Assignment 13

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Download latex-tikz codes from

https://github.com/utkarshsurwade/Matrix_Theory_EE5609/tree/master/codes

1 Problem

Find a 3×3 matrix for which the minimal polynomial is x^2.

2 Definitions

Characteristic Polynomial	For an $n \times n$ matrix $\mathbf{A}$, characteristic polynomial is defined by, $$p(x) = \left|x\mathbf{I} - \mathbf{A}\right|$$
Cayley-Hamilton Theorem	If $p(x)$ is the characteristic polynomial of an $n \times n$ matrix $\mathbf{A}$, then, $$p(\mathbf{A}) = \mathbf{0}$$
Minimal Polynomial	Minimal polynomial $m(x)$ is the smallest factor of characteristic polynomial $p(x)$ such that, $$m(\mathbf{A}) = 0$$ Every root of characteristic polynomial should be the root of minimal polynomial

TABLE 1: Definitions

3 Explanation

Statement	Solution
Assuming matrix $\mathbf{A}$ as follows:	Let us Consider 3×3 upper triangular matrix, $$\mathbf{A} = \begin{pmatrix} e & a & b \\ 0 & f & c \\ 0 & 0 & d \end{pmatrix}$$
Characteristic polynomial of $\mathbf{A}$	$$\lvert x\mathbf{I} - \mathbf{A} \rvert = \begin{pmatrix} x-e & -a & -b \\ 0 & x-f & -c \\ 0 & 0 & x-d \end{pmatrix}$$ $$= (x-e)(x-f)(x-d)$$
Given	The minimum polynomial is $$p(x) = x^2$$ Therefore p(x) must divide characteristic polynomial. This will be satisfied only if at least two values among e,f,d are zeros.
Characteristic polynomial when e=0 and f=0	$$\lvert x\mathbf{I} - \mathbf{A} \rvert = x^2(x-d)$$
Since $p(x) = x^2$ Hence $p(\mathbf{A}) = \mathbf{A}^2 = \mathbf{0}_{3\times 3}$	Therefore calculating $p(\mathbf{A})$ as follows: $$\begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & d \end{pmatrix}\begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & d \end{pmatrix} = \mathbf{0}_{3\times 3}$$ $$\begin{pmatrix} 0 & 0 & ac+bd \\ 0 & 0 & dc \\ 0 & 0 & d^2 \end{pmatrix} = \mathbf{0}_{3\times 3}$$

With entries a=0,d=0,e=0,f=0 The matrix **A** will be: For b=1,c=1	For $\mathbf{A}^2$ to be a zero matrix, either d=0,a=0 or d=0,c=0 $$\mathbf{A} = \begin{pmatrix} 0 & 0 & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix}$$ $$\mathbf{A} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$
Conclusion	Thus the matrix, $$\mathbf{A} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$ has the minimal polynomial as x^2.

TABLE 2: Solution summary