Assignment 17

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Download latex-tikz codes from

https://github.com/utkarshsurwade/Matrix_Theory_EE5609/tree/master/codes

1 Problem

Let $\mathbf{u}$ be a real n × 1 vector satisfying $\mathbf{u}^T\mathbf{u} = 1$, where $\mathbf{u}^T$ is the transpose of $\mathbf{u}$.Define $\mathbf{A} = \mathbf{I} - 2\mathbf{u}\mathbf{u}^T$ where $\mathbf{I}$ is the n^{th} order identity matrix.Which of the following statements are true?
1. $\mathbf{A}$ is singular
2. $\mathbf{A}^2 = \mathbf{A}$
3. Trace($\mathbf{A}$)=n-2
4. $\mathbf{A}^2 = \mathbf{I}$

2 Theorem 1.

Let $\mathbf{A}_{m\times n}$ and $\mathbf{B}_{n\times k}$ be matrices such that the product $\mathbf{AB}$ is well defines. Then

$$\text{rank}(\mathbf{AB}) \leq \min(\text{rank}(\mathbf{A}), \text{rank}(\mathbf{B})) \tag{2.0.1}$$

Proof: Matrix $\mathbf{A}$ can be treated as a linear transformation from $\mathbb{F}^n$ to $\mathbb{F}^m$.In that case rank of the matrix is the dimension of the image space of the transformation. If $\mathbf{T}$ is a linear transformation from $\mathbf{V}_1$ to $\mathbf{V}_2$ then clearly dim $\mathbf{T}(\mathbf{V}_1) \leq$ dim $(\mathbf{V}_1)$.Hence rank($\mathbf{AB}$) $\leq$ rank($\mathbf{B}$). Since row rank and column rank of a matrix are equal,

$$\text{Therefore } \text{rank}(\mathbf{AB}) \leq \min(\text{rank}(\mathbf{A}), \text{rank}(\mathbf{B})) \tag{2.0.2}$$

3 Explanation

<table>
<tr><th>Statement</th><th>Solution</th></tr>
<tr><td>1.</td><td>

$$\text{Let } \mathbf{u} = \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{pmatrix}$$

$$\text{Let } \mathbf{B} = \mathbf{u}\mathbf{u}^T$$

$$\therefore \mathbf{B} = \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{pmatrix} \begin{pmatrix} u_1 & u_2 & \dots & u_n \end{pmatrix}$$

$$\therefore \mathbf{B} = \begin{pmatrix} u_1^2 & u_1u_2 & \dots & u_1u_n \\ u_2u_1 & u_2^2 & \dots & u_2u_n \\ \vdots & \vdots & \ddots & \vdots \\ u_nu_1 & u_nu_2 & \dots & u_n^2 \end{pmatrix}$$

given that, $\mathbf{u}^T\mathbf{u} = 1$

$$\therefore \mathbf{u}^T\mathbf{u} = \begin{pmatrix} u_1 & u_2 & \dots & u_n \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{pmatrix}$$

$$\therefore \mathbf{u}^T\mathbf{u} = u_1^2 + u_2^2 + \dots + u_n^2$$

Since $\mathbf{u}$ is non-zero vector and $\mathbf{B} = \mathbf{u}\mathbf{u}^T$.
Hence $\mathbf{B}$ is a non-zero matrix.
Therefore Rank of $\mathbf{B}$ is at least 1.
From (2.0.2)

$$\text{rank}(\mathbf{B}) \leq \min(\text{rank}(\mathbf{u}), \text{rank}(\mathbf{u}^T))$$
$$\therefore \text{rank}(\mathbf{B}) \leq \min(1, 1)$$

So Rank of $\mathbf{B}$ is at most 1.
Hence Rank of $\mathbf{B}$ is equal to 1.
Therefore $\mathbf{B}$ has n-1 eigenvalues equal to 0.
Since the trace of a matrix is equal to the sum of its eigen values.
We know that trace of $\mathbf{B} = u_1^2 + u_2^2 + \dots + u_n^2 = 1$

$$\therefore \text{Trace of } \mathbf{B} = \lambda_1 + \lambda_2 + \dots + \lambda_{n-1} + \lambda_n$$
$$1 = 0 + 0 + \dots + \lambda_n$$
$$\therefore \lambda_n = 1$$

Therefore the eigen values of $\mathbf{B}$ are $\lambda_1 = 0, \lambda_2 = 0, \dots, \lambda_{n-1} = 0, \lambda_n = 1$
Hence the characteristic polynomial for $\mathbf{B} = x^{n-1}(x-1)$
Since $\mathbf{A} = \mathbf{I} - 2\mathbf{u}\mathbf{u}^T$

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	and we know the eigen values of $\mathbf{I}$ are $\lambda_1 = 1, \lambda_2 = 1, \ldots, \lambda_{n-1} = 1, \lambda_n = 1$ and we know the eigen values of $\mathbf{uu^T}$ are $\lambda_1 = 0, \lambda_2 = 0, \ldots, \lambda_{n-1} = 0, \lambda_n = 1$ $$\therefore \text{ The eigen values of } \mathbf{A} = \lambda_1 = 1, \lambda_2 = 1, \ldots, \lambda_{n-1} = 1, \lambda_n = -1 \tag{3.0.1}$$
Example	$$\text{Let } \mathbf{u} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \tag{3.0.2}$$ $$\text{then } \mathbf{u}^T = \begin{pmatrix} 1 & 0 & 0 \end{pmatrix} \tag{3.0.3}$$ $$\text{which satisfies } \mathbf{u}^T\mathbf{u} = 1 \tag{3.0.4}$$ $$\therefore \mathbf{uu}^T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{3.0.5}$$ $$\text{Since } \mathbf{A} = \mathbf{I} - 2\mathbf{uu}^T \tag{3.0.6}$$ $$\therefore \mathbf{A} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} - 2\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{3.0.7}$$ $$\therefore \mathbf{A} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{3.0.8}$$ $$\therefore \text{The eigen values of } \mathbf{A} = \lambda_1 = 1, \lambda_2 = 1, \lambda_3 = -1 \tag{3.0.9}$$
Conclusion	From (3.0.1) and (3.0.9) Since $\mathbf{A}$ does not have 0 as an eigen value Therefore $\mathbf{A}$ is not singular. Therefore the statement is false.
2.	For $\mathbf{A}^2 = \mathbf{A}$, we know that $p(x) = x^2 - x$ $\therefore$ minimal polynomial of $\mathbf{A}$ must divide x(x-1) $\therefore$ possible eigenvalues of $\mathbf{A}$ are 0 or 1 But from (3.0.1) and (3.0.9), we know that $\mathbf{A}$ has -1 as an eigen value Therefore $\mathbf{A}^2 = \mathbf{A}$ is false.
Conclusion	Therefore the statement is false.

3.	From equation (3.0.1) and (3.0.9), Trace of $\mathbf{A} = n - 2$
Conclusion	Therefore the statement is true.
4.	Since $\mathbf{A} = \mathbf{I} - 2\mathbf{u}\mathbf{u}^T$ $\mathbf{A}^2 = (\mathbf{I} - 2\mathbf{u}\mathbf{u}^T)(\mathbf{I} - 2\mathbf{u}\mathbf{u}^T)$ $\therefore \mathbf{A}^2 = \mathbf{I} - 2\mathbf{u}\mathbf{u}^T - 2\mathbf{u}\mathbf{u}^T + 4\mathbf{u}\mathbf{u}^T\mathbf{u}\mathbf{u}^T$ Since $\mathbf{u}^T\mathbf{u} = 1$ $\therefore \mathbf{A}^2 = \mathbf{I} - 2\mathbf{u}\mathbf{u}^T - 2\mathbf{u}\mathbf{u}^T + 4\mathbf{u}\mathbf{u}^T$ $\therefore \mathbf{A}^2 = \mathbf{I}$
Example	From (3.0.8) Since $\mathbf{A} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ $\therefore \mathbf{A}^2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ Therefore $\mathbf{A}^2 = \mathbf{I}$ is true.
Conclusion	Therefore the statement is true.

TABLE 2: Solution summary