# Matrix Theory (EE5609) Assignment 8

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***Abstract*—This document contains the solution SVD problem**

Download all python codes from

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https://github.com/utkarshsurwade/
    Matrix_Theory_EE5609/tree/master/codes
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and latex-tikz codes from

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https://github.com/utkarshsurwade/
    Matrix_Theory_EE5609/tree/master/
    Assignment8
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## 1 Problem

Find the foot of the perpendicular to the plane

$$6x - 2y - 3z + 2 = 0 \tag{1.0.1}$$

from the point $\begin{pmatrix}1\\3\\0\end{pmatrix}$ using SVD

## 2 Solution

The given plane equation is

$$\begin{pmatrix}6 & -2 & -3\end{pmatrix}\mathbf{x} = 0 \tag{2.0.1}$$

The equation of plane is

$$\mathbf{n}^T\mathbf{x} = c \tag{2.0.2}$$

Hence the normal vector $\mathbf{n}$ is,

$$\mathbf{n} = \begin{pmatrix}6\\-2\\-3\end{pmatrix} \tag{2.0.3}$$

Let, the normal vectors $\mathbf{m_1}$ and $\mathbf{m_2}$ to the normal vector $\mathbf{n}$ be,

$$\mathbf{m} = \begin{pmatrix}a\\b\\c\end{pmatrix} \tag{2.0.4}$$

$$\text{then, } \mathbf{m}^T\mathbf{n} = 0 \tag{2.0.5}$$

$$\implies \begin{pmatrix}a & b & c\end{pmatrix}\begin{pmatrix}6\\-2\\-3\end{pmatrix} = 0 \tag{2.0.6}$$

Let, a=1 and b=0 we get,

$$\mathbf{m_1} = \begin{pmatrix}1\\0\\2\end{pmatrix} \tag{2.0.7}$$

Let, a=0 and b=1,

$$\mathbf{m_2} = \begin{pmatrix}0\\1\\-\frac{2}{3}\end{pmatrix} \tag{2.0.8}$$

Now solving the equation,

$$\mathbf{Mx} = \mathbf{b} \tag{2.0.9}$$

Where,

$$\mathbf{M} = \begin{pmatrix}1 & 0\\0 & 1\\2 & -\frac{2}{3}\end{pmatrix}, \mathbf{b} = \begin{pmatrix}1\\3\\0\end{pmatrix} \tag{2.0.10}$$

To solve (2.0.9) we perform singular value decomposition on M given by,

$$\mathbf{M} = \mathbf{USV}^T \tag{2.0.11}$$

substituting the value of $\mathbf{M}$ from equation (2.0.11) to (2.0.9),

$$\implies \mathbf{USV}^T\mathbf{x} = \mathbf{b} \tag{2.0.12}$$

$$\implies \mathbf{x} = \mathbf{VS_+U}^T\mathbf{b} \tag{2.0.13}$$

where, $\mathbf{S_+}$ is Moore-Pen-rose Pseudo-Inverse of $\mathbf{S}$. Columns of $\mathbf{U}$ are eigenvectors of $\mathbf{MM}^T$, columns of $\mathbf{V}$ are eigenvectors of $\mathbf{M}^T\mathbf{M}$ and $\mathbf{S}$ is diagonal matrix of singular value of eigenvalues of $\mathbf{M}^T\mathbf{M}$. Calculating the eigenvectors for $\mathbf{M}^T\mathbf{M}$.

$$\mathbf{M}^T\mathbf{M} = \begin{pmatrix}1 & 0 & 2\\0 & 1 & -\frac{2}{3}\end{pmatrix}\begin{pmatrix}1 & 0\\0 & 1\\2 & -\frac{2}{3}\end{pmatrix} = \begin{pmatrix}5 & -\frac{4}{3}\\-\frac{4}{3} & \frac{13}{9}\end{pmatrix} \tag{2.0.14}$$

Eigenvalues corresponding to $\mathbf{M}^T\mathbf{M}$ is,

$$\left|\mathbf{M}^T\mathbf{M} - \lambda\mathbf{I}\right| = 0 \tag{2.0.15}$$

$$\implies \begin{pmatrix} 5-\lambda & -\frac{4}{3} \\ -\frac{4}{3} & \frac{13}{9}-\lambda \end{pmatrix} \tag{2.0.16}$$

$$\implies (\lambda - \frac{49}{9})(\lambda - 1) = 0 \tag{2.0.17}$$

$$\therefore \lambda_1 = \frac{49}{9}, \lambda_2 = 1, \tag{2.0.18}$$

Hence the eigenvectors corresponding to $\lambda_1$ and $\lambda_2$ respectively is,

$$\mathbf{v_1} = \begin{pmatrix} -3 \\ 1 \end{pmatrix}, \mathbf{v_2} = \begin{pmatrix} \frac{1}{3} \\ 1 \end{pmatrix} \tag{2.0.19}$$

Normalizing the eigenvectors we get,

$$\mathbf{v_1} = \frac{1}{\sqrt{10}}\begin{pmatrix} -3 \\ 1 \end{pmatrix} \tag{2.0.20}$$

$$\mathbf{v_2} = \frac{1}{\sqrt{10}}\begin{pmatrix} 1 \\ 3 \end{pmatrix} \tag{2.0.21}$$

$$\implies \mathbf{V} = \frac{1}{\sqrt{10}}\begin{pmatrix} -3 & 1 \\ 1 & 3 \end{pmatrix} \tag{2.0.22}$$

Now calculating the eigenvectors corresponding to $\mathbf{M}\mathbf{M}^T$

$$\mathbf{M}\mathbf{M}^T = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 2 & -\frac{2}{3} \end{pmatrix}\begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & -\frac{2}{3} \end{pmatrix} \tag{2.0.23}$$

$$\implies \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & -\frac{2}{3} \\ 2 & -\frac{2}{3} & \frac{40}{9} \end{pmatrix} \tag{2.0.24}$$

Eigenvalues corresponding to $\mathbf{M}\mathbf{M}^T$ is,

$$\left|\mathbf{M}\mathbf{M}^T - \lambda\mathbf{I}\right| = 0 \tag{2.0.25}$$

$$\implies \begin{pmatrix} 1-\lambda & 0 & 2 \\ 0 & 1-\lambda & -\frac{2}{3} \\ 2 & -\frac{2}{3} & \frac{40}{9}-\lambda \end{pmatrix} \tag{2.0.26}$$

$$\implies \lambda(\lambda - 1)(\lambda - \frac{49}{9}) = 0 \tag{2.0.27}$$

$$\therefore \lambda_3 = \frac{49}{9}, \lambda_4 = 1, \lambda_5 = 0 \tag{2.0.28}$$

Hence the eigenvectors corresponding to $\lambda_3$, $\lambda_4$ and $\lambda_5$ respectively is,

$$\mathbf{v_3} = \begin{pmatrix} \frac{9}{20} \\ \frac{-3}{20} \\ 1 \end{pmatrix}, \mathbf{v_4} = \begin{pmatrix} \frac{1}{3} \\ 1 \\ 0 \end{pmatrix}, \mathbf{v_5} = \begin{pmatrix} -2 \\ \frac{2}{3} \\ 1 \end{pmatrix} \tag{2.0.29}$$

Normalizing the eigenvectors we get,

$$\mathbf{v_3} = \begin{pmatrix} \frac{9}{\sqrt{490}} \\ \frac{-3}{\sqrt{490}} \\ \frac{20}{\sqrt{490}} \end{pmatrix} \tag{2.0.30}$$

$$\mathbf{v_4} = \begin{pmatrix} \frac{1}{\sqrt{10}} \\ \frac{3}{\sqrt{10}} \\ 0 \end{pmatrix} \tag{2.0.31}$$

$$\mathbf{v_5} = \begin{pmatrix} \frac{-6}{7} \\ \frac{2}{7} \\ \frac{3}{7} \end{pmatrix} \tag{2.0.32}$$

$$\implies \mathbf{U} = \begin{pmatrix} \frac{9}{\sqrt{490}} & \frac{1}{\sqrt{10}} & \frac{-6}{7} \\ \frac{-3}{\sqrt{490}} & \frac{3}{\sqrt{10}} & \frac{2}{7} \\ \frac{20}{\sqrt{490}} & 0 & \frac{3}{7} \end{pmatrix} \tag{2.0.33}$$

Now $\mathbf{S}$ corresponding to eigenvalues $\lambda_3$, $\lambda_4$ and $\lambda_5$ is as follows,

$$\mathbf{S} = \begin{pmatrix} \sqrt{\frac{49}{9}} & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \tag{2.0.34}$$

Now, Moore-Penrose Pseudo inverse of $\mathbf{S}$ is given by,

$$\mathbf{S}_+ = \begin{pmatrix} \frac{3}{7} & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \tag{2.0.35}$$

Hence we get singular value decomposition of $\mathbf{M}$ as,

$$\mathbf{M} = \frac{1}{\sqrt{10}}\begin{pmatrix} \frac{9}{\sqrt{490}} & \frac{1}{\sqrt{10}} & \frac{-6}{7} \\ \frac{-3}{\sqrt{490}} & \frac{3}{\sqrt{10}} & \frac{2}{7} \\ \frac{20}{\sqrt{490}} & 0 & \frac{3}{7} \end{pmatrix}\begin{pmatrix} \sqrt{\frac{49}{9}} & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} -3 & 1 \\ 1 & 3 \end{pmatrix}^T \tag{2.0.36}$$

Now substituting the values of (2.0.22), (2.0.35), (2.0.33) and (2.0.10) in (2.0.13),

$$\mathbf{U}^T\mathbf{b} = \begin{pmatrix} \frac{9}{\sqrt{490}} & \frac{1}{\sqrt{10}} & \frac{-6}{7} \\ \frac{-3}{\sqrt{490}} & \frac{3}{\sqrt{10}} & \frac{2}{7} \\ \frac{20}{\sqrt{490}} & 0 & \frac{3}{7} \end{pmatrix}^T\begin{pmatrix} 1 \\ 3 \\ 0 \end{pmatrix} \tag{2.0.37}$$

$$\implies \mathbf{U}^T\mathbf{b} = \begin{pmatrix} 0 \\ \sqrt{10} \\ 0 \end{pmatrix} \tag{2.0.38}$$

$$\mathbf{S}_+\mathbf{U}^T\mathbf{b} = \begin{pmatrix} \frac{3}{7} & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ \sqrt{10} \\ 0 \end{pmatrix} \tag{2.0.39}$$

$$\implies \mathbf{S}_+\mathbf{U}^T\mathbf{b} = \begin{pmatrix} 0 \\ \sqrt{10} \end{pmatrix} \tag{2.0.40}$$

$$\mathbf{V}\mathbf{S}_+\mathbf{U}^T\mathbf{b} = \frac{1}{\sqrt{10}} \begin{pmatrix} -3 & 1 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} 0 \\ \sqrt{10} \end{pmatrix} \tag{2.0.41}$$

$$\implies \mathbf{V}\mathbf{S}_+\mathbf{U}^T\mathbf{b} = \begin{pmatrix} 1 \\ 3 \end{pmatrix} \tag{2.0.42}$$

$\therefore$ from equation (2.0.13),

$$\mathbf{x} = \begin{pmatrix} 1 \\ 3 \end{pmatrix} \tag{2.0.43}$$

Verifying the solution using,

$$\mathbf{M}^T\mathbf{M}\mathbf{x} = \mathbf{M}^T\mathbf{b} \tag{2.0.44}$$

$$\implies \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & \frac{-2}{3} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & \frac{-2}{3} \end{pmatrix} \mathbf{x} = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & \frac{-2}{3} \end{pmatrix} \begin{pmatrix} 1 \\ 3 \\ 0 \end{pmatrix} \tag{2.0.45}$$

$$\implies \begin{pmatrix} 5 & \frac{-4}{3} \\ \frac{-4}{3} & \frac{13}{9} \end{pmatrix} \mathbf{x} = \begin{pmatrix} 1 \\ 3 \end{pmatrix} \tag{2.0.46}$$

Solving the augmented matrix we get,

$$\begin{pmatrix} 5 & \frac{-4}{3} & 1 \\ \frac{-4}{3} & \frac{13}{9} & 3 \end{pmatrix} \xleftrightarrow{R_1 \leftarrow \frac{R_1}{5}} \begin{pmatrix} 1 & \frac{-4}{15} & \frac{1}{5} \\ \frac{-4}{3} & \frac{13}{9} & 3 \end{pmatrix} \tag{2.0.47}$$

$$\xleftrightarrow{R_2 \leftarrow R_2 + \frac{4}{3}R_1} \begin{pmatrix} 1 & \frac{-4}{15} & \frac{1}{5} \\ 0 & \frac{49}{45} & \frac{49}{15} \end{pmatrix} \tag{2.0.48}$$

$$\xleftrightarrow{R_2 \leftarrow \frac{45}{49}R_2} \begin{pmatrix} 1 & \frac{-4}{15} & \frac{1}{5} \\ 0 & 1 & 3 \end{pmatrix} \tag{2.0.49}$$

$$\xleftrightarrow{R_1 \leftarrow R_1 + \frac{4}{15}R_2} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 3 \end{pmatrix} \tag{2.0.50}$$

$$\implies \mathbf{x} = \begin{pmatrix} 1 \\ 3 \end{pmatrix} \tag{2.0.51}$$

Hence from equations (2.0.43) and (2.0.51) we conclude that the solution is verified.