Assignment 16

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https://github.com/utkarshsurwade/Matrix_Theory_EE5609/tree/master/codes

1 Problem

Let $\mathbf{T} : \mathbb{R}^n \to \mathbb{R}^n$ be a linear map that satisfies $\mathbf{T}^2 = \mathbf{T} - \mathbf{I}_n$. Then which of the following are true?

1. $\mathbf{T}$ is invertible
2. $\mathbf{T} - \mathbf{I}_n$ is not invertible
3. $\mathbf{T}$ has a real eigen value
4. $\mathbf{T}^3 = -\mathbf{I}_n$

2 Definitions

Characteristic Polynomial	For an $n \times n$ matrix $\mathbf{A}$, characteristic polynomial is defined by, $$p(x) = \left\lvert x\mathbf{I} - \mathbf{A} \right\rvert$$
Cayley-Hamilton Theorem	If $p(x)$ is the characteristic polynomial of an $n \times n$ matrix $\mathbf{A}$, then, $$p(\mathbf{A}) = \mathbf{0}$$
Minimal Polynomial	Minimal polynomial $m(x)$ is the smallest factor of characteristic polynomial $p(x)$ such that, $$m(\mathbf{A}) = 0$$ Every root of characteristic polynomial should be the root of minimal polynomial

TABLE 1: Definitions

3 Explanation

Statement	Solution
1.	Given that $\mathbf{T} : \mathbb{R}^n \to \mathbb{R}^n$ Since $\mathbf{T}$ is a linear map from $\mathbb{R}^n$ to $\mathbb{R}^n$ therefore the matrix corresponding to it is of order $n \times n$. $$\text{Since } \mathbf{T}^2 = \mathbf{T} - \mathbf{I}_n$$ $$\therefore \mathbf{T}^2 - \mathbf{T} + \mathbf{I}_n = \mathbf{0}$$ $\implies p(x) = x^2 - x + 1$ will be annihilating polynomial. $\therefore p(\mathbf{T}) = \mathbf{T}^2 - \mathbf{T} + \mathbf{I}_n = \mathbf{0}$ We know that minimal polynomial always divides annihilating polynomial. $\therefore$ The roots of minimal polynomial are as follows: $$x = \frac{1 \pm \sqrt{3}i}{2} \qquad (3.0.1)$$ Therefore any eigenvalue of $\mathbf{T}$ is a root of its minimal polynomial. Since 0 is not a root of p(x), Therefore 0 is not an eigen value for $\mathbf{T}$. Since $\mathbf{T}$ is not invertible iff there exists an eigen value which is zero. $$\therefore \mathbf{T} \text{ is invertible.} \qquad (3.0.2)$$
Conclusion	Therefore the statement is true.
2.	From equation (3.0.1) , Since 1 is not a root of p(x), Therefore 1 is not an eigen value for $\mathbf{T}$. Therefore, 0 is not an eigen values of $\mathbf{T} - \mathbf{I}_n$. $$\therefore \mathbf{T} - \mathbf{I}_n \text{ is invertible.} \qquad (3.0.3)$$
Conclusion	Therefore the statement is false.

3.	From equation (3.0.1) , Therefore any eigenvalue of $\mathbf{T}$ is a root of its minimal polynomial. But the roots of minimal polynomial are not real. Therefore $\mathbf{T}$ cant have a real eigen value.
Conclusion	Therefore the statement is false.
4.	$$\text{Since } \mathbf{T}^2 = \mathbf{T} - \mathbf{I}_n \quad (3.0.4)$$ $$\mathbf{T}^3 = \mathbf{T}(\mathbf{T} - \mathbf{I}_n) \quad (3.0.5)$$ $$\therefore \mathbf{T}^3 = \mathbf{T}^2 - \mathbf{T} \quad (3.0.6)$$ $$\therefore \mathbf{T}^3 = -\mathbf{I}_n \quad (3.0.7)$$
Conclusion	Therefore the statement is true.

TABLE 2: Solution summary