Assignment 18

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Download latex-tikz codes from

https://github.com/utkarshsurwade/Matrix_Theory_EE5609/tree/master/codes

1 Problem

Which of the following matrices have Jordan canonical form equal to

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}?$$

1. $\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$
2. $\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$
3. $\begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$
4. $\begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$

2 Definitions

Characteristic Polynomial	For an $n \times n$ matrix $\mathbf{A}$, characteristic polynomial is defined by, $$p(x) = \left\lvert x\mathbf{I} - \mathbf{A} \right\rvert$$
Cayley-Hamilton Theorem	If $p(x)$ is the characteristic polynomial of an $n \times n$ matrix $\mathbf{A}$, then, $$p(\mathbf{A}) = \mathbf{0}$$
Minimal Polynomial	Minimal polynomial $m(x)$ is the smallest factor of characteristic polynomial $p(x)$ such that, $$m(\mathbf{A}) = 0$$ Every root of characteristic polynomial should be the root of minimal polynomial

TABLE 1: Definitions

3 Explanation

Statement	Solution
1.	Let $\mathbf{A} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ Since $\mathbf{A}$ is upper triangular matrix, $\therefore \lambda_1 = 0, \lambda_2 = 0, \lambda_3 = 0$ Therefore, $p(x) = (x)^3$ Solving $\mathbf{A}^3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ Solving $\mathbf{A}^2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ Since $\mathbf{A} \neq \mathbf{0}$ Therefore, $m(x) = (x)^2$
Justification	Hence, the Jordan form of $\mathbf{A}$ is a 3×3 matrix consisting of two block: one block of order 2 with principal diagonal value as $\lambda = 0$ and super diagonal of the block (i.e the set of elements that lies directly above the elements comprising the principal diagonal) contains 1. And one block of order 1 with $\lambda = 0$. Hence the required Jordan form of $\mathbf{A}$ is, $\therefore \mathbf{J} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$
Conclusion	Therefore option 1 is true.

2.	Let $\mathbf{A} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$ Since $\mathbf{A}$ is upper triangular matrix, $\therefore \lambda_1 = 0, \lambda_2 = 0, \lambda_3 = 0$ Therefore, $p(x) = (x)^3$ Solving $\mathbf{A}^3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ Solving $\mathbf{A}^2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ Since $\mathbf{A} \neq \mathbf{0}$ Therefore, $m(x) = (x)^2$
Justification	Hence, the Jordan form of $\mathbf{A}$ is a 3×3 matrix consisting of two block: one block of order 2 with principal diagonal value as $\lambda = 0$ and super diagonal of the block (i.e the set of elements that lies directly above the elements comprising the principal diagonal) contains 1. And one block of order 1 with $\lambda = 0$. Hence the required Jordan form of $\mathbf{A}$ is, $\therefore \mathbf{J} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$
Conclusion	Therefore option 2 is true.

3.	Let $\mathbf{A} = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ Since $\mathbf{A}$ is upper triangular matrix, $\therefore \lambda_1 = 0, \lambda_2 = 0, \lambda_3 = 0$ Therefore, $p(x) = (x)^3$ Solving $\mathbf{A}^3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ Solving $\mathbf{A}^2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ Since $\mathbf{A} \neq \mathbf{0}$ Therefore, $m(x) = (x)^2$
Justification	Hence, the Jordan form of $\mathbf{A}$ is a 3×3 matrix consisting of two block: one block of order 2 with principal diagonal value as $\lambda = 0$ and super diagonal of the block (i.e the set of elements that lies directly above the elements comprising the principal diagonal) contains 1. And one block of order 1 with $\lambda = 0$. Hence the required Jordan form of $\mathbf{A}$ is, $\therefore \mathbf{J} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$
Conclusion	Therefore option 3 is true.

4.	Let $\mathbf{A} = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$ Since $\mathbf{A}$ is upper triangular matrix, $\therefore \lambda_1 = 0, \lambda_2 = 0, \lambda_3 = 0$ Therefore, $p(x) = (x)^3$ Solving $\mathbf{A}^3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ Solving $\mathbf{A}^2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ Since $\mathbf{A}^2 \neq \mathbf{0}$ Therefore, $m(x) = (x)^3$
Justification	Hence, the Jordan form of $\mathbf{A}$ is a 3×3 matrix consisting of only one block with principal diagonal values as $\lambda = 0$ and super diagonal of the matrix (i.e the set of elements that lies directly above the elements comprising the principal diagonal) contains 1. Hence the required Jordan form of $\mathbf{A}$ is, $\therefore \mathbf{J} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$
Conclusion	Therefore option 4 is false.

TABLE 2: Solution

4 Summarization of Above Results

For given jordan form:	$\mathbf{A} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$
We have two blocks:	one block is of order 2. And one block is of order 1. And eigenvalues are all $\lambda = 0$ $\therefore$ Algebraic Multiplicity of 0 is 3. The rank of the matrix is 1. $\begin{aligned}\text{Geometric Multiplicity of } 0 &= n - \text{Rank}(\mathbf{A} - \lambda\mathbf{I}) \\ &= n - \text{Rank}(\mathbf{A}) \\ &= 2\end{aligned}$
1.	The eigenvalue order of 0 in the characteristic polynomial = 3. $\therefore$ Algebraic Multiplicity of 0 is 3. The eigenvalue order of 0 in the minimal polynomial = 2. The rank of the matrix is 1. $\therefore$ The Geometric Multiplicity of 0 = 2. Therefore the matrix gives the same jordan form
2.	The eigenvalue order of 0 in the characteristic polynomial = 3. $\therefore$ Algebraic Multiplicity of 0 is 3. The eigenvalue order of 0 in the minimal polynomial = 2. The rank of the matrix is 1. $\therefore$ The Geometric Multiplicity of 0 = 2. Therefore the matrix gives the same jordan form
3.	The eigenvalue order of 0 in the characteristic polynomial = 3. $\therefore$ Algebraic Multiplicity of 0 is 3. The eigenvalue order of 0 in the minimal polynomial = 2. The rank of the matrix is 1. $\therefore$ The Geometric Multiplicity of 0 = 2. Therefore the matrix gives the same jordan form
4.	The eigenvalue order of 0 in the characteristic polynomial = 3. $\therefore$ Algebraic Multiplicity of 0 is 3. The eigenvalue order of 0 in the minimal polynomial = 3. The rank of the matrix is 2. $\therefore$ The Geometric Multiplicity of 0 = 1. Therefore the matrix gives different jordan form

TABLE 3: Conclusion of above Results