

# Assignment 17

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[https://github.com/utkarshsurwade/Matrix\\_Theory\\_EE5609/tree/master/codes](https://github.com/utkarshsurwade/Matrix_Theory_EE5609/tree/master/codes)

## 1 PROBLEM

Let  $\mathbf{u}$  be a real  $n \times 1$  vector satisfying  $\mathbf{u}^T \mathbf{u} = 1$ , where  $\mathbf{u}^T$  is the transpose of  $\mathbf{u}$ . Define  $\mathbf{A} = \mathbf{I} - 2\mathbf{u}\mathbf{u}^T$  where  $\mathbf{I}$  is the  $n^{th}$  order identity matrix. Which of the following statements are true?

1.  $\mathbf{A}$  is singular
2.  $\mathbf{A}^2 = \mathbf{A}$
3.  $\text{Trace}(\mathbf{A}) = n - 2$
4.  $\mathbf{A}^2 = \mathbf{I}$

## 2 THEOREM 1.

Let  $\mathbf{A}_{m \times n}$  and  $\mathbf{B}_{n \times k}$  be matrices such that the product  $\mathbf{AB}$  is well defines. Then

$$\text{rank}(\mathbf{AB}) \leq \min(\text{rank}(\mathbf{A}), \text{rank}(\mathbf{B})) \quad (2.0.1)$$

Proof: Matrix  $\mathbf{A}$  can be treated as a linear transformation from  $\mathbb{F}^n$  to  $\mathbb{F}^m$ . In that case rank of the matrix is the dimension of the image space of the transformation. If  $\mathbf{T}$  is a linear transformation from  $\mathbf{V}_1$  to  $\mathbf{V}_2$  then clearly  $\dim \mathbf{T}(\mathbf{V}_1) \leq \dim (\mathbf{V}_1)$ . Hence  $\text{rank}(\mathbf{AB}) \leq \text{rank}(\mathbf{B})$ . Since row rank and column rank of a matrix are equal,

$$\text{Therefore } \text{rank}(\mathbf{AB}) \leq \min(\text{rank}(\mathbf{A}), \text{rank}(\mathbf{B})) \quad (2.0.2)$$

### 3 DEFINITIONS

Characteristic Polynomial	<p>For an <math>n \times n</math> matrix <math>\mathbf{A}</math>, characteristic polynomial is defined by,</p> $p(x) =  x\mathbf{I} - \mathbf{A} $
Cayley-Hamilton Theorem	<p>If <math>p(x)</math> is the characteristic polynomial of an <math>n \times n</math> matrix <math>\mathbf{A}</math>, then,</p> $p(\mathbf{A}) = \mathbf{0}$
Minimal Polynomial	<p>Minimal polynomial <math>m(x)</math> is the smallest factor of characteristic polynomial <math>p(x)</math> such that,</p> $m(\mathbf{A}) = \mathbf{0}$ <p>Every root of characteristic polynomial should be the root of minimal polynomial</p>

TABLE 1: Definitions

## 4 EXPLANATION

Statement	Solution
1.	$\text{Let } \mathbf{u} = \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{pmatrix}$ $\text{Let } \mathbf{B} = \mathbf{u}\mathbf{u}^T$ $\therefore \mathbf{B} = \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{pmatrix} \begin{pmatrix} u_1 & u_2 & \dots & u_n \end{pmatrix}$ $\therefore \mathbf{B} = \begin{pmatrix} u_1^2 & u_1u_2 & \dots & u_1u_n \\ u_2u_1 & u_2^2 & \dots & u_2u_n \\ \vdots & \vdots & \ddots & \vdots \\ u_nu_1 & u_nu_2 & \dots & u_n^2 \end{pmatrix}$ <p>given that, <math>\mathbf{u}^T\mathbf{u} = 1</math></p> $\therefore \mathbf{u}^T\mathbf{u} = \begin{pmatrix} u_1 & u_2 & \dots & u_n \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{pmatrix}$ $\therefore \mathbf{u}^T\mathbf{u} = u_1^2 + u_2^2 + \dots + u_n^2$ <p>Since <math>\mathbf{u}</math> is non-zero vector and <math>\mathbf{B} = \mathbf{u}\mathbf{u}^T</math>.  Hence <math>\mathbf{B}</math> is a non-zero matrix.  Therefore Rank of <math>\mathbf{B}</math> is at least 1.  From (2.0.2)</p> $\text{rank}(\mathbf{B}) \leq \min(\text{rank}(\mathbf{u}), \text{rank}(\mathbf{u}^T))$ $\therefore \text{rank}(\mathbf{B}) \leq \min(1, 1)$ <p>So Rank of <math>\mathbf{B}</math> is at most 1.  Hence Rank of <math>\mathbf{B}</math> is equal to 1.  Therefore <math>\mathbf{B}</math> has <math>n-1</math> eigenvalues equal to 0.  Since the trace of a matrix is equal to the sum of its eigen values.  We know that trace of <math>\mathbf{B} = u_1^2 + u_2^2 + \dots + u_n^2 = 1</math></p> $\therefore \text{Trace of } \mathbf{B} = \lambda_1 + \lambda_2 + \dots + \lambda_{n-1} + \lambda_n$ $1 = 0 + 0 + \dots + \lambda_n$ $\therefore \lambda_n = 1$ <p>Therefore the eigen values of <math>\mathbf{B}</math> are <math>\lambda_1 = 0, \lambda_2 = 0, \dots, \lambda_{n-1} = 0, \lambda_n = 1</math>  Hence the characteristic polynomial for <math>\mathbf{B} = x^{n-1}(x - 1)</math>  Since <math>\mathbf{A} = \mathbf{I} - 2\mathbf{u}\mathbf{u}^T</math></p>

	<p>and we know the eigen values of <math>\mathbf{I}</math> are <math>\lambda_1 = 1, \lambda_2 = 1, \dots, \lambda_{n-1} = 1, \lambda_n = 1</math>  and we know the eigen values of <math>\mathbf{uu}^T</math> are <math>\lambda_1 = 0, \lambda_2 = 0, \dots, \lambda_{n-1} = 0, \lambda_n = 1</math></p> <p><math>\therefore</math> The eigen values of <math>\mathbf{A} = \lambda_1 = 1, \lambda_2 = 1, \dots, \lambda_{n-1} = 1, \lambda_n = -1</math> (4.0.1)</p> <p>Since <math>\mathbf{A}</math> does not have 0 as an eigen value  Therefore <math>\mathbf{A}</math> is not singular.</p>
Conclusion	Therefore the statement is false.
2.	<p>For <math>\mathbf{A}^2 = \mathbf{A}</math> ,  we know that <math>p(x) = x^2 - x</math>  <math>\therefore</math> minimal polynomial of <math>\mathbf{A}</math> must divide <math>x(x-1)</math>  <math>\therefore</math> possible eigenvalues of <math>\mathbf{A}</math> are 0 or 1  But from (4.0.1), we know that <math>\mathbf{A}</math> has -1 as an eigen value  Therefore <math>\mathbf{A}^2 = \mathbf{A}</math> is false.</p>
Conclusion	Therefore the statement is false.
3.	<p>From equation (4.0.1) ,  Trace of <math>\mathbf{A} = n - 2</math></p>
Conclusion	Therefore the statement is true.
4.	<p>Since <math>\mathbf{A} = \mathbf{I} - 2\mathbf{uu}^T</math>  <math>\mathbf{A}^2 = (\mathbf{I} - 2\mathbf{uu}^T)(\mathbf{I} - 2\mathbf{uu}^T)</math>  <math>\therefore \mathbf{A}^2 = \mathbf{I} - 2\mathbf{uu}^T - 2\mathbf{uu}^T + 4\mathbf{uu}^T\mathbf{uu}^T</math>  Since <math>\mathbf{u}^T\mathbf{u} = 1</math>  <math>\therefore \mathbf{A}^2 = \mathbf{I} - 2\mathbf{uu}^T - 2\mathbf{uu}^T + 4\mathbf{uu}^T</math>  <math>\therefore \mathbf{A}^2 = \mathbf{I}</math></p>
Conclusion	Therefore the statement is true.

TABLE 2: Solution summary