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Primordial gravitational waves: Studying its effect on the Hubble tension.

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ABSTRACT
Primordial gravitational waves (GWs) are a key area of study when examining the Universe. We review the ongoing work connecting them to the Hubble constant H_0. Assuming the linearized version of Einsteins equations we derive an analytic expression relating parameters of primordial GWs to the H_0. We write this in terms of key parameters such as the tensor-to-scalar ratio r and the spectral index n_T. In doing so, we treat the evolution of primordial GWs radiatively. We then assume that primordial GWs behave as an extension to neutrinos and thus may contribute to their relativistic degrees of freedom $N_{\rm eff}^{\nu+{\rm GW}}$. We use this to compute an analytic relation for the H_0. We then examine recent analysis to show that primordial gravitational waves relax the H_0 tension. Studying the constraints on the spectral index for various models we examine values for an updates H_0. Our discussion includes data from Planck, BICEP2/Keck array , the laser interferometer gravitational-wave observatory (LIGO) and the baryonic acoustic oscillations survey (BAO). Finally we highlight extensions to different models which may further relax the ongoing Hubble tension. Such studies are important as observational data is likely to restrict many models. These constraints will set bounds for new fundamental physics.

Key words: cosmological parameters – gravitational waves – cosmic background radiation

1 INTRODUCTION

Over a century ago, Einstein created the theory of general relativity (GR), in this theory he postulates an equation relating matter to the curvature of space-time (Einstein 1918). Solving the linearized version of his equations, we obtain gravitational waves (GWs). He defines these waves to be both transverse and longitudinal perturbations on the fabric of spacetime. GWs are the mass quadrupole moment of an accelerating or asymmetric source (Maggiore 2000). They may also be generated from primordial fluctuations in the early Universe (Guzzetti et al. 2016). The underlying physics which generates gravitational waves is a common area of study today. Observationally studying them allows us to investigate multiple astrophysical phenomena. (Abbott et al. 2016).

Primordial gravitational waves arise from cosmic inflation and are orders of magnitude smaller than GWs from compact-object mergers. New projects probe the possibility of high-precision measurements which are able to detect them (Jennrich 2009; Krauss et al. 2010; Westbrook et al. 2020). Projects, such as Deci-hertz Interferometer Gravitational Wave Observatory (DECIGO) aim to directly detect gravitational waves that arise from cosmic inflation (Seto et al. 2001). With future data expected from DECIGO and other missions such as LiteBIRD (Matsumura et al. 2014) and CMB-S4 (Abazajian et al. 2019). The study of primordial gravitational waves has gathered interest in the cosmology community. An improved theoretical background will help us distinguish between various models of primordial gravitational waves. They are also able help us determine the rate of expansion of the Universe H_0 (Ade et al. 2016).

Experimentally, primordial GWs have yet to be detected (Alizadeh & Hirata 2012). They occur when Thompson scattering on free electron generates polarization in the cosmic microwave background (CMB). Quadrapoles from gravitational waves results in B-mode polarization of the CMB (Li & Cooray 2006).

Theoretically, inflationary GWs play an important role in the early Universe (Grishchuk 1974). By considering primordial perturbations to the metric tensor and expansion rate of the Universe during inflation, we are able to predict the primordial GWs spectrum. The tensorial perturbations help us determine the initial magnitude of GWs during inflation. The expansion rate, helps us determine how the density of primordial gravitational waves changes as its strength dissipates through cosmic expansion (Saikawa & Shirai 2018). The inflationary history of gravitational waves provides insight into different inflationary models. Hence their study is important.

Primordial gravitational waves can be separated into two forms, those from metric perturbations in inflation, and those from astrophysical events that occur between the end of inflation and big bang nucleosynthesis (BBN) (Ricciardone 2017). Some examples include; GWs produced during reheating (Dufaux et al. 2007), from phase transitions (Kamionkowski et al. 1994) or by primordial black holes (Clesse & García-Bellido 2017). We will ignore the second category as it has a smaller impact on the Hubble constant H_0

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(Chongchitnan & Efstathiou 2006; Meerburg et al. 2015).

Our bold assumption that primordial gravitational waves arise from predictions of inflation (Rees 1971) allow us to ignore the electroweak and first-order phase transitions (Huber et al. 2016; Dev & Mazumdar 2016). We use perturbations to the Friedmann–Lemaître–Robertson–Walker (FRW) metric to derive a dimensionless power spectrum which denotes the amplitude of perturbations at a given Fourier mode k (Baumann 2012). Standard slow-roll inflationary models correspond to a flat primordial GW background for many frequencies (Ricciardone 2017). We use the *tilt* of the primordial power spectrum to describe variations from a flat GW background. This is caused by frequency dependence in the inflationary Hubble parameter (Maggiore 2000).

$$h_0^2\Omega_{\rm gw}(f) \sim f^{n_T} \tag{1}$$

This frequency dependence can be described through the characteristic *tensor spectral index* n_T. It is an experimental parameters which observationalists can measure and report. It holds information that characterizes the primordial GW (Kinney 2003). Using the *consistency relation*, we are able to find the tensor-to-scalar ratio r.

$$r\left(k_*\right) \equiv \frac{A_{\rm T}\left(k_*\right)}{A_{\rm S}\left(k_*\right)} \tag{2}$$

This ratio determines the amplitudes between the tensorial power spectrum parameterization and its scalar counterpart for a given k_*. The tensorial amplitude A_t is set by the expansion rate which is close to constant during inflation (Misner et al. 1973). The scalar amplitude A_s is dependent on the spectral scalar index n_s where $n_s = 1$ represents perfect scale invariance (k-mode independence). Unfortunately, this convention is different to that of the tensorial index in which $n_T = 0$ represents perfect scale invariance (Baumann 2012).

The tensor-scalar-value r is highly dependent on the inflationary model. As a result, it has a large parameter space (Guzzetti et al. 2016). This ratio can be constrained with observational data which narrows the inflationary model selection (Barnaby et al. 2012; Ade et al. 2016).

A negative value of n_T corresponds to red-tilted values and positive values of n_T are associated with blue-tilted value. The tilt holds information about the equation of state of the Universe when modes enter the horizon (Guzzetti et al. 2016). In general, inflationary data from CMB polarization may predict blue-tilted spectral index $n_T < 0$ (Cabass et al. 2016).

Primordial gravitational waves have a similar effect on the CMB as massless neutrinos (Pagano et al. 2016). This occurs when the primordial GWs energy density perturbations are adiabatic (Pagano et al. 2016). Since they radiate similarly to neutrinos, primordial GWs are able to contribute relativistic degree's of freedom to the standard value of $N_{\rm eff}$.

$$\begin{aligned} N_{\rm eff}^{\rm new} &= N_{\rm eff} + N_{\rm eff}^{\rm GW} \\ &= 3.046 + N_{\rm eff}^{\rm GW} \end{aligned} \tag{3}$$

We aim to use $N_{\rm eff}^{\rm new}$ to determine an analytic solution to the Hubble constant H_0. We will do this by equating the energy densities at matter-radiation equality. We will then show how these contributions alleviate some of the H_0 tension. This adds to ongoing work in understanding the Hubble tension (Mali & Rogers Mali & Rogers).

2 THEORETICAL BACKGROUND

2.1 Single-field slow-roll inflation

Quantum fluctuations of fields describe the inflationary evolution of the early Universe. In its simplest form, we use a neutral scalar field and metric tensor (Baumann 2012). We begin with the standard FRW metric.

$$ds^2 = dt^2 - a^2(t)\left\{g_{ij}dx^i dx^j\right\} \tag{4}$$

Using $a(t)$ as the scale factor in non-conformal time (coordinate time). From here, we then apply small tensorial perturbations $h_{\mu\nu}$. Using the gauge ($h_{\mu 0} = 0$) we can recover the perturbations in only spacial components (Chongchitnan & Efstathiou 2006).

$$ds^2 = dt^2 - a^2(t)\left(\delta_{ij} + h_{ij}\right)dx^i dx^j \tag{5}$$

Following this, we impose the transverse ($\partial^i h_{ij}^T = 0$) and traceless ($h_i^i = 0$) conditions to obtain the fully perturbed metric solutions as shown below (Bruni et al. 1997).

$$g_{00} = -a(\tau)\left(1 + 2\sum_{n=1}^{+\infty}\frac{1}{n!}\Psi^{(n)}\right) \tag{6}$$

$$g_{0i} = a^2(\tau)\sum_{n=1}^{+\infty}\frac{1}{n!}\omega_i^{(n)} \tag{7}$$

$$g_{ij} = a^2(\tau)\left[1 - 2\left(\sum_{n=1}^{+\infty}\frac{1}{n!}\Phi^{(n)}\right)\right]\delta_{ij} + a^2(\tau)\sum_{n=1}^{+\infty}\frac{1}{n!}h_{ij}^{(n)} \tag{8}$$

Notice here, $\Phi^{(n)}$, $\omega_i^{(n)}$, $\Psi^{(n)}$ and $h_{ij}^{(n)}$ are the corresponding nth order perturbations. We will now show that these can be used, along with the slow-roll parameters to obtain the tensor-scalar ratio r. First, we define the Hubble expansion parameters for slow roll inflation (Lidsey et al. 1997).

$$\epsilon(\phi) \equiv \frac{3\dot\phi^2}{2}\left[V + \frac{1}{2}\dot\phi^2\right]^{-1} = \frac{m_{\rm Pl}^2}{4\pi}\left(\frac{H'(\phi)}{H(\phi)}\right)^2 \tag{9}$$

Here we have used textbook formulation $\dot H = -\frac{1}{2}\dot\phi^2$ to rewrite the slow-roll parameters (Hobson et al. 2006). We have also used the textbook definitions of the Klein-Gordon equation $3H\dot\phi \approx V'$ (Baumann 2012). In slow-roll approximation, we assume that these parameters are very small, hence $|\epsilon|, |\eta| < 1$.

$$\eta(\phi) \equiv -\frac{\ddot\phi}{H\dot\phi} = \frac{m_{\rm Pl}^2}{4\pi}\frac{H''(\phi)}{H(\phi)} = \epsilon - \frac{m_{\rm Pl}\epsilon'}{\sqrt{16\pi\epsilon}} \tag{10}$$

Physically, we enforce that the scalar perturbation must be positive $\dot\phi > 0$, hence $H'(\phi) < 0$, without this we would have accelerating expansion in the early Universe. Due to this we must take the negative root for $\epsilon(\phi)$. Formally, $\sqrt{\epsilon} = -\sqrt{m_{\rm Pl}^2/4\pi}H'/H$. The amount of inflation can be represented as an e-fold and can be written as the following (Baumann) (Baumann 2012).

$$N_{\rm tot} = \int_{t_I}^{t_E} H(t){\rm d}t = -\frac{4\pi}{m_{\rm Pl}^2}\int_{\phi}^{\phi_e} d\phi\frac{H(\phi)}{H'(\phi)} \tag{11}$$

Following this, we solve assuming an inflationary scale factor $a(\phi) \propto e^{f(\phi)}$ for some arbitrary function of ϕ. We can hence obtain the equation for $a(\phi)$. We do this by assuming that the scalar components of the fluctuations follow a power-law relationship.

$$a(\phi) = a_e\exp[-N(\phi)] \tag{12}$$

This formula connects two periods of inflation. It is used to determine how rapidly the scale passes outside of the Hubble radius. Evaluating the expression with a scale matching the Hubble radius $k = aH$. We obtain the following equation.

$$k(\phi) = a_e H(\phi) \exp[-N(\phi)] \tag{13}$$

Subsequently, we take the exponential derivative to find the characteristic relation between k and ϕ.

$$\begin{aligned} \frac{d\ln(k)}{d\phi} &= \frac{4\pi}{m_{Pl}^2}\left(\frac{H}{H'}(\epsilon - 1)\right) \\ &= \frac{4\pi}{m_{Pl}^2}\left(\frac{H}{H'}\left[\frac{m_{\rm Pl}^2}{4\pi}\left(\frac{H'}{H}\right)^2 - 1\right]\right) \end{aligned} \tag{14}$$

This equation shows us how primordial GWs leave the Hubble radius.

Using these parameters, we are able to obtain the scalar and tensorial primordial spectral amplitudes (Chongchitnan & Efstathiou 2006). This is done using quantum perturbations which are beyond the scope of this paper. In theory, one would apply the canonical quantization formulae to a quantum oscillator system in de Sitter space. This would then lead to zero-point fluctuations which is used to derive the Mukhanov-Sasaki equation. After applying the Minkowski initial condition, one would be able to obtain the power spectra below (Baumann 2012).

$$A_S(k) \equiv \frac{2}{5}\mathcal{P}_S^{1/2} \simeq \frac{4}{5}\frac{H^2}{m_{\rm Pl}^2\,|H'|}\bigg|_{k=aH} \tag{15}$$

$$A_T(k) \equiv \frac{1}{10}\mathcal{P}_T^{1/2} \simeq \frac{2}{5\sqrt{\pi}}\frac{H}{m_{\rm Pl}}\bigg|_{k=aH} \tag{16}$$

Here $A_s(k)$ and $A_R(k)$ represent scalar and tensor amplitudes of the power spectrum introduced in section **??**. We have simply shown the lowest order approximation above.

It is possible to write each as a power expansion. The scalar case written as follows (Ade et al. 2016).

$$n_{\rm s} - 1 + \frac{1}{2}\,{\rm d}n_{\rm s}/{\rm d}\ln k \ln(k/k_*) + \ldots \approx n_s - 1$$

Below is the equivalent tensorial formulation.

$$n_{\rm t} + \frac{1}{2}\,{\rm d}n_{\rm t}/{\rm d}\ln k \ln(k/k_*) + \ldots \approx n_t$$

Here we define the spectrum around a specific pivot point k_* in which the approximation is made. Combining all of this together, we are able to rewrite the tensorial and scalar power spectrum's. These are the forms most commonly seen in textbooks and the community.

$$P_s(k) = \frac{k^3}{2\pi^2}\,|\mathcal{T}(k,\tau)|^2 \approx A_s(k_*)^2\left(\frac{k}{k_*}\right)^{n_s-1} \tag{17}$$

$$\begin{aligned} P_t(k) &= \frac{k^3}{2\pi^2}\left(\left|h_k^+\right|^2 + \left|h_k^\times\right|^2\right) \\ &= A_t\,(k_*)^2\left(\frac{k}{k_*}\right)^{n_t(k_*)+\frac{1}{2}n_{t,\rm run}\ln(k/k_*)} \\ &\approx A_t(k_*)^2\left(\frac{k}{k_*}\right)^{n_t(k_*)} \end{aligned} \tag{18}$$

We are now able to resolve the scalar and spectral indices in terms of their slow-roll parameters. We do this by taking the first derivative of equations 15 and 16 and substituting it into equation 14. Taking the first order approximation $O(\eta, \epsilon)$, we find the following relations.

$$n_s - 1 \approx 2\eta - 4\epsilon \tag{19}$$

$$n_T \approx -2\epsilon \tag{20}$$

Using the new power spectra and their slow-roll approximations, we are able to finally determine the tensor-to-scalar ratio. This determines the variation of the scalar field during the inflationary period (Hirano 2019). We do this by using equations 19 and 20 to determine the amplitude ratio.

$$\begin{aligned} \frac{A_t(k_*)^2}{A_s(k_*)^2} &= \left(\frac{2}{5\sqrt{\pi}}\frac{H}{m_{\rm Pl}}\right)^2\left(\frac{4}{5}\frac{H^2}{m_{\rm Pl}^2\,|H'|}\right)^{-2} \\ &= \left(\frac{H' m_{Pl}}{2\sqrt{\pi}H}\right)^2 \\ &= \epsilon \end{aligned} \tag{21}$$

We then substitute this into the equation for the power spectrum.

$$r = \frac{P_t(k_*)}{P_s(k_*)} = 16\frac{A_t(k_*)^2}{A_s(k_*)^2} \approx 16\epsilon \tag{22}$$

We can further combine the result from equations 22 and 20 to obtain the tensor spectral index in terms of the tensor-scalar ratio.

$$n_T \approx -2\epsilon = -\frac{r}{8} \tag{23}$$

We have analytically arrived at the quantity which observationalists will measure and report. In order to equate this to H_0 we shift our focus to computing the energy density.

2.2 Energy Density

Now we must obtain the energy density of the gravitational waves. This will allow us to relate n_T and r to H_0. The energy density from the tensorial power spectrum can be obtained by applying density perturbations to the standard scalar field action (Baumann 2012; Meerburg et al. 2015).

$$S = \int {\rm d}^4x\sqrt{-g}\left[\frac{1}{2}M_{\rm pl}^2 R - \frac{1}{2}g^{\mu\nu}\partial_\mu\varphi\partial_\nu\varphi - V(\varphi)\right] \tag{24}$$

We now apply perturbations to the metric through h_{ij}, the gauge invariant transverse traceless quantity. Computing the Christoffels, Riemann and Ricci tensor, we are able to derive the perturbed stress-energy tensor.

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + \delta g_{\mu\nu} \tag{25}$$

As before, we use the FRW metric which defines $h_{i,j}$ in terms of the spacial components of the metric perturbations $\delta g_{ij} = a(t)^2 h_{ij}$. When computing the stress-energy-tensor (through the Ricci tensor) we ignore modes who's perturbations are larger horizon limit $k = a(t)H(t)$. This can be done since larger modes have been shown to have insignificant effect on the GW energy density (Meerburg et al. 2015).

$$R_{\mu\nu} = \bar{R}_{\mu\nu} + R_{\mu\nu}^{(1)} + R_{\mu\nu}^{(2)} + \ldots \tag{26}$$

The first and second orders terms are shown explicitly.

$$\begin{aligned} R_{\mu\nu}^{(1)} = \frac{1}{2}(&\bar{D}^\alpha\bar{D}_\mu h_{\nu\alpha} + \bar{D}^\alpha\bar{D}_\nu h_{\mu\alpha} - \\ &\bar{D}^\alpha\bar{D}_\alpha h_{\mu\nu} - \bar{D}_\nu\bar{D}_\mu h) \end{aligned} \tag{27}$$

$$\begin{aligned}R_{\mu\nu}^{(2)} =&\frac{1}{2}\bar{g}^{\rho\sigma}\bar{g}^{\alpha\beta}\left[\frac{1}{2}\bar{D}_\mu h_{\rho\alpha}\bar{D}_\nu h_{\sigma\beta}\right.\\ &+\left(\bar{D}_\rho h_{\nu\alpha}\right)\left(\bar{D}_\sigma h_{\mu\beta}-\bar{D}_\beta h_{\mu\sigma}\right)\\ &+h_{\rho\alpha}\left(\bar{D}_\nu\bar{D}_\mu h_{\sigma\beta}+\bar{D}_\beta\bar{D}_\sigma h_{\mu\nu}\right.\\ &\left.-\bar{D}_\beta\bar{D}_\nu h_{\mu\sigma}-\bar{D}_\beta\bar{D}_\mu h_{\nu\sigma}\right.\\ &+\left(\frac{1}{2}\bar{D}_\alpha h_{\rho\sigma}-\bar{D}_\rho h_{\alpha\sigma}\right)\\ &\left.\times\left(\bar{D}_\nu h_{\mu\beta}+\bar{D}_\mu h_{\nu\beta}-\bar{D}_\beta h_{\mu\nu}\right)\right.\end{aligned} \tag{28}$$

It can be shown that the the first and third terms of the $R_{\mu\nu}$ expansion contain lower order frequencies while the linear first term contains higher order frequencies (Misner et al. 1973). This is known as the *short-wave approximation*. We use this approximation to rewrite Einstein's equations.

$$\bar{R}_{\mu\nu}=-\left(R_{\mu\nu}^{(2)}\right)^{(Low)}+8\pi G\left(T_{\mu\nu}-\frac{1}{2}g_{\mu\nu}\right)^{(Low)} \tag{29}$$

Before calculating the stress-energy tensor, we eliminate the high frequency dependence in the $R_{\mu\nu}^{(2)}$ term (Refer to the short-wave approximation in Kip Thorne's text on gravitation) (Misner et al. 1973). Once again, this can be done since larger modes do not have much impact on the GW energy density (Meerburg et al. 2015).

$$\bar{R}_{\mu\nu}=R_{\mu\nu}^{(2)}+8\pi GT_{\mu\nu}-\frac{1}{2}g_{\mu\nu}T \tag{30}$$

Inverting this, we find the solution for the gravitational wave stress energy tensor.

$$T_{\mu\nu}^{GW}=\frac{-1}{8\pi G}R_{\mu\nu}^{(2)}-\frac{1}{2}\bar{g}_{\mu\nu}R^{(2)}+O(\delta g^3) \tag{31}$$

Converting to transverse traceless gauge, we are able to convert space-time indices to spacial ones $\{\mu,\nu\}\to\{i,j\}$. This can be described in terms of the covariant derivatives of the metric perturbations. Here we use equations 27 and 28 to rewrite the stress energy tensor. Notice that the covariant derivatives has now been converted to normal derivatives $D^\mu\to\partial^\mu$. We are able to do this by assuming the background spacetime is flat (Maggiore 2007).

$$\begin{aligned}R_{\mu\nu}^{(2)} =&\frac{1}{2}\left[\frac{1}{2}\partial_\mu h_{\alpha\beta}\partial_\nu h^{\alpha\beta}+h^{\alpha\beta}\partial_\mu\partial_\nu h_{\alpha\beta}\right.\\ &-h^{\alpha\beta}\partial_\nu\partial_\beta h_{\alpha\mu}-h^{\alpha\beta}\partial_\mu\partial_\beta h_{\alpha\nu}\\ &+h^{\alpha\beta}\partial_\alpha\partial_\beta h_{\mu\nu}+\partial^\beta h_\nu^\alpha\partial_\beta h_{\alpha\mu}\\ &-\partial^\beta h_\nu^\alpha\partial_\alpha h_{\beta\mu}-\partial_\beta h^{\alpha\beta}\partial_\nu h_{\alpha\mu}\\ &+\partial_\beta h^{\alpha\beta}\partial_\alpha h_{\mu\nu}-\partial_\beta h^{\alpha\beta}\partial_\mu h_{\alpha\nu}-\frac{1}{2}\partial^\alpha h\partial_\alpha h_{\mu\nu}\\ &\left.+\frac{1}{2}\partial^\alpha h\partial_\nu h_{\alpha\mu}+\frac{1}{2}\partial^\alpha h\partial_\mu h_{\alpha\nu}\right]\end{aligned} \tag{32}$$

Using this, and plugging back into equation 31 we obtain the energy-density of gravitational waves (by taking $\{0,0\}$ component of the stress energy tensor). Notice here that we are working to second order.

$$\begin{aligned}\rho_{\rm GW} &= T_{00}^{\rm GW}\\ &=\frac{1}{32\pi Ga^4}\delta^{ik}\delta^{j\ell}\left\langle\left(\partial_0-2H\right)\delta g_{ij}\left(\partial_0-2H\right)\delta g_{k\ell}\right\rangle\\ &+O\left(\delta g^3\right)\\ &=\frac{1}{32\pi G}\delta^{ik}\delta^{j\ell}\left\langle\dot{h}_{ij}\dot{h}_{k\ell}\right\rangle+O\left(\delta g^3\right)\end{aligned} \tag{33}$$

Hence we can then directly write the gravitational wave energy-density (Guzzetti et al. 2016).

$$\rho_{\rm gw}=\frac{1}{32\pi Ga^2}\left\langle h'_{ij}(\mathbf{x},\eta)h'^{ij}(\mathbf{x},\eta)\right\rangle \tag{34}$$

This can be defined in terms of a transfer function which allows us to obtain the final form of the energy-density of gravitational waves. This form matches most common textbooks (Baumann 2012).

$$\left\langle h_{ij}(\mathbf{x},\eta)h'^{ij}(\mathbf{x},\eta)\right\rangle=\int d\log k\Delta_h^2(\mathbf{x},\eta) \tag{35}$$

Below is the definition of the power spectrum. In it contains the transfer function $T(k,\eta)$ (Baumann 2012).

$$\Delta_h^2(\mathbf{x},\eta)=P_t(k)T(k,\eta) \tag{36}$$

Using this and equation 34 we are able to define the full form of the energy density.

$$\rho_{GW}=\int_0^k d\log k\frac{P_t(k)}{32\pi Ga^2}T'(k,\eta)^2 \tag{37}$$

2.3 Primordial Gravitational Waves Energy Density

Now that we have defined the gravitational wave density, we intend to show how it impacts the H_0 tensor-to-scalar ratio r. We will use both sections 2.1 and 2.2 to show this.

Notice that we are able to written the k-modes in terms of frequency to re-parameterise equation 22 (Meerburg et al. 2015).

$$\frac{f}{\rm Hz}=\frac{k}{2\pi a\tau_0}=\frac{1.6\times10^{-15}}{\rm Mpc^{-1}} \tag{38}$$

We will now rewrite equation 22. In order to do this, we must define the pivot scale (k_*). The pivot scale determines the scale of tensorial mode measurements. Typical values range from $k_*\approx 0.01-0.05\mathrm{Mpc}^{-1}$ (Ade et al. 2016). In our work we assume $k_*=0.01\mathrm{Mpc}^{-1}$. The choice of pivot scale has been studied in the literature and often warrants a full section (Cortês et al. 2015).

$$P_t(f)=rA_s\left(\frac{f}{f_*}\right)=rA_s\left(\frac{f/Hz}{1.6\times10^{-17}}\right) \tag{39}$$

As previously mentioned, since we are only considering lower frequencies (Misner et al. 1973) a sensible assumption to make is to consider frequencies in the UV bandwidth $k=k_{UV}$. As a result of this, we are able to re-write equation 37 in terms of k_{UV}.

$$\rho_{GW}=\int_0^{\mathbf{k}_{UV}} d\log k\frac{P_t(k)}{32\pi Ga^2}T'(k,\eta)^2 \tag{40}$$

The goal is to now define a suitable transfer function and integrate this equation to achieve the gravitational-wave density in terms of the total density.

The transfer function is typically a linear combination of Bessel and Neumann functions. They are normalized such that they approached unity as $k\to0$ (Turner et al. 1993; Guzzetti et al. 2016; Baumann 2012). We obtain them by imposing boundary conditions on the massless Klein-Gordon equation for plane waves (Watanabe & Komatsu 2006).

$$h''_{\lambda,\mathbf{k}}+\left(\frac{2a'}{a}\right)h'_{\lambda,\mathbf{k}}+k^2h_{\lambda,\mathbf{k}}=16\pi Ga^2\Pi_{\lambda,\mathbf{k}} \tag{41}$$

The general solution to this equation is $h_k^\lambda=h_{k,\rm prim}^\lambda$ (Watanabe & Komatsu 2006). We use the equation above to compute the transfer function. For values of k less than k_{eq}. $T(k < k_{eq}) = j_o(k\tau)$. Taking its derivative we find $T'(k < k_{eq}) = -kj_1(k\tau)$. We now applying this to equation 40 to obtain an analytic form of ρ_{GW}.

$$\begin{aligned} \rho_{\rm GW} &= \int_{k_{\rm IR}}^{k_{\rm UV}} d\log k \frac{P_t(k)}{32\pi G a^2}\left[T'(k,\tau)\right]^2 \\ &= \frac{A_s r}{32\pi G a^2}\int_{k_{\rm IR}}^{k_{\rm UV}} k dk \left(\frac{k}{k_*}\right)^{n_T} j_1^2(k\tau) \end{aligned} \tag{42}$$

Here the bounds of integration are fixed between k_{IR} and k_{UV}. k_{IR} is defined to be far inside the horizon (sub-horizon scales). Hence, we set $k_{IR} << H(t)a(t)$. Now, notice the k dependence of the derivative of the transfer function for small values of k.

$$\begin{aligned} T'(k,\tau)^2 &= (-kj_1(k\tau))^2 \\ &= k^2\left(\frac{\sin k\tau}{(k\tau)^2} - \frac{\cos k\tau}{k\tau}\right)^2 \\ &\approx k^2\left(\frac{k\tau}{(k\tau)^2} - \frac{1-\frac{1}{2}(k\tau)^2}{k\tau}\right)^2 \\ &= \frac{k^4\tau^4}{4} \end{aligned} \tag{43}$$

The k^4 dependence in the square transfer function in the $k \to 0$ limit highlights that the integral above in equation 42 should converge for values of $n_T \geq -4$ in the $k_{IR} \to 0$ limit. This should occur for any arbitrary value of k_{UV}. As a result of this, we are able to approximate the lower bound frequency in this limit. Hence, $k_{IR} = 0$. The upper bound k_{UV} varies depending on model choice. Here we restrict the value of k_{UV} to be larger than the Plank scale, this is a fair assumption as we do not expect primordial gravitational waves to be generated on scales smaller than this.

We now adopt the LIGO model in which scales of UV are 60 e-folds smaller than the horizon scale $a(t_o)H(t_o)$ (Liddle & Leach 2003). We further consider the models in which the UV energy dominates the total energy of the system, this enforces a positive spectral index $n_T > 0$. As a result of this we are able to take the limit $\frac{k_{IR}}{k_{UV}} \to 0$.

$$\begin{aligned} \rho_{\rm GW} &= \frac{A_s r}{32\pi G a^2}\int_{k_{\rm IR}}^{k_{\rm UV}} k dk \left(\frac{k}{k_*}\right)^{n_T} j_1^2(k\tau) \\ &= \frac{A_s r}{32\pi G a^2 k_*^{n_T}}\int_{\approx 0}^{k_{\rm UV}} k^{3+n_T} dk \left(\frac{\sin k\tau}{(k\tau)^2} - \frac{\cos k\tau}{k\tau}\right)^2 \\ &= \frac{A_s r}{32\pi G a^2}\left(\frac{k_{\rm UV}}{k_*}\right)^{n_t} \frac{1}{2n_t}\frac{1}{\tau^2} + O\left(1/(k_{\rm UV}\tau)\right) \end{aligned} \tag{44}$$

Notice above that we have used the assumption that since k_{UV} dominates we are able to take $\frac{1}{k\tau}$ to be very small. Hence we arrive at a closed form of the gravitational wave density.

$$\begin{aligned} \rho_{GW} &= \frac{A_s r}{32\pi G}\left(\frac{k_{UV}}{k_*}\right)^{n_T} \frac{1}{2n_T(a\tau)^2} \\ &= \frac{A_s r}{24 n_T}\left(\frac{k_{UV}}{k_*}\right)^{n_T} \rho_{tot} \end{aligned} \tag{45}$$

Notice above we have used the assumption that $\frac{1}{(a\tau)^2} = H^2 = 8\pi G\frac{\rho_{tot}}{3}$, which is true in the radiation dominated era (since $t \to \tau^2$). To do this, we must also assume flat curvature and negligible dark energy constant (Osano 2020).

2.4 Hubble Tension

Now that we have obtained the gravitational wave density, its time to use this, along with our knowledge of density in the radiation dominated era to work towards an analytic solution of H_o. We take the total energy density to be the sum of photon, baryon and primordial gravitational wave energy densities.

$$\rho_{tot} = \rho_\gamma + \rho_\nu + \rho_{GW} \tag{46}$$

Trivially, the neutrino density is dependent on the effective number of neutrinos (Baumann 2012). It is dependent on the non-instantaneous neutrino decoupling which still occurs while electron-positron annihilation takes place (Baumann 2012).

$$\rho_\nu = \frac{7}{8}N_{\rm eff}\left(\frac{4}{11}\right)^{4/3}\rho_\gamma \tag{47}$$

Subsequently, we are able to compute the total energy density in terms of $N_{\rm eff}$.

$$\rho_{tot} = \rho_\gamma\left(1 + \frac{7}{8}\left(\frac{4}{11}\right)^{\frac{4}{3}} N_{\rm eff}\right) \tag{48}$$

Here we have assumed that the GWs energy density is very small compared to the total energy density, hence we ignore it in the additive component. Using this we are able to rewrite the GW energy density in term of $N_{\rm eff}$.

$$\begin{aligned} \rho_{GW} &= \frac{A_s r}{24 n_T}\left(\frac{k_{UV}}{k_*}\right)^{n_T}\rho_\gamma\left(1 + \frac{7}{8}\left(\frac{4}{11}\right)^{\frac{4}{3}} N_{\rm eff}\right) \\ &= \frac{A_s(-8n_T)}{24 n_T}\left(\frac{k_{UV}}{k_*}\right)^{n_T}\rho_\gamma + \frac{7}{8}\left(\frac{4}{11}\right)^{\frac{4}{3}} N_{\rm eff}\left(\frac{k_{UV}}{k_*}\right)^{n_T}\rho_\gamma \\ &= -3.046\frac{7}{8}\left(\frac{4}{11}\right)^{\frac{4}{3}}\rho_\gamma + \frac{7}{8}\left(\frac{4}{11}\right)^{\frac{4}{3}}\rho_\gamma N_{\rm eff} \\ &= \frac{7}{8}\left(\frac{4}{11}\right)^{\frac{4}{3}}\left(N_{\rm eff} - 3.046\right)\rho_\gamma \end{aligned} \tag{49}$$

We now substitute the equation above into the total energy density in equation 45. We assume a primordial gravitational wave extension to the standard model. $N_{\rm eff} = 3.046 + N_{\rm eff}^{GW}$, i.e. we assume no other degrees of freedom beyond the standard model.

$$\begin{aligned} \frac{\rho_{\rm GW}}{\rho_{\rm tot}} &= \frac{A_s r}{24 n_t}\left(\frac{k_{\rm UV}}{k_*}\right)^{n_t} \\ &= \frac{\frac{7}{8}\left(\frac{4}{11}\right)^{4/3}\left(N_{\rm eff} - 3.046\right)\rho_\gamma}{\left(1 + \frac{7}{8}\left(\frac{4}{11}\right)^{4/3} N_{\rm eff}\right)\rho_\gamma} \end{aligned} \tag{50}$$

This can be inverted to solve for $N_{\rm eff}$.

$$
\begin{aligned}
N_{\rm eff} &= \frac{\frac{8}{7}\left(\frac{11}{4}\right)^{4/3}\left[\frac{A_s r}{24n_t}\left(\frac{k_{\rm UV}}{k_*}\right)^{n_t}\right]+3.046}{1-\left[\frac{A_s r}{24n_t}\left(\frac{k_{\rm UV}}{k_*}\right)^{n_t}\right]} \\
&\approx \left(\frac{8}{7}\left(\frac{11}{4}\right)^{4/3}\left[\frac{A_s r}{24n_t}\left(\frac{k_{\rm UV}}{k_*}\right)^{n_t}\right]+3.046\right) \\
&\times\left(1+\left[\frac{A_s r}{24n_t}\left(\frac{k_{\rm UV}}{k_*}\right)^{n_t}\right]\right) \\
&= 3.046+\left(3.046+\frac{8}{7}\left(\frac{11}{4}\right)^{4/3}\right)\frac{A_s r}{24n_t}\left(\frac{k_{\rm UV}}{k_*}\right)^{n_t} \\
&+ O\left(\left(\frac{A_s r}{24n_t}\left(\frac{k_{\rm UV}}{k_*}\right)\right)^2\right)
\end{aligned} \tag{51}
$$

Here we have Taylor expanded in terms of $\frac{A_s r}{24n_t}\left(\frac{k_{\rm UV}}{k_*}\right)$. We have then taken the leading order behaviour, in other words we are treating the gravitational energy density as small compared to the total energy density $\rho_{GW} << \rho_{\rm tot}$. This gives the final key result for $N_{\rm eff}$.

$$
N_{\rm eff} \approx 3.046+\left(3.046+\frac{8}{7}\left(\frac{11}{4}\right)^{4/3}\right)\frac{A_s r}{24n_t}\left(\frac{k_{\rm UV}}{k_*}\right)^{n_t} \tag{52}
$$

Referring back to equation 45, the short-wave approximation allows us to assume that the gravitational waves redshift analogous to radiation (Misner et al. 1973). We can compute the effect this has by investigating the primordial GWs during matter radiation equality ($a = a_{eq}$). Before we do this, we must compute the energy density for relativistic photons and neutrinos (Carroll & Ostlie 2017). Below is the equation for radiation without the addition of neutrinos.

$$
\rho_\gamma = \frac{1}{2} g_\gamma a T^4 \tag{53}
$$

We have defined a as the radiation constant , $g_r = 2$ for a photon and T to be the temperature of the CMB (i.e. black body photon). Following this, we apply the correction to g to obtain the relativistic version.

$$
g_* = 2\left(1+N_{\rm eff}\frac{7}{8}\left(\frac{4}{11}\right)^{4/3}\right) \tag{54}
$$

Substituting this in, we obtain the fully relativistic version.

$$
\rho_r = 1+N_{\rm eff}\frac{7}{8}\left(\frac{4}{11}\right)^{4/3} aT^4 \tag{55}
$$

Hence, using the Friedmann equation for a flat Universe, we find the Hubble constant as a function of effective degree's of freedom (Hobson et al. 2006).

$$
\begin{aligned}
\rho_{\rm m}(a_{\rm eq}) &= \rho_r(a_{\rm eq}) \\
\rho_{m,0}\left(\frac{a_0}{a_{eq}}\right)^3 &= \rho_{r,0}\left(\frac{a_0}{a_{eq}}\right)^4 \\
\frac{3H_0^2\Omega_m}{8\pi G}a_{eq} &= a_{\rm rad}T_{\gamma,0}^4\left(1+N_{\rm eff}\frac{7}{8}\left(\frac{4}{11}\right)^{4/3}\right)
\end{aligned} \tag{56}
$$

This equation analytically solves for the Hubble constant in terms of the parameters of the primordial gravitational waves. We are able to make this more explicit with a singular H_0 dependence below.

$$
\begin{aligned}
H_0^2 &= \frac{8\pi G}{3\Omega_m}\frac{a_{\rm rad}}{a_{eq}}T_{\gamma,0}^4\left(1+N_{\rm eff}\frac{7}{8}\left(\frac{4}{11}\right)^{4/3}\right) \\
&\approx \frac{8\pi G}{3\Omega_m}\frac{a_{\rm rad}}{a_{eq}}T_{\gamma,0}^4 \\
&\times\left\{1+\left[3.046+\left(3.046+\frac{8}{7}\left(\frac{11}{4}\right)^{4/3}\right)\frac{A_s r}{24n_t}\left(\frac{k_{\rm UV}}{k_*}\right)^{n_t}\right]\right. \\
&\left.\times\frac{7}{8}\left(\frac{4}{11}\right)^{4/3}\right\}
\end{aligned} \tag{57}
$$

We have thus shown, that the gravitational wave parameters directly influence the Hubble constant. The computed H_0 is model dependent on our choice of ultra-violet cutoff $k_{\rm UV}$. We now examine what this observationally looks like.

3 RESULTS AND DISCUSSION

In this section, we draw heavily from the results of Dr. Graef and Dr. Benetti (Graef et al. 2019). There they use the LIGO and VIRGO experiments to apply constraints on the tensor scalar r ratio and spectral index n_T. They investigate two models, varying the UV cut-off $k_{\rm UV}$ as highlighted in equation 57.

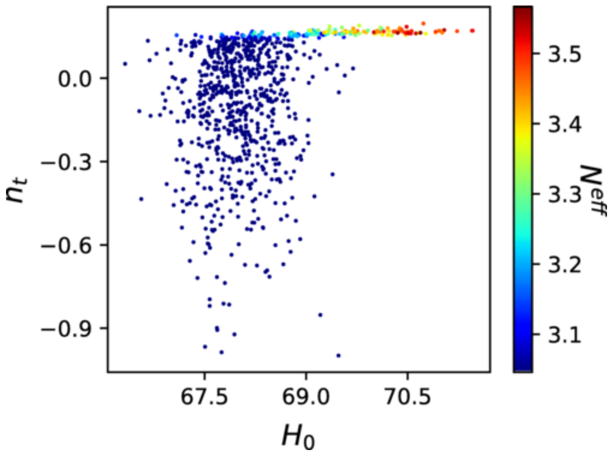
In the first model, they take a cut-off corresponding to the grand unified theory (GUT) scale, i.e. $k_{{\rm UV}/k_*} \approx 10^{56}$ and $f_{UV} \approx 10^{40} Hz$. In their second model, they take a 60 e-fold assumption in the power spectrum. This corresponds to $k_{{\rm UV}/k_*} \approx 10^{24}$ and $f_{UV} \approx 10^8$. This model matches the expected inflation of the Universe. They use data releases from; Planck and BICEP2 collaborations (BKP), LIGO and baryonic acoustic oscillations survey (BAO) to constrain the spectral index and tensor-to-scalar amplitudes (Aasi et al. 2015; Ade et al. 2016; Beutler et al. 2011; Riess et al. 2018; Ross et al. 2015; Anderson et al. 2014). The initial data is known as "base".

They then include measurements by Gaia space observatory and the Hubble space telescope (HST). Below we observe the main result from their paper which directly compares the effect of the spectral index n_T on the Hubble constant H_0. Tolman–Oppenheimer–Volkoff (TOV) equation It can be observed that the value of H_0 increases as the spectral index n_T increases. This occurrence exceeds the limits bounded by the standard model (Ade et al. 2016). The computed values of the Hubble constant H_0 can be found in the table below. Constraining the spectral index with the base + HST data suggests a higher value of the Hubble constant. Their best model (Model 2 - Inflation, base + HST) predicts $H_0 = 68.49 \pm 0.83$. This is closer to late-time measurements when compared with other early Universe predictions (Ade et al. 2016), but still far from early-time measurements.

Since the maximum-a-posteriori (MAP) values bring their H_0 closer to those measured from late Universe times such as cepheid's (Riess et al. 2018). Primordial GWs alleviates the ongoing Hubble tension (Graef et al. 2019).

Figure 1 demonstrates the effect of the n_T vs H_0 plot as determined by the relativistic degrees of freedom ($N_{\rm eff}$).

From this plot, it is clear that positive values of the spectral index are most likely to result in relaxing of the Hubble tension. This



[h]

Figure 1. Here, they use the base + HST dataset to plot the change in effective degrees of freedom on the Hubble constant. Colours closer to blue represent values of $N_{\rm eff}$ closer to the standard model. This plot was also taken directly from Graef et. al (Graef et al. 2019)

matches the possible CMB exclusion of red-tilted values $n_T < 0$. Constraints on n_T are only expected to get stronger with newer, more powerful GW detectors coming online in the coming years (Jennrich 2009; Krauss et al. 2010; Westbrook et al. 2020; Seto et al. 2001; Matsumura et al. 2014; Abazajian et al. 2019). In making the following predictions we have had to assert some assumptions. We will now state them explicitly for the reader. Firstly, we assume that the effect of primordial GWs are similar to that of neutrinos. We are hence able to add to their relativistic degrees of freedom. We also assume that GWs do not interact with matter or radiation. We then assume the consistency relation of standard inflation in which the tensor-to-scalar ratio r can be determined as the ratio of tensorial to scalar power spectrum amplitudes. Then, we compute the total density of the Universe, we assume that it is the sum of radiation density + gravitational waves and nothing else (stated in equation 46). Finally, we assume that the GW energy density evolves identically to photon radiation from inflation. Making these assumptions allows us to compute an analytic form of the Hubble constant H_0.

4 EXTENSIONS TO SLOW-ROLL THEORY FOR PRIMORDIAL GRAVITATIONAL WAVES

While the primary focus of this paper is to analyze the standard slow-roll case and its direct impact on the Hubble constant H_0. A review of primordial gravitational waves would be incomplete without a brief overview of extensions to the slow-roll scenario. The choice of inflationary model will determine the *new physics* the community adopts. Alternatives to the slow-roll scalar field are necessary to explore a wider parameter space of n_T and r. These alternatives can then be constrained through upcoming observational data. Our paper will not rigorously study this alternatives. Instead we will state them and leave their analysis of H_0 as an opportunity of further study to the reader.

The natural extension to this is to consider models other than the slow-roll case in GR. These models typically study physical properties which lead to a non-trivial GW power spectrum (Guzzetti et al. 2016). Examples of this include different scenarios for inflation such as solid, elastic, trapped or warm inflation (Endlich et al. 2013; Gruzinov 2004; Bartrum et al. 2014; Bastero-Gil et al. 2014, 2016; Pearce et al. 2016). Some work also highlights the possibility of primordial GWs during kination domination (Bernal & Hajkarim 2019). There has also been work on a modified gravity (MG) approach to determining primordial gravitational waves. This is done in order to achieve an accelerated expansionary rate in the early Universe. Theoretically we do this by including a matter Lagrangian $\mathcal{L}_{\rm matter}$ and using it cumulatively with the Einstein-Hilbert Lagrangian $\mathcal{L}_{\rm grav}$ in the action.

$$S = \int \sqrt{-g}\left(\mathcal{L}_{\rm grav} + \mathcal{L}_{\rm mat}\right) {\rm d}^4x \tag{58}$$

There has also been work studying inflation that is coupled to a scalar field. This typically includes an interaction term with an extra scalar field χ (Cook & Sorbo 2012).

$$\mathcal{L} = -\frac{1}{2}\partial_\mu\varphi\partial^\mu\varphi - V(\varphi) - \frac{1}{2}\partial_\mu\chi\partial^\mu\chi - \frac{g^2}{2}\left(\varphi - \varphi_0\right)^2\chi^2 \tag{59}$$

Another approach is one known as the effective field theory (EFT) method. This, like the previous example leads to cosmic acceleration, and a quasi de-Sitter Universe expansion (Tsujikawa 2015).

Finally there are additional possibilities for the emission of gravitational waves through classical production (GWs generated by an accelerating body or asymmetry during BBN) (Guzzetti et al. 2016). Here we require the presence of a source term. They are a growing field of study, with plenty of scope for extensions to improve our understanding of the Hubble H_0 tension.

5 CONCLUSION

In this paper we extend the standard model consisting only of primordial GWs energy density. We used this to analytically relate the properties of primordial GWs (r and n_T) to the Hubble constant H_0. In the early Universe, the effects of primordial GWs resulted in a larger contribution to the effective number of relativistic species $N_{\rm eff}$. This lead to a higher expansion rate of a Universe H_0. We obtained H_0 by setting bounds on r, n_T. We did this by examining data from various collaborations (Ade et al. 2016; Seto et al. 2001; Matsumura et al. 2014; Abazajian et al. 2019; Li & Cooray 2006).

Theoretically, we applied relativistic perturbations to the FRW metric. From this we obtained its energy density. After relating this to $N_{\rm eff}$ we treated GWs as an extension to neutrinos, something the standard model does not account for. Equating this to the total energy density $\rho_{\rm tot}$ we analytically expressed H_0 in terms of $N_{\rm eff}$, spectral index and ratios r, n_T.

We then studied data which demonstrates that a positive spectral index $n_T > 0$ could relax the Hubble tension. Following this, we discussed key results from Graef et. al in which they analysed LIGO and HST data against the base measurements predicted by ΛCDM. In those results, all models resulted in a greater H_0. The best improvement occurred for positive spectral indices $n_T > 0$ (blue tilted). It is clear that the current data prefers positively tilted spectral value.

[]

TABLE I. Confidence limits (68%) for the H_0 and n_t parameters using the two datasets discussed in the text.

		ΛCDM $+ r + n_t$	$k_{\rm UV} = 10^{22}$ [a] Model 2	$k_{\rm UV} = 10^{54}$ Model 1
Base	H_0 [b]	67.57 ± 0.53	67.55 ± 0.56	67.51 ± 0.54
	n_t	$0.46^{+0.36}_{-0.53}$	$0.04^{+0.38}_{-0.14}$	$-0.13^{+0.29}_{-0.11}$
Base + HST	H_0	67.91 ± 0.55	68.49 ± 0.83	68.35 ± 0.81
	n_t	$0.45^{+0.51}_{-0.47}$	$0.12^{+0.34}_{-0.13}$	$-0.07^{+0.29}_{-0.10}$

[a] $k_{\rm UV}$ is in units Mpc^{-1}.
[b] H_0 is in units kms^{-1} Mpc^{-1}.

Table 1. Table representing the computer maximum-a-posteriori values for the Hubble constant with their respective 1σ confidence interval. Both models in the base dataset decrease the value of H_0 compared to the standard CDM model. They tighten the Hubble tension.However, both models under the base + HST dataset relax the Hubble tension. This table was taken directly from Graef et. al (Graef et al. 2019)

The results of this paper showed that an analysis of H_0 using various models for GWs is possible. The assumptions we make about the UV cutoff and short-wave approximation may introduce other effects that clutter the constrains. As a result, one should interpret the results of this paper with reservations.

Finally we outline possible extensions to the current literature beyond the standard slow-roll approximation. As an open field of study, there is scope for analysis on a broader range of models and their effect on H_0. As new data from observationalists further constrain the value of $N_{\rm eff}$, we head towards an exciting time for fundamental physics. Primordial GWs are a viable way of relaxing the Hubble tension. In the coming years, improved observational data may provide evidence for them. (Lokken et al. 2022)

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DATA AVAILABILITY

No data was used in the derivation. All papers citepd were mentioned. No analysis was completed. This was a final project, hence is not subjected to peer review.

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