5. If X and Y are any two random variables such that $Y \le X$, then $E(Y) \le E(X)$.

Proof Given: $Y \le X \Rightarrow Y - X \le 0$

i.e. $X - Y \ge 0$

$\therefore$ $E(X - Y) \ge 0 \Rightarrow E(X) - E(Y) \ge 0$

i.e. $E(X) \ge E(Y) \Rightarrow E(Y) \le E(X)$

2.11 COVARIANCE (X, Y)

The covariance of the two random variables is denoted by

$$\rho(x, y) = \rho_{xy} = \text{Cov}(X, Y) \text{ which is defined as}$$

$$\text{Cov}(X, Y) = E(XY) - E(X)\,E(Y)$$

Important note:

(i) If X and Y are independent random variables, then Cov $(X, Y) = 0$.

(ii) If X and Y are any two random variables, then
$\text{Var}(aX \pm bY) = a^2\,\text{Var}(X) + b^2\,\text{Var}(Y) \pm 2ab\,\text{Cov}\,(X, Y)$

(iii) $\text{Cov}(aX + b, cY + d) = ab\,\text{Cov}(X, Y)$.

EXAMPLE 2.65 The number of hardware failures of a computer sytem in a week of operations has the following PMF:

Number of failures	0	1	2	3	4	5	6
Probability	0.18	0.28	0.25	0.18	0.16	0.04	0.01

Find the mean of the number of failures in a week. **[AU June '06]**

Solution We know that

$$E(X) = \sum_{x=0}^{6} xp(x)$$

$$= 0 \times 0.18 + 1 \times 0.28 + 2 \times 0.25 + 3 \times 0.18 + 4 \times 0.06 + 5 \times 0.04 + 6 \times 0.01$$

$$E(X) = 1.82$$

EXAMPLE 2.66 If X has the distribution function

$$f(x) = \begin{cases} 0, & x < 1 \\ \frac{1}{3}, & 1 \le x < 4 \\ \frac{1}{2}, & 4 \le x < 6 \\ \frac{7}{8}, & 6 \le x < 10 \\ 1, & x \ge 0 \end{cases}$$

Find

(i) the probability distribution function of X,
(ii) $P(2 < X < 6)$,
(iii) mean of X, and
(iv) variance of X. **[AU April/May '08]**

Solution

(i) The probaility distribution of X using $p(x) = P(X = x) = F(x_i) - F(x_{i-1})$

x	1	4	6	10
$P(X = x)$	1/3	1/6	1/3	1/6

(ii) $P(2 < X < 6) = P(X = 4) = \frac{1}{6}$

(iii) Mean of $X = E(X) = \Sigma xp(x) = 1\times\frac{1}{3} + 4\times\frac{1}{6} + 6\times\frac{1}{3} + 10\times\frac{1}{6} = \frac{14}{3}$

$$E(X^2) = \Sigma x^2 p(x) = 1\times\frac{1}{3} + 4^2\times\frac{1}{6} + 6^2\times\frac{1}{3} + 10^2\times\frac{1}{6} = \frac{95}{3}$$

$$\therefore \quad \text{Var}(X) = E(X^2) - [E(X)]^2 = \frac{89}{9}$$

EXAMPLE 2.67 If X and Y are independent random variables with means 2, 3 and variance 1, 2 respectively, find the mean and variance of the random variable $Z = 2X - 5Y$. **[AU May '07]**

Solution Given:

$$E(X) = 2,\ E(Y) = 3,\ \text{Var}(X) = 1,\ \text{Var}(Y) = 2,\ Z = 2X - 5Y$$

$$\therefore \quad E(Z) = E(2X - 5Y) = 2E(X) - 5E(Y)$$

$$E(Z) = (2 \times 2) - (5 \times 3) = -11$$

If X and Y are independent, then $\text{Var}(aX + bY) = a^2\,\text{Var}(X) + b^2\,\text{Var}(Y)$

$$\therefore \quad \text{Var}(2X - 5Y) = 4\,\text{Var}(X) + 25\,\text{Var}(Y)$$

$$= (4 \times 1) + (25 \times 2) = 54$$

EXAMPLE 2.68 If the range of X is the set $\{0, 1, 2, 3, 4\}$ and $P(X = x) = 0.2$, determine the mean and variance of the random variable. **[AU December '09]**

Solution Given:

x	0	1	2	3	4
$P(X = x)$	0.2	0.2	0.2	0.2	0.2

By definition,

$$E(X) = \sum_{x=0}^{4} xp(x) = 0.2(0+1+2+3+4)$$

$$= 0.2 \times 10 = 2$$

$$E(X^2) = \sum_{x=0}^{4} x^2 p(x) = 0.2(0+1+4+9+16) = 6$$

$$\therefore \quad \text{Var}(X) = E(X^2) - [E(X)]^2 = 6 - 4 = 2$$

EXAMPLE 2.69 A fair coin is tossed three times. Let X be the number of tails appearing. Find the probability distribution of X and also calculate $E(X)$. **[AU December '09]**

Solution Given: X denotes the number of tails. Since the coin is tossed 3 times, the number of tails may be 0, 1, 2 or 3

$$n(S) = 2^3 = 8.$$

$$S = \{HHH, HTH, THH, HHT, TTH, THT, HTT, TTT\}$$

Given:

x	0	1	2	3
$P(X = x)$	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$

The probability distribution is

$$F(x) = \begin{cases} 0, & x < 0 \\ \frac{1}{8}, & 0 \le x < 1 \\ \frac{4}{8}, & 1 \le x < 2 \\ \frac{7}{8}, & 2 \le x < 3 \\ 1, & x \ge 3 \end{cases}$$

$$E(X) = \sum_{x=0}^{3} xp(x) = 0\times\frac{1}{8} + 1\times\frac{3}{8} + 2\times\frac{3}{8} + 3\times\frac{1}{8} = \frac{3}{2}$$

EXAMPLE 2.70 Given the following probability distribution of X, compute

(i) $E(X)$,
(ii) $E(X^2)$,

(iii) $E(2X \pm 3)$, and
(iv) $\text{Var}(2X \pm 3)$

x	−3	−2	−1	0	1	2	3
$p(x)$	0.05	0.10	0.30	0	0.30	0.15	0.10

[AU June '07]

Solution We know that for a discrete random variable X

(i) $$E(X) = \sum_{i=1}^{n} x_i\, p(x_i)$$
$$= (-3)\,(0.05) - 2(0.1) - 1(0.30) + 0 + 1(0.30) + 2(0.15) + 3(0.10)$$
$$= 0.25$$

(ii) $$E(X^2) = \sum_{i=1}^{n} x_i^2\, P(x_i)$$
$$= (-3)^2\,(0.05) + (-2)^2\,(0.1) + (-1)^2\,(0.30) + 0 + (1)^2\,(0.30) + (2)^2\,(0.15) + (3)^2\,(0.10)$$
$$= 2.95$$

(iii) $$E(2X \pm 3) = 2E(X) \pm 3 = 2(0.25) \pm 3 = 0.5 \pm 3$$

(iv) $$\text{Var}(2X \pm 3) = 2^2\,\text{Var}(X)$$
$$\text{Var}(X) = E(X^2) - [E(X)]^2 = 2.95 - (0.25)^2 = 2.8875$$
$$\text{Var}(2X \pm 3) = 4(2.8875) = 11.55$$

EXAMPLE 2.71 The monthly demand for Allwyn watches is known to have the following probabilty distribution:

Demand	1	2	3	4	5	6	7	8
Probability	0.08	0.12	0.19	0.24	0.16	0.10	0.07	0.04

Find the expected demand for watches. Also compute the variance.

[AU December '06]

Solution We know that for a discrete random variable X

(i) $$E(X) = \sum_{i=1}^{8} x_i\, p(x_i)$$

$$= (1)\,(0.08) + 2(0.12) + 3(0.19) + 4(0.24) + 5(0.16) + 6(0.10) + 7(0.07) + 8(0.04)$$
$$= 4.06$$

(ii) $$E(X^2) = \sum_{i=1}^{8} x_i^2\, p(x_i)$$
$$= 1(0.08) + 4(0.12) + 9(0.19) + 16(0.24) + 25(0.16) + 36(0.10) + 49(0.07) + 64(0.04)$$
$$= 19.7$$

(iii) $$\text{Var}(X) = E(X^2) - [E(X)]^2 = 19.7 - (4.06)^2 = 19.7 - 16.48 = 3.22$$

EXAMPLE 2.72 Suppose that the random variable X is equal to the number of hits obtained by a certain base ball player in his next 3 bats and if $P(X = 1) = 0.3$, $P(X = 2) = 0.2$ and $P(X = 0) = 3P(X = 3)$, find $E(X)$.

[AU December '03]

Solution The random variable X is the number of hits obtained and $i = 3$ bats, $i = 0, 1, 2, 3$.

$$P(X = 0) + P(X = 1) + P(X = 2) + P(X = 3) = 1 \quad \text{(i)}$$
Given: $$P(X = 0) = 3\,P(X = 3) \quad \text{(ii)}$$

Substituting Eq. (ii) in Eq. (i), we get

$$3P(X = 3) + 0.3 + 0.2 + P(X = 3) = 1$$
i.e.
$$4P(X = 3) = 1 - 0.5$$
$$P(X = 3) = \frac{0.5}{4} = 0.125$$
$$\therefore \quad P(X = 0) = 3P(X = 3) = 3 \times 0.125 = 0.375$$

We know that

$$E(X) = \sum_{i=0}^{3} x_i\, p(x_i)$$
$$= (1)\,P(X = 1) + (2)\,P(X = 2) + (3)\,P(X = 3)$$
$$= 1(0.3) + 2(0.2) + 3(0.125)$$
$$= 1.075$$

EXAMPLE 2.73 The cumulative distribution function (CDF) of a random variable X is $F(x) = 1 - (1 + x)\,e^{-x}$, $x > 0$. Find the probability density function of X, mean and variance.

[AU June '07, May '06]

Solution Given: $F(x) = 1 - (1 + x)e^{-x}, \; 0 < x < \infty$

$$\text{PDF} = f(x) = \frac{d}{dx}[F(x)] = \frac{d}{dx}[1 - (1+x)e^{-x}]$$

$$= 0 + e^{-x} - (-xe^{-x} + e^{-x}) = xe^{-x}, \; x > 0$$

$$\text{Mean} = E(X) = \int_0^\infty xf(x)\,dx = \int_0^\infty x \cdot xe^{-x}dx = \int_0^\infty x^2e^{-x}dx$$

$$= \left[x^2(-e^{-x}) - (2x)(e^{-x}) + 2(-e^{-x})\right]_0^\infty$$

$$= 0 + 2 = 2$$

$$E(X^2) = \int_0^\infty x^2 f(x)dx = \int_0^\infty x^2 \cdot xe^{-x}dx = \int_0^\infty x^3e^{-x}dx$$

$$= \left[x^3(-e^{-x}) - (3x^2)(e^{-x}) + (6x)(-e^{-x}) - (6)(e^{-x})\right]_0^\infty$$

$$= 6$$

$$\therefore \quad \text{Var}(X) = E(X^2) - [E(X)]^2 = 6 - 4 = 2$$

EXAMPLE 2.74 If $dF = kx^2e^{-x}\,dx$, $x \geq 0$, find k, mean and variance. **[AU May '06]**

Solution Given: $\dfrac{dF}{dx} = kx^2\,e^{-x} = f(x), \; x \geq 0,$

(i) Since $f(x)$ is a PDF

$$\int_{-\infty}^{\infty} f(x)dx = 1$$

$$\therefore \quad \int_0^\infty kx^2e^{-x}dx = 1 \Rightarrow k\int_0^\infty x^2e^{-x}dx = 1$$

$$k\left[\frac{x^2e^{-x}}{-1} - \frac{2xe^{-x}}{1} + \frac{2e^{-x}}{-1}\right]_0^\infty = 1 \Rightarrow k(0+2) = 1 \Rightarrow k = \frac{1}{2}$$

(ii) $$E(X) = \int_0^\infty xf(x)dx = \int_0^\infty \frac{1}{2}x^3e^{-x}dx$$

$$= \frac{1}{2}\left[x^3\frac{e^{-x}}{-1} - 3x^2 \cdot \frac{e^{-x}}{1} + 6x\frac{e^{-x}}{-1} - \frac{6e^{-x}}{1}\right]_0^\infty$$

$$= \frac{1}{2}(6) = 3$$

(iii) $$E(X^2) = \int_0^\infty x^2 f(x)dx = \frac{1}{2}\int_0^\infty x^4e^{-x}dx$$

$$= \frac{1}{2}\left[\frac{x^4e^{-x}}{-1} - \frac{4x^3e^{-x}}{1} + \frac{12x^2e^{-x}}{-1} - \frac{24x\,e^{-x}}{1} + \frac{24e^{-x}}{-1}\right]_0^\infty$$

$$= 12$$

$$\therefore \quad \text{Var}(X) = E(X) - [E(X)]^2 = 12 - 9 = 3$$

EXAMPLE 2.75 Let X be a random variable with $E(X) = 10$ and $\text{Var}(X) = 25$. Find the positive values of a and b such that $Y = aX - b$ has expectation 0 and variance 1. **[AU June '06, May '07]**

Solution Given: $E(X) = 10$

and $\text{Var}(X) = 25$

Now, $E(Y) = E(aX - b) = 0$ (given)

$$aE(X) - b = 0 \Rightarrow a(10) - b = 0$$

$$10a - b = 0 \qquad \text{(i)}$$

Given: $\text{Var}(Y) = 1$

$$\text{Var}(aX - b) = a^2\,\text{Var}(X) = 1 \Rightarrow 25a^2 = 1$$

i.e. $a = \dfrac{1}{5}$

Substituting $a = \dfrac{1}{5}$ in Eq. (i), we get

$$b = 2$$

EXAMPLE 2.76 For the triangular distribution

$$f(x) = \begin{cases} x, & 0 < x \leq 1 \\ 2 - x, & 1 \leq x < 2 \\ 0, & \text{otherwise} \end{cases}$$

Find the mean and variance. **[AU May '06, June '07]**

Solution $$\text{Mean} = E(X) = \int_{-\infty}^{\infty} xf(x)dx = \int_0^1 x \cdot x\,dx + \int_1^2 x(2-x)dx$$

$$= \int_0^1 x^2dx + \int_1^2 (2x - x^2)dx$$

$$= \left[\frac{x^3}{3}\right]_0^1 + \left[2\left(\frac{x^2}{2}\right) - \frac{x^3}{3}\right]_1^2$$

$$= \frac{1}{3} + \left[\left(4 - \frac{8}{3}\right) - \left(1 - \frac{1}{3}\right)\right] = \frac{1}{3} + \frac{4}{3} - \frac{2}{3} = 1$$

$$E(X^2) = \int_{-\infty}^{\infty} x^2 f(x)dx$$

$$= \int_0^1 x^2 \cdot x\, dx + \int_1^2 x^2(2-x)dx$$

$$= \int_0^1 x^3 dx + \int_1^2 (2x^2 - x^3)dx$$

$$= \left[\frac{x^4}{4}\right]_0^1 + \left[2\left(\frac{x^3}{3}\right) - \frac{x^4}{4}\right]_1^2$$

$$= \frac{1}{4} + \left[\left(\frac{16}{3} - \frac{16}{4}\right) - \left(\frac{2}{3} - \frac{1}{4}\right)\right]$$

$$= \frac{1}{4} + \frac{16}{3} - 4 - \frac{2}{3} + \frac{1}{4} = \frac{7}{6}$$

$$\text{Var}(X) = E(X^2) - E(X) = \frac{7}{6} - (1)^2 = \frac{1}{6}$$

EXAMPLE 2.77 Find the value of

(i) c, and
(ii) mean of the following distribution given

$$f(x) = \begin{cases} c(x - x^2), & 0 < x < 1 \\ 0, & \text{otherwise} \end{cases}$$ **[AU December '06]**

Solution

(i) Since $f(x)$ is a PDF, $\int_{-\infty}^{\infty} f(x)dx = 1$

$$\int_0^1 c(x - x^2)dx = 1 \Rightarrow c\left[\frac{x^2}{2} - \frac{x^3}{3}\right]_0^1 = 1$$

$$c\left[\frac{1}{2} - \frac{1}{3}\right]_0^1 = 1 \Rightarrow c\left(\frac{3-2}{6}\right) = 1$$

$$\therefore \quad c = 6$$

(ii) Mean $= E(X) = \int_0^1 x \cdot 6(x - x^2)dx = \int_0^1 6(x^2 - x^3)dx$

$$= \left[\frac{6x^3}{3} - \frac{6x^4}{4}\right]_0^1 = \left[\frac{6}{3} - \frac{6}{4}\right] = 2 - \frac{3}{2} = \frac{1}{2}$$

EXAMPLE 2.78 If X has the probability density function

$$f(x) = \begin{cases} ke^{-3x}, & \text{for } x > 0 \\ 0, & \text{otherwise} \end{cases}$$

find k, $P(0.5 \le X \le 1)$ and the mean of X.

Solution Since $f(x)$ is a PDF, $\int_{-\infty}^{\infty} f(x)dx = 1$

$$\therefore \quad \int_0^{\infty} ke^{-3x}dx = 1 \Rightarrow k\left[\frac{e^{-3x}}{-3}\right]_0^{\infty} = 1 \Rightarrow \frac{k}{3} = 1, \text{ i.e. } k = 3$$

$$\therefore \quad f(x) = 3e^{-3x},\ x > 0$$

$$P(0.5 \le X \le 1) = \int_{0.5}^{1} f(x)dx = \int_{0.5}^{1} 3e^{-3x}dx$$

$$= 3\left[\frac{e^{-3x}}{-3}\right]_{0.5}^{1} = 3\left(\frac{e^{-1.5} - e^{-3}}{3}\right) = 0.17331$$

$$E(X) = \int_{-\infty}^{\infty} xf(x)dx = 3\int_0^{\infty} xe^{-3x}dx$$

$$= 3\left[x\left(\frac{e^{-3x}}{-3}\right) - 1\left(\frac{e^{-3x}}{9}\right)\right]_0^{\infty} = 3\left(\frac{0+1}{9}\right) = \frac{1}{3}$$

$$\text{Mean} = E(X) = \frac{1}{3}$$

EXAMPLE 2.79 Suppose the duration x in minutes of long distance calls from your home, follows exponential law with PDF $f(x) = \frac{1}{5}e^{-x/5}$, for $x > 0$, 0 otherwise. Find

(i) $P(X > 5)$,
(ii) $P(3 \le X \le 6)$,

(iii) mean of X, and
(iv) variance of X. **[AU November '05]**

Solution Given: $f(x)=\begin{cases}\frac{1}{5}e^{-\frac{x}{5}}, & x>0\\ 0, & \text{otherwise}\end{cases}$

(i) $P(X>5)=\int_5^\infty \frac{1}{5}e^{-\frac{x}{5}}dx=\frac{1}{5}\left[\frac{e^{-\frac{x}{5}}}{-\frac{1}{5}}\right]_5^\infty=e^{-1}=0.3679$

(ii) $P(3\le X\le 6)=\int_3^6 \frac{1}{5}e^{-\frac{x}{5}}dx=\frac{1}{5}\left[\frac{e^{-\frac{x}{5}}}{-\frac{1}{5}}\right]_3^6=e^{-\frac{3}{5}}-e^{-\frac{6}{5}}=0.2476$

(iii) $E(X)=\int_{-\infty}^{\infty} xf(x)\frac{1}{5}e^{-\frac{x}{5}}dx=\frac{1}{5}\int_{-\infty}^{\infty} xe^{-\frac{x}{5}}dx=\frac{1}{5}\left[x\left(\frac{e^{-\frac{x}{5}}}{-\frac{1}{5}}\right)-1\frac{e^{-\frac{x}{5}}}{\left(-\frac{1}{5}\right)^2}\right]_0^\infty$

$=0+\frac{1}{5}\times 25=5$

$E(X^2)=\frac{1}{5}\int_0^\infty x^2e^{-\frac{x}{5}}dx=\frac{1}{5}\left\{x^2\left(\frac{e^{-\frac{x}{5}}}{-\frac{1}{5}}\right)-2x\left[\frac{e^{-\frac{x}{5}}}{\left(-\frac{1}{5}\right)^2}\right]+2\left[\frac{e^{-\frac{x}{5}}}{\left(-\frac{1}{5}\right)^3}\right]\right\}_0^\infty$

$=\frac{1}{5}\times 250=50$

(iv) $\therefore$ $\text{Var}(X)=50-5^2=25$

EXAMPLE 2.80 Let X be a continuous random variable with PDF $f_x(x)=x^2$, $1<x<2$. Find $E(\log X)$. **[AU December '08]**

Solution By definition,

$$E(\log X)=\int_{-\infty}^{\infty}\log x\, f(x)dx$$

$$=\int_1^2 x^2\log x\, dx$$

$$=\left[\log x\frac{x^3}{3}\right]_1^2-\int_1^2\frac{x^3}{3}\times\frac{1}{x}dx$$

$$=\log 2\left(\frac{2^3}{3}\right)-\log 1\times\left(\frac{1}{3}\right)-\int_1^2\frac{x^2}{3}dx$$

$$=\frac{8}{3}\log 2-0-\frac{1}{3}\left[\frac{x^3}{3}\right]_1^2$$

$$=\frac{8}{3}\log 2-\frac{1}{9}(8-1)$$

$$=\frac{8}{3}\log 2-\frac{7}{9}$$

EXAMPLE 2.81 If a random variable X has the probability density function

$$f(x)=\begin{cases}\frac{1}{2}(x+1), & \text{if } -1<x<1\\ 0, & \text{elsewhere}\end{cases}$$

Find the mean and variance. **(AU December '07]**

Solution Mean $=E(X)=\int_{-\infty}^{\infty}xf(x)dx=\frac{1}{2}\int_{-1}^{1}x(x+1)dx$

$$=\frac{1}{2}\int_{-1}^{1}(x^2+x)dx=\frac{1}{2}\left[\frac{x^3}{3}+\frac{x^2}{2}\right]_{-1}^{1}$$

$$=\frac{1}{2}\left(\frac{1}{3}+\frac{1}{2}+\frac{1}{3}-\frac{1}{2}\right)=\frac{1}{3}$$

$$E(X^2)=\frac{1}{2}\int_{-1}^{1}x^2(x+1)dx=\frac{1}{2}\left[\frac{x^4}{4}+\frac{x^3}{3}\right]_{-1}^{1}$$

$$=\frac{1}{2}\left(\frac{1}{4}+\frac{1}{3}-\frac{1}{4}+\frac{1}{3}\right)=\frac{1}{3}$$

$$\text{Var}(X)=E(X^2)-[E(X)]^2=\frac{1}{3}-\frac{1}{9}=\frac{2}{9}$$

EXAMPLE 2.82 If $f(x)$ is a density function defined by $f(x)=ae^{-|x|}$, $-\infty<x<\infty$, find

(i) the value of a,
(ii) mean and variance. **[AU April '04, June '06, December '05]**

Solution

(i) If $f(x)$ is a PDF, then $\int_{-\infty}^{\infty} f(x)dx = 1$

$$\therefore \quad \int_{-\infty}^{\infty} f(x)dx = \int_{-\infty}^{\infty} ae^{-|x|}\,dx = 1$$

Since $|x|$ is an even function,

$$2\int_{0}^{\infty} ae^{-|x|}\,dx = 1$$

$$2\int_{0}^{\infty} ae^{-x}dx = 1 \qquad (\because |x| = x, \text{ if } x > 0)$$

$$2a\left[\frac{e^{-x}}{-1}\right]_0^{\infty} = 1 \Rightarrow 2a\cdot 1 = 1 \Rightarrow a = \frac{1}{2}$$

(ii) $\text{Mean} = E(X) = \int_{-\infty}^{\infty} xf(x)dx$

$$= \frac{1}{2}\int_{-\infty}^{\infty} xe^{-|x|}dx = 0 \quad [\because xe^{-|x|} \text{ is an odd function}]$$

$$E(X^2) = \int_{-\infty}^{\infty} x^2 f(x)dx = \frac{1}{2}\int_{-\infty}^{\infty} x^2 e^{-|x|}dx$$

$$= \frac{1}{2}\cdot 2\int_{0}^{\infty} x^2 e^{-x}dx \qquad [\because x^2e^{-|x|} \text{ is an even function}]$$

$$= \left[x^2\frac{e^{-x}}{-1} - 2x\frac{e^{-x}}{(-1)^2} + 2\frac{e^{-x}}{(-1)^3}\right]_0^{\infty}$$

$$= [0 - 0 - 2(0 - 1)] = 2$$

$$\therefore \text{Var}(X) = E(X^2) - E(X) = 2 - 0 = 2$$

EXAMPLE 2.83 The first four moments of the distribution about the value 4 of the variable are –1.5, 17, –30 and 108 respectively. Find the mean and moments about the mean.

Solution Given: $\mu_1' = -1.5,\ \mu_2' = 17,\ \mu_3' = -30,\ \mu_4' = 108$ and $A = 4$

$$\text{Mean} = \bar{X} = A + \mu_1' = 4 + (-1.5) = 2.5$$

The moments about mean are

$$\mu_1 = 0$$

$$\mu_2 = \mu_2' - \mu_1'^2 = 17 - (-1.5)^2 = 14.75$$

$$\mu_3 = \mu_3' - 3\mu_2'\mu_1' + 2\mu_1'^3 = -30 + 76.5 - 6.75 = 39.75$$

$$\mu_4 = \mu_4' - 4\mu_3'\mu_1' + 6\mu_1'^2\mu_2' - 3\mu_1'^4 = 108 - 180 + 229.5 - 15.1875 = 142.313$$

EXAMPLE 2.84 The first four moments of a distribution about $X = 4$ are 1, 4, 10 and 45 respectively. Show that the mean is 5, variance is 3, $\mu_3 = 0$, and $\mu_4 = 26$. **[AU November '04]**

Solution Given: $\mu_1' = 1, \mu_2' = 4, \mu_3' = 10, \mu_4' = 45$

and $A = 4$

$$E(X) = \text{mean} = \bar{X} = A + \mu_1' = 1 + 4 = 5$$

$$\mu_1 = 0$$

$$\mu_2 = \mu_2' - \mu_1'^2 = 4 - 1^2 = 3$$

$$\text{Var} = \mu_2 = 3$$

$$\mu_3 = \mu_3' - 3\mu_2'\mu_1' + 2\mu_1'^3$$

$$= 10 - 3 \times 4 \times 1 + 2 \times 1^3$$

$$= 10 - 12 + 2 = 0$$

$$\mu_4 = \mu_4' - 4\mu_3'\mu_1' + 6\mu_1'^2\mu_2' - 3\mu_1'^4$$

$$= 45 - 4 \times 10 \times 1 + 6 \times 1^2 \times 4 - 3 \times 1^4$$

$$= 45 - 40 + 24 - 3$$

$$= 26$$

EXAMPLE 2.85 Calculate the first four moments of the following distribution about the mean.

x_i	0	1	2	3	4	5	6	7	8
y_i	1	8	28	56	70	56	28	8	1

Solution

x_i	f_i	$d_i = x_i - 4$	f_id_i	$f_id_i^2$	$f_id_i^3$	$f_id_i^4$
0	1	–4	–4	16	–64	256
1	8	–3	–24	72	–216	648
2	28	–2	–56	112	–224	448
3	56	–1	–56	56	–56	56
4	70	0	0	0	0	0
5	56	1	56	56	56	56
6	28	2	56	112	224	448
7	8	3	24	72	216	648
8	1	4	4	16	64	256
	256		0	512	0	2816

$$N = \sum f_i = 256, \quad \mu_1' = \frac{1}{N}\sum_i f_i d_i = 0$$

$$\mu_2' = \frac{1}{N}\sum_i f_i d_i^2 = \frac{512}{256} = 2$$

$$\mu_3' = \frac{1}{N}\sum_i f_i d_i^3 = \frac{0}{256} = 0$$

$$\mu_4' = \frac{1}{N}\sum_i f_i d_i^4 = \frac{2816}{256} = 11$$

The moments about the mean are

$$\mu_1 = 0$$

$$\mu_2 = \mu_2' - \mu_1'^2 = 2 - 0 = 2$$

$$\mu_3 = \mu_3' - 3\mu_2'\mu_1' + 2\mu_1'^3 = 0 - 3(2)0 + 2(0) = 0$$

$$\mu_4 = \mu_4' - 4\mu_3'\mu_1' + 6\mu_1'^2\mu_2' - 3\mu_1'^4 = 11 - 0 = 11$$

2.12 MOMENT GENERATING FUNCTION (MGF): $M_X(t)$

The moment generating function of a random variable denoted by $M_X(t)$ is defined as

$$M_X(t) = E(e^{tX})$$

$$\therefore \quad M_X(t) = E(e^{tX}) = E\left[1 + \frac{tX}{1!} + \frac{(tX)^2}{2!} + \frac{(tX)^3}{3!} + \cdots\right]$$

$$= E(1) + \frac{t}{1!}E(X) + \frac{t^2}{2!}E(X^2) + \frac{t^3}{3!}E(X^3) + \cdots + \frac{t^r}{r!}E(X^r) + \cdots + \infty$$

$$\therefore \quad M_X(t) = 1 + \frac{t}{1!}\mu_1' + \frac{t^2}{2!}\mu_2' + \frac{t^3}{3!}\mu_3' + \cdots + \frac{t^r}{r!}\mu_r' + \cdots + \infty \tag{2.1}$$

$$= \sum_{r=0}^{\infty} \frac{t^r}{r!}\mu_r'$$

which gives the MGF in terms of moments.

$\therefore$ The coefficient of $\frac{t^r}{r!}$ in $M_X(t)$ is μ_r', where $r = 1, 2, 3\ldots$ and $\mu_r' = E(X^r)$, moment about the origin.

Moment generating function of X about any point $X = a$ is defined as

$$M_X(t) = E[e^{t(X-a)}] = E\left[1 + \frac{t}{1!}(X-a) + \frac{t}{2!}(X-a)^2 + \frac{t}{3!}(X-a)^3 + \cdots\right]$$

$$M_X(t) = 1 + \frac{t}{1!}\mu_1' + \frac{t}{2!}\mu_2' + \frac{t}{3!}\mu_3' + \cdots + \infty, \text{ where } E[(X-a)^r] = \mu_r'$$

Since $M_X(t)$ generates moments it is called *moment generating function.*

If X is a discrete random variable with PMF $p(x)$, then

$$M_X(t) = E(e^{tX}) = \sum_x e^{tx} p(x)$$

If X is a continuous random variable with PDF $f(x)$, then

$$M_X(t) = E(e^{tX}) = \int_{-\infty}^{\infty} e^{tx} f(x)dx$$

2.12.1 Moments Using Moment Generating Function

Differentiating Eq. (2.1) with respect to t and then putting $t = 0$, gives

$$\mu_1' = \left[\frac{d}{dt}M_X(t)\right]_{t=0}$$

$$\mu_2' = \left[\frac{d^2}{dt^2}M_X(t)\right]_{t=0}$$

In general,

$$\mu_r' = \left[\frac{d^r}{dt^r}M_X(t)\right]_{t=0}, \qquad r = 1, 2, 3,\ldots$$

Note: Moment generating function $M_X(t)$ is used to calculate the higher moments.

2.12.2 Limitations of Moment Generating Function

1. A random variable X may have no moments although its moment generating function exist.
   For example:

$$f(x) = \begin{cases} \dfrac{1}{x(x+1)}, & x = 1, 2, 3, \ldots \\ 0, & \text{otherwise} \end{cases}$$

2. A random variable X can have MGF and some or all moments, yet the MGF does not generate the moments.

For example:

$$P(X = \pm 2^x) = \frac{e^{-1}}{x!}, \; x = 0, 1, 2, \ldots$$

3. A random variable X can have all or some moments, but MGF does not exist, except perhaps at one point.
For example:

$$P(X = \pm 2^x) = \begin{cases} \dfrac{e^{-1}}{2x!}, & x = 0, 1, 2, \ldots \\ 0, & \text{otherwise} \end{cases}$$

2.12.3 Theorems on Moment Generating Function

THEOREM 5 $M_{ax}(t) = M_X(at)$, a being a constant.

Proof By definition,

$$M_{ax}(t) = E(e^{taX}) = E(e^{atX})$$

$$\therefore \quad M_{ax}(t) = M_X(at)$$

THEOREM 6 The moment generating function of the sum of n independent random variables is equal to the product of their respective moment generating functions, i.e.

$$M_{X_1 + X_2 + X_3 + \cdots + X_n}(t) = M_{X_1}(t)\, M_{X_2}(t) \ldots M_{X_n}(t)$$

Proof By definition,

$$M_{X_1 + X_2 + X_3 + \cdots + X_n}(t) = E\left[e^{t(X_1 + X_2 + X_3 + \cdots + X_n)}\right]$$

$$= E(e^{tX_1})\, E(e^{tX_2})\, E(e^{tX_3}) \ldots E(e^{tX_n})$$

(Since $X_1, X_2, \ldots, X_n$ are independent)

$$\therefore \quad M_{X_1 + X_2 + X_3 + \cdots + X_n}(t) = M_{X_1}(t)\, M_{X_2}(t) \ldots M_{X_n}(t)$$

Hence the proof.

2.12.4 Effect of Change of Origin and Scale on Moment Generating Function

Let a random variable X be transformed to a new variable U by changing both the origin and scale in X as

$$U = \frac{X - a}{h}$$

where a and h are constants.

The MGF of U (about origin) is given by

$$M_U(t) = E(e^{tU})$$

$$= E\left[e^{t\left(\frac{X-a}{h}\right)}\right]$$

$$= E\left(e^{\frac{tX}{h}} e^{-\frac{ta}{h}}\right) = e^{\frac{-at}{h}} E\left(e^{\frac{tX}{h}}\right)$$

$$\therefore \quad M_{\frac{X-a}{h}}(t) = e^{\frac{-at}{h}} M_X\left(\frac{t}{h}\right)$$

Note: If $Y = aX + b$, then $M_Y(t) = e^{bt} M_X(at)$

EXAMPLE 2.86 If X represents the outcome when a fair die is tossed, find the MGF of X and hence, find $E(X)$ and $\text{Var}(X)$. **[AU May '06, '07]**

Solution When a fair die is tossed

$$P(X = x) = \frac{1}{6}, \; x = 1, 2, 3, 4, 5, 6$$

$$\therefore \quad MX(t) = \sum_{X=1}^{6} e^{tx} p(X = x)$$

$$= \frac{1}{6} \sum_{x=1}^{6} e^{tx}$$

$$= \frac{1}{6}(e^t + e^{2t} + e^{3t} + e^{4t} + e^{5t} + e^{6t})$$

$$E(X) = \left[\frac{d}{dt} M_X(t)\right]_{t=0}$$

$$= \frac{1}{6}[e^t + 2e^{2t} + 3e^{3t} + 4e^{4t} + 5e^{5t} + 6e^{6t}]_{t=0}$$

$$= \frac{1}{6}(1 + 2 + 3 + 4 + 5 + 6) = \frac{21}{6} = \frac{7}{2}$$

$$\therefore \quad \text{Mean} = E(X) = \frac{7}{2}$$

$$E[X^2]_{t=0} = \left\{\frac{d^2}{dt^2}[M_X(t)]\right\}_{t=0}$$

$$= \frac{1}{6}(1 + 4 + 9 + 16 + 25 + 36) = \frac{91}{6}$$

$$\therefore \qquad \text{Var}(X) = E(X^2) - [E(X)]^2 = \frac{91}{6} - \frac{49}{4} = \frac{35}{12}$$

EXAMPLE 2.87 Find the MGF of the random variable X whose probability function $P(X = x) = \frac{1}{2^x}$, $x = 1, 2, 3, \ldots$. Hence find its mean. **[AU May '05]**

Solution By definition,

$$M_X(t) = E(e^{tX}) = \sum_{x=0}^{\infty} e^{tx} P(X = x)$$

$$= \sum_{x=1}^{\infty} e^{tx}\left(\frac{1}{2^x}\right) = \sum_{x=1}^{\infty}\left(\frac{e^t}{2}\right)^x$$

$$= \left[\frac{e^t}{2} + \left(\frac{e^t}{2}\right)^2 + \left(\frac{e^t}{2}\right)^3 + \cdots\right]$$

$$= \frac{e^t}{2}\left[1 + \frac{e^t}{2} + \left(\frac{e^t}{2}\right)^2 + \cdots\right]$$

$$= \frac{e^t}{2}\left(1 - \frac{e^t}{2}\right)^{-1} = \frac{e^t}{2}\left(\frac{2 - e^t}{2}\right)^{-1}$$

$$= \frac{e^t}{2}\left(\frac{2}{2 - e^t}\right)$$

$$= \frac{e^t}{2 - e^t}$$

$$\therefore \qquad M_X(t) = \frac{e^t}{2 - e^t}$$

$$\mu_1' = \text{mean} = \left[\frac{d}{dt} M_X(t)\right]_{t=0}$$

$$= \left[\frac{d}{dt}\left(\frac{e^t}{2 - e^t}\right)\right]_{t=0}$$

$$= \left[\frac{(2 - e^t)e^t - e^t(-e^t)}{(2 - e^t)^2}\right]_{t=0}$$

$$= \frac{(2-1)1 + 1}{(2-1)^2} = 2$$

$$E(X) = \text{mean} = 2$$

EXAMPLE 2.88 If the moments of a random variable X are defined by $E(X^r) = 0.6$, $r = 1, 2, \ldots$. Show that $P(X = 0) = 0.4$, $P(X = 1) = 0.6$, and $P(X \geq 2) = 0$.

Solution We know that

$$M_X(t) = \sum_{r=0}^{\infty} \frac{t^r}{r!}\mu_r'$$

where $\mu_r' = E(X^r) = 0.6$

$$\therefore \qquad M_X(t) = 1 + \sum_{r=1}^{\infty} \frac{t^r}{r!}\mu_r' = 1 + \sum_{r=1}^{\infty} \frac{t^r}{r!}(0.6)$$

$$= 1 + (0.6)\left(\frac{t}{1!} + \frac{t^2}{2!} + \frac{t^3}{3!} + \cdots\right)$$

$$= 1 + (0.6)(e^t - 1)$$

$$= 1 - 0.6 + 0.6e^t = 0.4 + 0.6e^t \qquad \text{(i)}$$

But by definition,

$$M_X(t) = E(e^{tx}) = \sum_{x=0}^{\infty} e^{tx} P(X = x)$$

$$M_X(t) = P(X = 0) + e^t P(X = 1) + e^{2t} P(X = 2) + e^{3t} P(X = 3) + \cdots \qquad \text{(ii)}$$

From Eqs. (i) and (ii), we have

$$0.4 + 0.6e^t = P(X = 0) + e^t P(X = 1) + e^{2t} P(X = 2) + e^{3t} P(X = 3) + \cdots$$

Equating the coefficients of like terms on both sides,

$$P(X = 0) = 0.4, \quad P(X = 1) = 0.6$$

$$P(X = 2) = P(X = 3) = \cdots = 0 \quad \Rightarrow \quad P(X \geq 2) = 0$$

EXAMPLE 2.89 Let X be a random variable with $E(X) = 1$ and $E[X(X-1)] = 4$. Find $\text{Var}(X/2)$ and $\text{Var}(2 - 3X)$.

Solution Given $E(X) = 1$, $E(X^2 - X) = 4 \Rightarrow E(X^2) - E(X) = 4$

$$\Rightarrow \qquad E(X^2) = 4 + E(X) = 4 + 1 = 5$$

$$\text{Var}(X) = E(X^2) - [E(X)]^2 = 5 - 1^2 = 4$$

By the property,

$$\text{Var}\left(\frac{X}{2}\right) = \frac{1}{4}\text{Var}(X) = \frac{1}{4} \times 4 = 1$$

$$\text{Var}(2 - 3X) = \text{Var}(2) + 3^2\,\text{Var}(X)$$

$$= 0 + 9 \times 4 = 36$$

EXAMPLE 2.90 Find the first four moments about the origin for a random variable X having density function $f(x) = \frac{4x(9-x^2)}{81}, 0 \le x \le 3$.

Solution Given: The density function

$$f(x) = \frac{4x(9-x^2)}{81}, 0 \le x \le 3$$

Using the definition,

$$\mu_1^1 = E(X) = \int_0^\infty x f(x)dx = \frac{4}{81}\int_0^3 x^2(9-x^2)dx = \frac{4}{81}\left[\frac{9x^3}{3} - \frac{x^5}{5}\right]_0^3 = \frac{8}{5}$$

$$\mu_2^1 = E(X^2) = \int_0^\infty x^2 f(x)dx = \frac{4}{81}\int_0^3 x^3(9-x^2)dx = \frac{4}{81}\left[\frac{9x^4}{4} - \frac{x^6}{6}\right]_0^3 = 3$$

$$\mu_3^1 = E(X^3) = \frac{4}{81}\int_0^3 x^4(9-x^2)dx = \frac{4}{81}\left[\frac{9x^5}{5} - \frac{x^7}{7}\right]_0^3 = \frac{299}{35}$$

$$\mu_4^1 = E(X^4) = \int_0^\infty x^4 f(x)dx = \frac{4}{81}\int_0^3 x^5(9-x^2)dx = \frac{4}{81}\left[\frac{9x^6}{6} - \frac{x^8}{8}\right]_0^3 = \frac{27}{2}$$

EXAMPLE 2.91 Find the MGF of a random variable whose moments are $\mu_r = (r+1)!\,2^r$. **[AU December '07]**

Solution By definition,

$$M_X(t) = \sum_{r=0}^{\infty} \frac{t^r}{r!}\mu_r'$$

$$= \sum_{r=0}^{\infty} \frac{t^r}{r!}(r+1)!\,2^r = \sum_{r=0}^{\infty}(r+1)(2t)^r \qquad [\because (r+1)! = (r+1)r!]$$

$$= 1 + 2(2t) + 3(2t)^2 + \cdots = (1-2t)^{-2} = \frac{1}{(1-2t)^2}$$

$$\therefore \quad M_X(t) = \frac{1}{(1-2t)^2}$$

EXAMPLE 2.92 If a random variable X has the MGF $M_X(t) = \frac{3}{3-t}$, find the standard deviation of X. **[AU December '04]**

Solution Given: $M_X(t) = \frac{3}{3-t} = \frac{3}{3\left(1-\frac{t}{3}\right)} = \frac{1}{1-\frac{t}{3}}$

$$M_X(t) = \sum_{r=0}^{\infty} \frac{t^r}{r!}\mu_r' = \left(1-\frac{t}{3}\right)^{-1} = 1 + \frac{t}{3} + \left(\frac{t}{3}\right)^2 + \cdots$$

$$1 + \frac{t}{1!}\mu_1' + \frac{t^2}{2!}\mu_2' + \cdots$$

$$= 1 + \frac{1}{3}\frac{t}{1!} + \frac{2!}{9}\frac{t^2}{2!} + \cdots$$

$\therefore$ Equating the coefficients of $\frac{t}{1!}$ and $\frac{t^2}{2!}$, we get

$$\mu_1' = \frac{1}{3}, \mu_2' = \frac{2}{9}$$

$$\therefore \quad \text{Var}(X) = \mu_2' - (\mu_1')^2$$

$$= \frac{2}{9} - \frac{1}{9} = \frac{1}{9}$$

$$\therefore \quad \text{Standard deviation of } X = \frac{1}{3}$$

EXAMPLE 2.93 Find the MGF of a random variable X having the density function

$$f(x) = \begin{cases} \frac{x}{2}, & 0 \le x \le 2 \\ 0, & \text{otherwise} \end{cases}$$

Solution By definition,

$$M_X(t) = E(e^{tX}) = \int_{-\infty}^{\infty} e^{tx} f(x)dx$$

$$= \int_0^2 e^{tx}\frac{x}{2}dx$$

$$= \frac{1}{2}\left[x\frac{e^{tx}}{t} - (1)\frac{e^{tx}}{t^2}\right]_0^2$$

$$= \frac{1}{2}\left[\left(2\frac{e^{2t}}{t} - 0\right) - \left(\frac{e^{2t}}{t^2} - \frac{1}{t^2}\right)\right]$$

$$= \frac{1}{2}\left(\frac{2e^{2t}}{t} - \frac{e^{2t}}{t^2} + \frac{1}{t^2}\right)$$

$$\therefore \quad M_X(t) = \frac{1}{2t^2}(1 + 2te^{2t} - e^{2t})$$

EXAMPLE 2.94 A random variable X has the PDF given by

$$f(x)=\begin{cases}2e^{-2x}, & x\geq 0\\ 0, & x<0\end{cases}$$

Find

(i) the MGF, and

(ii) the first four moments about the origin. **[AU December '07]**

Solution By definition,

$$M_X(t)=E(e^{tx})=\int_{-\infty}^{\infty}e^{tx}f(x)dx$$

$$M_X(t)=\int_0^{\infty}e^{tx}2e^{-2x}dx=2\int_0^{\infty}e^{-(2-t)x}dx=2\left[\frac{e^{-(2-t)x}}{-(2-t)}\right]_0^{\infty}$$

$$\therefore\qquad M_X(t)=\frac{2}{2-t}$$

We know that

$$M_X(t)=\sum_{r=0}^{\infty}\frac{t^r}{r!}\mu_r'=\frac{2}{2-t}=\frac{2}{2\left(1-\frac{t}{2}\right)}$$

$$\therefore\qquad 1+\frac{t}{1!}\mu_1'+\frac{t^2}{2!}\mu_2'+\frac{t^3}{3!}\mu_3'+\cdots=\left(1-\frac{t}{2}\right)^{-1}$$

$$=1+\frac{t}{2}+\left(\frac{t}{2}\right)^2+\left(\frac{t}{2}\right)^3+\left(\frac{t}{2}\right)^4+\cdots$$

$$=1+\frac{1}{2}\frac{t}{1!}+\frac{2!}{4}\frac{t^2}{2!}+\frac{3!}{8}\frac{t^3}{3!}+\frac{4!}{16}\frac{t^4}{4!}+\cdots$$

Equating the coefficients of $\frac{t}{1!},\frac{t^2}{2!},\frac{t^3}{3!},\cdots$, we get

$$\mu_1'=\frac{1}{2},\ \mu_2'=\frac{1}{2},\ \mu_3'=\frac{6}{8}=\frac{3}{4}\ \text{and}\ \mu_4'=\frac{4!}{2^4}=\frac{3}{2}$$

EXAMPLE 2.95 Show that the MGF of the random variable X having the PDF $f(x)=\begin{cases}\frac{1}{3}, & -1<x<2\\ 0, & \text{otherwise}\end{cases}$ is given by $M_X(t)=\begin{cases}\frac{e^{2t}-e^{-t}}{3t}, & t\neq 0\\ 1, & t=0\end{cases}$. **[AU December '04]**

Solution By definition,

$$M_X(t)=E(e^{tX})$$

$$=\int_{-\infty}^{\infty}e^{tx}f(x)dx$$

$$=\int_{-1}^{2}e^{tx}\frac{1}{3}dx$$

$$=\frac{1}{3}\left[\frac{e^{tx}}{t}\right]_{-1}^{2}$$

$$=\frac{1}{3}\left(\frac{e^{2t}-e^{-t}}{t}\right),\ t\neq 0$$

If $t=0$,

$$M_X(t)=\lim_{t\to 0}\frac{e^{2t}-e^{-t}}{3t}=\frac{0}{0}$$

$$=\lim_{t\to 0}\frac{2e^{2t}-e^{t}}{3}=\frac{3}{3}=1\qquad\text{(using L'Hospitals rule)}$$

$$\therefore\qquad M_X(t)=\frac{e^{2t}-e^{-t}}{3t},\quad t\neq 0$$

$$=1,\qquad t=0$$

EXAMPLE 2.96 Find the MGF of the random variable X having PDF

$$f(x)=\begin{cases}x, & \text{for } 0<x<1\\ 2-x, & \text{for } 1<x<2\\ 0, & \text{otherwise}\end{cases}$$ **[AU December '06, May '09]**

Solution We know that

$$M_X(t)=\int_0^{\infty}e^{tx}f(x)dx$$

$$=\int_0^1 e^{tx}f(x)dx+\int_1^2 e^{tx}f(x)dx$$

$$=\int_0^1 e^{tx}\cdot x\,dx+\int_1^2 e^{tx}(2-x)dx$$

$$=\left[x\left(\frac{e^{tx}}{t}\right)-\left(\frac{e^{tx}}{t^2}\right)\right]_0^1+\left[(2-x)\left(\frac{e^{tx}}{t}\right)-(-1)\left(\frac{e^{tx}}{t^2}\right)\right]_1^2$$

$$= \frac{e^t}{t} - \frac{e^t}{t^2} + \frac{1}{t^2} + \frac{e^{2t}}{t^2} - \frac{e^t}{t} - \frac{e^t}{t^2}$$

$$= \frac{e^{2t}}{t^2} + \frac{1}{t^2} - \frac{2e^t}{t^2}$$

$$= \frac{(e^t - 1)^2}{t^2}$$

EXAMPLE 2.97 Let X be a random variable with PDF

$$f(x) = \begin{cases} \frac{1}{3}e^{-\frac{x}{3}}, & x > 0 \\ 0, & \text{otherwise} \end{cases}$$

Find

(i) $P(X > 3)$,

(ii) MGF of X,

(iii) $E(X)$ and Var (X). **[AU June '07, December '05]**

Solution Using the definition,

(i) $$P(X > 3) = \int_3^\infty f(x)dx = \int_3^\infty \frac{1}{3}e^{-\frac{x}{3}}dx$$

$$= \frac{1}{3}\left[\frac{e^{-\frac{x}{3}}}{-\frac{1}{3}}\right]_3^\infty = -(0 - e^{-1}) = e^{-1} = \frac{1}{e}$$

(ii) $$M_X(t) = E(e^{tX}) = \int_0^\infty e^{tx} f(x)dx$$

$$= \int_0^\infty e^{tx}\frac{1}{3}e^{-\frac{x}{3}}dx = \frac{1}{3}\int_0^\infty e^{\left(t-\frac{1}{3}\right)x}dx$$

$$= \frac{1}{3}\int_0^\infty e^{-\left(\frac{1}{3}-t\right)x}dx = \frac{1}{3}\left[\frac{e^{-\left(\frac{1}{3}-t\right)x}}{-\left(\frac{1}{3}-t\right)}\right]_0^\infty$$

$$= \frac{1}{3}\left[0 - \frac{1}{-\left(\frac{1}{3}-t\right)}\right] = \frac{1}{3}\left[\frac{1}{\left(\frac{1-3t}{3}\right)}\right]$$

$$M_X(t) = \frac{1}{1-3t} = (1-3t)^{-1}$$

$$\frac{d}{dt}[M_X(t)] = -(1-3t)^{-2}(-3) = 3(1-3t)^{-2}$$

(iii) $$E(X) = \text{Mean} = \left[\frac{d}{dt}M_X(t)\right]_{t=0} = 3$$

$$\frac{d^2}{dt^2}[M_X(t)] = -6(1-3t)^{-3}(-3) = 18(1-3t)^{-3}$$

$$E(X^2) = \left[\frac{d^2}{dt^2}M_X(t)\right]_{t=0} = 18$$

$$\text{Var}(X) = E(X^2) - [E(X)]^2 = 18 - 3^2 = 9$$

EXAMPLE 2.98 Consider a discrete random variable X with a probability function

$$p(x) = \begin{cases} \frac{1}{x(x+1)}, & x = 1, 2 \cdots \\ 0, & \text{otherwise} \end{cases}$$

Show that $E(X)$ does not exist even though MGF exist.

Solution $$E(X) = \sum_{x=1}^\infty xp(x) = \sum_{x=1}^\infty \frac{1}{x+1} = \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots = \sum_{x=1}^\infty \frac{1}{x} - 1$$

But $\sum_{x=1}^\infty \frac{1}{x}$ is a divergent series.

$\therefore$ $E(X)$ does not exist and hence, no moment exists.

Now, MGF of X is given by

$$M_X(t) = \sum_{x=1}^\infty p(x)e^{tx} = \sum_{x=1}^\infty \frac{e^{tx}}{x(x+1)}$$

Using $z = e^t$

$$M_X(t) = \sum_{x=1}^\infty \frac{z^x}{x(x+1)} = \frac{z}{1\cdot 2} + \frac{z^2}{2\cdot 3} + \frac{z^3}{3\cdot 4} + \cdots$$

$$= z\left(1 - \frac{1}{2}\right) + z^2\left(\frac{1}{2} - \frac{1}{3}\right) + z^3\left(\frac{1}{3} - \frac{1}{4}\right) + \cdots$$

$$= \left(z + \frac{z^2}{2} + \frac{z^3}{3} + \cdots\right) - \frac{z}{2} - \frac{z^2}{3} - \frac{z^3}{4} \cdots$$

$$= -\log(1-z) - \frac{1}{z}\left(\frac{z^2}{2} + \frac{z^3}{3} + \frac{z^4}{4} \cdots\right)$$

$$= -\log(1-z) + 1 + \frac{1}{z}\log(1-z), \ |z| < 1$$

$$= 1 + \left(\frac{1}{z} - 1\right)\log(1-z), \ |z| < 1$$

$$M_X(t) = \begin{cases} 1 + (e^{-t} - 1)\log(1 - e^t), & t < 0 \\ 1, & \text{for } t = 0 \end{cases}$$

and $M_X(t)$ does not exist for $t > 0$.

EXAMPLE 2.99 A random variable X has density function given by

$$f(x) = \begin{cases} \frac{1}{k}, & \text{for } 0 < x < k \\ 0, & \text{otherwise} \end{cases}$$

Find

(i) MGF,
(ii) rth moment,
(iii) mean, and
(iv) variance.

[AU April '07]

Solution

$$\text{MGF} = M_X(t) = E(e^{tx}) = \int_0^k \frac{1}{k} e^{tx} dx = \frac{1}{k}\left[\frac{e^{tx}}{t}\right]_0^k = \frac{1}{kt}(e^{tk} - 1)$$

$$= \frac{1}{kt}\left[1 + \frac{(kt)}{1!} + \frac{(kt)^2}{2!} + \cdots - 1\right]$$

$$= 1 + \frac{(kt)}{2!} + \cdots + \frac{(kt)^r}{(r+1)!} + \cdots$$

$\therefore$

$$\mu_r' = \text{coefficient of } t^r = \frac{k^r}{(r+1)!}$$

When $r = 1$, we get

$$\mu_1' = \text{mean} = \frac{k}{2}$$

When $r = 2$, we get

$$\mu_2' = \frac{2k^2}{3!} = \frac{k^2}{3}$$

$$\text{Var} = \mu_2' - \mu_1'^2$$

$$= \frac{k^2}{3} - \frac{k^2}{4} = \frac{k^2}{12}$$

EXERCISES

1. Define discrete and continuous random variables with examples.
2. Define distribution function of a random variable.
3. If $F(x)$ is a distribution function, then what is the value of $F(x)$ at $x = \infty$ and at $x = -\infty$?
4. Define probability mass function.
5. What are the important two conditions that a PMF $p(x)$ should satisfy.
6. Define PDF of a continuous random variable X.
7. Define probability curve or probability distribution curve.
8. If $f(x)$ is a PDF of a continuous random variable X, then what is the value of $\int_{-\infty}^{\infty} f(x)dx$?
9. If X is a continuous random variable, prove that $P(X = c) = 0$.
10. If X is a continuous random variable for the following density function $f(x) = cx^2(1 - x)$, $0 < x < 1$, find
    (i) the constant c, and
    (ii) mean.

    $\left[\text{Ans. } c = 12, \text{ mean} = \frac{3}{5}\right]$
11. If the PDF of a random variable X is $f(x) = y_0(x - x^2)$ in $0 \le x \le 1$, find mean.

    $\left[\text{Ans. } \frac{1}{2}\right]$
12. Check whether the following are PDF or not
    (i) $f(x) = \lambda e^{-\lambda x}$, $x \ge 0$, $\lambda > 0$ **[Ans. PDF]**

(ii) $f(x)=\begin{cases}2x, & 0<x<1\\ 4-2x, & 1<x<2\\ 0, & \text{otherwise}\end{cases}$ [**Ans.** not a PDF]

13. Given the PDF of a random variable X is $f(x) = y_0e^{-b(x-a)}$, $a \le x < \infty$, $b > 0$, where y_0, a and b are constants, show that $y_0 = b$.
14. Find k, so that $f(x)$ given below may be a PDF

$$f(x)=\begin{cases}\dfrac{1}{k}, & \text{for } a\le x\le b\\ 0, & \text{otherwise}\end{cases}$$

Find also the distribution function.

$$\left[\text{Ans. } k=b-a,\ F(x)=0,\ \text{if } x<a \quad =\frac{x-b}{b-a},\ \text{if } a\le x\le b \quad =1,\ \text{if } x>b\right]$$

15. Define raw moment and central moment of a random variable.
16. Define expected value of a random variable X.
17. Define the mean and variance of a random variable.
18. If X is a discrete/continuous random variable prove that $E(aX + b) = aE(X) + b$ and $\text{Var}(aX) = a^2\,\text{Var}(X)$.
19. Define MGF of a discrete and continuous random variable X and state any two of its properties.
20. Define moments of a random variable X about any point and about origin.
21. If $M_X(t)$ is the MGF of a random variable X about the origin, then show that

$$\left[\frac{d^rM(t)}{dt^r}\right]_{t=0}$$

22. Find the relation between $M_X(t)$ and $M_Y(t)$ when $Y = aX + b$.
23. Define the raw and central moments of a random variable and state the relation between them.
24. The probability distribution of a random variable X is given by

x	0	1	2	3
$p(x)$	0.1	0.3	0.4	0.2

find $E(Y)$, where $Y = X^2 + X$.

25. The PMF of a random variable X is given by

$$p(x)=\frac{a\lambda^x}{x!},\ \lambda=0,1,2,\ldots$$

where λ is a positive value. Find
(i) $P(X = 0)$, and
(ii) $P(X > 2)$.

$$\left[\text{Ans. (i) } e^{-\lambda},\ \text{(ii) } 1-e^{-\lambda}-\lambda e^{-\lambda}-\frac{\lambda^2e^{-\lambda}}{2}\right]$$

26. The CDF $F(x)$ of a random variable X is given by

(i) $F(x)=\begin{cases}0, & x\le 0\\ c(1-e^{-x}), & x>0\end{cases}$

(ii) $F(x)=\begin{cases}0, & \text{if } x\le 1\\ c(x-1)^4, & \text{if } 1<x\le 3\\ 1, & \text{if } x>3\end{cases}$

Find the PDF $f(x)$.

$$\left[\text{Ans. (i) } f(x)=\begin{cases}0 & x\le 0\\ ce^{-x} & x>0\end{cases}\quad \text{(ii) } f(x)=0 \quad x\le 1;\ =4c(x-1)^3,\ 1<x\le 3;\ =0 \quad x>3\right]$$

27. A continuous random variable has the PDF $f(x) = kx^4$, $-1 < x < 0$.

Find the value of k and also $P\left(X>-\frac{1}{2}\,/\,X<-\frac{1}{4}\right)$.

$$\left[\text{Ans. } k=5,\ p=\frac{1}{33}\right]$$

28. If the probability distribution function of a random variable X is $f(x) = Ce^{-\alpha x}$, $0 \le x < \infty$, $\alpha > 0$, find
(i) the value of C,
(ii) mean, and
(iii) variance

$$\left[\text{Ans. (i) } C=\alpha,\ \text{(ii) mean}=\frac{1}{\alpha^2},\ \text{(iii) variance}=\frac{2}{\alpha^3}\right]$$

29. If the PDF of a random variable X is $f(x) = \frac{x}{2}$ in $0 \le x \le 2$, find $P(X > 1.5/X > 1)$.

[Ans. $\frac{7}{12}$]

30. Find $P(X = 0)$, if $P(X = 0) = P(X < 0) = P(X > 0)$.

[Ans. $\frac{1}{3}$]

31. A continuous distribution of a random variable X in the range $(-3, 3)$ is defined by

$$f(x) = \begin{cases} \frac{1}{16}(3+x)^2, & -3 \le x - 1 \\ \frac{1}{16}(2-6x^2), & -1 \le x \le 3 \\ \frac{1}{16}(3-x)^2, & 1 \le x \le 3 \end{cases}$$

Show that the area under the curve is unity and the mean is zero.

32. Verify that each of the following functions is a probability density function:

(i) $f(x) = 1 - |1 - x|$, for $0 < x < 2$.

(ii) $f(x) = \frac{1}{\pi} \frac{c}{c^2 + (x-a)^2}$, for $-\infty < x \le \infty$

(iii) $f(x) = \frac{1}{2\sigma} e^{\frac{-(x-\mu)}{\sigma}}$, for $-\infty < x < \infty$

(iv) $f(x) = \frac{1}{4} x e^{\frac{-x}{2}}$, for $-\infty < x < \infty$.

33. The amount of bread (in hundreds of kilos) that a bakery sells in a day is a random variable with PDF

$$f(x) = \begin{cases} cx, & \text{for } 0 \le x < 3 \\ c(6-x), & \text{for } 3 \le x < 6 \\ 0, & \text{otherwise} \end{cases}$$

(i) Find the value of c which makes $f(x)$ a PDF.

(ii) What is the probability that the number of kilos of bread that will be sold in a day is

(a) more than 300 kilos

(b) between 150 and 450 kilos.

[Ans. (i) $c = \frac{1}{9}$, (ii) $\frac{3}{4}$]

34. A continuous random variable X has a PDF $f(x) = 3x^2$, $0 \le x \le 1$. Find a and b such that

(i) $P(X \le a) = P(X > a)$, and

(ii) $P(X > b) = 0.05$.

[Ans. (i) $a = \left(\frac{1}{2}\right)^{\frac{1}{3}}$, (ii) $b = (0.95)^{\frac{1}{3}}$]

35. A continuous random variable X has a PDF $f(x) = kx^2e^{-x}$; $x \ge 0$. Find k, mean and variance.

[Ans. $k = \frac{1}{2}$, mean = 3, variance = 3]

36. A continuous random variable X that can assumes values between $x = 2$ and $x = 5$ has a density function given by $f(x) = \frac{2}{27}(1+x)$. Find $P(3 < X < 4)$.

[Ans. $\frac{1}{3}$]

37. Find the MGF if it exists, given the PDF

$$f(x) \begin{cases} xe^{-x}, & x > 0 \\ 0, & \text{otherwise} \end{cases}$$

[Ans. $\frac{1}{1-t}$]

38. Find the MGF of the random variable X with PDF

$$f(x) = \begin{cases} x, & 0 \le x < 1 \\ 2-x, & 1 \le x < 2 \\ 0, & \text{otherwise} \end{cases}$$

and also find μ_1, μ_2.

[Ans. $\frac{(e^t-1)^2}{t^2}$, $\mu_1 = 0 = \mu_2$]

39. Find the MGF of a random variable whose density function is given by $f(x) = \lambda e^{-\lambda(x-a)}$, $x \ge a$. Hence find its mean and variance.

[Ans. MGF $= \frac{\lambda e^{at}}{\lambda - t}$, mean $= \frac{a\lambda + 1}{\lambda}$, Var $= \frac{1}{\lambda^2}$]

40. A continuous random variable has the PDF $f(x) = \begin{cases} \frac{1}{2}(x+1), & -1 < x < 1 \\ 0, & \text{otherwise} \end{cases}$

Find the mean and variance.

[Ans. mean $= \frac{1}{3}$, Var $= \frac{2}{9}$]

41. Given the following probability distribution of X:

x	−3	−2	−1	0	1	2	3
$p(x)$	0.05	0.10	0.30	0	0.30	0.15	0.10

Compute

(i) $E(X)$,
(ii) $E(X^2)$,
(iii) $E(2X + 3)$, and
(iv) Var $(2X + 3)$

[**Ans.** (i) 0.25, (ii) 2.95, (iii) 0.5 + 3, and (iv) 11.55]

42. Let X be a random variable with $E(X) = 10$ and $\text{Var}(X) = 25$. Find the positive values of a and b such that $Y = aX - b$ has expectation 0 and variance 1.

$$\left[\textbf{Ans.}\ a = \frac{1}{5}, b = 2\right]$$

43. If $f(x) = \frac{1}{2}e^{-|x|}$, find MGF of X and, hence, find its mean and variance.

$$\left[\textbf{Ans.}\ \frac{1}{1-t^2}, \text{mean} = 0, \text{Var} = 2\right]$$

44. If the MGF of a random variable X is $\frac{2}{2-t}$, find the standard deviation of X.

$$\left[\textbf{Ans.}\ \frac{1}{2}\right]$$

45. Find the MGF of a random variable X whose density function is

$$f(x) = \begin{cases} \dfrac{1}{2\sqrt{x}}, & 0 < x < 1 \\ 0, & \text{otherwise} \end{cases}$$

[**Ans.** 1]

46. If $\bar{X}$ is the mean of n independent identically distributed variables, then show that

$$M_{\bar{X}}(t) = \left[M_X\left(\frac{t}{n}\right)\right]^n.$$

47. (i) If $Y = aX + b$, show that $\sigma_y = a\sigma_x$, and
(ii) If $Y = (X - \mu_X)/\sigma_X$, find σ_X and σ_Y.

[**Ans.** (ii) 0, 1]

48. Show that

(i) $E(aX + bY) = aE(X) + bE(Y)$, and
(ii) $\text{Var}(aX + bY) = a^2\,\text{Var}(X) + b^2\,\text{Var}(Y) + 2abC(X, Y)$, where $C(X, Y)$ is the covariance of (X, Y).

49. If two random variables are uncorrelated, prove that the variance of their sum is equal to the sum of their variances.

50. Let X be a random variable with mean value = 3 and variance = 2. Find the second moment of X about the origin. Another random variable Y is defined by $Y = -6x + 22$. Find the mean value of Y.

[**Ans.** 11, 4]

51. If the continuous random variable X has the density function $f(x) = 2xe^{-\frac{x^2}{2}}$, $x \geq 0$, and if $Y = X^3$, find the mean and variance Y.

[**Ans.** 1, 1]