Estimating Euler's Number (e) Using Probability Theory

Introduction

Euler's number e is a fundamental mathematical constant. In this report, we estimate using two probabilistic approaches.

Method 1: Geometric Mean of Uniform Samples

Concept

If we take a large number of samples from a uniform distribution , compute their geometric mean, and take its inverse, we obtain an estimate for . Repeating this process multiple times and averaging the results improves accuracy.

Implementation

```
n= 1e6        ## No.of observation from U(0,1) by each individual.

k= 1e2       ## No. of observer..

vec<- numeric(length = k)## Vector that will store the estimated value of e from each

for( i in 1:k){
  uni<- runif(n,0,1)         ##generation of 1e6 samples from U(0,1)

  yn<- prod(uni^(1/n))       ## Taking geometric mean of all samples
```

```r
  vec[i]<- 1/yn
}

e <- mean(vec)  ## mean of the values estimated by every individual.

cat("The value of e :",e)      ## Value of estimated e..
```

```
The value of e : 2.718274
```

Mathematical Proof of Method 1

Let $X_1, X_2, \ldots, X_n$ be independent and identically distributed (i.i.d.) random variables drawn from the uniform distribution $U(0,1)$. We define their geometric mean as:

$$Y_n = \left(\prod_{i=1}^{n} X_i\right)^{\frac{1}{n}}$$

Taking the natural logarithm on both sides:

$$\log Y_n = \frac{1}{n}\sum_{i=1}^{n} \log X_i$$

Since $X_i \sim U(0,1)$, we use the known expectation result:

$$\mathbb{E}[\log X] = \int_0^1 \log x \, dx = -1$$

By the Law of Large Numbers, as $n \to \infty$, the sample mean of $\log X_i$ converges to its expected value:

$$\mathbb{E}[\log Y_n] \approx -1$$

which implies:

$$\mathbb{E}[Y_n] \approx e^{-1}$$

Taking the reciprocal gives our estimator for e:

$$\hat{e} = \frac{1}{Y_n}$$

Thus, as n grows large, $\hat{e}$ provides an approximation to Euler's number e.

Explanation

- The geometric mean of a large number of independent uniform variables approximates ,
  so taking its inverse gives an estimate of .
- By repeating the experiment 100 times, we mitigate randomness and improve accuracy.

Method 2: Estimation Using Derangements of a Deck of Cards

Concept

- A derangement is a permutation where no element appears in its original position.
- The probability of a derangement of objects converges to , so the estimate of is given by the inverse of this probability.

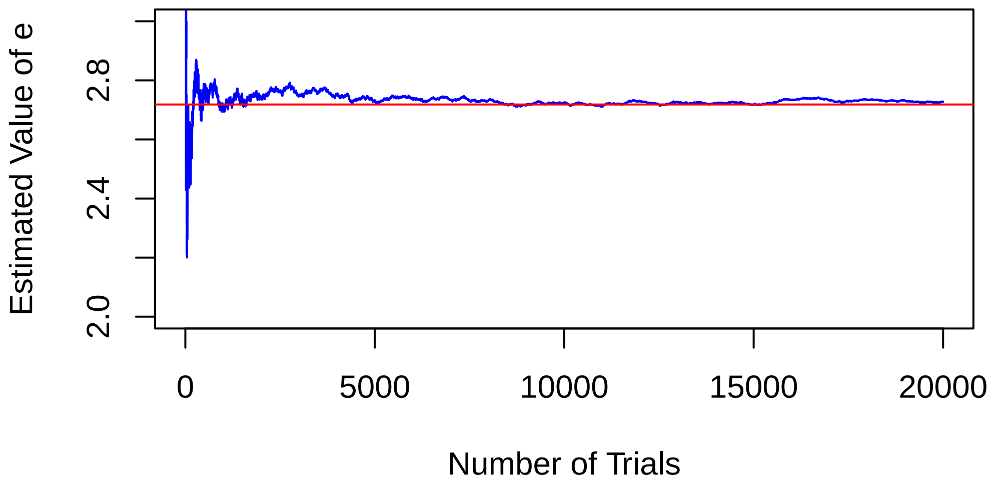
Implementation

```
is_derangement <- function(permutation) {
  all(permutation != seq_along(permutation))
  ## Check if the permutation is deranged
}

estimate_e <- function(deck_size, trials) {
  derangements <- 0
  cumulative_e <- numeric(trials)
  ## Store e estimates at each trial

  for (i in 1:trials) {
    deck <- sample(1:deck_size)
    ## Shuffle the cards

    if (is_derangement(deck)) {
      derangements <- derangements + 1
    }

    cumulative_e[i] <- ifelse(derangements > 0, i / derangements, Inf)
    #inverse of probability after each trial
  }
```

```r
  return(cumulative_e) #e estimates after each trial
}

cards <- 52  ## Standard 52 deck of cards (numbered 1 to 52)
trials <- 2*1e4  ## Number of trials
repeats <-100;   ## Repetitions for accuracy

vec_e<-numeric(length = repeats); #vector that stores e estimates

for(i in 1:repeats){
  cumulative_e <- estimate_e(cards,trials)
  vec_e[i] <- cumulative_e[length(cumulative_e)]
}

final_estimate <- mean(vec_e)

cat("The value of e:", final_estimate)
```

```
The value of e: 2.718365
```

```r
## Plot how estimate gets closer to real value
plot(1:trials, cumulative_e, type = "l", col = "blue",
     , ylim = c(2, 3), xlab = "Number of Trials", ylab = "Estimated Value of e")
abline(h = exp(1), col = "red")
```



Mathematical Proof of Method 2

A derangement is a permutation where no element appears in its original position. We define:

- D_n as the number of derangements of an n -element set.
- n! as the total number of permutations.

The probability of a random permutation being a derangement is:

$$P(D_n) = \frac{D_n}{n!}$$

Using the recurrence relation:

$$D_n = (n-1)(D_{n-1} + D_{n-2})$$

it follows that:

$$D_n = \left(\frac{n!}{e}\right)\sum_{k=0}^{n}\frac{(-1)^k}{k!}$$

Taking the limit:

$$\lim_{n\to\infty} \frac{D_n}{n!} = \frac{1}{e}$$

Thus, the probability of a derangement approaches $(\frac{1}{e})$. Given k independent trials and p observed derangements, the estimate of e is:

$$\hat{e} = \frac{k}{p}$$

which converges to e as $k \to \infty$.

Explanation

- The experiment repeatedly shuffles a deck of 52 cards and checks if it is a derangement.
- The probability of derangement D_n/n! so the inverse of the observed probability provides an estimate of e.
- The red line in the plot represents the true value of e showing convergence.

Conclusion

Both methods successfully estimate e. The geometric mean method relies on properties of uniform distributions, while the derangement method leverages combinatorial probabilities. The results illustrate how fundamental mathematical constants emerge naturally in probability theory.