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On isomorphisms of finite Cayley graphs— a survey

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Abstract

The isomorphism problem for Cayley graphs has been extensively investigated over the past 30 years. Recently, substantial progress has been made on the study of this problem, many long-standing open problems have been solved, and many new research problems have arisen. The results obtained, and methods developed in this area have also effectively been used to solve other problems regarding finite vertex-transitive graphs. The methods used in this area range from deep group theory, including the finite simple group classification, through to combinatorial techniques. This article is devoted to surveying results, open problems and methods in this area.
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The isomorphism problem for graphs, that is to decide whether or not two given graphs are isomorphic, is a fundamental problem in graph theory. Here we investigate this problem for Cayley graphs.

Cayley graphs are those whose vertices may be identified with elements of groups and adjacency relations may be defined by subsets of the groups. Cayley graphs stem from a type of diagram now called a *Cayley color diagram*, which was introduced by A. Cayley in 1878 as a graphic representation of abstract groups. Cayley color diagrams were used by Coxeter and Moser [18] to investigate groups given by generators and relations. A Cayley color diagram is a directed graph with colored edges, and gives rise to a Cayley graph if the colors on the edges are ignored. Cayley color diagrams were generalized to Schreier coset diagrams by Schreier in 1927, and both were investigated as *graphs* by Sabidussi [105]. Cayley graphs and their generalizations—vertex-transitive

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graphs have been an active topic in algebraic graph theory for a long time, see, for example, [8,13,101].

Cayley graphs contain long paths and have many other nice combinatorial properties (see [8]). They have been used to construct extremal graphs: see [85,86] for the constructions of Ramanujan graphs and expanders, see [2,88] for the constructions of graphs without short cycles. They have also been used to construct other combinatorial structures: see [30,47] for the constructions of various communication networks, and see [11,15] difference sets in design theory.

Cayley graphs have been used to analyze algorithms for computing with groups, see [8]. For infinite groups, Cayley graphs provide convenient metric diagrams for words in the corresponding groups, and underlie the study of growth of groups, see [8,41]. *Cayley maps* are Cayley graphs embedded into some surfaces, and provide pictorial representations of groups and group actions on surfaces. They have been extensively studied, see [12,44,73,107].

1. Cayley graphs and their isomorphism problem

Given a group G and a subset $S \subset G$, the *associated Cayley graph* $\text{Cay}(G, S)$ is the digraph (directed graph) Γ with vertex set G and with x connected to y if and only if $yx^{-1} \in S$. By definition, $\text{Cay}(G, S)$ has out-valency $|S|$, and further we observe the following simple properties:

- (i) The subset S contains the identity of G if and only if the Cayley graph $\text{Cay}(G, S)$ contains a self-loop at every vertex. Thus we will assume that S does not contain the identity.
- (ii) A Cayley graph $\text{Cay}(G, S)$ is connected if and only if $\langle S \rangle = G$, so that $\text{Cay}(\langle S \rangle, S)$ is a component of $\text{Cay}(G, S)$, where $\langle S \rangle$ is the subgroup of G generated by elements in S .
- (iii) If for any element $s \in S$, the inverse s^{-1} also lies in S , then the adjacency is symmetric and thus the Cayley graph $\text{Cay}(G, S)$ may be viewed as an *undirected* graph.

A fundamental problem naturally arises:

Isomorphism problem for Cayley graphs: Given two Cayley graphs $\text{Cay}(G, S)$ and $\text{Cay}(H, T)$, determine whether or not $\text{Cay}(G, S) \cong \text{Cay}(H, T)$.

Let Γ be a digraph with vertex set V and edge set E . Then Γ is called a *Cayley graph* if there exist a group G and a subset $S \subset G$ such that $\Gamma \cong \text{Cay}(G, S)$, that is, we may identify vertices of Γ with elements of the group G and edges of Γ with pairs (x, y) of elements $x, y \in G$ such that $yx^{-1} \in S$. In the case, the pair (G, S) is called a *Cayley representation* of the graph Γ .

We observe that a Cayley graph may have many different Cayley representations, that is, for a given graph Γ , there may exist different pairs (G, S) of groups G and subsets $S \subset G$ such that $\Gamma \cong \text{Cay}(G, S)$. For example, let $\Gamma = K_n$, the undirected complete graph of order n . Then $\Gamma \cong \text{Cay}(G, S)$, where G is an arbitrary group of order n , and S is

the set of all non-identity elements of G . Thus, the number of Cayley representations for Γ equals the number of non-isomorphic groups of order n .

Two Cayley representations (G, S) and (H, T) of a digraph $\Gamma = (V, E)$ is said to be *equivalent* if there exists a permutation $\sigma \in \text{Sym}(V)$ such that σ induces an isomorphism from the group G to the group H , and $S^\sigma = T$. Therefore, if (G, S) and (H, T) are equivalent Cayley representations of a graph Γ , then G and H are isomorphic groups. In particular, two Cayley representations (G, S) and (G, T) of a graph Γ are equivalent if and only if $S^\sigma = T$ for some $\sigma \in \text{Aut}(G)$.

Cayley representation problem of graphs: Given a Cayley graph Γ , determine all Cayley representations of Γ , that is, determine all pairs (G, S) of groups G and subsets $S \subset G$ such that $\Gamma \cong \text{Cay}(G, S)$.

A permutation on vertex set V is called an *automorphism* of Γ if it preserves the adjacency relation of Γ . All automorphisms of Γ form a group, called the *automorphism group* of Γ and denoted by $\text{Aut } \Gamma$. Assume that $\Gamma = \text{Cay}(G, S)$ for some group G , and identify V with G . Then $\text{Aut } \Gamma$ contains a subgroup \hat{G} , induced by right multiplication of the group G , that is, for $\hat{g} \in \hat{G}$, we define

$$\hat{g}: x \rightarrow xg \quad \text{for all } x \in V.$$

It is easily shown that, for any two vertices $x, y \in V$, the element $\hat{g} = \hat{x}^{-1}\hat{y} \in \hat{G}$ maps x to y , and thus \hat{G} acts transitively on V . In particular, $\text{Aut } \Gamma$ is transitive on V , and Cayley graphs are vertex-transitive graphs. It then follows that if a Cayley graph $\text{Cay}(G, S)$ is disconnected, then all components of $\text{Cay}(G, S)$ are isomorphic to the smaller Cayley graph $\text{Cay}(\langle S \rangle, S)$. Moreover, for any $x, y \in V$, $\hat{g} = \hat{x}^{-1}\hat{y}$ is the unique element of \hat{G} which maps x to y , and so \hat{G} is a *regular* subgroup of $\text{Aut } \Gamma$. This leads to a criterion for a digraph to be a Cayley graph.

Proposition 1.1. *A digraph Γ is a Cayley graph of a group G if and only if $\text{Aut } \Gamma$ contains a subgroup which is isomorphic to G and regular on V .*

This proposition was first proved by Sabidussi [105], also see [13, Lemma 16.3]. Thus, to solve the Cayley representation problem of digraphs, some relevant problems occur.

Problem 1.2. Let Γ be a digraph with vertex set V .

- (1) Determine the full automorphism group $\text{Aut } \Gamma$, and determine whether $\text{Aut } \Gamma$ has regular subgroups on V .
- (2) Assume that Γ is a Cayley graph. Determine all regular subgroups on V of $\text{Aut } \Gamma$.
- (3) Assume that Γ is a Cayley graph of a group G . Determine all inequivalent subsets S under $\text{Aut}(G)$ such that $\Gamma \cong \text{Cay}(G, S)$.

2. Isomorphic Cayley graphs over different groups

In this section, we study some classes of graphs and digraphs which have Cayley representations over non-isomorphic groups. The first is about complete d -partite

graphs. For a positive integer n , denote by \mathbb{S}_n the *symmetric group* of degree n , that is, the group of all permutations on $\{1, 2, \dots, n\}$.

Example 2.1. For positive integers m and d , denote by $K_{m;d}$ the undirected *complete d-partite graph* such that each part has size m .

Let G be a group of order n which has a subgroup G_0 of order m . Let S be the set of elements of G which are not in G_0 . Then the Cayley graph $\Gamma := \text{Cay}(G, S)$ is isomorphic to a complete d -partite graph $K_{m;d}$, where $n = dm$. It is clear that $\text{Aut } \Gamma \cong \mathbb{S}_m \wr \mathbb{S}_d$, the *wreath product* of \mathbb{S}_m by \mathbb{S}_d (see [21, Section 2.6] for definition). Thus the complete d -partite graph $K_{m;d}$ can be expressed as a Cayley graph for each group of order n which has a subgroup of order m .

Conversely, suppose that (G, S) is a Cayley representation of the graph $K_{m;d}$ for some group G and some subset $S \subset G$. It is clear that the d -parts of $K_{m;d}$ form an $\text{Aut } \Gamma$ -invariant partition \mathcal{B} of the vertex set. In particular, \mathcal{B} is \hat{G} -invariant. Thus for $B \in \mathcal{B}$, the setwise stabilizer \hat{G}_B is a subgroup of \hat{G} of order m , where $\hat{G}_B = \{\hat{g} \in \hat{G} \mid B^{\hat{g}} = B\}$. Therefore, we have a characterization for such groups G .

Proposition 2.2. *A group G has a Cayley graph isomorphic to $K_{m;d}$ if and only if G is of order md and has a subgroup of order m .*

The statement in Proposition 2.2 has an equivalent statement in language of permutation groups:

Proposition 2.2'. *Let $X \cong \mathbb{S}_m \wr \mathbb{S}_d$ be a permutation group on a set Ω of size md , in its natural imprimitive action. Then a group G can be embedded in X as a regular subgroup on Ω if and only if G has a subgroup of order m .*

The next example is about a Cartesian product of cycles of same length. For given two digraphs $\Gamma_1 = (V_1, E_1)$ and $\Gamma_2 = (V_2, E_2)$, the *Cartesian product* $\Gamma_1 \times \Gamma_2$ of Γ_1 and Γ_2 is the digraph with vertex set $V_1 \times V_2$ such that $((a_1, a_2), (b_1, b_2))$ is an edge if and only if either $(a_1, b_1) \in E_1$ and $a_2 = b_2$, or $(a_2, b_2) \in E_2$ and $a_1 = b_1$. For two groups F and G , by $F.G$ we mean an extension of F by G , and by $F \bowtie G$ we mean a semi-direct product of F by G , that is, a split extension. For a positive integer n , by C_n we mean an undirected *cycle* of length n . As usual, we denote by \mathbb{Z}_n a cyclic group of order n ; while by D_{2n} we mean a dihedral group of order $2n$, that is, $D_{2n} \cong \langle x, y \mid x^n = y^2 = 1, y^{-1}xy = x^{-1} \rangle$.

Example 2.3. (1) Let $G = \langle a \rangle \cong \mathbb{Z}_{2n}$, and let $H = \langle x, y \mid x^n = y^2 = 1, y^{-1}xy = x^{-1} \rangle$. Then $\text{Cay}(G, \{a, a^{-1}\}) \cong C_{2n} \cong \text{Cay}(H, \{y, xy\})$, and $\text{Aut}(C_{2n}) \cong D_{4n}$.

(2) Let Γ be a graph isomorphic to C_n^d , a Cartesian product of d copies of C_n , where $n = 3$ or $n \geq 5$. Then

$$\text{Aut } \Gamma \cong D_{2n} \wr \mathbb{S}_d = D_{2n}^d \cdot \mathbb{S}_d.$$

If n is odd, then Γ can be expressed as a Cayley graph of a group isomorphic to \mathbb{Z}_n^d ; while if n is even, then Γ can be expressed as a Cayley graph of an arbitrary group with the form $G_1 \times G_2 \times \cdots \times G_d$, where $G_i \cong \mathbb{Z}_n$ or D_n .

In the above example, the case where $m=2$ or 4 is omitted since it is quite special, which corresponds the so-called *hypercubes*.

Example 2.4. (1) Let $\Gamma \cong Q_3$, the cube of dimension 3. Then Γ can be represented as a Cayley graph for three groups: \mathbb{Z}_2^3 , $\mathbb{Z}_4 \times \mathbb{Z}_2$, and D_8 . In fact, let $G = \langle a, b, c \rangle \cong \mathbb{Z}_2^3$, $H = \langle x \rangle \times \langle y \rangle \cong \mathbb{Z}_4 \times \mathbb{Z}_2$, and $K = \langle u, v \mid u^4 = v^2 = 1, v^{-1}uv = u^{-1} \rangle$. Then $Cay(G, \{a, b, c\})$, $Cay(H, \{x, x^{-1}, y\})$ and $Cay(K, \{u, u^{-1}, v\})$ are all isomorphic to Q_3 .

(2) Let $\Gamma \cong Q_d$, the hypercube of dimension d . Then

$$\text{Aut } \Gamma \cong \mathbb{Z}_2^d \cdot S_d.$$

A natural problem is to determine all pairs (G, S) of groups G and subsets $S \subset G$ such that $Cay(G, S) \cong Q_d$. If a group G is of the form $G_1 \times \cdots \times G_l$ such that $G_i \cong \mathbb{Z}_2$, \mathbb{Z}_4 , or D_8 , and $|G|=2^d$, then there exists $S \subset G$ such that $Cay(G, S) \cong Q_d$. The problem of classifying all regular subgroups of $\text{Aut } \Gamma$ is unsettled.

Dixon [20] enumerated the Cayley graphs isomorphic to small hypercubes, and in particular, proved that for $d=3, 4, 5$ and 6 , there are, respectively, 4, 14, 45 and 238 such graphs.

The following example is about lexicographic product of graphs. For given digraphs $\Gamma_1 = (V_1, E_1)$ and $\Gamma_2 = (V_2, E_2)$, the *lexicographic product* $\Gamma_1[\Gamma_2]$ of Γ_2 by Γ_1 is the digraph with vertex set $V_1 \times V_2$ such that $((a_1, a_2), (b_1, b_2))$ is an edge if and only if either $(a_1, b_1) \in E_1$ or $a_1 = b_1$ and $(a_2, b_2) \in E_2$. For a digraph $\Gamma = (V, E)$, its *complement* $\bar{\Gamma}$ is the digraph with vertex set V such that, for any $u, v \in V$, u is connected in $\bar{\Gamma}$ to v if and only if u is not connected in Γ to v .

Example 2.5. Let $G = \langle a \rangle \cong \mathbb{Z}_{n^2}$, and let $H = \langle b, c \rangle \cong \mathbb{Z}_n \times \mathbb{Z}_n$. Let $S = a\langle a^n \rangle$ and $T = b\langle c \rangle$. Then

$$Cay(G, S) \cong C_n[\bar{K}_n] \cong Cay(H, T),$$

where C_n is a directed cycle of length n , and K_n is a complete graph with n vertices. Similarly, let $S' = a\langle a^n \rangle \cup a^{-1}\langle a^n \rangle$ and $T' = b\langle c \rangle \cup b^{-1}\langle c \rangle$. Then

$$Cay(G, S') \cong C_n[\bar{K}_n] \cong Cay(H, T'),$$

where C_n is an undirected cycle of length n .

The example shows that a circulant digraph $C_n[\bar{K}_n]$ may be a Cayley graph of a non-cyclic group. A characterization for such an isomorphism in the prime-square case was obtained by Joseph [55]. The result of Joseph is extended by Morris [92] to circulant graphs of odd prime-power as follows: Let p be an odd prime, and let $n = n_1 + n_2 + \cdots + n_k$. Then there exists a graph Γ which is a Cayley graph of \mathbb{Z}_{p^n} and $\mathbb{Z}_{p^{n_1}} \times \cdots \times \mathbb{Z}_{p^{n_k}}$ if and only if $\Gamma = \Gamma_1[\Gamma_2[\dots[\Gamma_k]\dots]]$, where Γ_i is a circulant of order p^{n_i} .

We end this section with a remark: Recent achievements in permutation groups and finite simple groups, for example, maximal factorizations of finite almost simple groups given in [82], provide a powerful tool for investigating Cayley graphs. A characterization of transitive subgroups of primitive permutation groups is given in [83]. This can be used to construct various very interesting examples of isomorphic Cayley graphs over non-isomorphic groups.

3. Cayley isomorphisms

In this section, we consider the isomorphism problem of Cayley graphs over the same group. Let G be a group, and let $\Gamma = \text{Cay}(G, S)$ for some subset $S \subset G$. Let σ be an automorphism of G . Then σ naturally acts on the vertex set $V = G$. Let $T = S^\sigma$. Then it is easily shown that σ induces an isomorphism from $\text{Cay}(G, S)$ to the Cayley graph $\text{Cay}(G, T)$. Such an isomorphism is called a *Cayley isomorphism*. However, it is of course possible for two Cayley graphs $\text{Cay}(G, S)$ and $\text{Cay}(G, T)$ to be isomorphic but no Cayley isomorphisms mapping S to T . Here we investigate the conditions under which $\text{Cay}(G, S) \cong \text{Cay}(G, T)$ if and only if $S^\sigma = T$ for some $\sigma \in \text{Aut}(G)$.

Definition 3.1. A Cayley graph $\text{Cay}(G, S)$ is called a *CI-graph* of G if, for any Cayley graph $\text{Cay}(G, T)$, whenever $\text{Cay}(G, S) \cong \text{Cay}(G, T)$ we have $S^\sigma = T$ for some $\sigma \in \text{Aut}(G)$. (CI stands for *Cayley isomorphism*.)

One long-standing open question about Cayley graphs is as follows:

Question 3.2. Which Cayley graphs for a group G are CI-graphs?

This question has been investigated under various conditions in the literature. Interest in this question stems from a conjecture of Ádám [1] that all circulant graphs were CI-graphs of the corresponding cyclic groups. This conjecture was disproved by Elspas and Turner [27], and however, the conjecture stimulated the investigation of CI-graphs. A lot of work has been devoted to seeking CI-graphs. Recently, substantial progress in this area has been made: many important open problems have been solved, many new significant and deep results have been obtained, and many related new problems have naturally arisen. The purpose of this article is to survey results, open problems and methods in the area.

We remark that whether a Cayley graph Γ of a group G is a CI-graph is dependent on not only the graph Γ but also the group G , as shown below.

Example 3.3. Let $p \geq 5$ be a prime, and let $\Gamma = C_p[C_p]$. By a similar argument to the construction given Example 2.5, we have $\Gamma \cong \text{Cay}(\mathbb{Z}_p^2, S) \cong \text{Cay}(\mathbb{Z}_{p^2}, T)$. Then by Godsil [38], Γ is a CI-graph of the elementary abelian group \mathbb{Z}_p^2 , and however, Γ is not a CI-graph of the cyclic group \mathbb{Z}_{p^2} , see Section 5.2.

The next definition collects several terms which are often used in the study of CI-graphs.

Definition 3.4. Let m be a positive integer, and let G be a finite group.

- (1) G is said to have the *m-DCI property* if all Cayley graphs of G of valency m are CI-graphs (DCI stands for *directed Cayley isomorphism*); G is said to have the *m-CI property* if all undirected Cayley graphs of G of valency m are CI-graphs.
- (2) G is called an *m-DCI-group* if all Cayley graphs of G of valency at most m are CI-graphs; G is called an *m-CI-group* if all undirected Cayley graphs of G of valency at most m are CI-graphs. In particular, G is called a *DCI-group* or a *CI-group* if G is a $|G|$ -DCI-group or a $|G|$ -CI-group, respectively.

3.1. Determining isomorphic classes of Cayley graphs

One of the principal motivations for investigating CI-graphs is to determine isomorphic classes of Cayley graphs. By the definition, if $\text{Cay}(G, S)$ is a CI-graph, then to decide whether or not $\text{Cay}(G, S)$ is isomorphic to $\text{Cay}(G, T)$, we only need to decide whether or not there exists an automorphism $\sigma \in \text{Aut}(G)$ which maps S to T , and usually, the latter is much easier. See [80,117] for such a determination for isomorphic classes of some families of Cayley graphs which are edge-transitive but not arc-transitive.

Let G be a group. Then for any Cayley graph $\Gamma = \text{Cay}(G, S)$ and any automorphism $\sigma \in \text{Aut}(G)$, Γ^σ is also a Cayley graph $\text{Cay}(G, S^\sigma)$. Thus $\text{Aut}(G)$ acting on G induces an action on the set of Cayley graphs of the group G . Let $\mathcal{ISO}(G, \Gamma)$ be the set of Cayley graphs of G which are isomorphic to Γ , that is, $\mathcal{ISO}(G, \Gamma) = \{\text{Cay}(G, T) \cong \Gamma \mid T \subset G\}$. Then by definition, Γ is a CI-graph of G if and only if $\text{Aut}(G)$ is transitive on $\mathcal{ISO}(G, \Gamma)$. The point-stabilizer of this action, that is, the subgroup of $\text{Aut}(G)$ fixing Γ , is equal to $\text{Aut}(G, S)$, where

$$\text{Aut}(G, S) = \{\sigma \in \text{Aut}(G) \mid S^\sigma = S\}.$$

Now $\text{Aut}(G, S)$ is a subgroup of automorphisms of Γ , and every element of $\text{Aut}(G, S)$ fixes the identity of G . Hence $\text{Aut}(G, S)$ is a subgroup of the vertex-stabilizer $(\text{Aut } \Gamma)_1$ of the vertex corresponding the identity of G . This subgroup has played an important role in the study of Cayley graphs, see for example [7,37,73].

3.2. Normalizers of regular subgroups

For a subgroup H of a group X , denote by $N_X(H)$ and $C_X(H)$ the normalizer and the centralizer of H in G , respectively, that is

$$N_X(H) = \{x \in X \mid x^{-1}Hx = H\}, \text{ and } C_X(H) = \{x \in X \mid xh = hx \text{ for all } h \in H\}.$$

Then we have the following important property, see [37, Lemma 2.1].

Lemma 3.5. Let $\Gamma = \text{Cay}(G, S)$ be a Cayley graph, and let \hat{G} be the regular subgroup on the vertex set $V = G$ induced by right multiplication of G . Then

$$N_{\text{Aut } \Gamma}(\hat{G}) = \hat{G} \rtimes \text{Aut}(G, S).$$

Therefore, although it is difficult to determine the full automorphism group $\text{Aut } \Gamma$, the subgroup $N_{\text{Aut } \Gamma}(\hat{G})$ of $\text{Aut } \Gamma$ may be directly readout from information about the group G . The subgroup $N_{\text{Aut } \Gamma}(\hat{G})$ acts on the graph Γ in a nice way, that is, acts by translation and conjugation. See [73,102,119] for more detailed study of Cayley graphs based on the subgroup $\text{Aut}(G, S)$.

Since every element of G induces an automorphism of Γ by conjugation, every element $g \in G$ acts on $V = G$ by conjugation, that is, $x^g = g^{-1}xg$ for all $x \in V = G$. Let

$$\text{Inn}(G, S) = \{\sigma \in \text{Inn}(G) \mid S^\sigma = S\}.$$

Then $\text{Inn}(G, S) \leqslant \text{Aut}(G, S)$. We claim that $\text{Inn}(G, S)$ exactly corresponds the centralizer $C_{\text{Aut } \Gamma}(\hat{G})$. Let \tilde{G} be the regular subgroup on V induced by left multiplication of G , that is, for each $\tilde{g} \in \tilde{G}$,

$$\tilde{g}: x \rightarrow g^{-1}x \quad \text{for all } x \in G.$$

Then $\hat{G}, \tilde{G} < \text{Sym}(V)$ centralizes each other. Let $C = \hat{G}\tilde{G}$, which is a transitive permutation group on G . Let i be the identity of G . Then $C_i = \text{Inn}(G)$, and since $C \cap \text{Aut } \Gamma \leqslant N_{\text{Aut } \Gamma}(\hat{G})$, we have that

$$C_i \cap \text{Aut } \Gamma = \text{Inn}(G) \cap \text{Aut}(G, S) = \text{Inn}(G, S).$$

Since $C_{\text{Sym}(V)}(\hat{G}) \cap \text{Aut } \Gamma = C_{\text{Aut } \Gamma}(\hat{G})$, we may conclude

Lemma 3.6. *The centralizer $C_{\text{Aut } \Gamma}(\hat{G})$ is uniquely determined by $\text{Inn}(G, S)$; more precisely,*

$$\hat{G}C_{\text{Aut } \Gamma}(\hat{G}) = \hat{G} \rtimes \text{Inn}(G, S).$$

Lemmas 3.5 and 3.6 are useful for constructing certain Cayley graphs, see Section 5.4. More examples can be found in [73].

4. A criterion for CI-graphs

A useful tool for the study of isomorphisms of Cayley graphs is the criterion given in Theorem 4.1, which was obtained by Babai [6] and also by Alspach and Parsons [5]. It transforms the graph theoretic problem of deciding whether a Cayley graph Γ of a group G is a CI-graph to the group theoretic problem of deciding whether all regular representations of G in $\text{Aut } \Gamma$ are conjugate. Group theoretic results can therefore be applied to study CI-graphs. Actually, the proofs of most important and deep results obtained so far in this area have involved in using this criterion. We shall discuss the criterion in detail in Theorem 4.1.

The next is the criterion for a Cayley graph to be a CI-graph mentioned above. Since it is a fundamental tool in the study of isomorphisms of finite Cayley graphs, here we present a proof which is essentially the same as Babai's proof given in [6].

Theorem 4.1. Let Γ be a Cayley graph of a finite group G . Then Γ is a CI-graph of G if and only if all regular subgroups of $\text{Aut } \Gamma$ isomorphic to G are conjugate.

Proof. Assume that Γ is a CI-graph of G . We still use \hat{G} to denote the regular representation of $\text{Aut } \Gamma$ induced by right multiplications of elements of G . Let \tilde{G} be an arbitrary subgroup of $\text{Aut } \Gamma$ which is isomorphic to G and acts regularly on the vertex set G . Let $\text{Sym}(G)$ be the symmetric group on G . Since all regular representations of G are permutationally isomorphic (refer to [21]), there exists an isomorphism $\varphi : \hat{G} \rightarrow \tilde{G}$ and a bijection $\rho \in \text{Sym}(G)$ such that, for all $g \in \hat{G}$ and $x \in G$, $(x^g)^{\rho} = (x^{\rho})^{g^{\varphi}}$. Thus $g\rho = \rho g^{\varphi}$, and so $\rho^{-1}g\rho = g^{\varphi}$, that is, $\hat{G}^{\rho} = \rho^{-1}\hat{G}\rho = \hat{G}^{\varphi} = \tilde{G}$. Let $\Sigma = \Gamma^{\rho^{-1}}$, that is, Σ is the graph with vertex set G such that x is adjacent to y in Σ if and only if x^{ρ} is adjacent to y^{ρ} in Γ (noting that ρ is a permutation on G). Then $\rho^{-1}(\text{Aut } \Sigma)\rho = \text{Aut } \Gamma \geq \hat{G}$, and so by Proposition 1.1, Σ is also a Cayley graph of G . Since Γ is a CI-graph, there exists $\alpha \in \text{Aut}(G)$ which induces an isomorphism from Γ to Σ . We still denote this graph isomorphism by α . Then $\Gamma^{\alpha} = \Sigma = \Gamma^{\rho^{-1}}$, and so $\Gamma^{\alpha\rho} = \Gamma$, that is $\beta := \alpha\rho$ is an automorphism of Γ . Now $\hat{G}^{\beta} = \hat{G}^{\alpha^{-1}\beta} = \hat{G}^{\rho} = \tilde{G}$, namely, \tilde{G} and \hat{G} are conjugate in $\text{Aut } \Gamma$. Since \tilde{G} is arbitrary, all regular subgroups of $\text{Aut } \Gamma$ isomorphic to G are conjugate.

Conversely, suppose that all regular subgroups of $\text{Aut } \Gamma$ isomorphic to G are conjugate. Let Σ be a Cayley graph of G which is isomorphic to Γ , and let φ be an isomorphism from Σ to Γ . Then $\varphi \in \text{Sym}(G)$, and the conjugation \hat{G}^{φ} is regular on G . Since $\hat{G} \leq \text{Aut } \Sigma$, we have $\hat{G}^{\varphi} \leq \text{Aut } \Gamma$. By the assumption, there exists $\beta \in \text{Aut } \Gamma$ such that $\hat{G}^{\varphi} = \hat{G}^{\beta}$. Now $\varphi\beta^{-1} \in \text{Sym}(G)$ and $\hat{G}^{\varphi\beta^{-1}} = \hat{G}$, that is, $\gamma := \varphi\beta^{-1}$ normalizes \hat{G} . Thus γ induces an automorphism of \hat{G} . Also, since φ is an isomorphism from Σ to Γ and $\beta^{-1} \in \text{Aut } \Gamma$, γ is also an isomorphism from Σ to Γ . Hence Γ is a CI-graph of G . \square

Now we give some examples to demonstrate the applications of the criterion given in Theorem 4.1. By the definition, both the complete graph $\text{Cay}(G, G \setminus \{1\})$ and its complement $\text{Cay}(G, \emptyset)$ are CI-graphs of G . We shall sometimes call them *trivial CI-graphs*. A digraph Γ is called a *directed graphical regular representation (DGRR)* of the group G if Γ is a Cayley graph of G and $\text{Aut } \Gamma = \hat{G}$. Hence, by Theorem 4.1, if Γ is a DGRR of G , then Γ is a CI-graph of G . The problem of classifying finite groups which have DGRR was solved, see [7, Theorem 2.1], and it was proved that if G does not have a DGRR then G is \mathbb{Z}_2^2 , \mathbb{Z}_2^3 , \mathbb{Z}_2^4 , Q_8 or \mathbb{Z}_3^2 . It is easy to see that all minimal Cayley graphs of the groups \mathbb{Z}_2^2 , \mathbb{Z}_2^3 , \mathbb{Z}_2^4 , Q_8 and \mathbb{Z}_3^2 are CI-graphs. Thus we have the following result, which is observed in [79].

Proposition 4.2. Any finite groups of order greater than 2 have non-trivial CI-graphs.

Next we employ the Sylow's Theorem to study CI-graphs of prime-power order. Let G be a p -group where p is a prime. Suppose that Γ is a connected Cayley graph of G of valency less than p . Let $A = \text{Aut } \Gamma$. Then $A = \hat{G}A_i$ such that $p \nmid |A_i|$, where i is the identity of G . Thus \hat{G} is a Sylow p -subgroup of A . By Sylow's Theorem, all regular

subgroups of A are conjugate, and so by Theorem 4.1, Γ is a CI-graph. This simple property was first observed by Babai in [6, Theorem 3.1], and is slightly extended for undirected graphs in [65, Theorem 3.1] as follows.

Proposition 4.3. *Let p be a prime and let G be a p -group. Then all connected Cayley graphs of valency less than p , and all connected undirected Cayley graphs of valency at most $(2p - 2)$ are CI-graphs. In particular, all Cayley graphs of the cyclic group \mathbb{Z}_p are CI-graphs.*

We remark that the family of groups \mathbb{Z}_p with p prime is the first known infinite family of (D)CI-groups, which was obtained by several people using the spectrum of graphs, see [22,27,113].

5. Constructing Cayley graphs which are not CI-graphs

Some examples of CI-graphs are given in Propositions 4.2 and 4.3. We here construct examples of Cayley graphs which are not CI-graphs.

5.1. Disconnected graphs

We first notice that for a finite group G and subsets $S, T \subset G$,

$$\text{Cay}(G, S) \cong \text{Cay}(G, T) \text{ if and only if } \text{Cay}(\langle S \rangle, S) \cong \text{Cay}(\langle T \rangle, T).$$

Now let H, L be two subgroups of a group G , and let $S \subset H$, $T \subset L$ be such that $\langle S \rangle = H$, and $\langle T \rangle = L$. Assume that $\text{Cay}(H, S) \cong \text{Cay}(L, T)$. Then $\text{Cay}(G, S) \cong \text{Cay}(G, T)$. If $\text{Cay}(G, S)$ is a CI-graph, then $S^\sigma = T$ for some $\sigma \in \text{Aut}(G)$. Thus $H^\sigma = \langle S \rangle^\sigma = \langle S^\sigma \rangle = \langle T \rangle = L$, that is, H is conjugate under $\text{Aut}(G)$ to L , and in particular, $H \cong L$. Therefore, we have

Lemma 5.1. *Let G be a finite group, and assume that $\text{Cay}(G, S)$ is a CI-graph of G .*

- (1) *For each $T \subset G$, if $\text{Cay}(\langle S \rangle, S) \cong \text{Cay}(\langle T \rangle, T)$, then $\langle S \rangle$ is conjugate under $\text{Aut}(G)$ to $\langle T \rangle$, in particular, $\langle S \rangle \cong \langle T \rangle$.*
- (2) *All subgroups of G isomorphic to $\langle S \rangle$ are conjugate under $\text{Aut}(G)$.*

Assume that G has two subgroups $H = \langle a \rangle \cong \mathbb{Z}_{p^2}$ and $K = \langle b \rangle \times \langle c \rangle \cong \mathbb{Z}_p \times \mathbb{Z}_p$, where p is a prime. Let $S_1 = a\langle a^p \rangle$ and $T_1 = b\langle c \rangle$; and let $S_2 = a\langle a^p \rangle \cup a^{-1}\langle a^p \rangle$ and $T_2 = b\langle c \rangle \cup b^{-1}\langle c \rangle$. It is easily shown that $\text{Cay}(H, S_i) \cong C_p[\overline{K_p}] \cong \text{Cay}(K, T_i)$ where $i = 1, 2$. As $\langle S_i \rangle \not\cong \langle T_i \rangle$, by Lemma 5.1, $\text{Cay}(G, S_i)$ is not a CI-graph. Now a little further analysis leads to the following result.

Proposition 5.2. *Let G be a finite group, and let p be a prime divisor of $|G|$. Assume that G has the p -DCI property or the $2p$ -CI property. Then a Sylow p -subgroup of G is of exponent p , cyclic or generalized quaternion.*

5.2. Lexicographic product

For a cyclic group $G \cong \mathbb{Z}_{p^r}$ with $p \geq 3$ and $r \geq 2$, let $S = a\langle a^p \rangle \cup \{a^p\}$ and $T = a\langle a^p \rangle \cup \{a^{2p}\}$. Then it is easily shown that $\text{Cay}(G, S) \cong \mathbb{C}_{p^r-1}[\mathbb{C}_p] \cong \text{Cay}(G, T)$, which are directed graphs of out-valency $p+1$. Suppose that there exists $\alpha \in \text{Aut}(G)$ is such that $S^\alpha = T$. Then $a^\alpha = a^{ip+1}$ for some i , and thus $(a^p)^\alpha = (a^\alpha)^p = (a^{ip+1})^p = a^{ip^2+p} = a^p \notin T$, which is a contradiction. So $\text{Cay}(G, S)$ is not a CI-graph of G . Using similar constructions, undirected Cayley graphs of valency $2(p+1)$ which are not CI-graphs of G can be constructed, see for example [3] or [79]. This leads to the following conclusion:

Proposition 5.3. *Let p be a prime, and let $G = \mathbb{Z}_{p^r}$.*

- (1) *If $p=2$ and $r \geq 3$, then there exist connected directed Cayley graphs of G of out-valency 3 which are not CI-graphs;
if $p > 2$ and $r \geq 2$, then there exist connected directed Cayley graphs of G of out-valency $p+1$ which are not CI-graphs.*
- (2) *If $p=2$ and $r \geq 4$, then there exist connected undirected Cayley graphs of G of valency 6 which are not CI-graphs;
if $p=3$ and $r \geq 3$, then there exist connected undirected Cayley graphs of G of valency 8 which are not CI-graphs;
if $p > 3$ and $r \geq 2$, then there exist connected undirected Cayley graphs of G of valency $2(p+1)$ which are not CI-graphs.*

5.3. Minimal Cayley graphs

All examples of non-CI-graphs $\text{Cay}(G, S)$ given in the previous two subsections are such that S contains all non-identity elements of a subgroup or a coset of a subgroup of G . In particular, S is not minimal generating subset of $\langle S \rangle$. By inspecting the structure of those Cayley graphs, it is quite clear why they have the non-CI-property.

A Cayley graph $\text{Cay}(G, S)$ is said to be *minimal* if it is connected but $\text{Cay}(G, S \setminus \{s\})$ is disconnected for any $s \in S$, in other words, S is a minimal generating subset of G . Similarly, an undirected Cayley graph $\text{Cay}(G, S)$ is said to be *minimal* if $\langle S \rangle = G$ and $\langle S \setminus \{s, s^{-1}\} \rangle \neq G$. It seems unlikely for minimal Cayley graphs not to be CI-graphs, since graph isomorphisms between minimal Cayley graphs would be expected to induce group automorphisms. For example, Xu in 1993 conjectured that minimal Cayley graphs are all CI-graphs (refer to [118, 119]). The first example of minimal Cayley graphs which are not CI-graphs are constructed in [62]. The following is an extension of the example which is given in [81].

Proposition 5.4. *Let $G = \langle a \rangle \times \langle x \rangle \times \langle e \rangle$ where $\text{o}(a) > 1$ is odd, $\text{o}(x) = 2^r \geq 4$ and $\text{o}(e) = 2$, and let $S = \{x, xe, ax^2\}$ and $T = \{x, xe, ax^2e\}$. Then $\text{Cay}(G, S) \cong \text{Cay}(G, T)$, and $S^\sigma \neq T$ for any $\sigma \in \text{Aut}(G)$. In particular, $\text{Cay}(G, S)$ is not a CI-graph.*

It is believed that such examples are very rare, and thus the following problem is formulated in [62].

Problem 5.5. Characterize minimal Cayley graphs which are not CI-graphs.

There have been some partial results about the problem. Huang and Meng [53] proved that all minimal Cayley graphs of cyclic groups are CI-graphs. The author [62, Theorem 3.1] proved that all minimal Cayley graphs of abelian groups of odd order are CI-graphs. Then this result was extended by Feng and Xu [33] to prove that all minimum Cayley graphs of abelian groups are CI-graphs, and by Feng and Gao [35] to prove that all minimal Cayley graphs of abelian groups with cyclic Sylow 2-subgroups are CI-graphs. A complete classification of finite abelian groups for which all minimal Cayley digraphs are CI-graphs is obtained in [81] and [90] independently that is, it is proved that all minimal Cayley digraphs of an abelian group G are CI-graphs if and only if either G is a 2-group, or the Sylow 2-subgroup cannot be written as $H \times \mathbb{Z}_2$ where $\exp(H) \geq 4$.

5.4. Group automorphisms

A Cayley graph $\text{Cay}(G, S)$ is said to be *minimum* if $\langle S \rangle = G$ and $|S|$ has the smallest size of generating sets of G ; while an undirected Cayley graph $\text{Cay}(G, S)$ is said to be *minimum* if $\langle S \rangle = G$ and $|S|$ has the smallest size of self-inverted generating sets of G . The Cayley graphs constructed in Proposition 5.4 are minimal but not minimum. The following proposition shows that not all minimum Cayley graphs are CI-graphs.

Proposition 5.6 (Li [66]). *For any prime p , there exist infinitely many finite groups G such that p is the smallest prime divisor of $|G|$ and G has connected Cayley graphs of out-valency p and connected undirected Cayley graphs of valency $2p$ which are not CI-graphs.*

Proof (Sketch). Let p be a prime, and let $d = d_1 d_2$ be such that $d_1 \geq \max\{3, p\}$ and $d_2 \geq 2$. By [54, p. 508], $p^d - 1$ has a primitive prime q_1 , and $p^{d_1} - 1$ has a primitive prime divisor q_2 . Now $q_1 \geq d + 1 > p$ and $q_2 \geq d_1 + 1 > p$. Let $l = q_1 q_2$, and let

$$G = \mathbb{Z}_p^d \rtimes \mathbb{Z}_l \leq \text{AGL}(1, p^d).$$

Then p is the smallest prime divisor of $|G|$, and \mathbb{Z}_p^d is a minimal normal subgroup of G . Further, $\varphi(l) = (q_1 - 1)(q_2 - 1) \geq dd_1 \geq 3d > 2d + 3$ (where $\varphi(l)$ is the number of positive integers less than l and coprime to l). Thus, there exists $k \in \{2, 3, \dots, l - 2\} \setminus \{p^i, -p^i \pmod{p} \mid 0 \leq i \leq d - 1\}$ such that k is coprime to l . Let a and z be elements of G of order p and l , respectively. Let

$$S_i = \{z^i, a^{-1}z^i a, \dots, a^{-p+1}z^i a^{p-1}\}$$

and let $\Gamma_i = \text{Cay}(G, S_i)$. It is known that $\text{Aut}(G) = \text{AGL}(1, p^d)$, and it was proved in [66] that $\Gamma_1 \cong \Gamma_i$. Suppose that Γ_1 is a CI-graph of G . Then there exists $\alpha \in \text{Aut}(G)$ such that $S_1^\alpha = S_k$. Since $\text{Aut}(G, S_1)$ is transitive on S_1 , we may assume that $z^\alpha = z^k$.

Thus $\alpha \in N_{\text{Aut}(G)}(\langle z \rangle) = \Gamma L(1, p^d)$, and it follows that α is a Frobenius automorphism of G , that is, $z^\alpha = z^{p^j}$ for some $j \in \{0, 1, \dots, d-1\}$. Therefore, $k = p^j \pmod{l}$, which contradicts the choice of k , and so Γ_1 is not a CI-graph.

Let $S = S_1 \cup S_1^{-1}$ and $T = S_k \cup S_k^{-1}$. Then $\text{Cay}(G, S) \cong \text{Cay}(G, T)$. Suppose that $\text{Cay}(G, S)$ is a CI-graph. Then there exists $\alpha \in \text{Aut}(G)$ such that $S^\alpha = T$. Since $\text{Aut}(G, S_1)$ is transitive on S_1 , we may assume that $z^\alpha = z^k$ or z^{-k} . Thus, we have $\alpha \in N_{\text{Aut}(G)}(\langle z \rangle) = \Gamma L(1, p^d)$, and it follows that α is a Frobenius automorphism of G , namely, $z^\alpha = z^{p^j}$ for some $j \in \{0, 1, \dots, d-1\}$. Therefore, $k = p^j$ or $-p^j \pmod{l}$, which also contradicts the choice of k , and so Γ is not a CI-graph. \square

Remark. The construction of Cayley graphs $\text{Cay}(G, S)$ given above involves in automorphisms of the group G . This is a quite typical method for constructing edge-transitive or arc-transitive Cayley graphs, see for example [73, 80, 117]. By Lemmas 3.5 and 3.6, such a Cayley graph $\text{Cay}(G, S)$ has a group of automorphisms of the base group G which acts on the graph $\text{Cay}(G, S)$ edge-transitively.

5.5. Coset graphs

Let X be a group, and let H be a core-free subgroup of X , that is, there is no non-trivial normal subgroup of X contained in H . Let $[X : H] = \{Hx \mid x \in X\}$, the set of right cosets of H in X . For a subset $S \subset G$, the *coset graph*

$$\Gamma := \text{Cos}(X, H, HSH)$$

is defined as the digraph with vertex set $[X : H]$ such that

$$Hx \text{ is connected to } Hy \quad \text{if and only if} \quad yx^{-1} \in HSH.$$

By definition, the neighborhood of the vertex H in Γ is equal to $\{Hsh \mid s \in S, h \in H\}$. It easily follows that Γ is undirected if and only if $HSH = HS^{-1}H$, where $S^{-1} = \{s^{-1} \mid s \in S\}$. Further, each element $x \in X$ induces an automorphism of the graph Γ by right multiplication:

$$x : Hu \rightarrow Hux, \quad \text{for all } Hu \in [X : H].$$

It is easily shown that this action of X on $[X : H]$ is transitive. Thus, $\text{Aut } \Gamma$ has a transitive subgroup isomorphic to X ; in particular, Γ is a vertex-transitive graph.

It is easily shown that all vertex-transitive graphs may be represented as coset graphs, see [105]. It is known that not all vertex-transitive graphs are Cayley graphs, for instance, Petersen graph is vertex-transitive but not a Cayley graph. By Proposition 1.1, if the group X has a subgroup G such that $X = GH$ and $G \cap H = 1$, then the coset graph $\text{Cos}(X, H, HSH)$ is a Cayley graph of the group G . This provides us a method for constructing Cayley graphs. Usually we have two different methods for constructing Cayley graphs Γ of a group G

(C.1) Find subsets S of G such that $\Gamma \cong \text{Cay}(G, S)$.

- (C.2) Find an overgroup X of G , a subgroup $H < X$ with $X = GH$ and $G \cap H = 1$, and a subset $S \subset X$ such that $\Gamma \cong \text{Cos}(X, H, HSH)$.

Remark. We remark that Cayley graphs constructed by method (C.2) are usually hard to construct by method (C.1). This will be demonstrated by an example below. It indicates that, Cayley graphs with certain restricted properties can be constructed only by method (C.2), see [73] for the constructions of various interesting Cayley graphs and Cayley maps.

Next we describe a beautiful Cayley graph of the alternating group A_5 which is constructed by method (C.2). The construction leads to proving that finite CI-groups are soluble, see Theorem 8.6.

Theorem 5.7. *The alternating group A_5 has a connected undirected Cayley graph of valency 29 which is not a CI-graph of A_5 .*

Proof (Sketch). Here we briefly describe the construction of the 29-valent graph. Let $X = PSL(2, 29)$. It follows from information given in the Atlas [17] that X has a subgroup $H \cong \mathbb{Z}_{29} \rtimes \mathbb{Z}_7$ and an involution g such that $\langle H, g \rangle = X$ and $H \cap H^g \cong \mathbb{Z}_7$. Let Γ be the coset graph $\text{Cos}(X, H, HgH)$. Then Γ is a connected X -arc transitive graph of order 60 and valency 29. It was proved in [69] that $\text{Aut } \Gamma \cong \mathbb{Z}_2 \times PSL(2, 29)$. By the Atlas [17], X has subgroups P, Q isomorphic to A_5 but not conjugate in X . Thus P and Q are not conjugate in $\text{Aut } \Gamma$. Now $(|H|, |A_5|) = 1$ and $|H||A_5| = |X|$. It follows that $H \cap P = H \cap Q = 1$ and hence that $X = PH = QH$. Consequently, both P and Q are regular on V . Thus, by Proposition 1.1, we may identify V with P in such a way that $\Gamma = \text{Cay}(P, S)$ for some $S \subset P$. Therefore, by Theorem 4.1, Γ is not a CI-graph of A_5 . \square

Remarks. (1) Let $\Gamma = \text{Cay}(A_5, S)$ be the graph constructed above. The graph is constructed in language of the group $PSL(2, 29)$, but it is much harder to write S explicitly in terms of elements of A_5 . Very recently, with the assistance of computer, the 29 elements of S may be written out in elements of A_5 , acting on $\{1, 2, 3, 4, 5\}$:

$$\begin{aligned} & (13542), (154), (14)(23), (13254), (12543), (15324), (245), (132), \\ & (152), (13)(45), (12435), (235), (15234), (15342), (125), (14)(25), \\ & (13)(25), (123), (14325), (23)(45), (14235), (253), (254), (13452), \\ & (12453), (145), (12)(35), (14523), (15)(24). \end{aligned}$$

However, the structure property of the graph Γ is determined by the overgroup $PSL(2, 29)$, and is hard to read out from the elements of S . For instance, if we would be only given the pair (A_5, S) , it would be hard to figure out whether Γ is a CI-graph of A_5 and whether Γ is arc-transitive. Again with the assistance of computer, we find another subset T of A_5 such that S is not equivalent to T under $\text{Aut}(A_5)$ but

$\text{Cay}(\text{A}_5, S) \cong \text{Cay}(\text{A}_5, T)$, where elements of T are

$$\begin{aligned} & (13245), (15243), (14253), (13)(24), (124), (15423), (354), (13425), \\ & (15)(34), (13524), (12)(45), (345), (153), (15)(23), (12)(34), (143), \\ & (134), (12534), (142), (24)(35), (14532), (234), (243), (135), (12354), \\ & (14352), (12345), (14)(35), (15432). \end{aligned}$$

(The author is grateful to C. Schneider for his help in computing the subsets S and T .)

(2) Recently it is shown in [15] that S and T are so called *relative difference sets* of the group A_5 , which are the only known examples of relative difference sets in *insoluble* groups.

(3) The A_5 -graph described above was actually constructed in 1994 (though published in 1998, namely [69]). In 1997, with the assistance of computer, Conder and the author [16] found a 5-valent Cayley graph of A_5 which is not a CI-graph.

6. Isomorphisms of connected Cayley graphs

As observed before, a disconnected Cayley graph is a disjoint union of isomorphic connected smaller Cayley graphs, that is, the Cayley graph $\text{Cay}(G, S)$ is a union of $|G|/|\langle S \rangle|$ vertex-disjoint copies of $\text{Cay}(\langle S \rangle, S)$. Thus, to determine isomorphic classes of Cayley graphs, we only need to consider the connected graph case.

6.1. Hall subgroups

Let π be a set of primes and π' the complement of π in the set of all primes. A group G is called a π -group if all prime divisors of $|G|$ lie in π .

The Sylow's Theorem is used in Proposition 4.3 to study CI-graphs of prime-power order. The method can be extended to the more general case. For a finite group F , by a result of Wielandt (see [104, Theorem 9.1.10]), all nilpotent Hall π -subgroups (if exist) of F are conjugate; by Gross [43], all Hall π -subgroups (if exists) of F of odd order are conjugate. Let G be a finite group which is of odd order or nilpotent, and let Γ be a Cayley graph of G . Let $A = \text{Aut } \Gamma$ and A_i the stabilizer of i in A , where i is the identity of G . Suppose that $(|G|, |A_i|) = 1$. Then G is a Hall π -subgroup of A , where π is the set of prime divisors of $|G|$. Since G is of odd order or nilpotent, all Hall π -subgroups of A are conjugate. By Theorem 4.1, Γ is a CI-graph of G . Thus we have

Theorem 6.1 (Li [62, Theorem 4.1]). *Let G be a finite group, and let Γ be a Cayley graph of G . Let $A = \text{Aut } \Gamma$ and let i be the identity of G . Assume that G is of odd order or nilpotent. If $(|G|, |A_i|) = 1$ then Γ is a CI-graph of G .*

For the case where G is cyclic, the result was obtained independently by Morris [92]. A natural question is whether, for arbitrary groups G , the condition $(|G|, |A_i|) = 1$ implies that Γ is a CI-graph. An immediate consequence of Theorem 6.1 is the following result.

Corollary 6.2. Suppose that G is a finite group which is of odd order or nilpotent. If p is the smallest prime divisor of $|G|$, then all connected Cayley graphs of G of out-valency less than p are CI-graphs.

6.2. Small valent Cayley graphs

By Proposition 5.6, the conclusion of Corollary 6.2 cannot be extended to Cayley graphs of out-valency p . Regarding this, some natural problems proposed in [66].

Problem 6.3. (1) Characterize finite groups G such that G has connected Cayley graphs of out-valency p which are not CI-graphs, where p is the smallest prime divisor of $|G|$.

(2) Characterize finite groups G such that G has undirected connected Cayley graphs of valency m which are not CI-graphs, where $p \leq m \leq 2p$ and p is the smallest prime divisor of $|G|$.

(3) In particular, are all undirected connected cubic Cayley graphs CI-graphs?

In [67], it is proved that all connected 2-valent Cayley graphs of the simple group $\mathrm{PSL}(2, q)$ are CI-graphs, and it is conjectured that all connected 2-valent Cayley graphs of finite simple groups are CI-graphs. The conjecture has been proved by Hirasaka and Muzychuk [50]:

Theorem 6.4. All connected Cayley graphs of valency 2 of finite simple groups are CI-graphs.

It was shown in [31] that, for most finite simple groups G , connected undirected cubic Cayley graphs of G are CI-graphs.

6.3. Abelian Cayley graphs

Let G be an abelian group with the identity i . Let $\Gamma = \mathrm{Cay}(G, S)$ be connected and let $A = \mathrm{Aut} \Gamma$. It is proved in [58] that either A_i is faithful on S , or S contains a coset of some nontrivial subgroup of G . This result is applied in [65] to prove the following result, which is an extension of Theorem 6.1 for abelian groups.

Theorem 6.5. Let G be an abelian group and let $\Gamma = \mathrm{Cay}(G, S)$. Let $A = \mathrm{Aut} \Gamma$. Suppose that G is abelian and $(|G|, |A_i|) = p$ where p is a prime. Then either

- (i) Γ is a CI-graph of G , or
- (ii) S contains a coset of some non-trivial subgroup of G .

There do exist examples of Cayley graphs $\mathrm{Cay}(G, S)$ which satisfies part (ii) of the theorem and are not CI-graphs, constructed as follows:

Example 6.6. Let p be a prime, and let $G = \langle a \rangle \cong \mathbb{Z}_{p^r k}$, where $r \geq 2$ and k is even and coprime to p . Set $S = a\langle a^{p^{r-1}k} \rangle \cup \{a^{p^{r-1}k}, a^2\}$. Let $\Gamma = \text{Cay}(G, S)$ and $A = \text{Aut } \Gamma$. Then it is easily checked that $|A_i| = p$, so $(|G|, |A_i|) = p$. Let $T = a\langle a^{p^{r-1}k} \rangle \cup \{a^{p^{r-1}k}, a^{2+2^{r-1}k}\}$. A straightforward calculation shows that $\text{Cay}(G, T) \cong \Gamma$. Suppose that there exists $\alpha \in \text{Aut}(G)$ such that $S^\alpha = T$. Then $a^\alpha = a$ or $a^{1+2^{r-1}k}$. Thus $(a^2)^\alpha = a^2$ or $(a^2)^\alpha = (a^{1+2^{r-1}k})^2 = a^2$, so $(a^2)^\alpha \notin T$, which is a contradiction. Hence Γ is not a CI-graph of G . Therefore, for any prime p , there exists a cyclic group G and a Cayley graph Γ of G such that $(|G|, |A_i|) = p$ and Γ is not a CI-graph of G .

However we do not know any other examples. On the other hand, for all known examples, if G is abelian and S contains no cosets of a subgroup of G , then $\text{Cay}(G, S)$ is a CI-graph of G . For these reasons, the following problem would be worth studying.

Problem 6.7. (1) Let G be an abelian group, and let $\Gamma = \text{Cay}(G, S)$ and $A = \text{Aut } \Gamma$. Determine Cayley graphs Γ such that $(|G|, |A_i|)$ is a prime and Γ is not a CI-graph.

(2) Does there exist an abelian group G and a connected Cayley graph $\text{Cay}(G, S)$ such that S contains no cosets of a subgroup of G and $\text{Cay}(G, S)$ is not a CI-graph of G ?

A group is said to be *homocyclic* if it is a direct product of cyclic groups of the same order. The following theorem extends Corollary 6.2 to the case $m = p$ for abelian groups, which was obtained by Meng and Xu in [120] for the case where $p = 2$, and by the author in [63] for the other case. (The size of a minimum generating subset of a group is called the *rank* of the group.)

Theorem 6.8. Let G be an abelian group and let p be the least prime divisor of $|G|$. Let G_p be a Sylow p -subgroup of G . Then all connected Cayley graphs of G of out-valency at most p are CI-graphs. Further, all connected Cayley graphs of G of out-valency $p + 1$ are CI-graphs if and only if one of the following holds:

- (i) G is of rank at least 3;
- (ii) G is of rank at most 2, and either G_p is homocyclic of rank 2, or $G_p \cong \mathbb{Z}_p$ or \mathbb{Z}_4 .

For the undirected case, it is proved in [19] that all connected Cayley graphs of abelian groups of valency 4 are CI-graphs; a complete classification is given in [91] of abelian groups G such that G have connected Cayley graphs of valency 5 which are not CI-graphs of G .

6.4. Normal Cayley graphs

A Cayley graph $\text{Cay}(G, S)$ is called a *normal Cayley graph* if \hat{G} is normal in $\text{Aut } \Gamma$ (refer to [119]). The next result is an immediate consequence of Theorem 4.1.

Corollary 6.9. Let G be a finite group, and let Γ be a Cayley graph of G . If \hat{G} is normal in $\text{Aut } \Gamma$, then Γ is a CI-graph if and only if \hat{G} is the unique regular subgroup of $\text{Aut } \Gamma$ which is isomorphic to G .

There indeed exist normal Cayley graphs which are not CI-graphs.

Example 6.10. Let $G = \langle a \rangle \cong \mathbb{Z}_{2^r}$ where $r \geq 3$. Set $S = \{a, a^{2^{r-1}+1}, a^2\}$, and let $\Gamma = \text{Cay}(G, S)$. It is easily proved that $\text{Aut } \Gamma = \hat{G} \rtimes \langle \sigma \rangle \cong \mathbb{Z}_{2^r} \rtimes \mathbb{Z}_2$, where $\hat{G} = \langle \hat{a} \rangle$ and $\hat{a}^\sigma = \hat{a}^{2^{r-1}+1}$, so that \hat{G} is normal in $\text{Aut } \Gamma$. Since $(\hat{a}\sigma)^2 = \hat{a}^{2^{r-1}+2}$, it follows that $\langle \hat{a}\sigma \rangle$ is a regular subgroup of $\text{Aut } \Gamma$. Therefore, by Corollary 6.9, Γ is not a CI-graph. Actually, letting $T = \{a, a^{2^{r-1}+1}, a^{2^{r-1}+2}\}$, it is easily shown that $\text{Cay}(G, S) \cong \text{Cay}(G, T)$ but $S^\sigma \neq T$ for any $\sigma \in \text{Aut}(G)$.

However, normal Cayley graphs which are not CI-graphs seem to be very rare, so we would like to formulate a problem as follows.

Problem 6.11. Characterize normal Cayley graphs which are not CI-graphs.

We remark that whether a Cayley graph $\Gamma = \text{Cay}(G, S)$ is a normal Cayley graph of G not only depends on the graph Γ but also depends on the choice of the base group G . For instance, by Example 2.4, the hypercube \mathbf{Q}_d may be represented as a Cayley graph for many different 2-groups; clearly, \mathbf{Q}_d is a normal Cayley graph of the elementary abelian group \mathbb{Z}_2^d , but not normal with respect to any other 2-groups.

We end this section with presenting a potential method for constructing normal Cayley graphs which are not CI-graphs.

Proposition 6.12. Let G be a finite non-abelian simple group such that G contains an element g which is not conjugate to g^{-1} . Let $S = \{x^{-1}gx \mid x \in G\}$, a full conjugacy class containing g . Then $\text{Cay}(G, S)$ is a normal Cayley graph and is not a CI-graph of G .

Proof. Now we have $\text{Aut}(G, S) \geq \text{Inn}(G)$. Let $\Gamma = \text{Cay}(G, S)$, and let $X = \hat{G} \cdot \text{Inn}(G)$. Then $X \leq \text{Aut } \Gamma$ is a primitive permutation group on the vertex set G , which is of O’Nan-Scott type HS, see [101]. Further, by Praeger [100], $\text{Aut } \Gamma$ is of type HS or SD. If $\text{Aut } \Gamma$ is of type SD, then it is easily shown that there exists $\sigma \in \text{Aut } \Gamma$ such that $S^\sigma = S^{-1}$, which is a contradiction since now $S \neq S^{-1}$. Thus $\text{Aut } \Gamma$ is also of type HS, so that $\text{Aut } \Gamma$ has two minimal normal subgroups, which are isomorphic to G and regular on the vertex set G , and are not conjugate in $\text{Aut } \Gamma$. By Corollary 6.9, Γ is not a CI-graph. \square

With this proposition, it is easy to construct normal Cayley graphs which are not CI-graphs. However, inspecting the proof of the proposition, we find that this method cannot produce undirected examples.

7. Isomorphisms of circulants

A Cayley graph or digraph of a cyclic group is called a *circulant* or a *circulant graph*. As mentioned before, investigating isomorphisms of circulants was the starting point of the investigation of the isomorphism problem for general Cayley graphs.

7.1. The cyclic CI-groups and DCI-groups

One of the most remarkable achievements regarding the isomorphism problem for circulant graphs is the complete classification of cyclic CI-groups, which was completed by Muzychuk [93,94] that is,

Theorem 7.1. (1) A cyclic group of order n is a DCI-group if and only if $n=k$, $2k$ or $4k$ where k is odd square-free.

(2) A cyclic group of order n is a CI-group if and only if either $n \in \{8, 9, 18\}$ or $n=k, 2k$ or $4k$ where k is odd square-free.

The work of seeking such a classification had been lasted for 30 years. The necessary condition of the theorem is not hard to show. By Proposition 5.3, if G is a cyclic CI-group then a Sylow p -subgroup of G is \mathbb{Z}_p , \mathbb{Z}_4 , \mathbb{Z}_8 or \mathbb{Z}_9 . A similar construction shows that if $r > 1$, then \mathbb{Z}_{8r} is not a 6-CI-group and \mathbb{Z}_{9r} is not a 8-CI-group. The order of a cyclic CI-group G is therefore $8, 9, 18, k, 2k$ or $4k$, where k is odd and square-free.

However, verifying whether cyclic groups of such orders are CI-groups was much more difficult. Many special cases were settled separately by many people (where p and q are distinct primes): $n = p$, Djokovic [22], Elspas and Turner [27] and Turner [113]; $n \leq 37$ and $n \neq 16, 24, 25, 27, 36$ —McKay [89]; $n = 2p$ —Babai [6]; $(n, \varphi(n)) = 1$ —Pálfy [98], $n = pq$ —Alspach and Parsons [5], and Godsil [38]; $n = 4p$, $p > 2$ —Godsil [38]. The proof of Theorem 7.1 given by Muzychuk consists of his two papers, namely [93,94]. The first one is for the case n is odd, and the second one is for the case n is even. The proof is highly technical, and the main argument in the proof is based on the technique of Schur rings.

7.2. Cyclic groups with the CI-property and with the DCI-property

On a different direction (with classifying cyclic CI-groups), the circulants of small valency have been investigated by many people. Toida [112] proved that cyclic groups are 3-CI-groups. Boesch and Tindell [14] conjectured that cyclic groups are 4-CI-groups. Sun [110] first proved this conjecture. Other proofs can be found in [19,34]. Cyclic groups are also 5-CI-groups, see [58]. For the directed case, Sun [109] proved that cyclic groups are 2-DCI-groups. However, by Proposition 5.3, \mathbb{Z}_8 is not a 3-DCI-group and \mathbb{Z}_{16} is not a 6-CI-group. A natural question is, for a positive integer m , which cyclic groups are m -DCI-groups, and which cyclic groups are m -CI-groups? From Proposition 5.3 it follows that if G is a cyclic m -DCI-group, $p \mid |G|$ and $p < m$, then a Sylow p -subgroup of G is either \mathbb{Z}_p or \mathbb{Z}_4 ; if G is a cyclic m -CI-group, $p \mid |G|$

and $2p < m$, then a Sylow p -subgroup of G is \mathbb{Z}_4 , \mathbb{Z}_8 , \mathbb{Z}_9 or \mathbb{Z}_p . We have shown that \mathbb{Z}_{8k} is not a 6-CI-group and \mathbb{Z}_{9k} is not a 8-CI-group. Thus we can obtain the candidates for cyclic m -DCI-groups and cyclic m -CI-groups. Here we have used the k -(D)CI property for all values of k with $1 \leq k \leq m$ to deduce the conclusion, the author [61] proved that, for directed graphs, the conclusion can be deduced only from the single m -DCI property:

Theorem 7.2. *Let G be a cyclic group, and let G_p be a Sylow p -group of G . Suppose that G has the m -DCI property. If $|G|$ is not a prime-square and $p+1 \leq m \leq (|G|-1)/2$, then $G_p \cong \mathbb{Z}_p$ or \mathbb{Z}_4 .*

We remark that Theorem 1.2 of [61] was unfortunately mis-stated and the statement of Theorem 7.2 is a correct version of Theorem 1.2 of [61].

Theorem 7.2 is only regarding the directed graph case. We conjecture a similar conclusion should hold for the undirected graph case, see [61, Problem 1.4]. For the prime-square order case, a complete characterization is obtained in [45,61]: for $1 \leq m \leq (p^2 - 1)/2$, it is proved in [61, Theorem 1.1] that \mathbb{Z}_{p^2} has the m -DCI property if and only if $m < p$ or $m \equiv 0$ or $-1 \pmod{p}$; while it is proved in [45] that \mathbb{Z}_{p^2} has the m -CI property if and only if one of the following holds: $p=2$ or 3 , m is odd, $[m/p]$ is odd, $m \leq p-1$, or $m=kp$ or $kp+(p-1)$ for some even positive integer k .

By Theorem 7.2, for a positive integer n which is not a prime-square, if \mathbb{Z}_n has the m -DCI property, then n can be written as $n=n_1n_2$ such that $(n_1, n_2)=1$, n_1 divides $4k$ where k is odd and square-free, and every prime divisor of n_2 is greater than m . If $n_1=1$ then every prime divisor of n is greater than m , and it follows from Theorem 6.1 that \mathbb{Z}_n is an m -DCI-group. On the other hand, if $n_2=1$ then n divides $4k$ where k is odd and square-free, and consequently \mathbb{Z}_n is a DCI-group by Theorem 7.1. In the general case, it is still open whether \mathbb{Z}_n is an m -DCI-group. We are inclined to think that this is true and we therefore propose the following conjecture.

Conjecture 7.3. *Let $n=n_1n_2$ be an integer such that $(n_1, n_2)=1$ and n_1 divides $4k$ where k is odd and square-free.*

- (1) *If every prime divisor of n_2 is greater than m , then \mathbb{Z}_n is an m -DCI-group.*
- (2) *If every prime divisor of n_2 is greater than $2m$, then \mathbb{Z}_n is a $2m$ -CI-group.*

If this conjecture is true, then the conclusion is a generalization of Muzychuk's result given in Theorem 7.1.

7.3. Special classes of circulant graphs

We conclude this section with a discussion on several special classes of circulant graphs. A nice description of edge-transitive circulant graphs is given in [72], and a consequence of the description shows that all edge-transitive circulant graphs are CI-graphs.

Let $G = \mathbb{Z}_n$ be a cyclic group of order n , and let \mathbb{Z}_n^* be the set of elements of G of order n . Toida [112] posed the following conjecture:

Conjecture 7.4. *If $S \subseteq \mathbb{Z}_n^*$ then $\text{Cay}(G, S)$ is a CI-graph.*

This conjecture was proved for the case where n is a prime-power by different people: Klin and Pöschel [56,57], Gol'lfand, Najmark and Pöschel [39]. Very recently, the conjecture was proved independently in [72,96].

8. Finite CI-groups and DCI-groups

A DCI-group is also a CI-group, but the converse is not true, for example, the Frobenius group of order $3p$ with p odd prime is a CI-group but not a DCI-group, refer to Theorems 8.7 and 8.8. A DCI-group has the m -DCI property and is an m -DCI-group for all $m \leq |G|$; while a CI-group has the m -CI property and is an m -CI-group for all $m \leq |G|$.

8.1. A result of Babai and Frankl

In 1977, Babai [6] initiated the study of general CI-groups. Then Babai and Frankl [9,10] proved the following remarkable result.

Theorem 8.1. *Let G be a finite CI-group.*

- (1) *If G is of odd order, then a Sylow 3-subgroup is \mathbb{Z}_3 , \mathbb{Z}_9 or \mathbb{Z}_{27} , and Sylow p -subgroups for $p \neq 3$ are elementary abelian. Moreover, either G is abelian, or G has an abelian normal subgroup of index 3.*
- (2) *If G is insoluble, then $G = U \times V$ where $(|U|, |V|) = 1$, U is a direct product of elementary abelian Sylow groups, and $V \cong \text{PSL}(2, 5), \text{SL}(2, 5), \text{PSL}(2, 13)$ or $\text{SL}(2, 13)$.*

A crucial property for the proof of Theorem 8.1 is that the property of being CI-groups is inherited by subgroups and quotient groups to characteristic subgroups, that is,

Lemma 8.2 (Babai and Frankl [9, Lemmas 3.2 and 3.5]). *Let G be a CI-group. Then every subgroup of G is a CI-group, and if N is a characteristic subgroup of G then G/N is a CI-group.*

Proof. Let H be a subgroup of G , and let $S, T \subseteq H$ satisfy $\text{Cay}(H, S) \cong \text{Cay}(H, T)$ so that $\text{Cay}(G, S) \cong \text{Cay}(G, T)$. Noting that either $\langle S \rangle = H$ or $\langle H \setminus S \rangle = H$, and that $\text{Cay}(H, S)$ is a CI-graph if and only if $\text{Cay}(H, H \setminus S)$ is a CI-graph, we may assume that $\langle S \rangle = \langle T \rangle = H$. Since G is a CI-group, there is $\sigma \in \text{Aut}(G)$ such that $S^\sigma = T$. Now $H^\sigma = \langle S^\sigma \rangle = \langle T \rangle = H$, so σ induces an automorphism of H . Thus H is a CI-group.

Let N be a characteristic subgroup of G . Let S_1, T_1 be subsets of G/N which do not contain the identity of G/N such that $\Sigma := \text{Cay}(G/N, S_1) \cong \Gamma := \text{Cay}(G/N, T_1)$. Let S, T be the full preimages of S_1, T_1 under $G \rightarrow G/N$, respectively. Then $\text{Cay}(G, S) = \Sigma[\bar{K}_{|N}] \cong \Gamma[\bar{K}_{|N}] = \text{Cay}(G, T)$. Since G is a CI-group, there exists $\sigma \in \text{Aut}(G)$ such that $S^\sigma = T$. As N is characteristic in G , σ fixes N , so σ induces an automorphism $\bar{\sigma}$ of G/N sending S_1 to T_1 . Thus G/N is a CI-group. \square

8.2. Elementary abelian groups

Due to Theorem 8.1, one of the main problems regarding CI-groups has been to determine which groups listed in Theorem 8.1 are really CI-groups. In particular, Babai and Frankl [9] asked whether elementary abelian groups are all CI-groups:

Question 8.3. For a prime p and a positive integer d , is the elementary abelian group \mathbb{Z}_p^d a CI-group?

Some ingenious graph constructions have been found which led to much progress on this topic. Nowitz [97] constructed a 32-valent Cayley graph of \mathbb{Z}_2^6 that is not a CI-graph, which provides a negative answer to the Babai and Frankl's question.

Theorem 8.4 (Nowitz [97]). *The elementary abelian group \mathbb{Z}_2^6 is not a CI-group.*

It is proved in [97] that there are two non-conjugate regular subgroups of $\text{Aut } \Gamma$ which are isomorphic to \mathbb{Z}_2^6 . Thus by Theorem 4.1, Γ is not a CI-graph. The proof in [97] is very technical and complicated. A simpler proof was given by Alspach [3]. With the assistance of computer, it was found in [16] that the Nowitz graph has the full automorphism group of order $2^{13}3$. \square

The status for Question 8.3 is that \mathbb{Z}_p^d for $d \leq 4$ and \mathbb{Z}_2^5 have been proved to be CI-groups: Godsil [28] for $d = 2$; Dobson [23] for $d = 3$; Conder and Li [16] for $p = 2$ and $d = 4, 5$; Hirasaka and Muzychuk [49] for $p > 2$ and $d = 4$.

Very recently, a remarkable construction given by Muzychuk [95] extends Theorem 8.4 to general prime p as in the following theorem.

Theorem 8.5 (Muzychuk [94,95]). *Let p be a prime, and let $n \geq 2p - 1 + \binom{2p-1}{p}$. Then the elementary abelian group \mathbb{Z}_p^n is not a CI-group.*

8.3. Candidates for finite CI-groups and DCI-groups

A natural question related to Theorem 8.1 is whether $A_5, \text{SL}(2, 5), \text{PSL}(2, 13)$ and $\text{SL}(2, 13)$ are CI-groups, see for example [116]. It is easy to exclude $\text{PSL}(2, 13)$ and so $\text{SL}(2, 13)$ from information given in the Atlas [17], see for example [75]. By Theorem 5.7, A_5 has a Cayley graph of valency 29 which is not a CI-graph. It then easily follows from Lemma 8.2 that $\text{SL}(2, 5)$ is not a CI-group. This deduces the following result:

Theorem 8.6 (Li [69]). *All finite CI-groups are soluble.*

With respect to this result, Theorem 8.1 (obtained in 1978) only provides candidates for CI-groups of odd order. An explicit list of candidates for CI-groups of even order was obtained 20 years later, as a corollary of a description of finite m -CI-groups obtained by a series of papers [75,76,77]. The later is dependent on the classification of finite simple groups, refer to Section 9. Very recently, Pálfy and the author [74] produce a more precise description for such groups, using Lemma 8.2 and induction on the group order. Next we briefly describe the description. In our list of candidates for CI-groups, most members contain a direct factor of groups defined as follows. Let M be an abelian group for which all Sylow subgroups are elementary abelian, and let $n \in \{2, 3, 4, 8, 9\}$ such that $(|M|, n) = 1$. Let

$$E(M, n) = M \rtimes \langle z \rangle$$

such that $\text{o}(z) = n$, and if $\text{o}(z)$ is even then z inverts all elements of M , that is, $x^z = x^{-1}$ for all $x \in M$; while if $\text{o}(z) = 3$ or 9 then $x^z = x^l$ for all $x \in M$, where l is an integer satisfying $l^3 \equiv 1 \pmod{\exp(M)}$ and $(l(l-1), \exp(M)) = 1$. Let \mathcal{CI} denote the class of the following finite groups G :

- (1) $G = U \times V$ with $(|U|, |V|) = 1$, where all Sylow subgroups of G are elementary abelian, or isomorphic to \mathbb{Z}_4 or \mathbb{Q}_8 ; moreover, U is abelian, and $V = 1, \mathbb{Q}_8, \mathbb{A}_4, \mathbb{Q}_8 \times E(M, 3), E(M, n)$ with $n \in \{2, 3, 4\}$, or $E(M, n) \times E(M', 3)$ where $n \in \{2, 4\}$ and $|M|, |M'|, 6$ are coprime pairwise.
- (2) G is one of the groups: $\mathbb{Z}_8, \mathbb{Z}_9, \mathbb{Z}_{18}, \mathbb{Z}_9 \rtimes \mathbb{Z}_2 (= \mathbb{D}_{18}), \mathbb{Z}_9 \rtimes \mathbb{Z}_4$ with center of order 2, $\mathbb{Z}_2^2 \rtimes \mathbb{Z}_9$ with center of order 3, $E(M, 8), \mathbb{Z}_2^k \times E(M, 9)$ with $k \geq 1$.

Then we have a description for finite CI-groups.

Theorem 8.7. *All finite CI-groups are in \mathcal{CI} .*

By definition, a DCI-group is also a CI-group, and thus by Theorem 8.7, we may obtain a description of finite DCI-groups, which was first obtained by Praeger, Xu and the author in [78].

Theorem 8.8. *If G is a DCI-group, then all Sylow subgroups of G are elementary abelian or isomorphic to \mathbb{Z}_4 or \mathbb{Q}_8 . Moreover, $G = U \times V$, where $(|U|, |V|) = 1$, U is abelian, and $V = 1, \mathbb{Q}_8, \mathbb{A}_4, E(M, 2)$, or $E(M, 4)$.*

8.4. The known examples for CI-groups

Due to Theorem 8.7, the problem of classifying CI-groups becomes the problem of determining which members of \mathcal{CI} are CI-groups. We further note that if $G \in \mathcal{CI}$ then $\text{Cay}(G, S)$ is a CI-graph of G if and only if $\text{Cay}(\langle S \rangle, S)$ is a CI-graph of $\langle S \rangle$. Unfortunately, even with this knowledge, it is still very difficult to obtain a complete classification of finite CI-groups. Actually, so far only very ‘few’ families of groups are proved to be CI-groups, which are listed in the next theorem.

Theorem 8.9. *The following groups are all known CI-groups: where p is a prime,*

- (i) \mathbb{Z}_n , where either $n \in \{8, 9, 18\}$, or $n \mid 4k$ and k is odd square-free;
- (ii) \mathbb{Z}_p^2 , \mathbb{Z}_p^3 , \mathbb{Z}_p^4 ;
- (iii) \mathbb{Z}_2^5 , Q_8 , A_4 , $\mathbb{Z}_2^2 \times \mathbb{Z}_3$, D_{18} , $\mathbb{Z}_9 \rtimes \mathbb{Z}_4$, $\mathbb{Z}_2^2 \rtimes \mathbb{Z}_9$;
- (iv) D_{2p} , F_{3p} (a Frobenius group of order $3p$);
- (v) $\mathbb{Z}_p \rtimes \mathbb{Z}_4$ (with center of order 2), $\mathbb{Z}_p \rtimes \mathbb{Z}_8$ (with center of order 4).

The studies on cyclic CI-groups is described in Section 4. Godsil [38] proved \mathbb{Z}_p^2 to be CI-groups, and Dobson [23] proved that \mathbb{Z}_p^3 are CI-groups. Dobson's proof is very technical, and a much simpler proof was given by Alspach and Nowitz, see [4]. Recently, Hirasaka and Muzychuk [49] proved that \mathbb{Z}_p^4 for $p > 2$ is a CI-group; while with the assistance of a computer, Conder and the author [16] proved that both \mathbb{Z}_2^4 and \mathbb{Z}_2^5 are CI-groups. Thus \mathbb{Z}_p^k for $k \leq 4$ are CI-groups. It is easily checked that Q_8 is a CI-group. It is proved that A_4 is a CI-group in [46]. The other groups in part (iii) are verified to be CI-groups in [16] with the assistance of computer. Babai [6] proved that D_{2p} is a CI-group. Dobson [24] proved that the Frobenius group F_{3p} is a CI-group, and actually he determined the isomorphism problem of general metacirculant graphs (not necessarily Cayley graphs) of order $3p$. Very recently, Pálfy and the author [74] proved that the group $\mathbb{Z}_p \rtimes \mathbb{Z}_4$ with center of order 2 and the group $\mathbb{Z}_p \rtimes \mathbb{Z}_8$ with center of order 4 are all CI-groups.

The candidates for CI-groups and DCI-groups are so restricted that it would be possible to obtain a complete classification of finite (D)CI-groups.

Problem 8.10. Give a complete classification of finite CI-groups and finite DCI-groups.

For this problem, determining whether the following members $G \in \mathcal{CI}$ are CI-groups would be crucial: $G = \mathbb{Z}_p^5$ with odd p , $G = \mathbb{Z}_p^2 \times \mathbb{Z}_q$ where p, q are different primes, and the groups G of which all Sylow subgroups are cyclic.

9. Finite m -DCI-groups and m -CI-groups

In 1979, Toida [112] proved that cyclic groups are 3-CI-groups. Then Sun proved that cyclic groups are 2-DCI-groups and 4-CI-groups in [109,110], respectively. In a series of papers [28,29,33], Fang and Xu completely classified abelian m -DCI-groups for $m \leq 3$. For $m \leq 5$, the abelian m -CI-groups are completely classified in [32,34]. The methods for these results are mainly combinatorial. This work led Xu [116] to propose the problem of studying m -(D)CI-groups. By employing the finite simple group classification and some other deep group theory results, the status of investigating m -(D)CI-groups has been significantly changed.

9.1. Finite 1-DCI-groups

By definition, an m -DCI-group is also an m -CI-group; an m -DCI-group is a k -DCI-group, and an m -CI-group is a k -CI-group for all $k \leq m$. It easily follows

from the definition that if G is a 1-DCI- or 2-CI-group then all cyclic subgroups of the same order are conjugate under $\text{Aut}(G)$. The classes of p -groups (p a prime) with the latter property were investigated by quite a few people (see [42,48,106,115]),

Theorem 9.1 (See Wilkens [115]). *Let p be a prime and let G be a p -group all subgroups of G order p are conjugate under $\text{Aut}(G)$. Then either G is homocyclic, or $p=2$ and for some $k \geq 1$:*

- (i) $Z(G) = G' = \Phi(G) = \mathbb{Z}_2^k$, and $Z(G) \setminus \{1\}$ consists of all the involutions of G ;
- (ii) G/G' is of order 2^k or 2^{2k} .

Dependent on the classification of 2-transitive affine permutation groups, a description for general 1-DCI-groups was obtained by Zhang [121]. A key property in Zhang's work is that if G is a 1-DCI-group and N is a characteristic subgroup of G , then N and G/N are also 1-DCI-groups. The following improved form was obtained in [76, Corollary 1.3].

Theorem 9.2. *Let G be a 1-DCI-group. Then all Sylow subgroups of G are 1-DCI-groups.*

- (1) *If G is of odd order, then $G = U \times V$, where $(|U|, |V|) = 1$, U is abelian, and $V = 1$ or $M \rtimes \mathbb{Z}_n$ with $(|M|, n) = 1$ and M is abelian.*
- (2) *If G is even order, then $G = U \times V$, where $(|U|, |V|) = 1$, U is a 1-DCI-group of odd order, and V is a Sylow 2-subgroup of G , or $V = \text{PSL}(2, q)$ or $\text{SL}(2, q)$ with $q \in \{5, 7, 8, 9\}$ or $\text{PSL}(3, 4)$, or $V = (L \times M) \rtimes ((H \rtimes K) \times \mathbb{Z}_n)$ where $6, |L|, |M|$ and n are pairwise coprime, and*
 - (i) $L \times M$ is abelian, $\langle M, H \rangle = M \times H$ and $\langle L, \mathbb{Z}_n \rangle = L \times \mathbb{Z}_n$,
 - (ii) $L \rtimes H \rtimes K = \mathbb{Z}_{5^u}^2 \rtimes \mathbb{Z}_{3^t} \rtimes \mathbb{Z}_{2^r}, \mathbb{Z}_{3^u}^4 \rtimes \mathbb{Z}_{5^t} \rtimes \mathbb{Z}_{2^r}$, or $\mathbb{Z}_q^2 \rtimes \mathbb{Q}_8 \rtimes \mathbb{Z}_{3^s}$ where $q \neq 1$, each prime divisor of q lies in $\{3, 5, 7, 11, 23\}$, and if $3 \mid q$ then $M = 1$ and $s = 0$.

9.2. Finite m -DCI-groups for $m \geq 2$

We note that an m -DCI-group has the k -DCI property for all $k \leq m$. Applying the k -DCI property for $2 \leq k \leq m$ to the groups listed in Theorem 9.2, an explicit list of groups which contain finite m -DCI-groups for $m \geq 2$ is produced in [78]. Let M be an abelian group all Sylow subgroups of which are homocyclic, and let m be the exponent of M . We define certain non-abelian extensions of such homocyclic groups M . Let r, s be non-negative integers such that $r + s \geq 1$, and suppose that there exists an integer l such that $1 < l < m$ and l has order e modulo m (that is, e is the least positive integer such that $l^e \equiv 1 \pmod{m}$ and we write $\text{o}(l \pmod{m}) = e$) and in addition r, s, e are as one of the lines in Table 1.

Definition 9.3. Let

$$E_e(M, 2^r 3^s) = M \rtimes \langle z \rangle = M \rtimes \mathbb{Z}_{2^r 3^s}, \text{ where } x^z = x^l \text{ for all } x \in M.$$

Table 1

r, s	e
$r = 0, s \geq 1$	3
$r \geq 1, s = 0$	2 or 4
$r \geq 1, s \geq 1$	6

Let $m \geq 2$ be an integer, and let $\mathcal{DCI}(m)$ denote the class of finite groups $G = U \times V$ which satisfies the following conditions:

- (i) a Sylow p -subgroup G_p of G is homocyclic or \mathbb{Q}_8 , and further, if $p = m$ then G_p is elementary abelian, cyclic or \mathbb{Q}_8 ; if $p < m$ then G_p is elementary abelian, \mathbb{Z}_4 or \mathbb{Q}_8 ;
- (ii) $(|U|, |V|) = 1$, U is nilpotent, and V is one of the following: A_4 , $\mathbb{Q}_8 \rtimes \mathbb{Z}_3$, $\mathbb{Z}_3^2 \rtimes \mathbb{Q}_8$, A_5 or $E_e(M, 2^r)$ with $r \geq 1$.

Then we have an explicit list for the candidates of m -DCI-groups for $m \geq 2$.

Theorem 9.4 (Li, et al. [78]). *Let G be an m -DCI-group for some $m \geq 2$. Then G is a member of $\mathcal{DCI}(m)$.*

The problem of determining finite m -DCI-groups is therefore reduced to the problem of determining which members G of $\mathcal{DCI}(m)$ are m -DCI-groups. A further reduction for the study of m -DCI-groups is given in the next theorem.

Theorem 9.5 (Li [71]). *Let $m \geq 2$ be an integer, and let G be a finite group which is a member of $\mathcal{DCI}(m)$. Then G is an m -DCI-group if and only if, for each subset $S \subset G$, $\text{Cay}(\langle S \rangle, S)$ is a CI-graph of $\langle S \rangle$.*

This result was used in [71] to prove that a finite group G is a 2-DCI-group if and only if G is a member of $\mathcal{DCI}(2)$. For special classes of groups, there are some further results: by Fang and Xu [28,29,33], an abelian group G is an m -DCI-group for $m \leq 3$ if and only if $G \in \mathcal{DCI}(m)$; by Qu and Yu [103], a dihedral group G is an m -DCI-group for $m \leq 3$ if and only if $G \in \mathcal{DCI}(m)$; by Ma [87], a di-cyclic group G is an m -DCI-group for $m \leq 3$ if and only if $G \in \mathcal{DCI}(m)$. Combining Theorem 9.5 and Theorem 6.8, an immediate consequence is that if G is an abelian member of $\mathcal{DCI}(m)$ then G is a $(p + 1)$ -DCI-group where p is the smallest prime divisor of $|G|$. However, it is still a very difficult problem to obtain a complete classification of general m -DCI-groups for $m \geq 3$.

Problem 9.6. Completely determine m -DCI-groups for certain small values of $m \geq 3$. In particular, give a complete classification of 3-DCI-groups.

9.3. Finite m -CI-groups

For undirected graphs, dependent on the classification of affine primitive permutation groups of rank 2 and 3, a description of finite 2-CI-groups is given in [76,77].

Table 2

L	$H \rtimes K$	Conditions
\mathbb{Z}_2^3	$\mathbb{Z}_{7^t} \rtimes \mathbb{Z}_{3^s}$	$t, s \geq 1$
$\mathbb{Z}_{5^{u_1} 7^{u_2} 11^{u_3}}^2$	$\mathbb{Z}_{3^r} \rtimes \mathbb{Z}_{2^r}$	$r, t \geq 1, u_1 + u_2 + u_3 \geq 1$
$\mathbb{Z}_{3^{u_1}}^4 \times \mathbb{Z}_{19^{u_2}}^2$	$\mathbb{Z}_{5^r} \rtimes \mathbb{Z}_{2^r}$	$r, t \geq 1, u_1 + u_2 \geq 1$
$\mathbb{Z}_{7^{u_1} 11^{u_2} 19^{u_3}}^2$	$\mathbb{Z}_{3^{t_1} 5^{t_2}} \rtimes \mathbb{Z}_{2^r}$	$r, t_1, t_2 \geq 1, u_1 + u_2 \geq 1, u_3 \geq 1,$ $[\mathbb{Z}_{19^{u_3}}, \mathbb{Z}_{3^{t_1}}] = 1, [\mathbb{Z}_{7^{u_1} 11^{u_2}}, \mathbb{Z}_{5^{t_2}}] = 1$
$\mathbb{Z}_{3^{u_1}}^6 \times \mathbb{Z}_{19^{u_2}}^2$	$\mathbb{Z}_{7^{t_1} 5^{t_2}} \rtimes \mathbb{Z}_{2^r}$	$r, t_1 \geq 1, u_1 + u_2 \geq 1, t_2 = 0 \Leftrightarrow u_2 = 0$ $[\mathbb{Z}_{3^{u_1}}, \mathbb{Z}_{5^{t_2}}] = 1, [\mathbb{Z}_{19^{u_2}}, \mathbb{Z}_{7^{t_1}}] = 1$
$\mathbb{Z}_{5^{u_1} 11^{u_2}}^2$	$\mathbb{Z}_{3^{t_1} 7^{t_2}} \rtimes \mathbb{Z}_{2^r}$	$u_1, t_1, t_2, r \geq 1, u_2 \geq 0, [\mathbb{Z}_{11^{u_2}}, \mathbb{Z}_{7^{t_2}}] = 1$
\mathbb{Z}_q^2	$\mathbb{Q}_8 \rtimes \mathbb{Z}_{3^s}$	$q \neq 1, \text{each prime divisor of } q \text{ lies in } \{3, 5, 7, 11, 23\},$ $\text{and if } 3 \text{ divides } q \text{ then } M = 1, s = 0 \text{ and } n = 1$

Theorem 9.7. Suppose that G is a finite 2-CI-group. Then a Sylow p -subgroup G_p of G is either homocyclic, or $p = 2$ and G_2 is elementary abelian, cyclic or generalized quaternion.

- (1) If $|G|$ is odd, then $G = U \times V$, where $(|U|, |V|) = 1$, U is abelian and $V = M \rtimes \mathbb{Z}_n$ with $(|M|, n) = 1$ and M is abelian.
- (2) If $|G|$ is even and G is not nilpotent, then $G = U \times V$, where $(|U|, |V|) = 1$, $|U|$ is odd, and either $V = \mathbb{A}_5$, $\text{PSL}(2, 8)$, $\text{SL}(2, 5)$, $\text{SL}(2, 7)$ or $\text{SL}(2, 9)$, or $V = (L \times M) \rtimes ((H \rtimes K) \times \mathbb{Z}_n)$ where $|L|, |H|, |K|, |M|$ and n are pairwise coprime, and
 - (i) $L \times M$ is abelian, $\langle M, H \rangle = M \times H$ and $\langle L, \mathbb{Z}_n \rangle = L \times \mathbb{Z}_n$,
 - (ii) $H \rtimes K$ is indecomposable, neither H nor K centralizes a Sylow subgroup of L , and $(L, H \rtimes K)$ satisfies one of the lines in Table 2.

Applying the k -CI property for $3 \leq k \leq m$ to the groups listed in Theorem 9.7, an explicit list of groups which contains all m -CI-groups for $m \geq 4$ was produced in [77].

Theorem 9.8. Let G be an m -CI-group for some $m \geq 4$ and G_p a Sylow p -subgroup of G for some prime p . Then $G = U \times V$, where $(|U|, |V|) = 1$, U is nilpotent, and either $V = 1$ or V is one of the following:

- (i) $\mathbb{Z}_3^2 \rtimes \mathbb{Q}_8$, \mathbb{A}_5 , $\text{SL}(2, 5)$;
- (ii) $\mathbb{Z}_2^2 \rtimes \mathbb{Z}_{3^s}$, $\mathbb{Q}_8 \rtimes \mathbb{Z}_{3^s}$ for some $s \geq 1$;
- (iii) $E_e(M, 2^r 3^s)$, where $e \in \{2, 3, 4, 6\}$, $r, s \geq 0$ and $r + s \geq 1$.

Moreover, if $p > [m/2]$ then G_p is homocyclic; if $p = [m/2]$ then G_p is elementary abelian, cyclic or \mathbb{Q}_8 ; if $p < [m/2]$ then either G_p is elementary abelian, or $G_p \cong \mathbb{Z}_4, \mathbb{Z}_8, \mathbb{Z}_9$ or \mathbb{Q}_8 .

By Proposition 5.3, if $m \geq 6$ then $r = 1, 2$ or 3 , and if $m \geq 8$ then $s = 1$ or 2 . Further, with the assistance of computer, Conder and the author [16] proved that A_5 is not a 5-CI-group, and that neither $Q_8 \rtimes \mathbb{Z}_3$ nor $\text{SL}(2, 5)$ is a 7-CI-group. Therefore, in the case where $m \geq 8$, G is soluble and we may have a finer description for m -CI-groups which is similar to that for CI-groups (see Theorem 8.7). However, as for m -DCI-groups, even with this result, it is still a very hard problem to obtain a complete classification of m -CI-groups for $m \geq 4$. Let $\mathcal{CI}(m)$ denote the class of groups which satisfy Theorem 9.8. Then \mathbb{Z}_2^6 is a member of $\mathcal{CI}(m)$, but by [97], \mathbb{Z}_2^6 is not a 31-CI-group. From [32,34], we know that for $m \leq 5$, an abelian group G is an m -CI-group if and only if $G \in \mathcal{CI}(m)$. We are particularly interested in the case of small values of m , and we think solving the following problem must be interest for understanding better m -(D)CI-groups.

Problem 9.9. (1) Completely determine m -CI-groups for certain small values of $m \geq 4$. In particular, give a complete classification of 4-CI-groups.

(2) For given a prime p and an integer $d \geq 4$, find the largest value of m such that \mathbb{Z}_p^d is an m -(D)CI-group.

10. Finite groups with the m -DCI or m -CI property

Finite m -DCI-groups have the i -(D)CI property, and m -CI-groups have the i -CI property for all $i \leq m$. They have very restricted structure shown as in Theorems 9.4 and 9.8. Naturally we would ask: if we only impose a single m -DCI or m -CI property on a group G , what can we say about the structure of G ? In [79], Praeger, Xu and the author initiated the study of finite groups with the m -(D)CI property, and proposed

Problem 10.1. Characterize finite groups with the m -DCI property or the m -CI property.

10.1. Simple groups with the m -CI-property

By the definition, finite groups with the 1-DCI property are exactly the 1-DCI-groups, which were characterized in Theorem 9.2. Finite groups G with the 1-CI property are exactly the finite groups G in which all involutions are conjugate under $\text{Aut}(G)$. Now let G be a finite group with the 2-CI property. From Lemma 5.1, it follows that a Sylow 2-subgroup of G is cyclic, generalized quaternion or elementary abelian. By Robinson [104, 10.2.2], a finite group with a cyclic or generalized quaternion Sylow 2-subgroup is not simple. Thus a Sylow 2-subgroup of a nonabelian simple 2-CI-group G must be elementary abelian. Therefore, by Suzuki [111, p. 582], G is one of the following groups:

$$\text{J}_1, \text{ Ree}(3^{2n+1}), \text{ PSL}(2, 2^n), \text{ or } \text{PSL}(2, q) \text{ with } q \equiv \pm 3 \pmod{8}.$$

It was easily proved in [75] that, for any element $z \in G$,

$$|\{\{z^i, z^{-i}\} \mid (\text{o}(z), i) = 1\}| \text{ divides } |\text{N}_{\text{Aut}(G)}(\langle z \rangle)/\text{C}_{\text{Aut}(G)}(\langle z \rangle)|.$$

Applying this divisibility condition to these simple groups G , calculations show that G must be A_5 or $\text{PSL}(2, 8)$. Further, we have

Theorem 10.2 (Conder and Li [16] and Li and Praeger [75]). *Suppose that G is a non-abelian simple group. Then*

- (i) G has the 2-CI property if and only if $G = A_5$ or $\text{PSL}(2, 8)$;
- (ii) G has the m -CI property for $m = 3, 4$ if and only if $G = A_5$;
- (iii) G does not have the 5-CI property.

10.2. Groups with the m -DCI-property but not the k -DCI-property

Several infinite families of finite groups are constructed in [70,60] which show the m -(D)CI property does not necessarily imply the k -(D)CI property for $k < m$. Such groups involve Frobenius groups $F(M, n) = M \rtimes \langle z \rangle$, where

- (i) M is an abelian group of odd order and all Sylow subgroups of M are homocyclic;
- (ii) $\langle z \rangle \cong \mathbb{Z}_n$ where $n \geq 2$, and $(|M|, n) = 1$;
- (iii) there exists an integer l such that for any $x \in M \setminus \{1\}$, $x^z = x^l$ and n is the least positive integer satisfying $l^n \equiv 1 \pmod{\text{o}(x)}$.

The following result is obtained in [70,60].

Theorem 10.3. (1) Let $G = F(M, q)$ and $m = q - 1$ where q is a prime. If $q \geq 3$ then G has the m -DCI property but does not have the k -DCI property for any $k < m$.

(2) Let $G = F(M, q)$ and $m = q - 1$, where $q \geq 5$ is a prime. Then G has the m -CI property but does not have the non-trivial k -CI property for any $k < m$.

(3) Let H be a group of odd order such that H has the 2-CI property and 3 divides $|H|$, and let $G = H \times A_4$. Then G has the 3-CI property but does not have the 2-CI property.

However, it is proved in [70] that the 3-DCI property implies the 1-DCI or the 2-DCI property. In the cases $m = 2$ and 4, we have a complete characterization.

Theorem 10.4 (Li [59,70]). *For $m = 2$ or 4, a finite group G has the m -DCI property but does not have the k -DCI property for all $k < m$ if and only if $G = F(M, m + 1)$ for some abelian group M .*

For $m \geq 5$, we formulate some problems as follows.

Problem 10.5. Let $m \geq 5$ be an integer.

- (1) Characterize finite groups which have the m -DCI property but not the k -DCI property for any k with $5 \leq k < m$.
- (2) Characterize finite groups which have the m -CI property but not the k -CI property for any k with $2 \leq k < m$.

This problem seems to be very difficult in general. It would be reasonable firstly to consider the case where G is of prime-power order or nilpotent. We conjecture that nilpotent groups which have the m -DCI property are homocyclic (refer to Theorem 10.8). Peter Neumann (1996, pri. comm.) posed a conjecture that if G is a group of odd order and has the m -DCI property then G is a semidirect product of a nilpotent group by a cyclic group.

10.3. Sylow subgroups

For general groups, we only have the following theorems which describe the structure of Sylow subgroups of groups with the m -DCI property and the m -CI property for ‘large’ values of m .

Theorem 10.6 (Li, Praeger and Xu [79]). *Let G be a finite group with the m -DCI property, let p be an odd prime dividing $|G|$ such that $p \leq m \leq 2p - 2$, and let G_p be a Sylow p -subgroup of G .*

- (1) *If $m = p$, then either G_p is of exponent p or G_p is cyclic.*
- (2) *If $p + 1 \leq m \leq 2p - 2$, then G_p is of exponent p .*

Further, if $p \leq m \leq 2p - 2$ and G_p is noncyclic, then all elements of G of order p are conjugate under $\text{Aut}(G)$.

Theorem 10.7 (Li, Praeger and Xu [79]). *Let G be a finite group with the $2m$ -CI property, let p be an odd prime dividing $|G|$ such that $p \leq m \leq (3p - 1)/2$, and let G_p be a Sylow p -subgroup of G .*

- (1) *If $m = p$ then either G_p is of exponent p or G_p is cyclic.*
- (2) *If $p = 3$ and $m = 4$ then G_3 is of exponent 3 or $G_3 \cong \mathbb{Z}_9$;
if $p = 3$ and $m = 5$ then G_3 is of exponent at most 9;
if $p \geq 5$ and $p + 1 \leq m < (3p - 1)/2$ then G_p is of exponent p .*

For abelian groups with the m -DCI property, we have a further characterization.

Theorem 10.8 (Li [68]). *If an abelian group G has the m -DCI property, then all Sylow subgroups of G are homocyclic.*

By Li, Praeger and Xu [79, Theorem 1.6], for abelian groups G and for $1 \leq m \leq 4$, the m -DCI property implies the k -DCI property for all $k < m$. On the other hand, by Li [68], \mathbb{Z}_{25} has the 9-DCI property but does not have the k -DCI property for $k = 6, 7$ or 8. It is asked in [68] whether, for abelian groups and $5 \leq m \leq 8$, the m -DCI property imply the k -DCI property for all $k < m$? It is easy to see that an abelian group with all Sylow subgroups homocyclic has the 1-DCI property, and therefore, by Theorem 10.8, for abelian groups the m -DCI property implies the 1-DCI property for any positive integer m . It seems that for abelian groups, with a few exceptions, the

m -DCI property implies the k -DCI property for all $k < m$. The following problem is naturally proposed in [68].

Problem 10.9. Classify the finite abelian groups which have the m -DCI property but do not have the k -DCI property for some $k < m$.

There should be similar results for groups with the m -CI property, and the following conjecture is thus proposed in [68]:

Conjecture 10.10. If G is an abelian group with the m -CI property for some integer $m \geq 2$, then all Sylow subgroups of G are homocyclic.

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