Assignment 15

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https://github.com/utkarshsurwade/Matrix_Theory_EE5609/tree/master/codes

1 Problem

Let $\mathbf{A}$ be a real matrix with characteristic polynomial $(x-1)^3$.Pick the correct statements from below:

1. $\mathbf{A}$ is necessarily diagonalizable.
2. If the minimal polynomial of $\mathbf{A}$ is $(x-1)^3$, then $\mathbf{A}$ is diagonalizale.
3. Characteristic polynomial of $\mathbf{A}^2$ is $(x-1)^3$.
4. If $\mathbf{A}$ has exactly two Jordan blocks, then $(\mathbf{A}-\mathbf{I})^2$ is diagonalizable.

2 Definitions

Characteristic Polynomial	For an $n \times n$ matrix $\mathbf{A}$, characteristic polynomial is defined by, $$p(x) = \left\lvert x\mathbf{I} - \mathbf{A} \right\rvert$$
Cayley-Hamilton Theorem	If $p(x)$ is the characteristic polynomial of an $n \times n$ matrix $\mathbf{A}$, then, $$p(\mathbf{A}) = \mathbf{0}$$
Minimal Polynomial	Minimal polynomial $m(x)$ is the smallest factor of characteristic polynomial $p(x)$ such that, $$m(\mathbf{A}) = 0$$ Every root of characteristic polynomial should be the root of minimal polynomial

TABLE 1: Definitions

3 Explanation

Statement	Solution
1.	Let $\mathbf{A} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}$ Since $\mathbf{A}$ is upper triangular matrix, $\therefore \lambda_1 = 1, \lambda_2 = 1, \lambda_3 = 1$ Therefore, $p(x) = (x-1)^3$ Soving $(\mathbf{A}-\mathbf{I})^3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ Soving $(\mathbf{A}-\mathbf{I})^2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ Soving $\mathbf{A}-\mathbf{I} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix}$ Since $\mathbf{A}-\mathbf{I} \neq \mathbf{0}$ Therefore, $m(x) = (x-1)^2$
Justification	Hence, the Jordan form of $\mathbf{A}$ is a 3×3 matrix consisting of two block: one block of order 2 with principal diagonal value as $\lambda = 1$ and super diagonal of the block (i.e the set of elements that lies directly above the elements comprising the principal diagonal) contains 1. And one block of order 1 with $\lambda = 1$. Hence the required Jordan form of $\mathbf{A}$ is, $\therefore \mathbf{J} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ A matrix is diagonalizable iff its jordan form is a diagonal matrix. Since $\mathbf{J}$ is not diagonizable therefore $\mathbf{A}$ is not diagonizable.
Conclusion	Therefore the statement is false.

2.	Let $\mathbf{A} = \begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}$ Since $\mathbf{A}$ is upper triangular matrix, $\therefore \lambda_1 = 1, \lambda_2 = 1, \lambda_3 = 1$ Therefore, $p(x) = (x-1)^3$ Soving $(\mathbf{A}-\mathbf{I})^3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ Soving $(\mathbf{A}-\mathbf{I})^2 = \begin{pmatrix} 0 & 0 & 4 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ Since $(\mathbf{A}-\mathbf{I})^2 \neq \mathbf{0}$ Therefore, $m(x) = (x-1)^3$
Justification	Hence, the Jordan form of $\mathbf{A}$ is a 3×3 matrix consisting of only one block with principal diagonal values as $\lambda_1 = 1$ and super diagonal of the matrix (i.e the set of elements that lies directly above the elements comprising the principal diagonal) contains 1. Hence the required Jordan form of $\mathbf{A}$ is, $\therefore \mathbf{J} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$ Since $\mathbf{J}$ is not diagonizable therefore $\mathbf{A}$ is not diagonizable.
Conclusion	Therefore the statement is false.
3.	Give that, $p(x)$ of $\mathbf{A} = (x-1)^3$ Hence the eigen values of $\mathbf{A} = 1, 1, 1$ Hence the eigen values of $\mathbf{A}^2 = 1^2, 1^2, 1^2$ or $1, 1, 1$ Therefore $p(x)$ of $\mathbf{A}^2 = (x-1)^3$
Conclusion	Therefore the statement is True.

4.	We know that jordan form of a matrix is similar to the original matrix Let $\mathbf{J}$ be the jordan form of the matrix $\mathbf{A}$ then, $$\begin{aligned} \mathbf{A} &= \mathbf{PJP}^{-1} \\ \mathbf{A}-\mathbf{I} &= \mathbf{PJP}^{-1}-\mathbf{I} \\ \mathbf{A}-\mathbf{I} &= \mathbf{P}(\mathbf{J}-\mathbf{I})\mathbf{P}^{-1} \\ (\mathbf{A}-\mathbf{I})^2 &= \mathbf{P}(\mathbf{J}-\mathbf{I})\mathbf{P}^{-1}\mathbf{P}(\mathbf{J}-\mathbf{I})\mathbf{P}^{-1} \\ (\mathbf{A}-\mathbf{I})^2 &= \mathbf{P}(\mathbf{J}-\mathbf{I})^2\mathbf{P}^{-1} \end{aligned}$$ Therefore $(\mathbf{A}-\mathbf{I})^2$ is similar to $(\mathbf{J}-\mathbf{I})^2$ Since $\mathbf{A}$ has exactly two jordan blocks and order of $\mathbf{A}$ is 3. $$\begin{aligned} \therefore \mathbf{J} &= \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \\ \mathbf{J}-\mathbf{I} &= \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\ (\mathbf{J}-\mathbf{I})^2 &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \end{aligned}$$ Since $(\mathbf{J}-\mathbf{I})^2$ is diagonal matrix. Therefore $(\mathbf{A}-\mathbf{I})^2$ is diagonalizable.
Conclusion	Therefore the statement is True.

TABLE 2: Solution summary