

Assignment 17

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https://github.com/utkarshsurwade/Matrix_Theory_EE5609/tree/master/codes

1 PROBLEM

Let \mathbf{u} be a real $n \times 1$ vector satisfying $\mathbf{u}^T \mathbf{u} = 1$, where \mathbf{u}^T is the transpose of \mathbf{u} . Define $\mathbf{A} = \mathbf{I} - 2\mathbf{u}\mathbf{u}^T$ where \mathbf{I} is the n^{th} order identity matrix. Which of the following statements are true?

1. \mathbf{A} is singular
2. $\mathbf{A}^2 = \mathbf{A}$
3. $\text{Trace}(\mathbf{A}) = n - 2$
4. $\mathbf{A}^2 = \mathbf{I}$

2 THEOREM 1.

Let $\mathbf{A}_{m \times n}$ and $\mathbf{B}_{n \times k}$ be matrices such that the product \mathbf{AB} is well defines. Then

$$\text{rank}(\mathbf{AB}) \leq \min(\text{rank}(\mathbf{A}), \text{rank}(\mathbf{B})) \quad (2.0.1)$$

Proof: Matrix \mathbf{A} can be treated as a linear transformation from \mathbb{F}^n to \mathbb{F}^m . In that case rank of the matrix is the dimension of the image space of the transformation. If \mathbf{T} is a linear transformation from \mathbf{V}_1 to \mathbf{V}_2 then clearly $\dim \mathbf{T}(\mathbf{V}_1) \leq \dim (\mathbf{V}_1)$. Hence $\text{rank}(\mathbf{AB}) \leq \text{rank}(\mathbf{B})$. Since row rank and column rank of a matrix are equal,

$$\text{Therefore } \text{rank}(\mathbf{AB}) \leq \min(\text{rank}(\mathbf{A}), \text{rank}(\mathbf{B})) \quad (2.0.2)$$

3 EXPLANATION

Statement	Solution
1.	$\text{Let } \mathbf{u} = \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{pmatrix}$ $\text{Let } \mathbf{B} = \mathbf{u}\mathbf{u}^T$ $\therefore \mathbf{B} = \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{pmatrix} \begin{pmatrix} u_1 & u_2 & \dots & u_n \end{pmatrix}$ $\therefore \mathbf{B} = \begin{pmatrix} u_1^2 & u_1u_2 & \dots & u_1u_n \\ u_2u_1 & u_2^2 & \dots & u_2u_n \\ \vdots & \vdots & \ddots & \vdots \\ u_nu_1 & u_nu_2 & \dots & u_n^2 \end{pmatrix}$ <p>given that, $\mathbf{u}^T\mathbf{u} = 1$</p> $\therefore \mathbf{u}^T\mathbf{u} = \begin{pmatrix} u_1 & u_2 & \dots & u_n \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{pmatrix}$ $\therefore \mathbf{u}^T\mathbf{u} = u_1^2 + u_2^2 + \dots + u_n^2$ <p>Since \mathbf{u} is non-zero vector and $\mathbf{B} = \mathbf{u}\mathbf{u}^T$. Hence \mathbf{B} is a non-zero matrix. Therefore Rank of \mathbf{B} is at least 1. From (2.0.2)</p> $\text{rank}(\mathbf{B}) \leq \min(\text{rank}(\mathbf{u}), \text{rank}(\mathbf{u}^T))$ $\therefore \text{rank}(\mathbf{B}) \leq \min(1, 1)$ <p>So Rank of \mathbf{B} is at most 1. Hence Rank of \mathbf{B} is equal to 1. Therefore \mathbf{B} has n-1 eigenvalues equal to 0. Since the trace of a matrix is equal to the sum of its eigen values. We know that trace of $\mathbf{B} = u_1^2 + u_2^2 + \dots + u_n^2 = 1$</p> $\therefore \text{Trace of } \mathbf{B} = \lambda_1 + \lambda_2 + \dots + \lambda_{n-1} + \lambda_n$ $1 = 0 + 0 + \dots + \lambda_n$ $\therefore \lambda_n = 1$ <p>Therefore the eigen values of \mathbf{B} are $\lambda_1 = 0, \lambda_2 = 0, \dots, \lambda_{n-1} = 0, \lambda_n = 1$ Hence the characteristic polynomial for $\mathbf{B} = x^{n-1}(x - 1)$ Since $\mathbf{A} = \mathbf{I} - 2\mathbf{u}\mathbf{u}^T$</p>

	<p>and we know the eigen values of \mathbf{I} are $\lambda_1 = 1, \lambda_2 = 1, \dots, \lambda_{n-1} = 1, \lambda_n = 1$ and we know the eigen values of \mathbf{uu}^T are $\lambda_1 = 0, \lambda_2 = 0, \dots, \lambda_{n-1} = 0, \lambda_n = 1$</p> <p>$\therefore$ The eigen values of $\mathbf{A} = \lambda_1 = 1, \lambda_2 = 1, \dots, \lambda_{n-1} = 1, \lambda_n = -1$ (3.0.1)</p> <p>Since \mathbf{A} does not have 0 as an eigen value Therefore \mathbf{A} is not singular.</p>
Conclusion	Therefore the statement is false.
2.	<p>For $\mathbf{A}^2 = \mathbf{A}$, we know that $p(x) = x^2 - x$ \therefore minimal polynomial of \mathbf{A} must divide $x(x-1)$ \therefore possible eigenvalues of \mathbf{A} are 0 or 1 But from (3.0.1), we know that \mathbf{A} has -1 as an eigen value Therefore $\mathbf{A}^2 = \mathbf{A}$ is false.</p>
Conclusion	Therefore the statement is false.
3.	<p>From equation (3.0.1) , Trace of $\mathbf{A} = n - 2$</p>
Conclusion	Therefore the statement is true.
4.	<p>Since $\mathbf{A} = \mathbf{I} - 2\mathbf{uu}^T$ $\mathbf{A}^2 = (\mathbf{I} - 2\mathbf{uu}^T)(\mathbf{I} - 2\mathbf{uu}^T)$ $\therefore \mathbf{A}^2 = \mathbf{I} - 2\mathbf{uu}^T - 2\mathbf{uu}^T + 4\mathbf{uu}^T\mathbf{uu}^T$ Since $\mathbf{u}^T\mathbf{u} = 1$ $\therefore \mathbf{A}^2 = \mathbf{I} - 2\mathbf{uu}^T - 2\mathbf{uu}^T + 4\mathbf{uu}^T$ $\therefore \mathbf{A}^2 = \mathbf{I}$</p>
Conclusion	Therefore the statement is true.

TABLE 2: Solution summary