Matrix Theory (EE5609) Assignment 12

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Abstract—**This document provides solutions to linear algebra.**

Download all python codes from

```
https://github.com/utkarshsurwade/
    Matrix_Theory_EE5609/tree/master/codes
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and latex-tikz codes from

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https://github.com/utkarshsurwade/
    Matrix_Theory_EE5609/tree/master/
    Assignment12
```

1 Problem

Let $\mathbf{N}$ be a 2×2 complex matrix such that $\mathbf{N}^2 = 0$. Prove that either $\mathbf{N} = 0$ or $\mathbf{N}$ is similar over $\mathbb{C}$ to

$$\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \tag{1.0.1}$$

2 Solution

$$\text{Let } \mathbf{N} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \tag{2.0.1}$$

$$\text{Since } \mathbf{N}^2 = 0 \tag{2.0.2}$$

If $\begin{pmatrix} a \\ c \end{pmatrix}, \begin{pmatrix} b \\ d \end{pmatrix}$ are linearly independent then $\mathbf{W_1}$ has rank two and $\mathbf{N}$ is diagonalizable to $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$.

$$\text{If } \mathbf{PNP}^{-1} = 0 \tag{2.0.3}$$

$$\text{then } \mathbf{N} = \mathbf{P}^{-1}\mathbf{0}\mathbf{P} = 0 \tag{2.0.4}$$

So in this case $\mathbf{N}$ itself is the zero matrix.
This contradicts the assumption that $\begin{pmatrix} a \\ c \end{pmatrix}, \begin{pmatrix} b \\ d \end{pmatrix}$ are linearly independent. Hence we can assume that $\begin{pmatrix} a \\ c \end{pmatrix}, \begin{pmatrix} b \\ d \end{pmatrix}$ are linearly dependent if both are equal to the zero vector then $\mathbf{N} = 0$.Therefore we can assume at least one vector is non-zero.

If $\begin{pmatrix} b \\ d \end{pmatrix}$ is the zero vector then $\mathbf{N} = \begin{pmatrix} a & 0 \\ c & 0 \end{pmatrix}$

$$\text{So } \mathbf{N}^2 = 0 \tag{2.0.5}$$

$$\implies a^2 = 0 \tag{2.0.6}$$

$$\therefore a = 0 \tag{2.0.7}$$

$$\text{Thus } \mathbf{N} = \begin{pmatrix} a & 0 \\ c & 0 \end{pmatrix} \tag{2.0.8}$$

In this case $\mathbf{N}$ is similar to $\mathbf{N} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ via the matrix $\mathbf{P} = \begin{pmatrix} c & 0 \\ 0 & 1 \end{pmatrix}$

Similary if $\begin{pmatrix} a \\ c \end{pmatrix}$ is the zero vector,

$$\text{Then } \mathbf{N}^2 = 0 \tag{2.0.9}$$

$$\implies d^2 = 0 \tag{2.0.10}$$

$$\therefore d = 0 \tag{2.0.11}$$

$$\text{Thus } \mathbf{N} = \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix} \tag{2.0.12}$$

In this case $\mathbf{N}$ is similar to $\mathbf{N} = \begin{pmatrix} 0 & 0 \\ b & 0 \end{pmatrix}$ via the matrix $\mathbf{P} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$,which is similar to $\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ as above. Therefore we can assume neither $\begin{pmatrix} a \\ c \end{pmatrix}$ or $\begin{pmatrix} b \\ d \end{pmatrix}$ is the zero vector.

Since they are linearly dependent we can assume,

$$\begin{pmatrix} b \\ d \end{pmatrix} = x\begin{pmatrix} a \\ c \end{pmatrix} \tag{2.0.13}$$

$$\therefore \mathbf{N} = \begin{pmatrix} a & ax \\ c & cx \end{pmatrix} \tag{2.0.14}$$

$$\therefore \mathbf{N}^2 = 0 \tag{2.0.15}$$

$$\implies a(a+cx) = 0 \tag{2.0.16}$$

$$c(a+cx) = 0 \tag{2.0.17}$$

$$ax(a+cx) = 0 \tag{2.0.18}$$

$$cx(a+cx) = 0 \tag{2.0.19}$$

We know that at least one of a or c is not zero. If a = 0 then since c ≠ 0, it must be that x = 0. So in this case $\mathbf{N} = \begin{pmatrix} 0 & 0 \\ c & 0 \end{pmatrix}$ which is similar to $\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ as before.

$$\text{If } a \neq 0 \tag{2.0.20}$$

$$\text{then } x \neq 0 \tag{2.0.21}$$

$$\text{else } a(a+cx) = 0 \tag{2.0.22}$$

$$\implies a = 0 \tag{2.0.23}$$

$$\text{Thus } a + cx = 0 \tag{2.0.24}$$

$$\text{Hence } \mathbf{N} = \begin{pmatrix} a & ax \\ \frac{-a}{x} & -a \end{pmatrix} \tag{2.0.25}$$

This is similar to $\begin{pmatrix} a & a \\ -a & -a \end{pmatrix}$ via $\mathbf{P} = \begin{pmatrix} \sqrt{x} & 0 \\ 0 & \frac{1}{\sqrt{x}} \end{pmatrix}$. And $\begin{pmatrix} a & a \\ -a & -a \end{pmatrix}$ is similar to $\begin{pmatrix} 0 & 0 \\ -a & 0 \end{pmatrix}$ via $\mathbf{P} = \begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix}$

And this finally is similar to $\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ as before.

Thus either $\mathbf{N} = 0$ or $\mathbf{N}$ is similar over $\mathbb{C}$ to $\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$.