###### V. Differentiation and Integration

1. Consider the function 

(a) Obtain finite difference approximations of ** with first order backward difference, second order central difference and 4th order central difference. Evaluate ** by the three methods at 20 equally spaced points in the interval [1,2]. Also evaluate the true value of ** at the same points. Plot ** vs. *x* and graphically compare the true values with the three approximations you have obtained, all in the same plot. Show them by different styles of lines.

(b) Start with h = 1 and do repeated interval halving for 10 times. For each h value, obtain the approximate derivative at x = 4. Also calculate the true derivative at x = 4. Now, compute the absolute value of the error for each h-value. Now, plot ln[error] vs. ln[h] and obtain the slope of the line. Repeat this procedure for each of the three methods mentioned in 1(a). What are the slopes of these lines?

2. Find  numerically using 5 points in the interval by a) Trapezoidal rule, b) Simpson’s rule, c) Gaussian Quadrature. Compute the % error in each of the three cases.

3. The following table is given for the values of *ex* :

*x* 0.00 0.25 0.50 0.75 1.00 1.25 1.50 1.75 2.00

*ex* 1.0000 1.2840 1.6487 2.1170 2.7183 3.4903 4.4817 5.7546 7.3891

a) Compute  using central difference scheme with *h* = 0.25, 0.50 and 1.00.

b) Using the values computed in (a), obtain an estimate with maximum possible accuracy for the derivative by successive application of Richardson’s extrapolation.

c) Compute absolute values of the true relative error for each computed value of the derivative.

4. We are interested in fitting a piecewise Lagrange Polynomial through a set of *N*+1 equispaced (regular grid) discrete points by taking three points at a time. The grid points are denoted as *x*0, *x*1, *x*2, …. *x*n and the corresponding functional values as *f*0, *f*1, *f*2, …. *f*n. Consider any three consecutive grid points *x*i, *x*i+1 and *x*i+2 where corresponding functional values are *f*i, *f*i+1 and *f*i+2, respectively.

a) Write the expression for the Lagrange Polynomial  through these three points.

b) Using (a), obtain the expressions for  and  at point *x*i.

c) What is the order of truncation errors for the expressions obtained in (b)?

d) Compare these expressions with the well-known Finite Difference expressions.

###### VI. ODE: Initial Value Problems

1. Solve the differential equation dy/dt = −100 y − 99 e−t with the initial condition y(0)=2 using, (a) Euler’s forward (explicit) method, and (b) Euler backward (implicit) method, to obtain the value of y at t=0.1. Use time steps of 0.01, 0.02 and 0.025. Find the analytical solution and compare the errors for these time steps.

2. Solve the differential equation dy/dx = x2y − 2y with y(0)=1 over the interval x=0 to 0.5, using (a) Heun’s method without iteration with h=0.25 and 0.125, (b) Heun’s method with iteration (with h=0.25 and stopping criterion 1%), (c) Classical 4th order Runge-Kutta method with h=0.125 and 0.25. Obtain the exact value of y at x=0.5 and perform an error analysis.

3. Solve the differential equation dy/dx = 10 sin(πx) with the initial condition y(0)=0 and step length of 0.2 using (a) the 4th order R-K method, (b) the Milne’s method and (c) 4th order Adams method to obtain the value of y at t=0.2, 0.4, 0.6, 0.8 and 1.0. (For the multi-step methods use the values obtained from the R-K method for start-up.)

###### System of ODEs and Boundary Value Problems

1. Solve the differential equation d2y/dx2 − dy/dx −2y + 2x = 3 with the boundary conditions y(0)=0 and y(0.5)=0.6967 using the direct method (use x = 0.25).

2. Consider a simple pendulum consisting of mass *m* attached to a string of length *l*. The equation of motion for the mass is  where positive ** is counterclockwise. For small angles **, sin ** ≈ ** and the linearized equation of motion is . The acceleration due to gravity is *g* = 9.81 m/sec2, and *l* = 0.6 m. Assume that the pendulum starts from rest with ** (*t* = 0) = 10o.

*l*

*m*

**

Solve the linearized equation for 0 ≤ *t* ≤ 2.0 using a time step *h* = 0.2 by:

a) Analytical method and obtain true solutions.

b) Euler explicit method and compute true absolute error at each time step.

c) Euler implicit method and compute true absolute error at each time step.

d) Trapezoidal method and compute true absolute error at each time step.

e) Graphically compare your results of (b), (c) and (d) with (a) and discuss in terms of the accuracy and stability of these numerical schemes. Which of the above three methods is most accurate for this equation and why?