Assignment 16

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https://github.com/utkarshsurwade/Matrix_Theory_EE5609/tree/master/codes

1 Problem

Let $\mathbf{T} : \mathbb{R}^n \rightarrow \mathbb{R}^n$ be a linear map that satisfies $\mathbf{T}^2 = \mathbf{T} - \mathbf{I}_n$. Then which of the following are true?

1. $\mathbf{T}^3 = -\mathbf{I}_n$
2. $\mathbf{T}$ is invertible
3. $\mathbf{T} - \mathbf{I}_n$ is not invertible
4. $\mathbf{T}$ has a real eigen value

2 Definitions

Characteristic Polynomial	For an $n \times n$ matrix $\mathbf{A}$, characteristic polynomial is defined by, $$p(x) = \left\lvert x\mathbf{I} - \mathbf{A} \right\rvert$$
Cayley-Hamilton Theorem	If $p(x)$ is the characteristic polynomial of an $n \times n$ matrix $\mathbf{A}$, then, $$p(\mathbf{A}) = \mathbf{0}$$
Minimal Polynomial	Minimal polynomial $m(x)$ is the smallest factor of characteristic polynomial $p(x)$ such that, $$m(\mathbf{A}) = 0$$ Every root of characteristic polynomial should be the root of minimal polynomial

TABLE 1: Definitions

3 Explanation

Statement	Solution
1.	Given that $\mathbf{T} : \mathbb{R}^n \to \mathbb{R}^n$ Since $\mathbf{T}$ is a linear map from $\mathbb{R}^n$ to $\mathbb{R}^n$ therefore the matrix corresponding to it is of order $n \times n$. Since $\mathbf{T}^2 = \mathbf{T} - \mathbf{I}_n$ (3.0.1) $\mathbf{T}^3 = \mathbf{T}(\mathbf{T} - \mathbf{I}_n)$ (3.0.2) $\therefore \mathbf{T}^3 = \mathbf{T}^2 - \mathbf{T}$ (3.0.3) $\therefore \mathbf{T}^3 = -\mathbf{I}_n$ (3.0.4)
Conclusion	Therefore the statement is true.
2.	From equation (3.0.4) , $\left\|\mathbf{T}^3\right\| = \left\|-\mathbf{I}_n\right\|$ (3.0.5) $\left\|\mathbf{T}\right\|^3 = \left\|-\mathbf{I}_n\right\|$ (3.0.6) $\left\|\mathbf{T}\right\| = \left\|-\mathbf{I}_n\right\|^{\frac{1}{3}}$ (3.0.7) Since $\left\|-\mathbf{I}_n\right\|$ is non-zero, therefore its cubic root will be non-zero. $\therefore \left\|\mathbf{T}\right\|$ is non-zero (3.0.8) $\therefore \mathbf{T}$ is invertible. (3.0.9)
Conclusion	Therefore the statement is true.

3.	$$\text{Since } \mathbf{T}^2 = \mathbf{T} - \mathbf{I}_n \quad (3.0.10)$$ $$\therefore \left\lvert\mathbf{T}^2\right\rvert = \left\lvert\mathbf{T} - \mathbf{I}_n\right\rvert \quad (3.0.11)$$ $$\therefore \left\lvert\mathbf{T}\right\rvert^2 = \left\lvert\mathbf{T} - \mathbf{I}_n\right\rvert \quad (3.0.12)$$ Since $\left\lvert\mathbf{T}\right\rvert$ is non-zero from equation (3.0.8) Therefore $\left\lvert\mathbf{T}\right\rvert^2$ is non-zero. Therefore $\left\lvert\mathbf{T} - \mathbf{I}_n\right\rvert$ is non-zero. Therefore $\mathbf{T} - \mathbf{I}_n$ is Invertible.
Conclusion	Therefore the statement is false.
4.	$$\text{Since } \mathbf{T}^2 = \mathbf{T} - \mathbf{I}_n$$ $$\therefore \mathbf{T}^2 - \mathbf{T} + \mathbf{I}_n = \mathbf{0}$$ $\implies x^2 - x + 1$ will be annihilating polynomial. We know that minimal polynomial always divides annihilating polynomial. Therefore m(x) can have following values: Case 1: $m(x) = x - \frac{1+\sqrt{3}i}{2}$ *or* Case 2: $m(x) = x - \frac{1-\sqrt{3}i}{2}$ *or* Case 3: $m(x) = x^2 - x + 1$ Since all the roots of minimal polynomial are the roots of characteristic polynomial. Therefore, for all three cases $\mathbf{T}$ has no real eigen values.
Conclusion	Therefore the statement is false.

TABLE 2: Solution summary