JOURNAL OF NUMBER THEORY 25, 360–378 (1987)

On the Enumeration of Finite Groups*

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Communicated by H. Zassenhaus

Received November 13, 1985

Let $G(n)$ denote the number of finite groups of order n (up to isomorphism). We prove that for n squarefree, $G(n) = \Omega(n^{1-\varepsilon})$ for any $\varepsilon > 0$, and that for almost all squarefree integers n, $\log G(n) = (1+o(1))(\log\log n) \sum_{p|n} (\log p)/(p-1)$. If we let $F_k(x)$ be the number of $n \leqslant x$ such that $G(n) = k$, then we prove $F_k(x) = (c(a) + o(1))\, x/(\log\log\log x)^{a+1}$ for $k = 2^a$, and $c(a)$ is an appropriate constant, as $x \to \infty$. If $k \neq 2^a$, then we show that $F_k(x) = O(x/(\log\log x)^{1-\varepsilon})$.

1. INTRODUCTION

Let $G(n)$ denote the number of groups (up to isomorphism) of order n. With the recent classification of finite simple groups, we know that

$$G(n) \leqslant n^{c(\log n)^2}$$

for some $c > 0$. (See Neumann [9].) This upper bound can be significantly improved if we confine our attention to certain classes of groups. For example, if n is squarefree, it was shown in [7] that

$$G(n) \leqslant \varphi(n),$$

* Research partially supported by NSERC Grant U0237.

0022-314X/87 $3.00

where φ is the Euler φ-function. Moreover, it was proved in [8] that

$$\sum_{n \leqslant x} \mu^2(n) \log G(n) = (1 + o(1))\, cx \log\log x$$

as $x \to \infty$, for a certain constant c.

This was established by utilising the following beautiful formula due to Hölder [6]. Let $V_p(n)$ denote the number of prime divisors of n which are $\equiv 1 \pmod p$. Then for n squarefree.

$$G(n) = \sum_{d|n} \prod_{p|d} \left(\frac{p^{V_p(n/d)} - 1}{p-1} \right), \tag{1}$$

where the inner product runs over prime divisors p of d.

Formula (1) has other applications. It will be the essential ingredient in the proofs of the main theorems of this paper.

THEOREM 1. *For n squarefree,*

$$G(n) = \Omega(n^{1-\varepsilon})$$

for every $\varepsilon > 0$.

Remark. This theorem shows that the estimate $G(n) \leqslant \varphi(n)$ for n squarefree, is nearly best possible.

THEOREM 2. *For almost all squarefree n,*

$$\log G(n) = (1 + o(1))(\log\log n) \sum_{p/n} \frac{\log p}{p-1},$$

or in other words,

$$\frac{\log G(n)}{\log\log n}$$

has a distribution function.

Remark. A weaker version of this result was proved in [7].

The main interest in formula (1) is that it can be utilised in obtaining information concerning the distribution of the values of $G(n)$. To this end, let us define

$$F_k(x) = \operatorname{card}(n \leqslant x \colon G(n) = k).$$

The following theorem shows that $G(n)$ is a power of 2 more often than any other value.

THEOREM 3. *Let a be a nonnegative integer.*

(i) *if $k \neq 2^a$, then* $F_k(x) \ll \dfrac{x}{(\log\log x)^{1-\varepsilon}}$.

(ii) *if $k = 2^a$, then for* $c(a) = e^{-\gamma}/a!$

$$F_k(x) = \frac{(c(a)+o(1))x}{(\log\log\log x)^{a+1}}.$$

Remarks. (1) With a little more care, (i) can be improved to

$$o\left(\frac{x}{\log\log x}\right).$$

(2) It is conceivable that if $k = 3$,

$$F_3(x) \gg \frac{x}{(\log\log x)^{1+o(1)}},$$

if hopeless, but obvious, conjectures concerning the distribution of primes are assumed. It is too early to predict the behaviour of $F_k(x)$ when k is not a power of 2.

(3) It should be noted that the constants implicit in (i) and (ii) above depend on k.

(4) Spiro has recently found an infinite set S, which includes the Fibonacci numbers, such that for any $\varepsilon > 0$, and any $k \in S$, $F_k(x) \gg_{k,\varepsilon} x(\log x)^{-\varepsilon}$. In particular, this holds for $k = 3$.

2. AN Ω RESULT FOR $G(n)$

In this section, we prove Theorem 1. Let N and D be defined by

$$\log N = \sum_{p \leq x} \log p,$$

and

$$\log D = \sum_{p \leq y} \log p,$$

where $y \leq x$, and x, y shall be chosen later. Utilising the explicit formula (1), we find that

$$G(N) \geq \prod_{p \mid D} \left(\frac{p^{v_p(N/D)} - 1}{p-1}\right).$$

If we denote by $\pi(x, q)$, the number of primes $p \leqslant x$, $p \equiv 1 \pmod q$, then it is easily seen that

$$V_p(N/D) = \pi(x, p) - \pi(y, p).$$

By elementary estimates, it follows that

$$\log G(N) \geqslant \sum_{p \leqslant y} \{\pi(x, p) - \pi(y, p)\} \log p + O(y).$$

We need the following lemmas:

LEMMA 1.

$$\sum_{p \leqslant x} \pi(x, p) \log p = (1 + o(1))x \qquad as \quad x \to \infty.$$

Proof. It is easy to see that

$$\sum_{q \leqslant x} \log(q-1) = \sum_{p \leqslant x} \{\pi(x, p) + \pi(x, p^2) + \cdots\} \log p,$$

where the sum on the left-hand side of the above equation ranges over primes $q \leqslant x$.

Using the trivial estimate,

$$\pi(x, p^\alpha) \leqslant x/p^\alpha,$$

we find

$$\sum_{\alpha \geqslant 2} \sum_{(\log x)^4 \leqslant p^\alpha \leqslant x} \pi(x, p^\alpha) \log p = O\left(\frac{x}{(\log x)^2}\right).$$

On the other hand, by the Brun–Titchmarsh inequality,

$$\sum_{\alpha \geqslant 2} \sum_{p^\alpha \leqslant (\log x)^4} \pi(x, p^\alpha) \log p = O\left(\frac{x}{\log x}\right).$$

This proves that

$$\sum_{p \leqslant x} \pi(x, p) \log p = x + O\left(\frac{x}{\log x}\right)$$

since by the prime number theorem

$$\sum_{q \leqslant x} \log(q-1) = x + O\left(\frac{x}{\log x}\right).$$

LEMMA 2. *There is an absolute constant $c>0$, such that*

$$\sum_{p \leqslant x^{1-\varepsilon}} \pi(x, p) \log p \gtrsim (1-c\varepsilon)x$$

as $x \to \infty$, for any $\varepsilon > 0$.

Proof. By Lemma 1, we see that it suffices to show that

$$\sum_{x^{1-\varepsilon} < p \leqslant x} \pi(x, p) \log p \ll \varepsilon x.$$

Indeed,

$$\sum_{x^{1-\varepsilon} < p \leqslant x} \pi(x, p) \log p \leqslant \log x \sum_{x^{1-\varepsilon} < p \leqslant x} \pi(x, p).$$

The last sum can be written as

$$\sum_{t \leqslant x^{\varepsilon}} N(x, t),$$

where $N(x, t)$ is the number of solutions p of $p-1=qt$, where p and q are prime numbers. By any sieve method,

$$N(x, t) = O\left(\frac{x}{\varphi(t) \log^2(x/t)}\right).$$

This estimate now yields the desired result, as

$$\sum_{t \leqslant x^{\varepsilon}} \frac{1}{\varphi(t)} = O(\varepsilon \log x).$$

We can now complete the proof of our theorem. We find by Lemma 1, that

$$\log G(N) \geqslant \sum_{p \leqslant y} \pi(x, p) \log p + O(y).$$

Choosing $y = x^{1-\varepsilon}$, yields, by Lemma 2,

$$\log G(N) \geqslant (1-c\varepsilon)x + O(x^{1-\varepsilon})$$

as $x \to \infty$. By the prime number theorem,

$$\log N = (1+o(1))x,$$

and hence,

$$\log G(N) \geqslant (1 - c\varepsilon + o(1)) \log N,$$

as desired.

3. Proof of Theorem 2

We want to establish that $\log G(n)$ has a distribution function for squarefree n. Recall that in [7], it was shown that for square-free n,

$$G(n) \leqslant \prod_{p \mid n} p^{V_p(n)}.$$

Therefore,

$$\log G(n) \leqslant \sum_{p \mid n} V_p(n) \log p.$$

Consider the set

$$L_1 = \{n \leqslant x \colon V_p(n) \geqslant 1 \text{ for some } p \mid n,\ p > (\log\log x)\}.$$

We begin by showing that $|L_1| = o(x)$. We need

LEMMA 3. *Let p be a prime. Then*

$$\sum_{\substack{q < x \\ q \equiv 1 \pmod{p}}} \frac{1}{q} \leqslant \frac{C(\log\log x + \log p)}{p}$$

for some absolute constant C.

Proof. See [3].

LEMMA 4.

$$|L_1| = O\left(\frac{x}{\log\log\log x}\right).$$

Proof. Clearly, the size of L_1 is bounded by

$$\sum_{\substack{q \equiv 1 \pmod{p} \\ p > (\log\log x)}} \frac{x}{pq}$$

and by Lemma 3, this is dominated by

$$\sum_{p > (\log\log x)} \frac{x(\log\log x + \log p)}{p^2} = O\left(\frac{x}{\log\log\log x}\right)$$

as desired.

We may therefore assume that $V_p(n)=0$ for all $p \mid n$, with $p>\log\log n$, so that for almost all squarefree n,

$$G(n) \leqslant \sum_{\substack{p \mid n \\ p<\log\log n}} V_p(n) \log p. \tag{2}$$

We show next that $V_p(n) \leqslant 2\,(\log\log n)/p$ for almost all n, uniformly for $p<\log\log n$.

LEMMA 5. *Let P be a set of primes satisfying $\sum_{p \leqslant x, p \in P}(1/p)=t_x$, with $t_x \to \infty$ as $x \to \infty$.*

Set $\omega_P(n)=\sum_{p \mid n, p \in P} 1$ and fix $\varepsilon>0$. The number of integers $n<x$ for which the inequalities:

$$(1-\varepsilon)t_x < \omega_P(n) < (1+\varepsilon)t_x$$

do not hold is less than

$$x \exp(-\eta t_x),$$

where $\eta=\eta(\varepsilon)$ is a positive constant which depends only on ε.

Proof. This result follows easily from the method of Hardy and Ramanujan [5] combined with Brun's method. As the derivation closely follows the method of [5], we suppress the details. (The referee informs us that a sharper version of this lemma appears in K. K. Norton, *Illinois J. Math.* **20** (1976), 681–705.)

COROLLARY. *Uniformly for $p<(\log\log x)^{1-\varepsilon}$,*

$$(1-\varepsilon)\frac{\log\log x}{p-1} < V_p(n) < (1+\varepsilon)\frac{\log\log x}{p-1}$$

is satisfied for all $n \leqslant x$ with at most $O(x/(\log\log x)^A)$ exceptions, for any $A>0$.

Proof. By Lemma 5,

$$(1-\varepsilon)\frac{\log\log x}{p-1} < V_p(n) < (1+\varepsilon)\frac{\log\log x}{p-1}$$

is satisfied for all $n \leqslant x$ apart from

$$O\left(x \exp\left(-\eta \frac{\log\log x}{p}\right)\right)$$

exceptions. We sum this over $p<(\log\log x)^{1-\varepsilon}$ to obtain the desired result.

LEMMA 6. *The number of $n \leqslant x$ divisible by a prime p in the range* $(\log\log x)^{1-\varepsilon} < p < \log\log x$ *is* $O(\varepsilon x)$.

Proof. The number of such $n \leqslant x$ is clearly bounded by

$$\sum_p{}' \frac{x}{p},$$

where the dash on the sum indicates that p is in the specified range.

Using the elementary fact

$$\sum_{p \leqslant x} \frac{1}{p} = \log\log x + B + O\left(\frac{1}{\log x}\right),$$

The result follows immediately.

From (2), we find from the preceding that apart from $O(\varepsilon x)$ squarefree numbers $n \leqslant x$, we have

$$\log G(n) \leqslant \sum_{\substack{p \mid n \\ p < (\log\log n)^{1-\varepsilon}}} (1 + o(1)) \frac{\log\log n}{p-1} \log p.$$

For the lower bound, set

$$d = \prod_{\substack{p \mid n \\ p < (\log\log n)^{1-\varepsilon}}} p.$$

With the exception of $o(x)$ of the $n \leqslant x$,

$$\sum_{\substack{q \mid n \\ q < (\log\log n)^{1-\varepsilon}}} 1 \ll \log\log\log\log n.$$

Therefore,

$$V_p(n/d) = V_p(n) + O(\log\log\log\log n).$$

Now, by Hölder's formula (1) and the corollary to Lemma 5, we get

$$\begin{aligned}
\log G(n) &\geqslant \sum_{\substack{p \mid n \\ p < (\log\log n)^{1-\varepsilon}}} (V_p(n/d) - 1) \log p \\
&\geqslant \sum_{\substack{p \mid n \\ p < (\log\log n)^{1-\varepsilon}}} V_p(n) \log p \\
&\quad + O((\log\log\log n)(\log\log\log\log n)^2) \\
&\geqslant (1 + o(1))(\log\log x) \sum_{\substack{p \mid n \\ p < (\log\log n)^{1-\varepsilon}}} (\log p)/(p-1)
\end{aligned}$$

except possibly for $o(x)$ of the squarefree $n \leqslant x$.

Let $S_\varepsilon(x)$ denote the set of integers n with $\sqrt{x} \leqslant n \leqslant x$ for which

$$\sum_{\substack{p \mid n \\ p > (\log\log n)^{1-\varepsilon}}} \frac{\log p}{p-1} > \varepsilon.$$

Then

$$\varepsilon\, |S_\varepsilon(x)| < \sum_{\sqrt{x} \leqslant n \leqslant x} \sum_{\substack{p \mid n \\ p > (\log\log n)^{1-\varepsilon}}} \frac{\log p}{p-1} \leqslant x \sum_{p > (\log\log x)^{1-\varepsilon}} \frac{\log p}{p(p-1)}$$
$$\leqslant x \frac{\log\log\log x}{(\log\log x)^{1-\varepsilon}}.$$

Thus, we have

$$\sum_{p \mid n} \frac{\log p}{p-1} - \varepsilon \leqslant \sum_{\substack{p \mid n \\ p < (\log\log n)^{1-\varepsilon}}} \frac{\log p}{p-1} < \sum_{p \mid n} \frac{\log p}{p-1}$$

except for $0(\varepsilon x)$ of the $n \leqslant x$. The function

$$\sum_{p \mid n} \frac{\log p}{p-1}$$

is additive, and by Theorem 5.1 of [2], it has a *continuous* distribution function. Thus, the same can be said of

$$\sum_{\substack{p \mid n \\ p < (\log\log n)^{1-\varepsilon}}} \frac{\log p}{p-1}$$

and of $(\log G(n))/\log\log n$. This completes the proof of Theorem 2.

4. An Upper Bound for $F_k(x)$

Let

$$F_k = \{n \leqslant x\colon G(n) = k\}$$

and denote by p_n the smallest prime divisor of n. Let $\varepsilon > 0$ be arbitrary. Let us write

$$F_k = S_1 \cup S_2,$$

where in S_1, $p_n < (\log\log x)^{1-\varepsilon}$ and in S_2, $p_n > (\log\log x)^{1-\varepsilon}$. We use Brun's method to estimate $|S_1|$.

LEMMA 7. *Let c be a fixed positive integer and suppose that $p < (\log\log x)^{1-\varepsilon}$. Then*

$$\operatorname{card}(n \leqslant x\colon p \mid n, \textit{ and } V_p(n) \leqslant c)$$

is

$$O\left(\frac{x}{p^{1+c}} (\log\log x)^c \exp\left(-\frac{\log\log x}{p}\right)\right).$$

Proof. By Brun's sieve, the number with $V_p(n) = c$ is

$$\ll \frac{x}{p} \sum_{\substack{q_1, \ldots, q_c \equiv 1 \pmod p \\ q_1 \cdots q_c < x/p}} \frac{1}{q_1 \cdots q_c} \prod_{\substack{r \equiv 1 \pmod p \\ r < \xi}} \left(1 - \frac{1}{r}\right),$$

where $\xi = x^{1/\log\log x}$. It follows that this is bounded by

$$\ll \frac{x}{p} \left\{ \sum_{\substack{q \leqslant x \\ q \equiv 1 \pmod p}} \frac{1}{q} \right\}^c \exp\left(-\frac{\log\log x}{p}\right).$$

By well-known estimates (see [3]) it follows that the above is

$$\ll \frac{x}{p^{1+c}} (\log\log x)^c \exp\left(-\frac{\log\log x}{p}\right).$$

Proof of Theorem 3(i). Let $p = p_n$ and suppose that $V_p(n) \geqslant k+1$. It follows from Theorem 1.1 of [8] that

$$G(n) \geqslant \frac{p^{V_p(n)} - 1}{p-1} \geqslant 2^{V_p(n) - 1} \geqslant 2^k > k.$$

Hence, if $G(n) = k$, we must have $V_p(n) \leqslant k$. By Lemma 7 (or by the corollary to Lemma 5), it follows that for any $A > 0$,

$$|S_1| = O\left(\frac{x}{(\log\log x)^A}\right).$$

Now we write

$$S_2 = S_3 \cup S_4,$$

where S_3 consists of squarefree numbers and S_4 consists of those elements with a squared prime factor. As $p_n > (\log\log x)^{1-\varepsilon}$, for elements of S_2, we find

$$|S_4| \leqslant \sum_{p > (\log\log x)^{1-\varepsilon}} \frac{x}{p^2} = O\left(\frac{x}{(\log\log x)^{1-\varepsilon}}\right).$$

It remains to estimate S_3. If $n \in S_3$, then for any $p \mid n$, $V_p(n) \leqslant 1$. For if $V_p(n) \geqslant 2$ then

$$k = G(n) \geqslant \frac{p^{V_p(n)} - 1}{p-1} \geqslant p_n > (\log\log x)^{1-\varepsilon}$$

a contradiction for sufficiently large x.

For the sake of convenience, we introduce for every natural number n, *the graph of* n, denoted by $g(n)$. The vertices of this graph are the prime divisors of n, and two prime divisors p, q of n are joined if $p \mid (q-1)$. If $g(n)$ has connected components given by $g(n_i)$, then it follows from Hölder's formula that

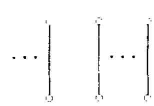
$$G(n) = \prod G(n_i)$$

when n is squarefree.

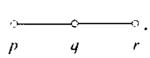
The fact that $V_p(n) \leqslant 1$, means that for $n \in S_3$, $g(n)$ consists of disjoint segments of the type

If the subgraph

does not appear in $g(n)$, then $g(n)$ consists of



in which case $G(n)$ is a power of 2, contrary to hypothesis. Hence, for $n \in S_3$, $g(n)$ contains the subgraph



Thus,

$$|S_3| \leqslant \sum_{\substack{p > (\log\log x)^{1-\varepsilon} \\ q \equiv 1 (\mathrm{mod}\, p) \\ r \equiv 1 (\mathrm{mod}\, q)}} \frac{x}{pqr}.$$

Lemma 3 applies, and we get

$$|S_3| \ll \sum_{\substack{p > (\log\log x)^{1-\varepsilon} \\ q \equiv 1 (\mathrm{mod}\, p)}} \frac{x}{pq^2} (\log\log x + \log q) = \Sigma_1 + \Sigma_2 \qquad \text{(say)}.$$

Then,

$$|\Sigma_1| \ll \sum_{\substack{p > (\log\log x)^{1-\varepsilon} \\ t \geqslant 1}} \frac{x \log\log x}{p^3 t^2}.$$

The latter sum is easily seen to be

$$O\left(\frac{x}{(\log\log x)^{1-2\varepsilon}}\right).$$

The term Σ_2 with the $(\log q)/q^2$ term is handled similarly. This completes the proof of (i).

Remark. The above argument shows that

$$|S_1| = O\left(\frac{x}{(\log\log x)^A}\right)$$

for any $k > 1$, and any $A > 0$. It is equally clear that

$$|S_4| = O\left(\frac{x}{(\log\log x)^{1-\varepsilon}}\right)$$

for any $k \geqslant 1$.

5. The Asymptotic Formula for $F_k(x)$, $k = 2^a$

By the remarks following the proof of (i), it is clear that we need to establish an asymptotic formula for the number of squarefree $n \leqslant x$, whose graph $g(n)$ has exactly a connected components of the form

together with a finite set of disjoint vertices. The case $k = 1$, when $a = 0$, has already been dealt with in Erdös [3]. We begin by considering the case $k = 2$, corresponding to $a = 1$. We must enumerate squarefree numbers $n \leqslant x$ of the form $n = pqm$, where $(m, \varphi(m)) = 1$, $q \equiv 1 \pmod p$ and $(pm, \varphi(pm)) = 1$, $(qm, \varphi(qm)) = 1$. For any fixed pair of primes p, q, with $q \equiv 1 \pmod p$, let $A_{pq}(x)$ denote the number of squarefree $n \leqslant x$ satisfying the above conditions. It is also clear that we need only consider $n \in S_3$, by the remarks at the end of the last section. Hence we may take $p > (\log\log x)^{1-\varepsilon}$, and assume that all prime divisors of m are greater than $(\log\log x)^{1-\varepsilon}$.

LEMMA 8.

$$\sum_{p > (\log\log x)^{1+\epsilon}} A_{pq}(x) = O\left(\frac{x}{(\log\log x)^{\epsilon}}\right).$$

Proof. By Lemma 3, we have

$$\sum_{p > (\log\log x)^{1+\epsilon}} \frac{x}{pq} \ll \sum_{p > (\log\log x)^{1+\epsilon}} x\,\frac{(\log\log x + \log p)}{p^2},$$

and this latter sum is $O(x/(\log\log x)^{\epsilon})$ by an easy computation.

The lemma shows that we may take p satisfying $(\log\log x)^{1-\epsilon} < p < (\log\log x)^{1+\epsilon}$ in the following discussion. We need:

LEMMA 9. *Let p be a prime $< (\log x)^c$ (where c is an arbitrary constant). Then*

$$\sum_{\substack{q < x \\ q \equiv 1 \pmod{p}}} \frac{1}{q} = \frac{\log\log x}{p-1} + O\left(\frac{\log p}{p}\right).$$

Proof. This is a straight forward consequence of the Siegel–Walfisz theorem and the Brun–Titchmarsh inequality on the number of primes in an arithmetic progression.

Remark. The referee informs us that the result is true without the restriction on p (see, e.g., the paper of Norton mentioned earlier or C. Pomerance, *J. Reine Angew. Math.* **293/294** (1977), 217–222).

COROLLARY. *Let $\xi = x^{1/\log\log x}$. Then*

$$\sum_{\substack{\xi < q < x \\ q \equiv 1 \pmod{p}}} \frac{1}{q} = O\left(\frac{\log\log\log x}{p}\right)$$

uniformly for $p < (\log\log x)^A$, for any constant $A > 0$.

Proof. As

$$\sum_{\substack{q < \xi \\ q \equiv 1 \pmod{p}}} \frac{1}{q} = \frac{\log\log x - \log\log\log x}{p-1} + O\left(\frac{\log p}{p}\right)$$

the result follows easily from Lemma 9.

We may also take $q < \xi$ as our next lemma shows.

LEMMA 10.

$$\sum_{\substack{q>\xi \\ q\equiv 1 (\operatorname{mod} p) \\ p>(\log\log x)^{1-\varepsilon}}} A_{pq}(x) \ll \left(\frac{x}{(\log\log x)^{1-\varepsilon}}\right).$$

Proof. Clearly $A_{pq}(x) \leqslant x/pq$. Since $p < (\log\log x)^{1+\varepsilon}$, the corollary to Lemma 9 implies that the above sum is bounded by

$$\sum_{\substack{x\geqslant q>\xi \\ q\equiv 1 (\operatorname{mod} p) \\ p>(\log\log x)^{1-\varepsilon}}} \frac{x}{pq} = O\left(\sum_{p>(\log\log x)^{1-\varepsilon}} \frac{x \log\log\log x}{p^2}\right)$$
$$= O\left(\frac{x}{(\log\log x)^{1-\varepsilon}}\right).$$

We state the following version of Brun's sieve for the sake of convenience.

LEMMA 11. *The number of* $n \leqslant x$ *not divisible by any of the primes* $p_1, ..., p_s$, *where* $p_i < \xi$ *is*

$$(1+o(1))x \cdot \prod_{i=1}^{s} \left(1-\frac{1}{p_i}\right).$$

Proof. See Halberstam and Richert [4].

Below, we shall sometimes write l_m for $\log_m x$, the m-fold iterate of $\log x$.

LEMMA 12. *For* $(\log\log x)^{1-\varepsilon} < p < (\log\log x)^{1+\varepsilon}$,

$$\sum_{q<\xi} A_{pq}(x) = (1+o(1)) \frac{xe^{-\gamma}\log\log x}{p^2 \log\log\log x}$$
$$\times \exp\left(-\frac{\log\log x}{p}\right) + O\left(\frac{x l_3 l_2}{p^3}\right).$$

Proof. By Lemma 11, the number of $pqm \leqslant x$ with all the prime factors of $m > (\log\log x)^{1-\varepsilon}$ and no prime factor $s < \xi$ which is $\equiv 1 \pmod p$ is

$$(1+o(1)) \frac{x}{pq} \prod_{r<(\log\log x)^{1-\varepsilon}} \left(1-\frac{1}{r}\right) \prod_{\substack{s<\xi \\ s\equiv 1 (\operatorname{mod} p)}} \left(1-\frac{1}{s}\right)$$

and by familiar estimates of number theory, we find

$$(1-\varepsilon)\, A_{pq}(x) \lesssim (1+o(1))\, \frac{x}{pq}\, \frac{e^{-\gamma}}{(\log\log\log x)} \exp\left(-\frac{\log\log x}{p}\right).$$

If we exclude those $m \leqslant x/pq$ which have a prime factor $s \equiv 1 \pmod q$ or a prime factor $r \equiv 1 \pmod p$ with $r > \xi$, we find that

$$A_{pq}(x) \gtrsim (1+o(1))\, \frac{xe^{-\gamma}}{pq \log\log\log x} \exp\left(-\frac{\log\log x}{p}\right) - E_{pq}(x),$$

where

$$E_{pq}(x) \leqslant \frac{x}{pq} \left\{ \sum_{s \equiv 1 (q)} \frac{1}{s} + \sum_{\substack{r \equiv 1 (p) \\ x \geqslant r \geqslant \xi}} \frac{1}{r} \right\}$$

$$\ll \frac{x}{pq} \left\{ \frac{l_2 + \log q}{q} + \frac{l_3}{p} \right\}$$

by Lemma 3 and the corollary to Lemma 9. Summing over $q < \xi$, we find

$$\sum_{q<\xi} E_{pq}(x) = O\left(\frac{x \log\log x}{p^3}\right) + O\left(\frac{xl_3}{p^3}(l_2 + \log p)\right),$$

and therefore

$$\sum_{q<\xi} A_{pq}(x) = (1+o(1))\, \frac{xe^{-\gamma} \log\log x}{p^2 \log\log\log x} \times \exp\left(-\frac{\log\log x}{p}\right) + O\left(\frac{xl_3 l_2}{p^3}\right).$$

This completes the proof of the lemma.

Proof of Theorem 3(ii). We can now give the asymptotic formula for $F_2(x)$. Indeed, by Lemma 12, we have

$$F_2(x) = \sum_p{}' (1+o(1))\, \frac{xe^{-\gamma} \log\log x}{p^2 \log\log\log x} \times \exp\left(-\frac{\log\log x}{p}\right) + O\left(\frac{x}{l_2^{1-2\varepsilon}}\right),$$

where the dash on the sum indicates that

$$(\log\log x)^{1-\varepsilon} < p < (\log\log x)^{1+\varepsilon}.$$

Let $y = \log\log x$. By partial summation, the sum

$$\sum \frac{1}{p^2} \exp(-y/p)$$

can be replaced by the integral

$$\int_{y^{1-\varepsilon}}^{y^{1+\varepsilon}} \exp\left(-\frac{y}{t}\right) \frac{dt}{t^2 \log t}$$

Making the substitution $u = y/t$, it becomes transformed into

$$\frac{1}{y} \int_{y^{-\varepsilon}}^{y^{\varepsilon}} \exp(-u) \frac{du}{(\log y/u)}$$

and integrating by parts shows that it is

$$\frac{1}{y \log y} + O\left(\frac{1}{y(\log y)^2}\right).$$

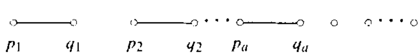
Thus,

$$F_2(x) = (1 + o(1)) \frac{e^{-\gamma} x}{(\log\log\log x)^2}.$$

In the general case of $k = 2^a$, the main contribution comes from squarefree $n < x$ which have the form

$$n = (p_1, q_1) \cdots (p_a q_a) m,$$

with $q_i \equiv 1 \pmod{p_i}$, $(m, \varphi(m)) = 1$ and the graph of n is isomorphic to



where the last set of disjoint vertices correspond to the prime divisors of m. By the preceding results, we make take $(\log\log x)^{1-\varepsilon} < p_i < (\log\log x)^{1+\varepsilon}$, for $1 \leqslant i \leqslant a$. Furthermore, we may take $q_i < \xi$ for $1 \leqslant i \leqslant a$, by the method of proof of Lemma 10. Hence, by Brun's sieve, the number of integers $n \leqslant x$ of the form

$$n = (p_1 q_1) \cdots (p_a q_a) m.$$

with $(m, \varphi(m)) = 1$, $q_i \equiv 1 (\bmod\, p_i)$, $1 \leqslant i \leqslant a$, and no other relations in $g(n)$, is

$$(1 + o(1)) \frac{x e^{-\gamma}}{(p_1 q_1) \cdots (p_a q_a)} \frac{1}{\log\log\log x} \exp\left(-l_2 \left(\frac{1}{p_1} + \cdots + \frac{1}{p_a}\right)\right)$$

as $x \to \infty$. We sum this over the distinguished pairs of primes (p_i, q_i) to obtain

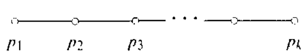
$$(1+o(1))\frac{xe^{-\gamma}(\log\log x)^a}{a!\,(\log\log\log x)}$$
$$\times\left(\exp\left(-\log\log x\left(\frac{1}{p_1}+\cdots+\frac{1}{p_a}\right)\right)\Big/ p_1^2\cdots p_a^2\right),$$

where the $a!$ is to take into account the different orderings of the a prime pairs. In order to evaluate the asymptotic behaviour of the sum of this expression over the primes p_i in the interval $(\log\log x)^{1-\varepsilon} < p_i < (\log\log x)^{1+\varepsilon}$, we again use partial summation to write the sum as a product of **a** integrals. Each of the integrals is of the type considered in the case $\mathbf{a}=1$. Applying the same method to each in turn, we find that

$$F_{2^a}(x) = (1+o(1))\frac{c(a)x}{(\log\log\log x)^{a+1}}.$$

In fact, the above yields $c(a) = e^{-\gamma}/a!$, which holds for $a \geqslant 0$. This completes the proof of Theorem 3(ii).

We include here the following curious observation. Suppose the graph of n, $g(n)$ has connected component



consisting of a chain of length k. We claim that $G(n) = F_k$, where F_k denotes the kth Fibonacci number. Indeed, if $n = p_1 \cdots p_k$, then

$$G(n) = \sum_{\substack{d \mid n \\ p_1 \mid d}} \prod_{p \mid d}\left(\frac{p^{\nu_p(n/d)}-1}{p-1}\right) + \sum_{\substack{d \mid n \\ p_1 \nmid d}} \prod_{p \mid d}\left(\frac{p^{\nu_p(n/d)}-1}{p-1}\right).$$

The second sum is $G(n/p_1)$, whereas the product in the first sum vanishes unless $d \mid (n/p_2)$. As $p_1 \mid d$ in the first sum, we find that this sum is $G(n/p_1 p_2)$. An easy induction argument utilising $G(n) = G(n/p_1) + G(n/p_1 p_2)$ now gives the result.

6. Concluding Remarks

If n is squarefree and $G(n) = 3$, then it is easy to see that $n = pqrm$ where $q \equiv 1 \pmod p$, $r \equiv 1 \pmod q$, $(m, \varphi(m)) = 1$ and no other relations hold. If there are at least $cx/(\log x)^2$ primes $p \leqslant x$ such that $2p+1$ is also prime, then it is easy to see from the preceding discussion that for at least

$$\frac{cx}{(\log\log x)^{1-\varepsilon}}$$

numbers $n \leqslant x$, we have $G(n) = 3$. This should not be expected for all values of k.

Concerning the size of $G(n)$ for squarefree n, we ask the following: is it true $G(n) = o(\varphi(n))$ as n runs over squarefree integers? (see the Note added in proof). In this connection, it will be recalled that in [7], it was shown that if

$$f(n) = \prod_{p \mid n} (n, p-1),$$

then

$$G(n) \leqslant f(n)$$

for all squarefree n. It is curious to note that $G(n) = \varphi(n)$ can hold only for finitely many squarefree integers. Indeed, from the above, we find that for each $p \mid n$, $(p-1)$ also divides n in such a case. We claim that n must be composed of $2, 3, 6, 43$ only and a quick computation yields that $n = 2, 6, 42, 1806$ are the only solutions. To see this, suppose that a prime $p \neq 2, 3, 7, 43$ divides such an n. Then letting p be the least such prime, we find $(p-1) \mid n$. But then $p-1$ must be composed of 2, 3, 7 or 43.

An immediate check of the corresponding squarefree products gives the result. This elegant elementary result appeared earlier (see Dyer–Bennet [1]) in a different context. It is likely that our question has an affirmative solution.

We have proved that for n squarefree,

$$\frac{\log G(n)}{\log\log n}$$

has a continuous distribution function. The above function has the same distribution as

$$\frac{\log f(n)}{\log\log n}.$$

It would be desirable to obtain nontrivial upper and lower bounds for

$$\sum_{n \leqslant x} f(n)$$

and

$$\sum_{n \leqslant x} \mu^2(n)\, G(n).$$

The second sum would be more difficult. Using the methods of [7] and [8] and Theorem 2, it follows that for any $c > 0$,

$$x(\log x)^c \ll \sum_{n \leqslant x} \mu^2(n)\, G(n) \ll x^2/\log\log\log x.$$

After this paper was written, Ram Murty and Srinivasan proved that

$$\sum_{n \leqslant x} \mu^2(n)\, G(n) \ll x^2(\log x)^{-c\log\log\log x}$$

for some $c > 0$. Carl Pomerance informs us that for $c > \frac{15}{23}$, he can prove that

$$x^{1+c} \ll \sum_{n \leqslant x} \mu^2(n)\, G(n) \ll x^{2-(\log\log\log x)/(\log\log x)}.$$

He can also give a heuristic argument to show that the upper bound is essentially best possible.

Note added in proof. The fact that $G(n) = o(\varphi(n))$ has been subsequently proved by M. Ram Murty and S. Srinivasan. In a forthcoming paper entitled, "On the number of groups of squarefree order," they show that for square free n,

$$G(n) = O(\varphi(n)/(\log n)^{A\log\log\log n})$$

for some constant $A > 0$.

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