state

Taylor approximation of F

Let's approximate $F(y^*_{t-1}, u_t, u_{t+1}, \sigma) = F(r)$ around $\bar{r}$ at 1st order:

$$F(r) \approx F(\bar{r}) + F_x\ \hat{x} + F_u\ \hat{u} + F_{u_+}\ \hat{u}_+ + F_\sigma\ \hat{\sigma}$$

where $\hat{x} = (x - \bar{x})$, $\hat{u} = (u - 0) = u$, $\hat{u}_+ = (u_+ - 0) = u_+$, $\hat{\sigma} = (\sigma - 0) = \sigma$

The dynamic model implies that $E_t F(r) = 0$. So let's take the conditional expectation and set to 0.

Taylor approximation of F

$$0 = E_t F(r) \approx 0 + F_x \ \hat{x} + F_u \ u + F_{u_+} \ E_t \sigma \varepsilon_+ + F_\sigma \ \sigma$$

$$0 \approx F_x \ \hat{x} + F_u \ u + \left(F_\sigma + F_{u_+} \ E_t \varepsilon_+ \right) \sigma$$

This equation needs to be satisfied for *any* value of $\hat{x}$, u and σ.

Therefore:

$$F_x = 0 \quad \text{and} \quad F_u = 0 \quad \text{and} \quad F_\sigma + F_{u_+} E_t \varepsilon_+ = 0$$

Taylor approximation of F

$$F_x = 0 \quad \text{and} \quad F_u = 0 \quad \text{and} \quad F_\sigma + F_{u_+} E_t \varepsilon_+ = 0$$

We have 3 (multivariate) equations and 3 (multivariate) unknowns!

We can recover:

- g_x from $F_x = f_z z_x = f_{y_-^*} + f_{y_0} g_x + f_{y_+^{**}} g_x^{**} g_x^* = 0$
- g_u from $F_u = f_z z_u = f_{y_0} g_u + f_{y_+^{**}} g_x^{**} g_u^* + f_u = 0$
- g_σ from $F_\sigma + F_{u_+} E_t \varepsilon_+ = f_z z_\sigma + f_z z_{u_+} E_t \varepsilon_+ = f_{y_0} g_\sigma + f_{y_+^{**}} (g_x^{**} g_\sigma^* + g_\sigma^{**}) + f_{y_+^{**}} g_u^{**} E_t \varepsilon_+ = 0$

Recovering g_σ

Recovering g_σ

First order Taylor expansion of

$$F = f(\underbrace{x}_{y_-^*}, \underbrace{g(x,u,\sigma)}_{y_0}, \underbrace{g^{**}(g^*(x,u,\sigma),u_+,\sigma)}_{y_+^{**}}, u)$$

with respect to σ yields:

$$F_\sigma = f_{y_0} g_\sigma + f_{y_+^{**}}(g_x^{**} g_\sigma^* + g_\sigma^{**})$$

with respect to u_+ yields:

$$F_{u_+} = f_{y_+^{**}} g_u^{**}$$

Recovering g_σ

$$F_\sigma + F_{u_+} E_t \varepsilon_+ = f_{y_0} g_\sigma + f_{y_+^{**}} (g_x^{**} g_\sigma^{*} + g_\sigma^{**}) + f_{y_+^{**}} g_u^{**} E_t \varepsilon_+ = 0$$

Let's introduce auxiliary perturbation matrices:

$$A = f_{y_0} + \left(\underbrace{0}_{n \times n^{static}} \; \vdots \; \underbrace{f_{y_+^{**}} g_x^{**}}_{n \times n^{spred}} \; \vdots \; \underbrace{0}_{n \times n^{fwrd}} \right)$$

$$B = \left(\underbrace{0}_{n \times n^{static}} \; \vdots \; \underbrace{0}_{n \times n^{pred}} \; \vdots \; \underbrace{f_{y_+^{**}}}_{n \times n^{sfwrd}} \right)$$

Recovering g_σ

$$f_{y_0} g_\sigma + f_{y_+^{**}}(g_x^{**} g_\sigma^* + g_\sigma^{**}) + f_{y_+^{**}} g_u^{**} E_t \varepsilon_+ = 0$$

$$(A + B) g_\sigma + f_{y_+^{**}} g_u^{**} E_t \varepsilon_+ = 0$$

$$g_\sigma = (A + B)^{-1} f_{y_+^{**}} g_u^{**} E_t \varepsilon_+$$

Of course, we know that $E_t \varepsilon_{t+1} = 0$ which implies:

$$g_\sigma = 0$$

Certainty Equivalence $g_\sigma = 0$

When we derived the optimality conditions (aka model equations) agents do take into account the effect of future uncertainty when optimizing their objective functions.

BUT: the policy function is independent of the size of the stochastic innovations:

$$y_t = g_x y^*_{t-1} + g_u u_t$$

Future uncertainty does not matter for the decision rules of the agents!

Certainty equivalence is a result of the first-order perturbation approximation, we can break it with e.g. higher-order perturbation approximation

Recovering g_u

Recovering g_u

First order Taylor expansion of

$$F = f(\underbrace{x}_{y_-^*}, \underbrace{g(x, u, \sigma)}_{y_0}, \underbrace{g^{**}(g^*(x, u, \sigma), u_+, \sigma)}_{y_+^{**}}, u)$$

with respect to u yields:

$$F_u = f_{y_0} g_u + f_{y_+^{**}} g_x^{**} g_u^* + f_u = A g_u + f_u$$

$F_u = 0$ implies:

$$g_u = -A^{-1} f_u$$

Recovering g_u

$$g_u = -A^{-1} f_u$$

This is a linear equation which requires the inverse of A

$$A = f_{y_0} + \begin{pmatrix} 0 & \vdots & f_{y_+^{**}} g_x^{**} & \vdots & 0 \end{pmatrix}$$

Once we know g_x, we can easily compute g_u.

Recovering g_x

Quadratic Equation

First order Taylor expansion of

$$F = f(\underbrace{x}_{y_-^*}, \underbrace{g(x, u, \sigma)}_{y_0}, \underbrace{g^{**}(g^*(x, u, \sigma), u_+, \sigma)}_{y_+^{**}}, u)$$

with respect to x and setting it to zero yields:

$$F_x = f_{y_-^*} + f_{y_0} g_x + f_{y_+^{**}} g_x^{**} g_x^* \stackrel{!}{=} 0$$

This is a *quadratic equation*, but the unknown g_x is a matrix!

It is generally impossible to solve this equation analytically, but there are several ways to deal with this as this boils down to solving so-called *Linear Rational Expecations Models*

Linear Rational Expectations Model

Re-consider original dynamic model:

$$E_t f(y^*_{t-1}, y_t, y^{**}_{t+1}, u_t) = 0$$

Take first-order Taylor expansion:

$$f_{y^*_-}\hat{y}^*_{t-1} + f_{y_0}\hat{y}_t + f_{y^{**}_+}E_t\hat{y}^{**}_{t+1} + f_u u_t = 0$$

In the literature this is known as a *Linear Rational Expectations Model*

Linear Rational Expectations Model

$$f_{y_-^*}\hat{y}^*_{t-1} + f_{y_0}\hat{y}_t + f_{y_+^{**}}E_t\hat{y}^{*}_{t+1} + f_u u_t = 0$$

From the policy functions we know:

$$\hat{y}_t = g_x\hat{y}^*_{t-1} + g_u u_t \quad \text{and} \quad \hat{y}^*_t = g_x^*\hat{y}^*_{t-1} + g_u^* u_t$$

$$E_t\hat{y}^{**}_{t+1} = g_x^{**}\hat{y}^*_t + g_u^{**}E_t u_{t+1} = g_x^{**}\hat{y}^*_t = g_x^{**}(g_x^*\hat{y}^*_{t-1} + g_u^* u_t) = g_x^{**}g_x^*\hat{y}^*_{t-1} + g_x^{**}g_u^* u_t$$

Connection to perturbation:

$$\underbrace{\left(f_{y_-^*} + f_{y_0}g_x + f_{y_+^{**}}g_x^{**}g_x^*\right)}_{F_x=0}\hat{y}^*_{t-1} = -\underbrace{\left(f_{y_0}g_u + f_{y_+^{**}}g_x^{**}g_u^* + f_u\right)}_{F_u=0}u_t = 0$$

Structural State-Space System

$$f_{y_-^*}\hat{y}_{t-1}^{*} + f_{y_0^{static}}\hat{y}_t^{static} + f_{y_0^{pred}}\hat{y}_t^{pred} + f_{y_0^{**}}\hat{y}_t^{**} + f_{y_+^{**}}E_t\hat{y}_{t+1}^{**} = -f_u u$$

$$\underbrace{\begin{pmatrix} f_{y_0^{static}} & f_{y_0^{pred}} & 0 & f_{y_+^{**}} \end{pmatrix}}_{:=\tilde{D}} \underbrace{\begin{pmatrix} \hat{y}_t^{static} \\ \hat{y}_t^{pred} \\ \hat{y}_t^{mixed} \\ E_t\hat{y}_{t+1}^{**} \end{pmatrix}}_{:=\tilde{Y}_t} = \underbrace{\begin{pmatrix} 0 & -f_{y_-^*} & -f_{y_0^{**}} \end{pmatrix}}_{:=\tilde{E}} \underbrace{\begin{pmatrix} \hat{y}_{t-1}^{static} \\ \hat{y}_{t-1}^{*} \\ \hat{y}_t^{**} \end{pmatrix}}_{:=\tilde{Y}_{t-1}} + \underbrace{-f_u u_t}_{\tilde{U}_t}$$

$$\tilde{D} \cdot \tilde{Y}_t = \tilde{E} \cdot \tilde{Y}_{t-1} + \tilde{U}_t$$

Getting rid of static variables

We usually have *a lot of static* variables

To compute g_x^{**} and g_x^*, we don't need the entries corresponding to the static variables in the Jacobian, so we can reduce the size of the matrices considerably

From a theoretical point of view this step is optional

From a numerical point of view this enables one to deal with very large models efficiently

Getting rid of static variables

Let S be the submatrix of f_{y_0} with columns for static endogenous variables only

Do a QR decomposition: $S = Q_s R_s$

Q_s is orthogonal ($Q_s Q_s' = I$ and $Q_s' = Q_s^{-1}$) and R_s is upper triangular

$$Q_s' f_{y_-^*} \hat{y}_{t-1}^{*} + Q_s' f_{y_0^{static}} \hat{y}_t^{static} + Q_s' f_{y_0^{pred}} \hat{y}_t^{pred} + Q_s' f_{y_0^{**}} \hat{y}_t^{**} + Q_s' f_{y_+^{**}} E_t \hat{y}_{t+1}^{**} = -Q_s' f_u u_t$$

By construction, columns of static variables in $Q_s' f_{y_0}$ are zero in their *lower part*

Getting rid of static variables

Denote the <u>*lower rows*</u> of $Q_s' f_{y_-^*}$, $Q_s' f_{y_0}$, $Q_s' f_{y_+^{**}}$, $Q_s' f_u$ by $f_{y_-^*}^Q$, $f_{y_0}^Q$, $f_{y_+^{**}}^Q$, f_u^Q

We can focus on the reduced system:

$$f_{y_-^*}^Q \hat{y}_{t-1}^* + f_{y_0^{pred}}^Q \hat{y}_t^{pred} + f_{y_0^{**}}^Q \hat{y}_t^{**} + f_{y_+^{**}}^Q E_t \hat{y}_{t+1}^{**} = -f_u^Q u_t$$

Static endogenous variables do not appear anymore

Again this step is theoretically optional, but numerically more efficient

Structural State-Space System

$$f^Q_{y^*_-}\hat{y}^*_{t-1} + f^Q_{y_0^{pred}}\hat{y}^{pred}_t + f^Q_{y_0^{**}}\hat{y}^{**}_t + f^Q_{y_+^{**}}E_t\hat{y}^{**}_{t+1} = -f^Q_u u_t$$

$$\underbrace{\begin{pmatrix} f^Q_{y_0^{pred}} & 0 & f^Q_{y_+^{**}} \\ 0 & I & 0 \end{pmatrix}}_{:=D}\begin{pmatrix} \hat{y}^{pred}_t \\ \hat{y}^{mixed}_t \\ E_t\hat{y}^{**}_{t+1} \end{pmatrix} = \underbrace{\begin{pmatrix} -f^Q_{y^*_-} & -f^Q_{y_0^{mixed}} & -f^Q_{y_0^{fwrd}} \\ 0 & I & 0 \end{pmatrix}}_{:=E}\begin{pmatrix} \hat{y}^*_{t-1} \\ \hat{y}^{mixed}_t \\ \hat{y}^{fwrd}_t \end{pmatrix} - f^Q_u u_t$$

$$D\underbrace{\begin{pmatrix} \hat{y}^*_t \\ E_t\hat{y}^{**}_{t+1} \end{pmatrix}}_{:=Y_t} = E\underbrace{\begin{pmatrix} \hat{y}^*_{t-1} \\ \hat{y}^{**}_t \end{pmatrix}}_{:=Y_{t-1}} + \underbrace{-f^Q_u u_t}_{:=U_t}$$

Stability

$$D \cdot Y_t = E \cdot Y_{t-1} + U_t$$

D and E are by construction square matrices

IF D is invertible, then:

$$Y_t = (D^{-1}E)Y_{t-1} + D^{-1}U_t$$

$$= (D^{-1}E)^0 D^{-1}U_t + (D^{-1}E)^1 D^{-1}U_{t-1} + (D^{-1}E)^2 D^{-1}U_{t-2} + (D^{-1}E)^3 D^{-1}U_{t-3} + \ldots$$

Stable solution if and only if Eigenvalues of $(D^{-1}E)$ are inside unit circle

BUT: D is typically singular and non-invertible!

Generalized Schur Decomposition

Instead of inverse we'll use a Schur decomposition on matrix pencil $< D, E >$:

$$D = QTZ \quad \text{and} \quad E = QSZ$$

Q is orthogonal: $Q' = Q^{-1}$ and $Q'Q = QQ' = I$

Z is orthogonal: $Z' = Z^{-1}$ and $Z'Z = ZZ' = I$

T is upper triangular and S is quasi-upper triangular

MATLAB: qz(D,E) provides the transposed decomposition $D = Q'TZ' \quad \text{and} \quad E = Q'SZ'$

Generalized Eigenvalues

Enforce stability: look at *Generalized Eigenvalues* of D and E:

$$\lambda_i D v_i = E v_i$$

which can be found on the diagonal of S and T: $\lambda_i = \dfrac{S_{ii}}{T_{ii}}$

If $T_{ii} = 0$, then: $S_{ii} > 0 \rightarrow \lambda_i = \infty$ and $S_{ii} < 0 \rightarrow \lambda_i = -\infty$

Schur Decomposition on Structural State-Space System

$$QTZ\begin{pmatrix} \hat{y}_t^{*} \\ E_t\hat{y}_{t+1}^{**} \end{pmatrix} = QSZ\begin{pmatrix} \hat{y}_{t-1}^{*} \\ \hat{y}_t^{**} \end{pmatrix} - f_u^Q u_t$$

Inserting the policy functions we can simplify:

$$TZ\begin{pmatrix} I \\ g_x^{**} \end{pmatrix} g_x^{*}\hat{y}_{t-1}^{*} = SZ\begin{pmatrix} I \\ g_x^{**} \end{pmatrix} \hat{y}_{t-1}^{*} + \widetilde{U}_t$$

where $\widetilde{U}_t$ collects all terms involving u_t.

Note that $\widetilde{U}_t = 0$, because $F_u = 0$ (see previous slide on Linear Rational Expectations Solution)

Re-ordering of Schur decomposition

Order stable generalized eigenvalues $\lambda_i < 1$ in the upper left corner of T and S:

$$\begin{pmatrix} T_{11} & T_{12} \\ 0 & T_{22} \end{pmatrix} \begin{pmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{pmatrix} \begin{pmatrix} I \\ g_x^{**} \end{pmatrix} g_x^{*} \hat{y}_{t-1}^{*} = \begin{pmatrix} S_{11} & S_{12} \\ 0 & S_{22} \end{pmatrix} \begin{pmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{pmatrix} \begin{pmatrix} I \\ g_x^{**} \end{pmatrix} \hat{y}_{t-1}^{*}$$

T_{11} and S_{11} are square matrices and contain stable generalized eigenvalues

T_{22} and S_{22} are square matrices and contain unstable generalized eigenvalues

Impose Stability

$$\begin{pmatrix} T_{11} & T_{12} \\ 0 & T_{22} \end{pmatrix} \begin{pmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{pmatrix} \begin{pmatrix} I \\ g_x^{**} \end{pmatrix} g_x^{*} \hat{y}_{t-1}^{*} = \begin{pmatrix} S_{11} & S_{12} \\ 0 & S_{22} \end{pmatrix} \begin{pmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{pmatrix} \begin{pmatrix} I \\ g_x^{**} \end{pmatrix} \hat{y}_{t-1}^{*}$$

We ***DON'T WANT*** an explosive solution, so we rule this out by imposing:

$$\begin{pmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{pmatrix} \begin{pmatrix} I \\ g_x^{**} \end{pmatrix} = \begin{pmatrix} \mathrm{XXX} \\ 0 \end{pmatrix}$$

such that the lower (explosive) rows are always zero:

$$0 \cdot XXX + T_{22} \cdot 0 = 0 \cdot XXX + S_{22} \cdot 0 = 0$$

Impose Stability

$$Z \begin{pmatrix} I \\ g_x^{**} \end{pmatrix} = \begin{pmatrix} XXX \\ 0 \end{pmatrix}$$

Pre-multiply by Z':

$$\underbrace{Z'Z}_{I} \begin{pmatrix} I \\ g_x^{**} \end{pmatrix} = \begin{pmatrix} Z'_{11} & Z'_{21} \\ Z'_{12} & Z'_{22} \end{pmatrix} \begin{pmatrix} XXX \\ 0 \end{pmatrix}$$

Focusing on the upper rows we get

$$XXX = (Z'_{11})^{-1}$$

Recovering g_x^{**}

$$\begin{pmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{pmatrix} \begin{pmatrix} I \\ g_x^{**} \end{pmatrix} = \begin{pmatrix} (Z'_{11})^{-1} \\ 0 \end{pmatrix}$$

From the lower rows we can recover g_x^{**}:

$$Z_{21}I + Z_{22}g_x^{**} = 0$$

$$g_x^{**} = -(Z_{22})^{-1}Z_{21}$$

Blanchard & Khan (1980) conditions

1. Order condition:
   Squareness of Z_{22}, i.e. requirement to have as many explosive Eigenvalues as forward or mixed endogenous variables

2. Rank condition:
   Invertibility of Z_{22}, i.e. full rank of Z_{22}

Blanchard & Khan (1980) conditions

Provided that the rank condition is satisfied, three cases are possible:

UNIQUE STABLE SOLUTION

Number of forward or mixed variables == Number of explosive Eigenvalues

INDETERMINACY

Number of forward or mixed variables > Number of explosive Eigenvalues

EXPLOSIVENESS

Number of forward or mixed variables < Number of explosive Eigenvalues

Recovering g_x^*

Combining $\begin{pmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{pmatrix} \begin{pmatrix} I \\ g_x^{**} \end{pmatrix} = \begin{pmatrix} (Z_{11}')^{-1} \\ 0 \end{pmatrix}$ with

$$\begin{pmatrix} T_{11} & T_{12} \\ 0 & T_{22} \end{pmatrix} \begin{pmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{pmatrix} \begin{pmatrix} I \\ g_x^{**} \end{pmatrix} g_x^* \hat{y}_{t-1}^* = \begin{pmatrix} S_{11} & S_{12} \\ 0 & S_{22} \end{pmatrix} \begin{pmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{pmatrix} \begin{pmatrix} I \\ g_x^{**} \end{pmatrix} \hat{y}_{t-1}^* \text{ we get}$$

$$\begin{pmatrix} T_{11} & T_{12} \\ 0 & T_{22} \end{pmatrix} \begin{pmatrix} (Z_{11}')^{-1} \\ 0 \end{pmatrix} g_x^* = \begin{pmatrix} S_{11} & S_{12} \\ 0 & S_{22} \end{pmatrix} \begin{pmatrix} (Z_{11}')^{-1} \\ 0 \end{pmatrix}$$

From the first rows we can recover g_x^*:

$$T_{11}(Z_{11}')^{-1} g_x^* = S_{11}(Z_{11}')^{-1}$$

$$g_x^* = (Z_{11}') T_{11}^{-1} S_{11} (Z_{11}')^{-1}$$

Recovering g_x^{static}

$$Q_s' f_{y_-^*} y_{t-1}^* + Q_s' f_{y_0^{static}} \hat{y}_t^{static} + Q_s' f_{y_0^{pred}} \hat{y}_t^{pred} + Q_s' f_{y_0^{**}} \hat{y}_t^{**} + Q_s' f_{y_+^{**}} E_t y_{t+1}^{**} = -Q_s' f_u u_t$$

Now we focus on the n^{static} lines in $Q' f_{y_0}$ (notation is with inverted hat):

$$\check{f}^Q_{y_-^*} \hat{y}_{t-1}^* + \check{f}^Q_{y_0^{static}} \hat{y}_t^{static} + \check{f}^Q_{y_0^{ns}} \hat{y}_t^{ns} + \check{f}^Q_{y_+^{**}} E_t \hat{y}_{t+1}^{**} = -\check{f}^Q_u u_t$$

$\hat{y}_t^s$ are static and $\hat{y}_t^{ns}$ are non-static (i.e. $\hat{y}_t^*$ and $\hat{y}_t^{**}$ combined) variables

Recovering g_x^{static}

Mixed variables appear both in g_x^{**} and g_x^{*}; the corresponding lines will be equal, let's call these $\hat{y}_t^{nonstatic}$

Now we focus on the n^{static} lines in $Q' f_{y_0}$ (notation is with inverted hat):

$$\check{f}^Q_{y_-^*}\hat{y}^*_{t-1} + \check{f}^Q_{y_0^{static}} g_x^{static}\hat{y}^*_{t-1} + \check{f}^Q_{y_0^{nonstatic}} g_x^{nonstatic}\hat{y}^*_{t-1} + \check{f}^Q_{y_+^{**}} g_x^{**} g_x^{*}\hat{y}^*_{t-1} = 0$$

$$g_x^{static} = -\left(\check{f}^Q_{y_0^{static}}\right)^{-1}\left(\check{f}^Q_{y_-^*} + \check{f}^Q_{y_0^{nonstatic}} g_x^{nonstatic} + \check{f}^Q_{y_+^{**}} g_x^{**} g_x^{*}\right)$$

Summary

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Policy function / decision rule: $y_t = \bar{y} + g_x(y^*_{t-1} - \bar{y}^*) + g_u u_t$

1. do a QR decomposition on dynamic Jacobian to get rid of static variables

2. set up the D and E matrices and do a QZ/Schur decomposition with re-ordering

3. $g_x^{**} = -(Z_{22})^{-1} Z_{21} \qquad g_x^{*} = (Z'_{11}) T_{11}^{-1} S_{11} (Z'_{11})^{-1}$
   $g_x^{static} = -\left(\check{f}^Q_{y_0^{static}}\right)^{-1} \left(\check{f}^Q_{y_-^*} + \check{f}^Q_{y_0^{nonstatic}} g_x^{nonstatic} + \check{f}^Q_{y_+^{**}} g_x^{**} g_x^{*}\right)$

4. $g_u = -A^{-1} f_u$, where $A = f_{y_0} + \begin{pmatrix} 0 & \vdots & f_{y_+^{**}} g_x^{**} & \vdots & 0 \end{pmatrix}$