Assignment 12

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Download latex-tikz codes from

https://github.com/utkarshsurwade/Matrix_Theory_EE5609/tree/master/codes

1 Problem

Let $\mathbf{N}$ be a 2×2 complex matrix such that $\mathbf{N}^2 = 0$. Prove that either $\mathbf{N} = 0$ or $\mathbf{N}$ is similar over $\mathbb{C}$ to

$$\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \tag{1.0.1}$$

2 Explanation

Statement	Solution
Proof that $\mathbf{N} = 0$	$$\text{Let } \mathbf{N} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \tag{2.0.1}$$ $$\text{Since } \mathbf{N}^2 = 0 \tag{2.0.2}$$ If $\begin{pmatrix} a \\ c \end{pmatrix}, \begin{pmatrix} b \\ d \end{pmatrix}$ are linearly independent then $\mathbf{N}$ is diagonalizable to $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$. $$\text{If } \mathbf{PNP}^{-1} = 0 \tag{2.0.3}$$ $$\text{then } \mathbf{N} = \mathbf{P}^{-1}\mathbf{0P} = 0 \tag{2.0.4}$$ So in this case $\mathbf{N}$ itself is the zero matrix. This contradicts the assumption that $\begin{pmatrix} a \\ c \end{pmatrix}, \begin{pmatrix} b \\ d \end{pmatrix}$ are linearly independent. $\therefore$ we can assume that $\begin{pmatrix} a \\ c \end{pmatrix}, \begin{pmatrix} b \\ d \end{pmatrix}$ are linearly dependent if both are equal to the zero vector $$\text{then } \mathbf{N} = 0. \tag{2.0.5}$$

Assuming $\begin{pmatrix} b \\ d \end{pmatrix}$ as the zero vector	Therefore we can assume at least one vector is non-zero. Therefore $\mathbf{N} = \begin{pmatrix} a & 0 \\ c & 0 \end{pmatrix}$ $$\text{So } \mathbf{N}^2 = 0 \quad (2.0.6)$$ $$\implies a^2 = 0 \quad (2.0.7)$$ $$\therefore a = 0 \quad (2.0.8)$$ $$\text{Thus } \mathbf{N} = \begin{pmatrix} a & 0 \\ c & 0 \end{pmatrix} \quad (2.0.9)$$ In this case $\mathbf{N}$ is similar to $\mathbf{N} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ via the matrix $\mathbf{P} = \begin{pmatrix} c & 0 \\ 0 & 1 \end{pmatrix}$
Assuming $\begin{pmatrix} a \\ c \end{pmatrix}$ as the zero vector	Therefore $\mathbf{N} = \begin{pmatrix} 0 & b \\ 0 & d \end{pmatrix}$ $$\text{Then } \mathbf{N}^2 = 0 \quad (2.0.10)$$ $$\implies d^2 = 0 \quad (2.0.11)$$ $$\therefore d = 0 \quad (2.0.12)$$ $$\text{Thus } \mathbf{N} = \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix} \quad (2.0.13)$$ In this case $\mathbf{N}$ is similar to $\mathbf{N} = \begin{pmatrix} 0 & 0 \\ b & 0 \end{pmatrix}$ via the matrix $\mathbf{P} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, which is similar to $\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ as above.
Hence	we can assume neither $\begin{pmatrix} a \\ c \end{pmatrix}$ or $\begin{pmatrix} b \\ d \end{pmatrix}$ is the zero vector.

Consequences of linear independence	Since they are linearly dependent we can assume, $$\begin{pmatrix} b \\ d \end{pmatrix} = x \begin{pmatrix} a \\ c \end{pmatrix} \quad (2.0.14)$$ $$\therefore \mathbf{N} = \begin{pmatrix} a & ax \\ c & cx \end{pmatrix} \quad (2.0.15)$$ $$\therefore \mathbf{N}^2 = 0 \quad (2.0.16)$$ $$\implies a(a+cx) = 0 \quad (2.0.17)$$ $$c(a+cx) = 0 \quad (2.0.18)$$ $$ax(a+cx) = 0 \quad (2.0.19)$$ $$cx(a+cx) = 0 \quad (2.0.20)$$
Proof that $\mathbf{N}$ is similar over $\mathbb{C}$ to $\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$	We know that at least one of a or c is not zero. If a = 0 then c ≠ 0, it must be that x = 0. So in this case $\mathbf{N} = \begin{pmatrix} 0 & 0 \\ c & 0 \end{pmatrix}$ which is similar to $\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ as before. $$\text{If } a \neq 0 \quad (2.0.21)$$ $$\text{then } x \neq 0 \quad (2.0.22)$$ $$\text{else } a(a+cx) = 0 \quad (2.0.23)$$ $$\implies a = 0 \quad (2.0.24)$$ $$\text{Thus } a + cx = 0 \quad (2.0.25)$$ $$\text{Hence } \mathbf{N} = \begin{pmatrix} a & ax \\ \frac{-a}{x} & -a \end{pmatrix} \quad (2.0.26)$$ This is similar to $\begin{pmatrix} a & a \\ -a & -a \end{pmatrix}$ via $\mathbf{P} = \begin{pmatrix} \sqrt{x} & 0 \\ 0 & \frac{1}{\sqrt{x}} \end{pmatrix}$. And $\begin{pmatrix} a & a \\ -a & -a \end{pmatrix}$ is similar to $\begin{pmatrix} 0 & 0 \\ -a & 0 \end{pmatrix}$ via $\mathbf{P} = \begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix}$ And this finally is similar to $\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ as before.
Conclusion	Thus either $\mathbf{N} = 0$ or $\mathbf{N}$ is similar over $\mathbb{C}$ to $\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$.

TABLE 1: Solution summary