ESO 208a November 8, 2014

Quiz-4

Solution. First, it is convenient to express the ODEs in the functional format of Eq. (20.46) as

$$\frac{dx}{dt} = f_1(t, x, v) = v$$

$$\frac{dv}{dt} = f_2(t, x, v) = g - \frac{c_d}{m}v^2$$

The first step in obtaining the solution is to solve for all the slopes at the beginning of the interval:

$$k_{1,1} = f_1(0, 0, 0) = 0$$

$$k_{1,2} = f_2(0, 0, 0) = 9.81 - \frac{0.25}{68.1}(0)^2 = 9.81$$

where $k_{i,j}$ is the ith value of k for the jth dependent variable. Next, we must calculate the first values of x and v at the midpoint of the first step:

$$x(1) = x(0) + k_{1,1}\frac{h}{2} = 0 + 0\frac{2}{2} = 0$$

$$v(1) = v(0) + k_{1,2}\frac{h}{2} = 0 + 9.81\frac{2}{2} = 9.81$$

which can be used to compute the first set of midpoint slopes:

$$k_{2,1} = f_1(1, 0, 9.81) = 9.8100$$
$$k_{2,2} = f_2(1, 0, 9.81) = 9.4567$$

These are used to determine the second set of midpoint predictions:

$$x(1) = x(0) + k_{2,1}\frac{h}{2} = 0 + 9.8100\frac{2}{2} = 9.8100$$

$$v(1) = v(0) + k_{2,2}\frac{h}{2} = 0 + 9.4567\frac{2}{2} = 9.4567$$

which can be used to compute the second set of midpoint slopes:

$$k_{3,1} = f_1(1, 9.8100, 9.4567) = 9.4567$$
$$k_{3,2} = f_2(1, 9.8100, 9.4567) = 9.4817$$

These are used to determine the predictions at the end of the interval:

$$x(2) = x(0) + k_{3,1}h = 0 + 9.4567(2) = 18.9134$$
$$v(2) = v(0) + k_{3,2}h = 0 + 9.4817(2) = 18.9634$$

which can be used to compute the endpoint slopes:

$$k_{4,1} = f_1(2, 18.9134, 18.9634) = 18.9634$$
$$k_{4,2} = f_2(2, 18.9134, 18.9634) = 8.4898$$

The values of k can then be used to compute [Eq. (20.44)]:

$$x(2) = 0 + \frac{1}{6}[0 + 2(9.8100 + 9.4567) + 18.9634]\,2 = 19.1656$$

$$v(2) = 0 + \frac{1}{6}[9.8100 + 2(9.4567 + 9.4817) + 8.4898]\,2 = 18.7256$$

Proceeding in a like manner for the remaining steps yields the values displayed in Table 20.4. In contrast to the results obtained with Euler's method, the fourth-order RK predictions are much closer to the true values. Further, a highly accurate, nonzero value is computed for distance on the first step.

TABLE 20.4 Distance and velocity of a free-falling bungee jumper as computed numerically with the fourth-order RK method.

t	x_{true}	v_{true}	x_{RK4}	v_{RK4}	$\varepsilon_t\ (x)$	$\varepsilon_t\ (v)$
0	0	0	0	0		
2	19.1663	18.7292	19.1656	18.7256	0.004%	0.019%
4	71.9304	33.1118	71.9311	33.0995	0.001%	0.037%
6	147.9462	42.0762	147.9521	42.0547	0.004%	0.051%
8	237.5104	46.9575	237.5104	46.9345	0.000%	0.049%
10	334.1782	49.4214	334.1626	49.4027	0.005%	0.038%