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Primordial gravitational waves: Studying its effect on the Hubble tension.*

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I. INTRODUCTION

Over a century ago, Einstein postulated the theory of general relativity (GR), in this theory he solves the his linearized equations to obtain solutions for gravitational waves (GW) [1]. He defines these waves to be the strain of both transverse and longitudinal perturbations on the fabric of spacetime. There, they are found to be the mass quadrupole moment of a GW emitting source [2]. The underlying physics which generates gravitational waves is still an area of study today. Their recent detection allows for the investigation of multiple astrophysical phenomena. [3]. While current detections are insensitive, new projects probe the possibility of high-precision measurements [4–6]. Other projects, such as Deci-hertz Interferometer Gravitational Wave Observatory (DECIGO) aim to directly detect gravitational waves that arise from cosmic inflation [7]. These are known as measurements of primordial gravitational waves. With future data expected from missions such as LiteBIRD [8] and CMB-S4 [9], the study of primordial gravitational waves has gathered interest in the cosmology community. There is scope for an improved theoretical background of primordial gravitational waves. Doing so will help us distinguish between various models of primordial gravitational waves. They can also help us determine the rate of expansion H_0.

Thompson scattering on free electron generates polarization in the cosmic microwave background (CMB). Quadrapoles from gravitational waves results in B-mode polarization of the CMB [10]. Its detection will confirm the existence of gravitational waves [11].

Theoretically, the inflationary GW predictions play an important role in the early universe [12]. By considering the power spectrum of primordial perturbations to the metric tensor as well as the expansion rate of the universe from recombination till today, we are able to predict the spectrum associated with primordial GWs. The tensorial perturbations help us determine the initial magnitude of GWs during inflation. The latter, the expansion rate, helps us determine how the density of primordial gravitational waves changes as its strength dissipates through cosmic expansion [13]. The inflationary history of gravitational waves gives us insight into the early universe as well as different inflationary models which may govern it. Primordial gravitational waves can be separated into two forms, those from metric perturbations in inflation, and those from astrophysical events that occur between the end of inflation and big bang nucleosynthesis (BBN) [14]. Examples of this may include GWs produced during reheating [15], by phase transitions [16] or primordial black holes [17]. While an important area of study, we will ignore the second category as they play a smaller role in determining the Hubble constant H_0.

Since we are boldly assuming that primordial gravitational waves arise from predictions of cosmological inflation [18] and are not a result of the electroweak or first-order phase transitions [19, 20]. Perturbations to the the Friedmann–Lemaître–Robertson–Walker (FRW) metric can be used to derive a dimensionless power spectrum which denotes the amplitude of perturbations at a given mode k [21]. Standard slow-roll inflationary models correspond to a flat primordial GW background for many frequencies [14]. We use the *tilt* of the primordial power spectrum to describe any variations from a flat GW background. This occurs due to a frequency dependence in the inflationary Hubble parameter [2].

$$h_0^2 \Omega_{\mathrm{gw}}(f) \sim f^{n_T} \tag{1}$$

This frequency dependence can be describes through the *tensor spectral index* n_T. Using the *consistency relation*, we are able to find the tensor-to-scalar ration.

$$r\left(k_*\right) \equiv \frac{A_{\mathrm{T}}\left(k_*\right)}{A_{\mathrm{S}}\left(k_*\right)} \tag{2}$$

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This ratio determines the ratio of amplitudes between the tensorial power spectrum parameterization and its scalar counterpart for a given k_*. The tensorial amplitude A_t is set by the expansion rate which is close to constant during inflation [22].

The tensor-scalar-value r is highly dependent on the inflationary model. As a result, it has a large parameter space [23]. This ratio can be constrained using observational data which narrows the inflationary model selection [24, 25].

A negative value of n_T corresponds to red-tilted values and positive values of n_T are associated with blue-tilted value. In general, inflationary data from CMB polarization may predict blue-tilted spectral index $n_T < 0$ [26].

Primordial gravitational waves have a similar effect on the CMB as massless neutrinos. This occurs when the primordial GWs energy density perturbations are adiabatic [27]. This criteria which makes them behave similar to neutrinos allows us to contribute relativistic degree's of freedom to the standard value of $N_{\rm eff}$.

$$\begin{aligned} N_{\rm eff}^{new} &= N_{\rm eff} + N_{\rm eff}^{\rm GW} \\ &= 3.046 + N_{\rm eff}^{\rm GW} \end{aligned} \tag{3}$$

We aim to use non-standard contributions to this to determine an analytic solution to the Hubble constant H_0 using the respective energy densities at matter-radiation equality. We will show that these contributions may alleviate some of the H_0 tension, but do not eradicate it. This adds to ongoing work in relaxing the Hubble tension [28].

II. THEORETICAL BACKGROUND

A. Single-field slow-roll inflation

Quantum fluctuations of fields can describe the inflationary evolution of the early Universe. In its simplest form, we use a neutral scalar field and metric tensor [21]. We begin with the standard FRW metric.

$$ds^2 = dt^2 - a^2(t)\left\{g_{ij}dx^i dx^j\right\} \tag{4}$$

Using $a(t)$ as the scale factor in non-conformal time (coordinate time). From here, we then apply small tensorial perturbations $h_{\mu v \nu}$ using the gauge ($h_{\mu 0} = 0$) we can recover the perturbations in only spacial components [29].

$$ds^2 = dt^2 - a^2(t)\left(\delta_{ij} + h_{ij}\right)dx^i dx^j \tag{5}$$

Following this, we impose the transverse ($\partial^i h_{ij}^T = 0$) and traceless ($h_i^i = 0$) conditions to obtain the fully perturbed metric solutions as shown below.

$$g_{00} = -a(\tau)\left(1 + 2\sum_{n=1}^{+\infty}\frac{1}{n!}\Psi^{(n)}\right) \tag{6}$$

$$g_{0i} = a^2(\tau)\sum_{n=1}^{+\infty}\frac{1}{n!}\omega_i^{(n)} \tag{7}$$

$$g_{ij} = a^2(\tau)\left[1 - 2\left(\sum_{n=1}^{+\infty}\frac{1}{n!}\Phi^{(n)}\right)\right]\delta_{ij} + a^2(\tau)\sum_{n=1}^{+\infty}\frac{1}{n!}h_{ij}^{(n)} \tag{8}$$

Notice here, $\Phi^(n)$, $\omega_i^{(n)}$, $\Psi^{(n)}$ and $h_{ij}^{(n)}$ are the corresponding nth order perturbation. Our goal is to show that these can be used, along with the slow-roll parameters to obtain the widely used tensor-scalar ratio r. We can do this by first defining the Hubble expansion parameters for slow roll inflation.

$$\epsilon(\phi) \equiv \frac{3\dot{\phi}^2}{2}\left[V + \frac{1}{2}\dot{\phi}^2\right]^{-1} = \frac{m_{\rm Pl}^2}{4\pi}\left(\frac{H'(\phi)}{H(\phi)}\right)^2 \tag{9}$$

$$\eta(\phi) \equiv -\frac{\ddot{\phi}}{H\dot{\phi}} = \frac{m_{\rm Pl}^2}{4\pi}\frac{H''(\phi)}{H(\phi)} = \epsilon - \frac{m_{\rm Pl}\epsilon'}{\sqrt{16\pi\epsilon}} \tag{10}$$

Enforcing the the scalar perturbation must be positive $\dot{\phi} > 0$ hence $H'(\phi) < 0$. Due to this we must take the negative root for $\epsilon(\phi)$. Formally, $\sqrt{\epsilon} = -\sqrt{m_{\rm Pl}^2/4\pi}H'/H$. Applying the slow-roll approximation now, we assume that these parameters are very small, hence $\epsilon, \eta < 1$. Values beyond this would violate the strong energy condition. The amount of inflation can be represented as an e-fold and can be written as the following (Baumann) [21].

$$N_{\rm tot} = \int_{t_I}^{t_E} H(t){\rm d}t = -\frac{4\pi}{m_{\rm Pl}^2}\int_{\phi}^{\phi_e} d\phi \frac{H(\phi)}{H'(\phi)} \tag{11}$$

Following this, we solve assuming an inflationary scale factor $a(\phi) \propto e^{b(t)}$. We can hence obtain the equation for $a(\phi)$. Here we are following the common assumption that the scalar components of the fluctuations follow a power-law relationship.

$$a(\phi) = a_e \exp[-N(\phi)] \tag{12}$$

Evaluating the expression to have a scale which matches the Hubble radius $k = aH$ results in the following equation, this equation can finally be differentiated to give the characteristic relation between k and ϕ.

$$k(\phi) = a_e H(\phi)\exp[-N(\phi)] \tag{13}$$

Subsequently, we take the exponential derivative to obtain the final form solution.

$$\frac{d\ln(k)}{d\phi} = \frac{4\pi}{m_{Pl}^2}\left(\frac{H}{H'}(\epsilon - 1)\right) \tag{14}$$

Using these parameters, one is able to obtain the scalar and tensorial primordial spectral amplitudes [29].

These are obtained using quantum perturbations which are outside the scope of this paper.

$$A_S(k) \equiv \frac{2}{5}\mathcal{P}_S^{1/2} \simeq \frac{4}{5}\frac{H^2}{m_{\mathrm{P1}}^2\left|H'\right|}\bigg|_{k=aH} \tag{15}$$

$$A_T(k) \equiv \frac{1}{10}\mathcal{P}_T^{1/2} \simeq \frac{2}{5\sqrt{\pi}}\frac{H}{m_{\mathrm{Pl}}}\bigg|_{k=aH} \tag{16}$$

Here $A_s(k)$ and $A_R(k)$ represent scalar and tensor amplitudes of the power spectrum. We have simply shown the lowest order approximation above. It is possible to write each as a power expansion. The scalar case written as follows [25].

$$n_{\mathrm{s}} - 1 + \frac{1}{2}\ \mathrm{d}n_{\mathrm{s}}/\mathrm{d}\ln k \ln\left(k/k_*\right) + \ldots \approx n_s - 1$$

Below is the equivalent tensorial formulation.

$$n_{\mathrm{t}} + \frac{1}{2}\ \mathrm{d}n_{\mathrm{t}}/\mathrm{d}\ln k \ln\left(k/k_*\right) + \ldots \approx n_t$$

Here we define the spectrum around a specific pivot point k_* in which the approximation is made. Combining all of this together, we are able to write the tensorial and scalar power spectrum's.

$$P_s(k) = \frac{k^3}{2\pi^2}\left|\mathcal{R}_k\right|^2 \approx A_s(k_*)^2\left(\frac{k}{k_*}\right)^{n_s-1} \tag{17}$$

$$P_t(k) = \frac{k^3}{2\pi^2}\left(\left|h_k^+\right|^2 + \left|h_k^\times\right|^2\right) \approx A_t(k_*)^2\left(\frac{k}{k_*}\right)^{n_t(k_*)} \tag{18}$$

We are now able to resolve the scalar and spectral indices in terms of their slow-roll parameters. This is done by taking the first derivative of equations 15 and 16 and substituting it into equation 14. Taking the first order approximation $O(\eta, \epsilon)$, we find the following relations.

$$n_s - 1 \approx 2\eta - 4\epsilon \tag{19}$$

$$n_T \approx -2\epsilon \tag{20}$$

Using the new power spectra and their slow-roll approximations, we are able to finally determine the tensor-to-scalar ratio which determines the variation of the scalar field during the inflationary period[30].

$$r = \frac{P_t(k_*)}{P_s(k_*)} = 16\frac{A_t(k_*)^2}{A_s(k_*)^2} \approx 16\epsilon \tag{21}$$

Above we have used the approximation defined in equations 19 and 20 $\frac{A_t(k_*)^2}{A_s(k_*)^2} \approx \epsilon$. We can further combine the result from equations 21 and 20 to obtain the tensor-scalar ratio in terms of the tensor spectral index.

$$n_T \approx -\frac{r}{8} \tag{22}$$

B. Density

Now we must obtain the density of the gravitational waves and include it into our total density calculations. The energy density from the tensorial power spectrum can be obtained by applying density perturbations to the initial scalar field action [21, 31].

$$S = \int \mathrm{d}^4x\sqrt{-g}\left[\frac{1}{2}M_{\mathrm{pl}}^2 R - \frac{1}{2}g^{\mu\nu}\partial_\mu\varphi\partial_\nu\varphi - V(\varphi)\right] \tag{23}$$

Following this, since we are keen on the relationship between spacetime-curvature and gravitational waves, we apply perturbations to the metric through h_{ij}, the gauge invariant transverse traceless quantity as before.

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + \delta g_{\mu\nu} \tag{24}$$

As before, the background is taken through the FRW metric which defines $h_{i,j}$ in terms of the special components of the metric perturbations $\delta g_{ij} h_{ij}$. When computing the stress-energy-tensor (through the Ricci tensor) we ignore modes who's perturbations are larger horizon limit $k = a(t)H(t)$. This can be done since larger modes have been shown to have insignificant effect on the GW energy density [31].

$$R_{\mu\nu} = \bar{R}_{\mu\nu} + R_{\mu\nu}^{(1)} + R_{\mu\nu}^{(2)} + \ldots \tag{25}$$

The first and second orders terms can be shown explicitly.

$$\begin{aligned} R_{\mu\nu}^{(1)} = \frac{1}{2}(&\bar{D}^\alpha\bar{D}_\mu h_{\nu\alpha} + \bar{D}^\alpha\bar{D}_\nu h_{\mu\alpha} - \\ &\bar{D}^\alpha\bar{D}_\alpha h_{\mu\nu} - \bar{D}_\nu\bar{D}_\mu h) \end{aligned} \tag{26}$$

$$\begin{aligned} R_{\mu\nu}^{(2)} = &\frac{1}{2}\bar{g}^{\rho\sigma}\bar{g}^{\alpha\beta}\left[\frac{1}{2}\bar{D}_\mu h_{\rho\alpha}\bar{D}_\nu h_{\sigma\beta}\right. \\ &+ \left(\bar{D}_\rho h_{\nu\alpha}\right)\left(\bar{D}_\sigma h_{\mu\beta} - \bar{D}_\beta h_{\mu\sigma}\right) \\ &+ h_{\rho\alpha}\left(\bar{D}_\nu\bar{D}_\mu h_{\sigma\beta} + \bar{D}_\beta\bar{D}_\sigma h_{\mu\nu}\right. \\ &- \bar{D}_\beta\bar{D}_\nu h_{\mu\sigma} - \bar{D}_\beta\bar{D}_\mu h_{\nu\sigma} \\ &+ \left(\frac{1}{2}\bar{D}_\alpha h_{\rho\sigma} - \bar{D}_\rho h_{\alpha\sigma}\right) \\ &\times \left(\bar{D}_\nu h_{\mu\beta} + \bar{D}_\mu h_{\nu\beta} - \bar{D}_\beta h_{\mu\nu}\right) \end{aligned} \tag{27}$$

It can be shown that the the first and third terms contain lower order frequencies while the linear first term contains higher order frequencies [22]. Hence we are able to obtain the lower frequency perturbations in the form of Einsteins equations.

$$\bar{R}_{\mu\nu} = -\left(R_{\mu\nu}^{(2)}\right)^{(Low)} + 8\pi G\left(T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}\right)^{(Low)} \tag{28}$$

This equation is used when calculating the stress-energy tensor which we will need for momentum energy density. The convention in the community is to average the low-frequency perturbations over multiple wavelengths.

This is done to eliminate the high-frequency dependence in the $R^{(2)}_{\mu\nu}$ term.

We now invert the Einstein equation to obtain the gravitational-waves stress-energy tensor (Refer to the short-wave approximation in Kip Thorne's text) [22].

$$\bar{R}_{\mu\nu} = \langle R^{(2)}_{\mu\nu}\rangle + 8\pi G \,\langle T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T\rangle \tag{29}$$

Inverting this, we find the solution for the gravitational wave stress energy tensor.

$$T^{GW}_{\mu\nu} = \frac{-1}{8\pi G}\,\langle R^{(2)}_{\mu\nu} - \frac{1}{2}\bar{g}_{\mu\nu} R^{(2)}\rangle + O(\delta g^3) \tag{30}$$

This quantity is then converted to the transverse traceless gauge. This can be described in terms of the covariant derivatives of the metric perturbations. Here we use equations 26 and 27 to rewrite the stress energy tensor. Notice now that the covariant derivatives have been $D^\mu \to \partial^\mu$(converted to normal derivatives) since we assume that the background spacetime is flat [32].

$$\begin{aligned} R^{(2)}_{\mu\nu} =& \frac{1}{2}\bigg[\frac{1}{2}\partial_\mu h_{\alpha\beta}\partial_\nu h^{\alpha\beta} + h^{\alpha\beta}\partial_\mu\partial_\nu h_{\alpha\beta} \\ &- h^{\alpha\beta}\partial_\nu\partial_\beta h_{\alpha\mu} - h^{\alpha\beta}\partial_\mu\partial_\beta h_{\alpha\nu} \\ &+ h^{\alpha\beta}\partial_\alpha\partial_\beta h_{\mu\nu} + \partial^\beta h^\alpha_\nu \partial_\beta h_{\alpha\mu} \\ &- \partial^\beta h^\alpha_\nu \partial_\alpha h_{\beta\mu} - \partial_\beta h^{\alpha\beta}\partial_\nu h_{\alpha\mu} \\ &+ \partial_\beta h^{\alpha\beta}\partial_\alpha h_{\mu\nu} - \partial_\beta h^{\alpha\beta}\partial_\mu h_{\alpha\nu} - \frac{1}{2}\partial^\alpha h \partial_\alpha h_{\mu\nu} \\ &+ \frac{1}{2}\partial^\alpha h \partial_\nu h_{\alpha\mu} + \frac{1}{2}\partial^\alpha h \partial_\mu h_{\alpha\nu}\bigg] \end{aligned} \tag{31}$$

Using this, and plugging back into equation 30 we obtain the energy-density of gravitational waves by taking (0,0) component of the stress energy tensor.

$$\begin{aligned} \rho_{\mathrm{GW}} &= T^{\mathrm{GW}}_{00} \\ &= \frac{1}{32\pi G a^4}\delta^{ik}\delta^{j\ell}\left\langle\left(\partial_0 - 2H\right)\delta g_{ij}\left(\partial_0 - 2H\right)\delta g_{k\ell}\right\rangle \\ &+ \mathcal{O}\left(\delta g^3\right) \\ &= \frac{1}{32\pi G}\delta^{ik}\delta^{j\ell}\left\langle \dot{h}_{ij}\dot{h}_{k\ell}\right\rangle + \mathcal{O}\left(\delta g^3\right) \end{aligned} \tag{32}$$

Hence we can then directly calculate the gravitational wave energy-density [23].

$$\rho_{\mathrm{gw}} = \frac{1}{32\pi G a^2}\left\langle h'_{ij}(\mathbf{x},\eta) h'^{ij}(\mathbf{x},\eta)\right\rangle \tag{33}$$

Notice that this is exactly the power spectrum integral. This can be defined in terms of a transfer function To get the final form of the energy-density of gravitational waves.

$$\langle\langle h_{ij}(\mathbf{x},\eta) h'^{ij}(\mathbf{x},\eta)\rangle\rangle = \int d\log k \Delta^2_h(\mathbf{x},\eta) \tag{34}$$

Above is the definition of the power spectrum. In it contains the transfer function $T(k,\eta)$.

$$\Delta^2_h(\mathbf{x},\eta) = P_t(k) T(k,\eta) \tag{35}$$

Using this and equation 33 we are able to define the full form of the energy density.

$$\rho_{GW} = \int_0^k d\log k \frac{P_t(k)}{32\pi G a^2} T'(k,\eta)^2 \tag{36}$$

C. Primordial Gravitational Waves Energy Density

Now that we have defined both tensor-scalar-ratio amplitudes as well as the gravitational wave density, we intend to show how they impact the H0 tension. We will use both sections II A and II B to show this.

Notice that we are able to written the k-modes in terms of frequency to re-parameterise equation 21 [31].

$$\frac{f}{\mathrm{Hz}} = \frac{k}{2\pi a \eta_0} = \frac{1.6\times 10^{-15}}{\mathrm{Mpc}^{-1}} \tag{37}$$

Hence, below we show the equivalent form of 21. The choice of pivot scale has been studied in the literature and often warrants a full section [33]. The pivot scale measured the tensorial mode which decorrelates the uncertainties between A_t and n_t. Here we simply assume $k_* = 0.01\mathrm{Mpc}^{-1}$.

$$P_t(f) = r A_s\left(\frac{f}{f_*}\right) = r A_s\left(\frac{f/Hz}{1.6\times 10^{-17}}\right) \tag{38}$$

As previously mentioned, since we are only considering lower frequencies [22] a sensible assumption to make is to consider frequencies in the UV bandwidth $k = k_{UV}$. As a result of this, we are able to re-write equation 36 in terms of k_{UV}.

$$\rho_{GW} = \int_0^{\mathbf{k}_{UV}} d\log k \frac{P_t(k)}{32\pi G a^2} T'(k,\eta)^2 \tag{39}$$

The goal is to now define a suitable transfer function and integrate this equation to achieve the gravitational-wave density in terms of the total density.

The transfer function is typically a linear combination of Bessel and Neumann functions, we obtain them by imposing boundary conditions on the massless Klein-Gordon equation for plane waves [34].

$$h''_{\lambda,\mathbf{k}} + \left(\frac{2a'}{a}\right) h'_{\lambda,\mathbf{k}} + k^2 h_{\lambda,\mathbf{k}} = 16\pi G a^2 \Pi_{\lambda,\mathbf{k}} \tag{40}$$

This is used to compute the transfer function. For values of k less than k_eq. $T(k < k_{eq}) = j_o(k\tau)$. Taking its derivative we find $T'(k < k_{eq}) = -k j_1(k\tau)$. We now applying this to equation 39 to obtain an analytic form of ρ_{GW}.

$$\begin{aligned}\rho_{\rm GW} &= \int_{k_{\rm IR}}^{k_{\rm UV}} d\log k \frac{P_t(k)}{32\pi G a^2}\left[T'(k,\tau)\right]^2 \\ &= \frac{A_s r}{32\pi G a^2}\int_{k_{\rm IR}}^{k_{\rm UV}} kdk\left(\frac{k}{k_*}\right)^{n_T} j_1^2(k\tau)\end{aligned} \tag{41}$$

Here the bounds of integration are fixed between k_{IR} and k_{UV}. k_{IR} is defined to be far inside the horizon (sub-horizon scales). Hence, we set $k_{IR} << H(t)a(t)$. Now, notice the k dependence of the derivative of the transfer function for small values of k.

$$\begin{aligned}T'(k,\tau)^2 &= (-kj_1(k\tau))^2 \\ &= k^2\left(\frac{\sin k\tau}{(k\tau)^2} - \frac{\cos k\tau}{k\tau}\right)^2 \\ &\approx k^2\left(\frac{k\tau}{(k\tau)^2} - \frac{1-\frac{1}{2}(k\tau)^2}{k\tau}\right)^2 \\ &= \frac{k^4\tau^4}{4}\end{aligned} \tag{42}$$

The k^4 dependence in the square transfer function in the $k \to 0$ limit highlights that the integral above in equation 41 should converge for values of $n_T >= -4$ in the $k_{IR} \to 0$ limit. This should occur for any arbitrary value of k_{UV}. As a result of this, we are able to approximate the lower bound frequency in this limit. Hence, $k_{IR} = 0$. The upper bound k_{UV} varies depending on model choice. Here we restrict the value of k_{UV} to be larger than the plank scale, this is a fair assumption as we do not expect primordial gravitational waves to be generated on scales smaller than this.

We now adopt the LIGO model in which scales of UV are 60 e-folds smaller than the horizon scale $a(t_o)H(t_o)$ [35]. We further consider the models in which the UV energy dominates the total energy of the system, this enforces a positive spectral index $n_T > 0$. As a result of this we are able to take the limit $\frac{k_{IR}}{k_{UV}} \to 0$.

$$\begin{aligned}\rho_{\rm GW} &= \frac{A_s r}{32\pi G a^2}\int_{k_{\rm IR}}^{k_{\rm UV}} kdk\left(\frac{k}{k_*}\right)^{n_T} j_1^2(k\tau) \\ &= \frac{A_s r}{32\pi G a^2 k_*^{n_T}}\int_{\approx 0}^{k_{\rm UV}} k^{3+n_T}dk\left(\frac{\sin k\tau}{(k\tau)^2} - \frac{\cos k\tau}{k\tau}\right)^2 \\ &= \frac{A_s r}{32\pi G a^2}\left(\frac{k_{\rm UV}}{k_*}\right)^{n_t}\frac{1}{2n_t}\frac{1}{\tau^2} + \mathcal{O}\left(1/(k_{\rm UV}\tau)\right)\end{aligned} \tag{43}$$

Notice above that we have used the assumption that since k_{UV} dominates we are able to take $\frac{1}{k\tau}$ to be very small. Hence we arrive at a closed form of the gravitational wave density.

$$\begin{aligned}\rho_{GW} &= \frac{A_s r}{32\pi G}\left(\frac{k_{UV}}{k_*}\right)^{n_T}\frac{1}{2n_T(a\tau)^2} \\ &= \frac{A_s r}{24 n_T}\left(\frac{k_{UV}}{k_*}\right)^{n_T}\rho_{tot}\end{aligned} \tag{44}$$

Notice above we have used the assumption that $\frac{1}{(a\tau)^2} = H^2 = 8\pi G\frac{\rho_{tot}}{3}$, which is true in the radiation dominated era (since $t \to \tau^2$). To do this, we must also assume flat curvature and negligible dark energy constant [36].

D. Hubble Tension

Now that we have obtained the gravitational wave density, its time to use this, along with our knowledge of density in the radiation dominated era to work towards an analytic solution of H_o. We take the total energy density to be the sum of photon, baryon and primordial gravitational wave energy densities.

$$\rho_{tot} = \rho_\gamma + \rho_\nu + \rho_{GW} \tag{45}$$

Trivially, the neutrino density is dependent on the effective number of neutrinos [21]. It is dependent on the non-instantaneous neutrino decoupling still occurred while electron-positron annihilation was occurring.

$$\rho_\nu = \frac{7}{8}N_{\rm eff}\left(\frac{4}{11}\right)^{4/3}\rho_\gamma \tag{46}$$

Subsequently, we are able to compute the total energy density in terms of $N_{\rm eff}$.

$$\rho_{tot} = \rho_\gamma\left(1 + \frac{7}{8}\left(\frac{4}{11}\right)^{\frac{4}{3}}N_{\rm eff}\right) \tag{47}$$

Using this we are able to rewrite the GW energy density in term of $N_{\rm eff}$.

$$\begin{aligned}\rho_{GW} &= \frac{A_s r}{24n_T}\left(\frac{k_{UV}}{k_*}\right)^{n_T}\rho_\gamma\left(1 + \frac{7}{8}\left(\frac{4}{11}\right)^{\frac{4}{3}}N_{\rm eff}\right) \\ &= \frac{A_s(-8n_T)}{24n_T}\left(\frac{k_{UV}}{k_*}\right)^{n_T}\rho_\gamma + \frac{7}{8}\left(\frac{4}{11}\right)^{\frac{4}{3}}N_{\rm eff}\left(\frac{k_{UV}}{k_*}\right)^{n_T}\rho_\gamma \\ &= -3.046\frac{7}{8}\left(\frac{4}{11}\right)^{\frac{4}{3}}\rho_\gamma + \frac{7}{8}\left(\frac{4}{11}\right)^{\frac{4}{3}}\rho_\gamma N_{\rm eff} \\ &= \frac{7}{8}\left(\frac{4}{11}\right)^{\frac{4}{3}}\left(N_{\rm eff} - 3.046\right)\rho_\gamma\end{aligned} \tag{48}$$

We now substitute the equation above into the total energy density in equation 44. We assume a primordial gravitational wave extension to the standard model. $N_{\rm eff} = 3.046 + N_{\rm eff}^{GW}$, i.e. we assume no other degrees of freedom beyond the standard model.

$$\begin{aligned}\frac{\rho_{\rm GW}}{\rho_{\rm tot}} &= \frac{A_s r}{24n_t}\left(\frac{k_{\rm UV}}{k_*}\right)^{n_t} \\ &= \frac{\frac{7}{8}\left(\frac{4}{11}\right)^{4/3}\left(N_{\rm eff} - 3.046\right)\rho_\gamma}{\left(1 + \frac{7}{8}\left(\frac{4}{11}\right)^{4/3}N_{\rm eff}\right)\rho_\gamma}\end{aligned} \tag{49}$$

Hence we are able to invert this and solve for N_{eff}.

$$\begin{aligned}
N_{\text{eff}} &= \frac{\frac{8}{7}\left(\frac{11}{4}\right)^{4/3}\left[\frac{A_s r}{24n_t}\left(\frac{k_{\text{UV}}}{k_*}\right)^{n_t}\right]+3.046}{1-\left[\frac{A_s r}{24n_t}\left(\frac{k_{\text{UV}}}{k_*}\right)^{n_t}\right]} \\
&\approx \left(\frac{8}{7}\left(\frac{11}{4}\right)^{4/3}\left[\frac{A_s r}{24n_t}\left(\frac{k_{\text{UV}}}{k_*}\right)^{n_t}\right]+3.046\right) \\
&\times\left(1+\left[\frac{A_s r}{24n_t}\left(\frac{k_{\text{UV}}}{k_*}\right)^{n_t}\right]\right) \\
&= 3.046+\left(3.046+\frac{8}{7}\left(\frac{11}{4}\right)^{4/3}\right)\frac{A_s r}{24n_t}\left(\frac{k_{\text{UV}}}{k_*}\right)^{n_t} \\
&+ O\left(\left(\frac{A_s r}{24n_t}\left(\frac{k_{\text{UV}}}{k_*}\right)\right)^2\right)
\end{aligned} \tag{50}$$

Here we have expanded in terms of $\frac{A_s r}{24n_t}\left(\frac{k_{\text{UV}}}{k_*}\right)$. And taken the leading order behaviour, in other words we are treating the gravitational energy density as small compared to the total energy density $\rho_{GW} << \rho_{\text{tot}}$. This gives the final key result for N_{eff}.

$$N_{\text{eff}} \approx 3.046+\left(3.046+\frac{8}{7}\left(\frac{11}{4}\right)^{4/3}\right)\frac{A_s r}{24n_t}\left(\frac{k_{\text{UV}}}{k_*}\right)^{n_t} \tag{51}$$

Referring back to equation 44, the short-wave approximation allows us to assume that the gravitational waves redshift analogous to radiation. We can compute the effect this has by investigating the primordial GWs during matter radiation equality. At that point a_{eq} we assume the matter and radiation density equality. Before we do this, we must compute the energy density for relativistic photons and neutrinos [37]. Below is the equation for radiation without the addition of neutrinos.

$$\rho_\gamma = \frac{1}{2} g_\gamma a T^4 \tag{52}$$

We have defined a as the radiation constant , $g_r = 2$ for a photon and T to be the temperature of the CMB (i.e. black body photon). Following this, we apply the correction to g to obtain the relativistic version.

$$g_* = 2\left(1+N_{\text{eff}}\frac{7}{8}\left(\frac{4}{11}\right)^{4/3}\right) \tag{53}$$

Substituting this in, we obtain the fully relativistic version.

$$\rho_r = 1+N_{\text{eff}}\frac{7}{8}\left(\frac{4}{11}\right)^{4/3} aT^4 \tag{54}$$

Hence, using the Friedmann equation for a flat universe, we find the Hubble constant as a function of effective degree's of freedom [38].

$$\begin{aligned}
\rho_{\text{m}}(a_{\text{eq}}) &= \rho_r(a_{\text{eq}}) \\
\rho_{m,0}\left(\frac{a_0}{a_{eq}}\right)^3 &= \rho_{r,0}\left(\frac{a_0}{a_{eq}}\right)^4 \\
\frac{3H_0^2\Omega_m}{8\pi G}a_{eq} &= a_{\text{rad}}T_{\gamma,0}^4\left(1+N_{\text{eff}}\frac{7}{8}\left(\frac{4}{11}\right)^{4/3}\right)
\end{aligned} \tag{55}$$

This equation analytically solves for the Hubble constant in terms of the parameters of the primordial gravitational waves. We are able to make this more explicit by solving for H_0.

$$\begin{aligned}
H_0^2 &= \frac{8\pi G}{3\Omega_m}\frac{a_{\text{rad}}}{a_{eq}}T_{\gamma,0}^4\left(1+N_{\text{eff}}\frac{7}{8}\left(\frac{4}{11}\right)^{4/3}\right) \\
&\approx \frac{8\pi G}{3\Omega_m}\frac{a_{\text{rad}}}{a_{eq}}T_{\gamma,0}^4 \\
&\times\left\{1+\left[3.046+\left(3.046+\frac{8}{7}\left(\frac{11}{4}\right)^{4/3}\right)\frac{A_s r}{24n_t}\left(\frac{k_{\text{UV}}}{k_*}\right)^{n_t}\right]\right. \\
&\left.\times\frac{7}{8}\left(\frac{4}{11}\right)^{4/3}\right\}
\end{aligned} \tag{56}$$

We have thus shown, that the gravitational wave parameters directly influence the Hubble constant. This is model dependent on out choice of ultra-violet cutoff k_{UV}.

III. RESULTS AND DISCUSSION

In this section, we draw heavily from the results of Dr. Graef and Dr. Benetti [39]. There they use the LIGO and VIRGO experiments to apply constraints on the tensor scalar r ratio and spectral index n_T. They investigate two models, varying the UV cut-off k_{UV} as highlighted in equation 56.

In the first model, they take a cut-off corresponding to the grand unified theory (GUT) scale, i.e. $k_{\text{UV}/k_*} \approx 10^{56}$ and $f_{UV} \approx 10^{40}Hz$. In their second model, they take a 60 e-fold assumption in the power spectrum. This corresponds to $k_{\text{UV}/k_*} \approx 10^{24}$ and $f_{UV} \approx 10^8$. This model matches the expected inflation of the universe. They use data releases from;Planck and BICEP2 collaborations (BKP), LIGO and baryonic acoustic oscillations survey (BAO) to constrain the spectral index and tensor-to-scalar amplitudes.

The initial data is known as "base". They then include measurements by Gaia and the Hubble space telescope (HST). Below we observe the main result from their paper which directly compares the effect of the spectral index n_T on the Hubble constant H_0.

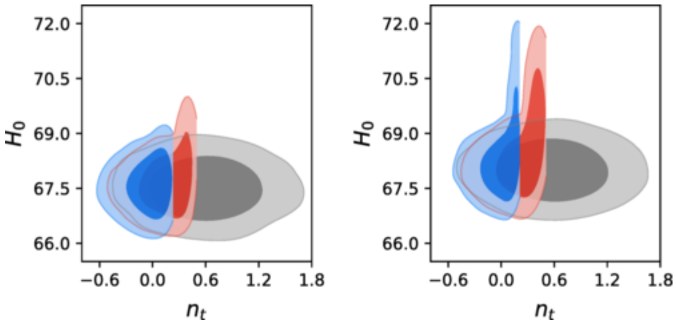

FIG. 1. Plot representing the effect of the spectral index on the Hubble constant. The left plot uses the base dataset, the right uses the base + HST dataset. The over-plotted gray contours represent a standard CDM model with primordial GWs. The blue plot represents ultraviolet scales associated with the grand unification theory (Model 1 - GUT). The red plot represents ultraviolet scales corresponding to a 60 e-fold power law spectrum (Model 2 - Inflation). The shaded regions highlight the 1σ and 2σ confidence intervals. This plot was taken directly from Graef et. al [39]

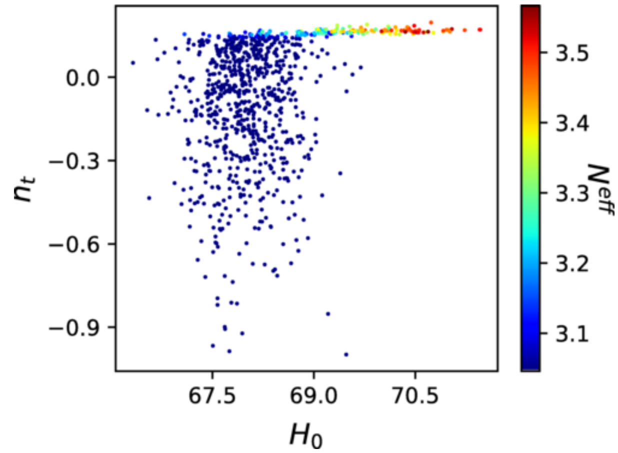

FIG. 2. Here, they use the base + HST dataset to plot the change in effective degrees of freedom on the Hubble constant. Colours closer to blue represent values of $N_{\rm eff}$ closer to the standard model. This plot was also taken directly from Graef et. al [39]

It can be observed that the value of H_0 increases as the spectral index n_T increases. This occurrence exceeds the limits bounded by the standard model. The computed values of the Hubble constant H_0 can be found in the table below. Constraining the spectral index with the base + HST data suggests a higher value of the Hubble constant.

The maximum-a-posteriori values bring their H_0 closer to those measured from late universe times such as cepheid's [40]. Hence, their work alleviates the ongoing Hubble tension [39].

Figure 2 demonstrates the effect of the n_T vs H_0 plot as determined by the relativistic degrees of freedom.

From this plot, it is clear that positive values of the spectral index are most likely to result in relaxing of the Hubble tension. This matches the possible CMB exclusion of red-tilted values $n_T < 0$. Constraints on n_T are only expected to get stronger with newer, more powerful GW detectors coming online in the coming years.

In making the following predictions we have had to assert some assumptions. We will now state them explicitly for the reader. Firstly, we assume that the effect of primordial GWs are similar to that of neutrinos. We are hence able to add to their relativistic degrees of freedom. We then assume the consistency relation of standard inflation in which the tensor-to-scalar ratio r can be determined as the ratio of tensorial to scalar power spectrum amplitudes. Then, when computing the total density of the universe, we assume that it is the sum of radiation density + gravitational waves and nothing else (stated in equation 45). Finally, we assume that the GW energy density evolves the same way as photon radiation from recombination. Making these assumptions allows us to compute an analytic form of the Hubble constant H_0.

IV. EXTENSIONS TO SLOW-ROLL THEORY FOR PRIMORDIAL GRAVITATIONAL WAVES

While the primary focus of this paper is to analyse the standard slow-roll case and its direct impact on the Hubble constant H_0, a review of primordial gravitational waves would be incomplete without a brief overview of extensions to the slow-roll scenario.

(p.450/61) All other mores in GR (not single field slow roll)G-Inflation. Potential-driven G-inflation, EFT approach. Vacuum inflation fluctuations. Fluctuations of extra scalar fields. Pseudo-scalar inflation + gauge field. Scalar inflation + pseudo scalar + gauge. Scalar inflation + scalar field. Chaotic inflation. Hybrid inflation

V. CONCLUSION

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TABLE I. Confidence limits (68%) for the H_0 and n_t parameters using the two datasets discussed in the text.

			$k_{\mathrm{UV}} = 10^{22}$ [a]	$k_{\mathrm{UV}} = 10^{54}$
		ΛCDM $+ r + n_t$	Model 2	Model 1
Base	H_0 [b]	67.57 ± 0.53	67.55 ± 0.56	67.51 ± 0.54
	n_t	$0.46^{+0.36}_{-0.53}$	$0.04^{+0.38}_{-0.14}$	$-0.13^{+0.29}_{-0.11}$
Base + HST	H_0	67.91 ± 0.55	68.49 ± 0.83	68.35 ± 0.81
	n_t	$0.45^{+0.51}_{-0.47}$	$0.12^{+0.34}_{-0.13}$	$-0.07^{+0.29}_{-0.10}$

[a] k_{UV} is in units Mpc^{-1}.
[b] H_0 is in units $\mathrm{kms}^{-1}\,\mathrm{Mpc}^{-1}$.

TABLE I. This table was taken directly from Graef et. al [39]

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None

[1] A. Einstein, On gravitational waves sitzungsber. preuss, Acad. Wiss **1**, 154 (1918).

[2] M. Maggiore, Gravitational wave experiments and early universe cosmology, Phys. Rept. **331**, 283 (2000), arXiv:gr-qc/9909001.

[3] B. P. Abbott, R. Abbott, T. Abbott, M. Abernathy, F. Acernese, K. Ackley, C. Adams, T. Adams, P. Addesso, R. Adhikari, *et al.*, Gw151226: observation of gravitational waves from a 22-solar-mass binary black hole coalescence, Physical review letters **116**, 241103 (2016).

[4] O. Jennrich, Lisa technology and instrumentation, Classical and Quantum Gravity **26**, 153001 (2009).

[5] L. M. Krauss, S. Dodelson, and S. Meyer, Primordial gravitational waves and cosmology, Science **328**, 989 (2010).

[6] B. Westbrook, C. Raum, S. Beckman, A. T. Lee, N. Farias, T. Sasse, A. Suzuki, E. Kane, J. E. Austermann, J. A. Beall, *et al.*, Detector fabrication development for the litebird satellite mission, in *Space Telescopes and Instrumentation 2020: Optical, Infrared, and Millimeter Wave*, Vol. 11443 (SPIE, 2020) pp. 915–936.

[7] N. Seto, S. Kawamura, and T. Nakamura, Possibility of direct measurement of the acceleration of the universe using 0.1 hz band laser interferometer gravitational wave antenna in space, Physical Review Letters **87**, 221103 (2001).

[8] T. Matsumura, Y. Akiba, J. Borrill, Y. Chinone, M. Dobbs, H. Fuke, A. Ghribi, M. Hasegawa, K. Hattori, M. Hattori, *et al.*, Mission design of litebird, Journal of Low Temperature Physics **176**, 733 (2014).

[9] K. Abazajian, G. Addison, P. Adshead, Z. Ahmed, S. W. Allen, D. Alonso, M. Alvarez, A. Anderson, K. S. Arnold, C. Baccigalupi, *et al.*, Cmb-s4 science case, reference design, and project plan, arXiv preprint arXiv:1907.04473 (2019).

[10] C. Li and A. Cooray, Weak lensing of the cosmic microwave background by foreground gravitational waves, Physical Review D **74**, 023521 (2006).

[11] E. Alizadeh and C. M. Hirata, How to detect gravitational waves through the cross correlation of the galaxy distribution with the cmb polarization, Physical Review D **85**, 123540 (2012).

[12] L. P. Grishchuk, Amplification of gravitational waves in an istropic universe, Zh. Eksp. Teor. Fiz. **67**, 825 (1974).

[13] K. Saikawa and S. Shirai, Primordial gravitational waves, precisely: The role of thermodynamics in the standard model, Journal of Cosmology and Astroparticle Physics **2018** (05), 035.

[14] A. Ricciardone, Primordial gravitational waves with lisa, in *Journal of Physics: Conference Series*, Vol. 840 (IOP Publishing, 2017) p. 012030.

[15] J.-F. Dufaux, A. Bergman, G. Felder, L. Kofman, and J.-P. Uzan, Theory and numerics of gravitational waves from preheating after inflation, Physical Review D **76**, 123517 (2007).

[16] M. Kamionkowski, A. Kosowsky, and M. S. Turner, Gravitational radiation from first-order phase transitions, Physical Review D **49**, 2837 (1994).

[17] S. Clesse and J. García-Bellido, Detecting the gravitational wave background from primordial black hole dark matter, Physics of the dark universe **18**, 105 (2017).

[18] M. Rees, Effects of very long wavelength primordial gravitational radiation, Monthly Notices of the Royal Astronomical Society **154**, 187 (1971).

[19] S. J. Huber, T. Konstandin, G. Nardini, and I. Rues, Detectable gravitational waves from very strong phase

transitions in the general nmssm, Journal of cosmology and astroparticle physics **2016**, 036 (2016).
[20] P. B. Dev and A. Mazumdar, Probing the scale of new physics by advanced ligo/virgo, Physical Review D **93**, 104001 (2016).
[21] D. Baumann, Cosmology, Part III Mathematical Tripos (2012).
[22] C. W. Misner, K. S. Thorne, and J. A. Wheeler, *Gravitation* (W. H. Freeman, San Francisco, 1973).
[23] M. C. Guzzetti, N. Bartolo, M. Liguori, and S. Matarrese, Gravitational waves from inflation, La Rivista del Nuovo Cimento **39**, 399 (2016).
[24] N. Barnaby, J. Moxon, R. Namba, M. Peloso, G. Shiu, and P. Zhou, Gravity waves and non-gaussian features from particle production in a sector gravitationally coupled to the inflaton, Physical Review D **86**, 103508 (2012).
[25] P. Ade, N. Aghanim, M. Arnaud, F. Arroja, M. Ashdown, J. Aumont, C. Baccigalupi, M. Ballardini, A. Banday, R. Barreiro, *et al.*, Planck 2015 results-xx. constraints on inflation, Astronomy & Astrophysics **594**, A20 (2016).
[26] G. Cabass, L. Pagano, L. Salvati, M. Gerbino, E. Giusarma, and A. Melchiorri, Updated constraints and forecasts on primordial tensor modes, Physical Review D **93**, 063508 (2016).
[27] L. Pagano, L. Salvati, and A. Melchiorri, New constraints on primordial gravitational waves from planck 2015, Physics Letters B **760**, 823 (2016).
[28] U. Mali and K. K. Rogers, Machine learning the visible counterparts to gravitational waves: kilonovae, .
[29] S. Chongchitnan and G. Efstathiou, Prospects for direct detection of primordial gravitational waves, Physical Review D **73**, 083511 (2006).
[30] K. Hirano, Inflation with very small tensor-to-scalar ratio, arXiv preprint arXiv:1912.12515 (2019).
[31] P. D. Meerburg, R. Hložek, B. Hadzhiyska, and J. Meyers, Multiwavelength constraints on the inflationary consistency relation, Physical Review D **91**, 103505 (2015).
[32] M. Maggiore, *Gravitational Waves. Vol. 1: Theory and Experiments*, Oxford Master Series in Physics (Oxford University Press, 2007).
[33] M. Cortês, A. R. Liddle, and D. Parkinson, Tensors, bicep2 results, prior dependence, and dust, Physical Review D **92**, 063511 (2015).
[34] Y. Watanabe and E. Komatsu, Improved calculation of the primordial gravitational wave spectrum in the standard model, Physical Review D **73**, 123515 (2006).
[35] A. R. Liddle and S. M. Leach, How long before the end of inflation were observable perturbations produced?, Physical Review D **68**, 103503 (2003).
[36] B. Osano, Evolution of cosmological total energy density and transient periods in cosmology, arXiv preprint arXiv:2002.08875 (2020).
[37] B. W. Carroll and D. A. Ostlie, *An introduction to modern astrophysics* (Cambridge University Press, 2017).
[38] M. P. Hobson, G. P. Efstathiou, and A. N. Lasenby, *General relativity: an introduction for physicists* (Cambridge University Press, 2006).
[39] L. L. Graef, M. Benetti, and J. S. Alcaniz, Primordial gravitational waves and the H_0-tension problem, Phys. Rev. D **99**, 043519 (2019).
[40] A. G. Riess, S. Casertano, W. Yuan, L. Macri, B. Bucciarelli, M. G. Lattanzi, J. W. MacKenty, J. B. Bowers, W. Zheng, A. V. Filippenko, *et al.*, Milky way cepheid standards for measuring cosmic distances and application to gaia dr2: implications for the hubble constant, The Astrophysical Journal **861**, 126 (2018).