Matrix Theory (EE5609) Assignment 11

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***Abstract*—This document provides solutions to ugc problems**

Download all python codes from

```
https://github.com/utkarshsurwade/
    Matrix_Theory_EE5609/tree/master/codes
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and latex-tikz codes from

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https://github.com/utkarshsurwade/
    Matrix_Theory_EE5609/tree/master/
    Assignment11
```

1 Problem

Let $\mathbf{A}$ be a real symmetric matrix. Then we can conclude that

1. $\mathbf{A}$ does not have 0 as an eigenvalue
2. All eigenvalues of $\mathbf{A}$ are real
3. If $\mathbf{A}^{-1}$ exists, then $\mathbf{A}^{-1}$ is real and symmetric
4. $\mathbf{A}$ has at least one positive eigenvalue

2 Solution

<table>
<tr><th>Options</th><th>Solutions</th></tr>
<tr><td>1.</td><td>
Let us consider $\mathbf{A}$ as follows:

$$\mathbf{A} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \tag{2.0.1}$$

Eigenvalues corresponding to $\mathbf{A}$ are,

$$\left|\mathbf{A} - \lambda\mathbf{I}\right| = 0 \tag{2.0.2}$$
$$\implies \begin{pmatrix} 1-\lambda & 1 \\ 1 & 1-\lambda \end{pmatrix} = 0 \tag{2.0.3}$$
$$\implies \lambda(\lambda - 2) = 0 \tag{2.0.4}$$
$$\therefore \lambda_1 = 0, \lambda_2 = 2 \tag{2.0.5}$$

Therefore symmetric matrix $\mathbf{A}$ can have 0 as an eighenvalue.
</td></tr>
<tr><td>2.</td><td>
Let $\mathbf{v}$ be an eigenvector with respect to eigenvalue λ for matrix $\mathbf{A}$ where $\mathbf{v} \neq 0$ and

$$\mathbf{A}\mathbf{v} = \lambda\mathbf{v} \tag{2.0.6}$$
$$\implies \mathbf{v}^H\mathbf{A}\mathbf{v} = \mathbf{v}^H(\lambda\mathbf{v}) = \lambda\mathbf{v}^H\mathbf{v} \tag{2.0.7}$$

where H denotes Hermitian conjugation. Taking Hermitian transpose, we get

$$(\lambda\mathbf{v}^H\mathbf{v})^H = (\mathbf{v}^H\mathbf{A}\mathbf{v})^H \tag{2.0.8}$$
$$\therefore \overline{\lambda}\mathbf{v}^H\mathbf{v} = \mathbf{v}^H\mathbf{A}^H\mathbf{v} = \mathbf{v}^H\mathbf{A}\mathbf{v} \tag{2.0.9}$$
$$\therefore (\lambda - \overline{\lambda})\mathbf{v}^H\mathbf{v} = 0 \tag{2.0.10}$$
$$\text{since } \mathbf{v} \neq 0, \ \therefore \lambda - \overline{\lambda} = 0 \tag{2.0.11}$$
$$\therefore \lambda = \overline{\lambda} \text{ is real} \tag{2.0.12}$$

Hence eigenvalues of a Hermitian matrix (in particular real symmetric matrix) are real.
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<tr><td>3</td><td>
$$\text{Since } \mathbf{A}^{-1}\mathbf{A} = \mathbf{I} \tag{2.0.13}$$
$$\therefore (\mathbf{A}^{-1}\mathbf{A})^T = \mathbf{I}^T \tag{2.0.14}$$
$$\implies (\mathbf{A}^T)(\mathbf{A}^{-1})^T = \mathbf{I} \tag{2.0.15}$$
$$\text{Since } \mathbf{A}^T = \mathbf{A} \tag{2.0.16}$$
$$\therefore \ \mathbf{A}(\mathbf{A}^{-1})^T = \mathbf{I} \tag{2.0.17}$$
$$\therefore \ \mathbf{A}^{-1} = (\mathbf{A}^{-1})^T \tag{2.0.18}$$

Since $\mathbf{A}$ is real and the inverse of a matrix is unique, therefore $\mathbf{A}^{-1}$ is real and symmetric.
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4.	Let us consider $\mathbf{A}$ as follows: $$\mathbf{A} = \begin{pmatrix} -6 & 2 \\ 2 & -9 \end{pmatrix} \quad (2.0.19)$$ Eigenvalues corresponding to $\mathbf{A}$ are, $$\left\|\mathbf{A} - \lambda\mathbf{I}\right\| = 0 \quad (2.0.20)$$ $$\implies \begin{pmatrix} -6-\lambda & 2 \\ 2 & -9-\lambda \end{pmatrix} = 0 \quad (2.0.21)$$ $$\implies (\lambda+5)(\lambda+10) = 0 \quad (2.0.22)$$ $$\therefore \lambda_1 = -5, \lambda_2 = -10 \quad (2.0.23)$$ Therefore symmetric matric $\mathbf{A}$ can have all negative eigenvalues.