CALCULUS AND LINEAR ALGEBRA

MATHEMATICS-I

(18MAB101T)

DEPARTMENT OF MATHEMATICS

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Introduction

Matrices find many applications in scientific field and useful in many practical real life problem. For example:

- It is useful in the study of electrical circuits, quantum mechanics and optics
- Matrices play a role in calculation of battery power outputs, resistor conversion of electrical energy into another useful energy using Kirchhoff law of voltage and current
- Matrices can play a vital role in the projection of three dimensional images into two dimensional screens, creating the realistic decreeing motion
- It is useful in wave equation associated with transmitting power through transmission lines
- It can be used to crack or deformities in a solid

Introduction

- In machine learning we often have to deal with structural data, which is generally represented by matrix
- Car designers analyze eigenvalues in order to damp out noise so that the occupant have a quite ride
- It is also used in structural analysis to calculate buckling margins of safty
- Matrices are used in the ranking of web pages in the Google search
- It can also be used in generalization of analytical motion like experimental and derivatives to their high dimensional
- The usages of matrices in computer side application are encryption of message codes with the help of encryptions in the transmission of sensitive and private data
- Matrices are also used in robotics and automation in terms of base elements for the robot movements which are programmed with the calculation of matrices

Definition: Let A be a square matrix. If there exists a scalar λ and non-zero column matrix X such that $AX = \lambda X$, then the scalar λ is called an eigenvalue/characteristic value/latent value of A and X is called the corresponding eigenvector of A.

How to find: We can obtain the eigenvalues and eigenvectors through the following steps:

Step 1: Write the characteristic equation as
$| A - \lambda I | = \lambda^n - S_1\lambda^{n-1} + S_2\lambda^{n-2} + ... + (-1)^n S_n = 0, \quad n = 2, 3, 4\cdots\cdots,$
where

$S_1 =$ sum of the main diagonal elements of A.

$S_2 =$ sum of the of minor of main diagonal elements of A $\cdots\cdots\cdots$

$S_n =$ determinant of A i.e $| A |$.

Step 2: Find the eigenvalues by factorizing the characteristic equation as $(\lambda_1 - a_1)(\lambda_2 - a_2)\cdots\cdots(\lambda_n - a_n) = 0$ or by synthetic division.

Step 3: Find the eigenvectors X for each value of λ from the linear system of equation $(A - \lambda_i I)X = 0, \quad i = 1, 2, 3\cdots\cdots$

Example: Find the eigenvalues and eigenvectors of $A = \begin{bmatrix} 2 & -2 & 3 \\ 1 & 1 & 1 \\ 1 & 3 & -1 \end{bmatrix}$

Solution:

$$| A - \lambda I | = \begin{vmatrix} 2-\lambda & -2 & 3 \\ 1 & 1-\lambda & 1 \\ 1 & 3 & -1-\lambda \end{vmatrix} = \lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$$

$S_1 = 2 + 1 - 1 = 2,$

$$S_2 = \begin{vmatrix} 1 & 1 \\ 3 & -1 \end{vmatrix} + \begin{vmatrix} 2 & 3 \\ 1 & -1 \end{vmatrix} + \begin{vmatrix} 2 & -2 \\ 1 & 1 \end{vmatrix} = -4 - 5 + 4 = -5$$

$$S_3 = |A| = \begin{vmatrix} 2 & -2 & 3 \\ 1 & 1 & 1 \\ 1 & 3 & -1 \end{vmatrix} = 6 \quad \Rightarrow \quad \lambda^3 - 2\lambda^2 - 5\lambda - 6 = 0$$

Which can be factorize as
$(\lambda - 1)(\lambda - 3)(\lambda + 2) = 0 \quad \Rightarrow \quad \lambda = 1, \quad -2, \quad 3.$

<u>**Eigenvector for $\lambda = 1$:**</u>
$$\begin{bmatrix} 2-1 & -2 & 3 \\ 1 & 1-1 & 1 \\ 1 & 3 & -1-1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

i.e. $$\begin{bmatrix} 1 & -2 & 3 \\ 1 & 0 & 1 \\ 1 & 3 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

$$\begin{aligned} x_1 - 2x_2 + 3x_3 &= 0 \\ x_1 + 0 + x_3 &= 0 \\ x_1 + 3x_2 - 2x_3 &= 0. \end{aligned}$$

$$\Rightarrow \quad \frac{x_1}{-3} = -\frac{x_2}{-3} = \frac{x_1}{3} \quad \Rightarrow \quad \frac{x_1}{-1} = \frac{x_2}{1} = \frac{x_1}{1} \quad \Rightarrow \quad X_1 = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}.$$

Eigenvector for $\lambda = -2$: $\begin{bmatrix} 4 & -2 & 3 \\ 1 & 3 & 1 \\ 1 & 3 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$$\begin{aligned} 4x_1 - 2x_2 + 3x_3 &= 0 \\ x_1 + 3x_2 + x_3 &= 0. \end{aligned}$$

Solving the above equation as $x_3 = -(x_1 + 3x_3) \quad \Rightarrow x_1 - 11x_2 = 0$, then we get $X_2 = \begin{bmatrix} 11 \\ 1 \\ -14 \end{bmatrix}$.

Eigenvector for $\lambda = 3$: $\begin{bmatrix} -1 & -2 & 3 \\ 1 & -2 & 1 \\ 1 & 3 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$$\begin{aligned} -x_1 - 2x_2 + 3x_3 &= 0 \\ x_1 - 2x_2 + x_3 &= 0 \\ x_1 + 3x_2 - 4x_3 &= 0. \end{aligned}$$

$$\Rightarrow \quad \frac{x_1}{5} = -\frac{x_2}{-5} = \frac{x_1}{5} \quad \Rightarrow \quad \frac{x_1}{1} = \frac{x_2}{1} = \frac{x_1}{1} \quad \Rightarrow \quad X_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

Example: Find the eigenvalues and eigenvectors of $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$

Solution:

$$| A - \lambda I | = \begin{vmatrix} 1-\lambda & 0 & 0 \\ 0 & 1-\lambda & 1 \\ 0 & 1 & 1-\lambda \end{vmatrix} = \lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$$

$$S_1 = 1+1+1 = 3, \quad S_2 = \begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} + \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} + \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 0+1+1 = 2$$

$$S_3 = | A | = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{vmatrix} = 0 \Rightarrow \lambda^3 - 3\lambda^2 + 2\lambda = 0 \Rightarrow \lambda(\lambda-1)(\lambda-2) = 0$$

$\Rightarrow \lambda = 0, \quad 1, \quad 2.$

Eigenvector for $\lambda = 0$:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{aligned} -x_1 - 2x_2 + 3x_3 &= 0 \\ x_1 - 2x_2 + x_3 &= 0 \end{aligned}$$

$$\begin{aligned} x_1 + 0x_2 + 0x_3 &= 0 \\ 0x_1 + x_2 + x_3 &= 0 \\ 0x_1 + x_2 + x_3 &= 0. \end{aligned}$$

$\Rightarrow \quad x_1 = 0$ and $x_2 = -x_3$. If we take $x_3 = k \quad \Rightarrow \quad x_2 = -k$

$$\Rightarrow \quad X_1 = \begin{bmatrix} 0 \\ -k \\ k \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}.$$

Eigenvector for $\lambda = 1$: $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$$\begin{aligned} 0x_1 + 0x_2 + 0x_3 &= 0 \\ 0x_1 + 0x_2 + x_3 &= 0 \\ 0x_1 + x_2 + 0x_3 &= 0 \end{aligned}$$

$\Rightarrow \quad x_2 = 0$ and $x_3 = 0$. Taking $x_1 = k \quad \Rightarrow \quad X_2 = \begin{bmatrix} k \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$.

Eigenvector for $\lambda = 2$: $\begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 1 \\ 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$$\begin{aligned} -x_1 + 0x_2 + 0x_3 &= 0 \\ 0x_1 - x_2 + x_3 &= 0 \\ 0x_1 + x_2 - x_3 &= 0 \end{aligned}$$

$\Rightarrow \; x_1 = 0$ and $x_2 = x_3$. If $x_3 = k \;\Rightarrow\; x_2 = k \Rightarrow X_3 = \begin{bmatrix} 0 \\ k \\ k \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$.

Example: Find the eigenvalues and eigenvectors of $A = \begin{bmatrix} 2 & 1 & 1 \\ 2 & 3 & 2 \\ 3 & 3 & 4 \end{bmatrix}$

Solution: Here $\mid A - \lambda I \mid = \lambda^3 - 9\lambda^2 + 15\lambda - 7 = 0 \Rightarrow \lambda = 1, \quad 1, \quad 7,$ some eigenvalues are repeated. Therefore we find the eigenvectors as:

Eigenvector for $\lambda = 7$: $\begin{bmatrix} -5 & 1 & 1 \\ 2 & -4 & 2 \\ 3 & 3 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$$\Rightarrow \quad \frac{x_1}{12-6} = -\frac{x_2}{-6-6} = \frac{x_3}{6+12} \quad \Rightarrow \quad \frac{x_1}{1} = \frac{x_2}{2} = \frac{x_3}{3} \quad \Rightarrow \quad X_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}.$$

Eigenvector for $\lambda = 1$: $\begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 3 & 3 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

Here we observe that all rows are linearly dependent

$$\Rightarrow \quad x_1 + x_2 + x_3 = 0.$$

Now we will construct two linearly independent eigenvectors from the same equation assuming the followings:

Assume $x_1 = 0 \quad \Rightarrow x_3 = -x_2 \quad$ hence $\quad X_2 = \begin{bmatrix} 0 \\ k \\ -k \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}.$

Similarly assuming

$x_2 = 0 \quad \Rightarrow x_3 = -x_1 \quad$ hence $\quad X_3 = \begin{bmatrix} k \\ 0 \\ -k \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}.$

Symmetric Matrix: A real matrix A is said to be symmetric if $A = A^T$, where T stands for transpose.
Orthogonal Matrix: Let X_1 and X_2 be two column matrices of same order. Then X_1 and X_2 are said to be orthogonal if $X_1^T X_2 = 0$
Example: Find the eigenvalues and eigenvectors of

$$A = \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$$

Solution: Here we can see that $A = A^T$, which implies it is a symmetric matrix.
Now $| A - \lambda I | = \lambda^3 - 12\lambda^2 + 36\lambda - 32 = 0 \Rightarrow \lambda = 2, \ \ 2, \ \ 8$, some eigenvalues are repeated. Therefore we find the eigenvectors as:

<u>**Eigenvector for $\lambda = 8$:**</u> $$\begin{bmatrix} -2 & -2 & 2 \\ -2 & -5 & -1 \\ 2 & -1 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \quad \frac{x_1}{25-1} = -\frac{x_2}{10+2} = \frac{x_3}{2+10} \quad \Rightarrow \quad \frac{x_1}{2} = \frac{x_2}{-1} = \frac{x_3}{1} \quad \Rightarrow \quad X_1 = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}.$$

Eigenvector for $\lambda = 2$: $\begin{bmatrix} 4 & -2 & 2 \\ -2 & 1 & -1 \\ 2 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ Here one can observe that all rows are linearly dependent $\Rightarrow \quad -2x_1 + x_2 - x_3 = 0.$

Assume $x_1 = 0 \quad \Rightarrow x_3 = x_2$ hence $X_2 = \begin{bmatrix} 0 \\ k \\ k \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}.$

For the nest eigenvalue $\lambda = 2$, we consider $X_3 = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$

As the matrix A is symmetric, so the eigenvectors are orthogonal.

$$\therefore X_1^T X_3 = 0 \quad \Rightarrow \begin{bmatrix} 2 & -1 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = 0 \quad \Rightarrow 2a - b + c = 0. \text{ again}$$

$$X_2^T X_3 = 0 \quad \Rightarrow \begin{bmatrix} 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = 0 \quad \Rightarrow b + c = 0.$$

Solving the above two equations we get $a = b = -c \quad \Rightarrow X_3 = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$.

Example: Find the eigenvalues and eigenvectors of

$$A = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}.$$

Property 1: Every square matrix and it's transpose has same eigenvalues.

Example: If $A = \begin{bmatrix} 1 & -2 \\ -5 & 4 \end{bmatrix} \quad \Rightarrow \lambda^2 - 5\lambda - 6 = 0 \quad \Rightarrow \lambda = 6, \quad -1.$

$A^T = \begin{bmatrix} 1 & -5 \\ -2 & 4 \end{bmatrix} \quad \Rightarrow \lambda^2 - 5\lambda - 6 = 0 \quad \Rightarrow \lambda = 6, \quad -1.$

Property 2: If $\lambda_1,\ \lambda_2,\ \lambda_3, \cdots\cdots \lambda_n$ are the eigenvalues of the matrix A then $\frac{1}{\lambda_1},\ \frac{1}{\lambda_2},\ \frac{1}{\lambda_3}, \cdots\cdots \frac{1}{\lambda_n}$ are the eigenvalues of A^{-1}.

Proof: Let λ be the eigenvalue of a matrix $A \quad \Rightarrow \quad AX = \lambda X$, where X is an eigenvector $X \neq 0$. If we multiply A^{-1} with $AX = \lambda X$ as below:

$A^{-1}AX = A^{-1}\lambda X \quad \Rightarrow IX = \lambda A^{-1}X \quad \Rightarrow \frac{1}{\lambda}X = A^{-1}X.$

$\therefore\ \frac{1}{\lambda}$ is the eigenvalue of A^{-1}.

Property 3: If $\lambda_1,\ \lambda_2,\ \lambda_3, \cdots\cdots \lambda_n$ are the eigenvalues of the matrix A, then $\lambda_1^2,\ \lambda_2^2,\ \lambda_3^2, \cdots\cdots \lambda_n^2$ are the eigenvalues of A^2.

Proof: Let λ be the eigenvalue of a matrix A.
$\therefore AX = \lambda X$, where X is an eigenvector $X \neq 0$. If we multiply A with $AX = \lambda X$ as below:

$$AAX = A\lambda X \quad \Rightarrow A^2X = \lambda AX \quad \Rightarrow A^2X = \lambda^2 X\ .$$

$\therefore \lambda^2$ is the eigenvalue of A^2.

Property 4: If $\lambda_1,\ \lambda_2,\ \lambda_3, \cdots\cdots \lambda_n$ are the eigenvalues of the matrix A, then $k\lambda_1,\ k\lambda_2,\ k\lambda_3, \cdots\cdots k\lambda_n$ are the eigenvalues of kA.

Proof: Let λ be the eigenvalue of a matrix A.
$\therefore AX = \lambda X \quad \Rightarrow \quad kAX = k(\lambda X) = (k\lambda)X.$

$\therefore k\lambda$ is the eigenvalue of kA.

Property 5: The eigenvalues of a real symmetric matrix are all real.

Proof: Let λ be the eigenvalue of a matrix A.

$$AX = \lambda X \tag{1}$$

Taking conjugate on both sides of (1) we get $\bar{A}\bar{X} = \bar{\lambda}\bar{X}$. As A is real

$\therefore A = \bar{A} \quad \Rightarrow A\bar{X} = \bar{\lambda}\bar{X}$. Taking transpose on both side one can get

$(A\bar{X})^T = (\bar{\lambda}\bar{X})^T \quad \Rightarrow \quad \bar{X}^T A^T = \bar{\lambda}^T \bar{X}^T \quad \Rightarrow \bar{X}^T A = \bar{\lambda}\bar{X}^T$

($\because A$ is symmetric $A = A^T$ and λ is a scalar). Now post multiply by X

$\bar{X}^T AX = \bar{\lambda}\bar{X}^T X \quad \Rightarrow \bar{X}^T \lambda X = \bar{\lambda}\bar{X}^T X \quad \Rightarrow \lambda\bar{X}^T X = \bar{\lambda}\bar{X}^T X \quad \Rightarrow \lambda = \bar{\lambda}$.

Property 6: If $\lambda_1, \ \lambda_2, \ \lambda_3, \cdots\cdots \lambda_n$ are the eigenvalues of the matrix A, then trace of A=sum of eigenvalues $= \lambda_1 + \lambda_2 + \lambda_3, \cdots\cdots + \lambda_n$ and product of eigenvalues of A=$\mid A \mid$ i.e $\mid A \mid = \lambda_1.\lambda_2.\lambda_3, \cdots\cdots \lambda_n$.

Property 6: Eigenvalues of a triangular matrix are just the diagonal elements of the matrix.

Proof: Let $A = \begin{bmatrix} a_{11} & 0 & 0 \\ a_{21} & a_{22} & 0 \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$

$$\Rightarrow |\ A - \lambda I\ | = \begin{vmatrix} a_{11} - \lambda & 0 & 0 \\ a_{21} & a_{22} - \lambda & 0 \\ a_{31} & a_{32} & a_{33} - \lambda \end{vmatrix} = 0$$

$$\Rightarrow (a_{11} - \lambda)(a_{22} - \lambda)(a_{33} - \lambda) = 0.$$

$$\Rightarrow \lambda = a_{11}, \quad a_{22}, \quad a_{33}.$$

Example: Find the sum and product of the eigenvalues of a matrix
$A = \begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & 6 \\ -1 & -2 & 0 \end{bmatrix}$.

Proof: We know sum of eigenvalues of A=Sum of he leading diagonal elements of A=trace of A=-2+1+0=-1.

Product of the eigenvalues=$|\ A\ | = -2(0-12) - 2(0-6) - 3(-4+1) = 45$.

Example: Two of the eigenvalues of $A = \begin{bmatrix} 3 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{bmatrix}$ are 3 and 6.

Find the eigenvalues of A^{-1}.

Solution: Let $\lambda_1,\ \lambda_2,\ \lambda_3$ are eigenvalues of A.
$\Rightarrow \lambda_1 + \lambda_2 + \lambda_3 = 3 + 5 + 3 = 11$

As $\lambda_1 = 3, \quad \lambda_2 = 6 \quad \Rightarrow \lambda_3 = 2$

$\therefore$ Eigenvalues of A^{-1} are $\frac{1}{2}, \quad \frac{1}{3}, \quad \frac{1}{6}$.

Example: If 2 and 3 are eigenvalues of $A = \begin{bmatrix} 3 & 10 & 5 \\ -2 & -3 & -4 \\ 3 & 5 & 7 \end{bmatrix}$. Find the eigenvalues of A^{-1} and A^3.

Solution: Let $\lambda_1, \ \lambda_2, \ \lambda_3$ are eigenvalues of A.
$\Rightarrow \lambda_1 + \lambda_2 + \lambda_3 = 3 + 2 + \lambda_3 = 3 - 3 + 7 = 7 \quad \Rightarrow \lambda_3 = 2$

$\therefore$ Eigenvalues of A^{-1} are $\frac{1}{2}, \quad \frac{1}{2}, \quad \frac{1}{3}$.

and eigenvalues of A^3 are $2^3, \quad 2^3, \quad 3^3$.

Example: Find the constant a and b such that $\begin{bmatrix} a & 4 \\ 1 & b \end{bmatrix}$ matrix has 3 and -2 as eigenvalues.

Solution: $a + b = 3 - 2 = 1$ and $ab - 4 = 3 \times -2 = -6$

$\therefore b = 1 - a \quad \Rightarrow \quad a(1-a) - 4 = -6 \quad \Rightarrow a(1-a) = -2$

$\Rightarrow a = 2, \ -1 \quad \Rightarrow b = -1, \ 2.$

Example: Two eigenvalues of $A = \begin{bmatrix} 4 & 6 & 6 \\ 1 & 3 & 2 \\ -1 & -5 & -2 \end{bmatrix}$ are equal and they are double the third. Find the eigenvalues of A^2.

Solution: Let the third eigenvalue is λ. Therefore the three eigenvalues are $\lambda, \ 2\lambda, \ 2\lambda$. $\quad \Rightarrow \lambda + 2\lambda + 2\lambda = 4 + 3 - 2 \quad \Rightarrow 5\lambda = 5 \quad \Rightarrow \quad \lambda = 1$

$\therefore$ The eigenvalues are $1, \ 2, \ 2$ and eigenvalues of A^2 are $1, \ 4, \ 4$.

Statement: Every square matrix satisfies it's own characteristics equation.
i.e If A is any $n \times n$ matrix and
$\lambda^n - S_1\lambda^{n-1} + S_2\lambda^{n-2} - S_3\lambda^{n-3} \cdots\cdots + (-1)^n S_n = 0$
is the characteristic equation then
$A^n - S_1 A^{n-1} + S_2 A^{n-2} - S_3 A^{n-3} \cdots\cdots + (-1)^n S_n = 0.$

Example: Verify Cayley-Hamilton theorem and hence find A^{-1} for
$$A = \begin{bmatrix} 8 & -8 & 2 \\ 4 & -3 & -2 \\ 3 & -4 & 1 \end{bmatrix}.$$

Solution: The characteristic equation can be obtain from
$$\begin{vmatrix} 8-\lambda & -8 & 2 \\ 4 & -3-\lambda & -2 \\ 3 & -4 & 1-\lambda \end{vmatrix} = 0 \quad \Rightarrow \lambda^3 - 6\lambda^2 + 11\lambda - 6 = 0$$

Now we need to show that $A^3 - 6A^2 + 11A - 6I = 0$. For that we find the followings:

$$A^2 = A.A = \begin{bmatrix} 8 & -8 & 2 \\ 4 & -3 & -2 \\ 3 & -4 & 1 \end{bmatrix} \begin{bmatrix} 8 & -8 & 2 \\ 4 & -3 & -2 \\ 3 & -4 & 1 \end{bmatrix} = \begin{bmatrix} 26 & -32 & -2 \\ 14 & -15 & -4 \\ 11 & -16 & 3 \end{bmatrix}$$

$$A^3 = A^2.A = \begin{bmatrix} 26 & -32 & -2 \\ 14 & -15 & -4 \\ 11 & -16 & 3 \end{bmatrix} \begin{bmatrix} 8 & -8 & 2 \\ 4 & -3 & -2 \\ 3 & -4 & 1 \end{bmatrix} = \begin{bmatrix} 74 & -104 & 10 \\ 40 & -51 & -24 \\ 33 & -52 & 13 \end{bmatrix}$$

Now $A^3 - 6A^2 + 11A - 6I = \begin{bmatrix} 74 & -104 & 10 \\ 40 & -51 & -24 \\ 33 & -52 & 13 \end{bmatrix} - \begin{bmatrix} 156 & -192 & -12 \\ 84 & -90 & -24 \\ 66 & -93 & 18 \end{bmatrix} +$

$$\begin{bmatrix} 88 & -88 & -22 \\ 44 & -33 & -22 \\ 33 & 44 & 11 \end{bmatrix} - \begin{bmatrix} 6 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 6 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = 0.$$

Finding A^{-1}: Let us premultiply the equation $A^3 - 6A^2 + 11A - 6I = 0$ by A^{-1}, then we get: $A^2 - 6A + 11I - 6A^{-1} = 0 \quad \Rightarrow 6A^{-1} = [A^2 - 6A + 11I]$.

$$\Rightarrow 6A^{-1} = \begin{bmatrix} 26 & -32 & -2 \\ 14 & -15 & -4 \\ 11 & -16 & 3 \end{bmatrix} - \begin{bmatrix} 48 & -48 & -12 \\ 24 & -18 & -12 \\ 18 & -24 & 6 \end{bmatrix} + \begin{bmatrix} 11 & 0 & 0 \\ 0 & 11 & 0 \\ 0 & 0 & 11 \end{bmatrix}$$

$$\Rightarrow A^{-1} = \frac{1}{6} \begin{bmatrix} -11 & 16 & 10 \\ -10 & 14 & 8 \\ -7 & 8 & 8 \end{bmatrix}.$$

Example: Using Cayley-Hamilton theorem find the inverse of
$$A = \begin{bmatrix} 2 & 1 \\ 1 & -5 \end{bmatrix}.$$

Solution: The characteristic equation can be obtain from

$$\begin{vmatrix} 2-\lambda & 1 \\ 1 & -5-\lambda \end{vmatrix} = 0 \quad \Rightarrow \lambda^2 + 3\lambda - 11 = 0 \quad \Rightarrow A^2 + 3A - 11I = 0.$$

$$\Rightarrow A + 3I = 11A^{-1} \qquad \Rightarrow A^{-1} = \frac{1}{11}[A + 3I] = \frac{1}{11}\begin{bmatrix} 5 & 1 \\ 1 & -2 \end{bmatrix}.$$

Example: Verify Cayley-Hamilton theorem for the matrix $A = \begin{bmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{bmatrix}$ and use it to find A^{-1} and A^4.

Solution: The characteristic equation can be obtain from

$$\begin{vmatrix} 1-\lambda & 2 & -2 \\ -1 & 3-\lambda & 0 \\ 0 & -2 & 1-\lambda \end{vmatrix} = 0 \quad \Rightarrow \lambda^3 - 5\lambda^2 + 9\lambda - 1 = 0$$

$$A^2 = A.A = \begin{bmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 12 & -4 \\ -4 & 7 & 2 \\ 2 & -8 & 1 \end{bmatrix}$$

$$A^3 = A^2.A = \begin{bmatrix} -1 & 12 & -4 \\ -4 & 7 & 2 \\ 2 & -8 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{bmatrix} = \begin{bmatrix} -13 & 42 & -2 \\ -11 & 9 & 10 \\ 10 & -22 & -3 \end{bmatrix}$$

Now $A^3 - 5A^2 + 9A - I = \begin{bmatrix} -13 & 42 & -2 \\ -11 & 9 & 10 \\ 10 & -22 & -3 \end{bmatrix} - 5 \begin{bmatrix} -1 & 12 & -4 \\ -4 & 7 & 2 \\ 2 & -8 & 1 \end{bmatrix} +$

$9 \begin{bmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = 0.$

Multiplying by A^{-1} gives $A^{-1} = A^2 - 5A + 9I = \begin{bmatrix} 3 & 2 & 6 \\ 1 & 1 & 2 \\ 2 & 2 & 5 \end{bmatrix}$

Multiplying by A gives $A^4 = 5A^3 - 9A^2 + A = \begin{bmatrix} -55 & 104 & 24 \\ -20 & -15 & 32 \\ 32 & -42 & -23 \end{bmatrix}$.

Diagonalizable Matrix: A matrix A is diagonalizable if there exists an invertible matrix P such that $P^{-1}AP = D$. Where D is diagonal matrix.

Remark 1: A square matrix A of order n is diagonalizable if and only if it has n linearly independent eigenvectors.

Remark 2: A square matrix A of order n has always n linearly independent eigenvectors when it's eigenvalues are distinct.

Remark 3: If $P^{-1}AP = D$, then $A^m = PD^mP^{-1}$

Orthogonal Transformation: Let A be a square symmetric matrix. Let N be the other square matrix whose columns are normalized eigenvectors of A. Then the transformation of the form $N^TAN = D$ is called orthogonal reduction/orthogonal transformation and D is called diagonal matrix.

Note: This is possibly only for real matrix.

Example: Diagonolize the matrix $A = \begin{bmatrix} 3 & 1 & 1 \\ 1 & 3 & -1 \\ 1 & -1 & 3 \end{bmatrix}$ by means of orthogonal transformations.

Solution: The characteristic equation of A is given by $\mid A - \lambda I \mid = 0$. i.e

$$\begin{vmatrix} 3-\lambda & 1 & 1 \\ 1 & 3-\lambda & -1 \\ 1 & -1 & 3-\lambda \end{vmatrix} = 0 \quad \Rightarrow \quad \lambda^3 - 9\lambda^2 + 24\lambda - 16 = 0.$$

Solving this we get $\lambda = 1, \quad 4, \quad 4$

<u>**Eigenvector for $\lambda = 1$:**</u> $\quad \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$

Which gives $\frac{x_1}{3} = -\frac{x_2}{3} = \frac{x_3}{-3} \Rightarrow X_1 = \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix} \equiv \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$.

<u>**Eigenvector for $\lambda = 4$:**</u> $\begin{bmatrix} -1 & 1 & 1 \\ 1 & -1 & -1 \\ 1 & -1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$-x_1 + x_2 + x_3 = 0 \quad \Rightarrow x_3 = 0, \quad x_1 = x_2 \Rightarrow X_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$.

As the eigenvalue $\lambda = 4$ is repeated so the other vector can be evaluated as below. Let us consider $X_3 = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$

Then find $X_1^T X_3 = \begin{bmatrix} -1 & 1 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = 0 \;\Rightarrow\; -a + b + c = 0.$

Similarly $X_2^T X_3 = \begin{bmatrix} 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = 0 \;\Rightarrow a + b = 0 \;\Rightarrow a = -b.$

Solving the above two equations we get

$a = -b$ and $c = -2b \;\Rightarrow\; X_3 = \begin{bmatrix} -1 \\ 1 \\ -2 \end{bmatrix}$. The normalized vectors are

$$\tilde{X}_1 = \begin{bmatrix} -1/\sqrt{3} \\ 1/\sqrt{3} \\ -1/\sqrt{3} \end{bmatrix} \quad \tilde{X}_2 = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{bmatrix} \quad \tilde{X}_3 = \begin{bmatrix} -1/\sqrt{6} \\ 1/\sqrt{6} \\ -2/\sqrt{6} \end{bmatrix}$$

Diagonalisation of Matrix by Orthogonal Transformation Unit-I

$$\Rightarrow N = \begin{bmatrix} \tilde{X}_1 & \tilde{X}_2 & \tilde{X}_3 \end{bmatrix} = \begin{bmatrix} -1/\sqrt{3} & 1/\sqrt{2} & -1/\sqrt{6} \\ 1/\sqrt{3} & 1/\sqrt{2} & 1/\sqrt{6} \\ 1/\sqrt{3} & 0 & -2/\sqrt{6} \end{bmatrix}.$$ Now

$$D = N^T A N$$

$$= \begin{bmatrix} -\frac{1}{\sqrt{3}} & 1/\sqrt{3} & 1/\sqrt{3} \\ 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ -1/\sqrt{6} & 1/\sqrt{6} & -\frac{2}{\sqrt{6}} \end{bmatrix} \begin{bmatrix} 3 & 1 & 1 \\ 1 & 3 & -1 \\ 1 & -1 & 3 \end{bmatrix} \begin{bmatrix} -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & 0 & -\frac{2}{\sqrt{6}} \end{bmatrix}$$

$$D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix}.$$

Example: Reduce the matrix $A = \begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$ to a diagonal form using orthogonal transformations.

Solution: The characteristic equation of A is given by $\mid A - \lambda I \mid = 0$. i.e

$$\begin{vmatrix} 8-\lambda & -6 & 2 \\ -6 & 7-\lambda & -4 \\ 2 & -4 & 3-\lambda \end{vmatrix} = 0 \quad \Rightarrow \quad \lambda^3 - 18\lambda^2 + 45\lambda = 0.$$

Solving this we get $\lambda = 0, \;\; 3, \;\; 15$

Eigenvectors are given as for

$$\lambda = 0 \;\; X_1 = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}, \text{ for } \;\lambda = 3 \;\; X_2 = \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}, \;\text{ for } \;\lambda = 15 \;\; X_3 = \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}.$$

$\therefore$ The normalized vectors are

$$\tilde{X}_1 = \begin{bmatrix} 1/3 \\ 2/3 \\ 2/3 \end{bmatrix} \quad \tilde{X}_2 = \begin{bmatrix} 2/3 \\ 1/3 \\ 2/3 \end{bmatrix} \quad \tilde{X}_3 = \begin{bmatrix} 2/3 \\ -2/3 \\ 1/3 \end{bmatrix}.$$

$$\text{Hence } N = \begin{bmatrix} \tilde{X}_1 & \tilde{X}_2 & \tilde{X}_3 \end{bmatrix} = \begin{bmatrix} 1/3 & 2/3 & 2/3 \\ 2/3 & 1/3 & 2/3 \\ 2/3 & -2/3 & 1/3 \end{bmatrix}.$$

$$D = N^T A N = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 15 \end{bmatrix}.$$

Definition: An homogeneous expression of second degree in any number of unknowns is called quadratic form.

Example: $Q = x_1^2 - 2x_2^2 + 4x_3^2 - 3x_1x_2 + 4x_2x_3 + 6x_1x_3$ is Q.F in three unknowns $x_1, \quad x_2, \quad x_3$.

Matrix representation: Let $Q = ax^2 + 2hxy + by^2$, then

$$Q = \begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} a & h \\ h & b \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = X^T A X$$

where $X = \begin{bmatrix} x \\ y \end{bmatrix}$ and $A = \begin{bmatrix} a & h \\ h & b \end{bmatrix}$ (A is symmetric matrix).

Note : The diagonal entries of the the symmetric matrix A are the square terms in Q.

Note : The non-diagonal entries of the the symmetric matrix A are the half of the product terms in Q.

Example: Write the quadratic form as product of matrices
$Q = x_1^2 - 2x_2^2 + 3x_3^2 - 4x_1x_2 + 5x_2x_3 + 6x_1x_3$.

Solution: The required form is

$$Q = \begin{bmatrix} x_1 & x_2 & x_{x_1} \end{bmatrix} \begin{bmatrix} 1 & -2 & 3 \\ -2 & -2 & 5/2 \\ 3 & 5/2 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_{x_1} \end{bmatrix} = X^T AX.$$

Example: Write the quadratic form where $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 9 \\ 3 & 9 & 3 \end{bmatrix}$.

Solution: The required form is
$Q = x_1^2 + 4x_2^2 + 3x_3^2 + 4x_1x_2 + 6x_1x_3 + 18x_2x_3$.

Canonical Form: The transformed Q.F is called as canonical form.

Index: The number of positive terms in the canonical form is called the index of the form and it is denoted by p.

Rank: The number of non-zero eigenvalues is called the rank of the form and it is denoted by r.

Signature: The difference between positive terms p and negative terms $(r-p)$ in the canonical form is called the signature of the form and it is denoted by $p-(r-p)=2p-r$.

Positive Definite: A Q.F $Q = X^T AX$ in n variables where $\mid A \mid \neq 0$ is called positive definite $Q = X^T AX > 0$ i.e $r = p = n$.

Positive Semi-Definite: A Q.F $Q = X^T AX$ in n variables where $\mid A \mid \neq 0$ is called positive semi-definite $r = p < n$.

Negative Definite: A Q.F $Q = X^T AX$ in n variables where $\mid A \mid \neq 0$ is called negative definite $p = 0$, and $r = n$.

Negative Semi-Definite: A Q.F $Q = X^T AX$ in n variables where $\mid A \mid \neq 0$ is called negative semi-definite $p = 0$, and $r < n$.

Indefinite: A Q.F $Q = X^T AX$ in n variables where $\mid A \mid \neq 0$ is called indefinite if non of he above thing happened.

The following steps are followed to construct the canonical form:

Step 1: First write the Q.F as $Q = X^T AX$.

Step 2: Find the eigenvalues and corresponding eigenvectors of A.

Step 3: Normalize the eigenvectors as $\bar{X}_1, \bar{X}_2, \bar{X}_3$ and write the normalized modal matrix $P = \begin{bmatrix} \bar{X}_1 & \bar{X}_2 & \bar{X}_3 \end{bmatrix}$.

Step 4: Find $P^T AP = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$.

Step 5: Assume the transformation $X = PY$ where $Y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$. Then write $Q = \lambda_1 y_1^2 + \lambda_2 y_2^2 + \lambda_3 y_3^2$.

If we take $X = PY \quad \Rightarrow \quad Q = X^T A X = (PY)^T A (PY) = Y^T P^T A P Y$

$$= Y^T (P^T A P) Y = Y^T \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} Y = \lambda_1 y_1^2 + \lambda_2 y_2^2 + \lambda_3 y_3^2.$$

As P is orthogonal $\Rightarrow X = PY$ is orthogonal.

Example: Reduce the Q.F $Q = 3x_1^2 + 5x_2^2 + 3x_3^2 - 2x_1x_2 - 2x_2x_3 + 2x_3x_1$ to a diagonal canonical form and hence find it's nature, rank, index and signature.

Solution: Given

$$Q = \begin{bmatrix} x_1 & x_2 & x_{x_1} \end{bmatrix} \begin{bmatrix} 3 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_{x_1} \end{bmatrix} = X^T AX.$$

The characteristic equation of A is given by $\mid A - \lambda I \mid = 0$. i.e

$$\begin{vmatrix} 3-\lambda & -1 & 1 \\ -1 & 5-\lambda & -1 \\ 1 & -1 & 3-\lambda \end{vmatrix} = 0 \quad \Rightarrow \quad \lambda^3 - 11\lambda^2 + 36\lambda - 36 = 0.$$

Solving this we get $\lambda = 2, \quad 3, \quad 6$

Eigenvectors are given as for $\lambda = 2 \quad \Rightarrow X_1 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$,

for $\lambda = 3 \quad \Rightarrow X_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, for $\lambda = 6 \quad \Rightarrow X_3 = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$.

$\therefore$ The normalized vectors are

$$\bar{X}_1 = \begin{bmatrix} -1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{bmatrix} \quad \bar{X}_2 = \begin{bmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix} \quad \tilde{X}_3 = \begin{bmatrix} 1/\sqrt{6} \\ -2/\sqrt{6} \\ 1/\sqrt{6} \end{bmatrix}.$$

Hence $P = \begin{bmatrix} \bar{X}_1 & \bar{X}_2 & \bar{X}_3 \end{bmatrix} = \begin{bmatrix} -1/\sqrt{2} & 1/\sqrt{3} & 1/\sqrt{6} \\ 0 & 1/\sqrt{3} & -2/\sqrt{6} \\ 1/\sqrt{2} & 1/\sqrt{3} & 1/\sqrt{6} \end{bmatrix}.$

Now P is orthogonal. Let $X = PY$ be orthogonal transformation then

$$Q = X^T AX = (PY)^T A(PY) = Y^T P^T APY = Y^T (P^T AP)Y$$

As $D = P^T AP = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 6 \end{bmatrix}.$

$$\Rightarrow Q = \begin{bmatrix} y_1 & y_2 & y_3 \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 6 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} \quad \Rightarrow Q = 2y_1^2 + 3y_2^2 + 6y_3^2.$$

Here the number of non-zero eigenvalues are 3, therefore the rank $r = 3$.

Again the number of positive eigenvalues are 3, therefore index $p = 3$.

$\therefore$ The signature is, $2p - r = 6 - 3 = 3$.

The quadratic form is positive definite, since all the eigenvalues are positive.

Example: Reduce the Q.F $Q = x_1^2 + 2x_2x_3$ into a canonical form by means of an orthogonal transformation. Determine it's nature, rank, index and signature.

Solution: Given $Q = \begin{bmatrix} x_1 & x_2 & x_{x_1} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_{x_1} \end{bmatrix} = X^T AX.$

The characteristic equation of A is given by $\mid A - \lambda I \mid = 0$. i.e

$$\begin{vmatrix} 1-\lambda & 0 & 0 \\ 0 & 0-\lambda & 1 \\ 0 & 1 & 0-\lambda \end{vmatrix} = 0 \quad \Rightarrow \quad \lambda^3 - \lambda^2 - \lambda + 1 = 0.$$

Solving this we get $\lambda = -1, \; 1, \; 1.$

Eigenvectors are given as for $\lambda = -1 \quad \Rightarrow X_1 = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}.$

For $\lambda = 1 \quad \Rightarrow X_2 = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$. As the matrix is symmetric and the eigenvalues are repeated, so the third eigenvalue is orthogonal to other two and can be determined consider $X_3 = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$

$$\Rightarrow X_1^T X_3 = 0 \quad \Rightarrow \begin{bmatrix} 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = 0 \quad \Rightarrow -b + c = 0 \quad \Rightarrow b = c.$$

again $X_2^T X_3 = 0 \quad \Rightarrow \begin{bmatrix} -1 & 1 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = 0 \quad \Rightarrow -a + b + c = 0.$

Solving the above two equations we get

$$a = 2c, \quad \text{and} \quad b = c \quad \Rightarrow X_3 = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}.$$

$\therefore$ The normalized vectors are

$$\bar{X}_1 = \begin{bmatrix} 0 \\ -1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} \quad \bar{X}_2 = \begin{bmatrix} -1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix} \quad \tilde{X}_3 = \begin{bmatrix} 2/\sqrt{6} \\ 1/\sqrt{6} \\ 1/\sqrt{6} \end{bmatrix}.$$

Hence $P = \begin{bmatrix} \bar{X}_1 & \bar{X}_2 & \bar{X}_3 \end{bmatrix} = \begin{bmatrix} 0 & -1/\sqrt{3} & 2/\sqrt{6} \\ -1/\sqrt{2} & 1/\sqrt{3} & 1/\sqrt{6} \\ 1/\sqrt{2} & 1/\sqrt{3} & 1/\sqrt{6} \end{bmatrix}.$

Now P is orthogonal. Let $X = PY$ be orthogonal transformation then

$$Q = X^T AX = (PY)^T A(PY) = Y^T P^T APY = Y^T (P^T AP)Y$$

As $D = P^T AP = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$

$$\Rightarrow Q = \begin{bmatrix} y_1 & y_2 & y_3 \end{bmatrix} \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} \quad \Rightarrow Q = y_1^2 + y_2^2 - y_3^2.$$

Here the number of non-zero eigenvalues are 3, therefore the rank $r = 3$.

Again the number of positive eigenvalues are 2, therefore index $p = 2$.

$\therefore$ The signature is, $2p - r = 4 - 3 = 1$.

As one eigenvalue is negative and two eigenvalues are positive, the given quadratic form is indefinite.

ORTHOGONAL REDUCTION

If A is a real symmetric matrix, then the eigen vectors of A are not only Linearly independent but also pairwise Orthogonal.

The Normalised eigen vector of A is formed by divide each element of the eigen vector X, by the square-root of the sum of the squares of all the elements of X.

Let N be the Normalised modal matrix whose columns are the normalised eigen vectors of A. Then N is an Orthogonal matrix and by property $N^T = N^{-1}$.

If A be a real symmetric matrix, then there exists an Orthogonal matrix N such that

$N^T AN = N^{-1} AN = D$ is known as Orthogonal Reduction (or) Orthogonal Transformation.

Example-1

Diagonalise $A = \begin{bmatrix} 2 & 1 & -1 \\ 1 & 1 & -2 \\ -1 & -2 & 1 \end{bmatrix}$ ***by orthogonal transformation.***

Solution

The characteristic equation of A is $|A - \lambda I| = 0$

$$\text{ie } a_0 + a_1\lambda + a_2\lambda^2 + a_3\lambda^3 = 0$$

$$a_0 = |A| = 2(1-4) - 1(1-2) - 1(-2+1) = -4$$

$$a_1 = (-1)\left[\begin{vmatrix} 1 & -2 \\ -2 & 1 \end{vmatrix} + \begin{vmatrix} 2 & -1 \\ -1 & 1 \end{vmatrix} + \begin{vmatrix} 2 & 1 \\ 1 & 1 \end{vmatrix}\right]$$

$$= -[-3 + 1 + 1] = 1$$

$$a_2 = (-1)^2 [2 + 1 + 1] = 4$$

$$a_3 = (-1)^3 = -1.$$

ie $\lambda^3 - 4\lambda^2 - \lambda + 4 = 0$

$(\lambda + 1)(\lambda - 1)(\lambda - 4) = 0$

$\therefore$ The eigen values of A are $-1, 1, 4$.

Let $X = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ be the eigen vector corresponding to the eigen value λ, we have $(A - \lambda I)X = 0$

ie $\begin{pmatrix} 2-\lambda & 1 & -1 \\ 1 & 1-\lambda & -2 \\ -1 & -2 & 1-\lambda \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

Case (i) When $\lambda = -1$. The eigen vector is given by

$$\begin{pmatrix} 3 & 1 & -1 \\ 1 & 2 & -2 \\ -1 & -2 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Taking the first two equations, we get

$$\frac{x_1}{0} = \frac{x_2}{5} = \frac{x_3}{5}$$

$$\Rightarrow X_1 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}.$$

The normalised eigen vector of X_1 is $P_1 = \begin{pmatrix} 0 \\ \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}$

Case (ii) When $\lambda = 1$. The eigen vector is given by

$$\begin{pmatrix} 1 & 1 & -1 \\ 1 & 0 & -2 \\ -1 & -2 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Taking the last two equations, we get

$$\frac{x_1}{-4} = \frac{x_2}{2} = \frac{x_3}{-2}$$

$$\Rightarrow X_2 = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}$$

The normalised eigen vector of X_2 is

$$P_2 = \begin{pmatrix} \frac{2}{\sqrt{6}} \\ -\frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \end{pmatrix}$$

Case (iii) When $\lambda = 4$. The eigen vector is given by

$$\begin{pmatrix} -2 & 1 & -1 \\ 1 & -3 & -2 \\ -1 & -2 & -3 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Taking the last two equations, we get

$$\frac{x_1}{5} = \frac{x_2}{5} = \frac{x_3}{-5}$$

$$X_3 = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$$

The normalised eigen vector of X_3 is

$$P_3 = \begin{pmatrix} \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{3}} \end{pmatrix}$$

Clearly $X_1^T X_2 = X_2^T X_3 = X_3^T X_1 = 0$.

The normalised modal matrix is $N = (P_1\ P_2\ P_3)$

$$N = \begin{pmatrix} 0 & \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{3}} \end{pmatrix}$$

The orthogonal transformation is $N^T AN = D$.

$$\text{Consider } AN = \begin{pmatrix} 2 & 1 & -1 \\ 1 & 1 & -2 \\ -1 & -2 & 1 \end{pmatrix} \begin{pmatrix} 0 & \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{3}} \end{pmatrix} = \begin{pmatrix} 0 & \frac{2}{\sqrt{6}} & \frac{4}{\sqrt{3}} \\ -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} & \frac{4}{\sqrt{3}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & -\frac{4}{\sqrt{3}} \end{pmatrix}$$

$$N^T(AN) = \begin{pmatrix} 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{2}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} \end{pmatrix} \begin{pmatrix} 0 & \frac{2}{\sqrt{6}} & \frac{4}{\sqrt{3}} \\ -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} & \frac{4}{\sqrt{3}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & -\frac{4}{\sqrt{3}} \end{pmatrix} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{pmatrix} = D(-1, 1, 4)$$

Therefore $N^T AN = D$.

Example-2

Diagonalise the matrix $\begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$ *by an orthogonal transformation.*

Solution

The characteristic equation of A is $|A - \lambda I| = 0$

ie $a_0 + a_1\lambda + a_2\lambda^2 + a_3\lambda^3 = 0$

$a_0 = |A| = 6(9-1) + 2(-6+2) + 2(2-6)$

$= 48 - 8 - 8 = 32.$

$a_1 = (-1)\left[\begin{vmatrix} 3 & -1 \\ -1 & 3 \end{vmatrix} + \begin{vmatrix} 6 & 2 \\ 2 & 3 \end{vmatrix} + \begin{vmatrix} 6 & -2 \\ -2 & 3 \end{vmatrix}\right] = -[8 + 14 + 14] = -36.$

$a_2 = (-1)^2\,[6 + 3 + 3] = 12$

$a_3 = (-1)^3 = -1$

ie $32 - 36\lambda + 12\lambda^2 - \lambda^3 = 0$

$\therefore \quad \lambda^3 - 12\lambda^2 + 36\lambda - 32 = 0$

$\lambda = 2$ is a root.

$$\begin{array}{c|cccc} 2 & 1 & -12 & 36 & -32 \\ & 0 & 2 & -20 & 32 \\ \hline & 1 & -10 & 16 & \boxed{0} \end{array}$$

$$\lambda^2 - 10\lambda + 16 = 0$$

$$(\lambda - 2)(\lambda - 8) = 0$$

$\therefore$ The Eigen values are 2, 2, 8.

Let $X = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ be the eigen vector corresponding to the eigen value λ. Then we have

$$(A - \lambda I)X = 0$$

$$\begin{pmatrix} 6-\lambda & -2 & 2 \\ -2 & 3-\lambda & -1 \\ 2 & -1 & 3-\lambda \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Case (i) When $\lambda = 8$

$$\begin{pmatrix} -2 & -2 & 2 \\ -2 & -5 & -1 \\ 2 & -1 & -5 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\frac{x_1}{24} = \frac{x_2}{-12} = \frac{x_3}{12}$$

$$X_1 = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}$$

Case (ii) When $\lambda = 2$

$$\begin{pmatrix} 4 & -2 & 2 \\ -2 & 1 & -1 \\ 2 & -1 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow 2x - y + z = 0$$

$$-y + z = 0$$

$$y = 1,\ z = 1$$

$$X_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

Let $X_3 = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$ be a new eigen vector which is orthogonal to both X_1 and X_2.

X_3 is orthogonal to X_1

$$\Rightarrow 2a - b + c = 0 \qquad \text{...(1)}$$

X_3 is orthogonal to X_2

$$\Rightarrow b + c = 0 \qquad \text{...(2)}$$

Solving (1) & (2), we get

$$\frac{a}{-2} = \frac{b}{-2} = \frac{c}{2}$$

$\therefore$ The third eigen vector for $\lambda = 2$ is $X_3 = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$.

Clearly $X_1^T X_2 = X_2^T X_3 = X_3^T X_1 = 0.$

The Modal Matrix is $P = \begin{pmatrix} 2 & 0 & 1 \\ -1 & 1 & 1 \\ 1 & 1 & -1 \end{pmatrix}.$

The Normalised modal matrix is

$$N = \begin{pmatrix} \frac{2}{\sqrt{6}} & 0 & \frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{3}} \end{pmatrix}$$

The orthogonal transformation is $N^T A N = D$

$$\text{Now } N^T A N = \begin{pmatrix} \frac{2}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} \end{pmatrix} \begin{pmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{pmatrix} \begin{pmatrix} \frac{2}{\sqrt{6}} & 0 & \frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{3}} \end{pmatrix}$$

$$= \begin{pmatrix} 8 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix} = D\,(8, 2, 2)$$

$$\therefore N^T A N = D\,(8, 2, 2)$$

CANONICAL FORM OF SUM OF THE SQUARES FORM USING LINEAR TRANSFORMATION

When a quadratic form is linearly transformed then the transformed quadratic of new variable is called canonical form of the given quadratic form.

When $X'AX$ is linearly transformed then the transformed quadratic $Y'BY$ is called the canonical form of the given quadratic $X'AX$.

If $B = P'AP = \text{Diag } (\lambda_1, \lambda_2, \lambda_3 \ldots\ldots \lambda_n)$ than $X'AX = Y'BY = \sum_{i=1}^{n} \lambda_i Y_i^2$

Remarks. (1) λ_i (eigen values) can be positive or negative or zero.

(2) If Rank $(A) = r$, then the quadratic form $X'AX$ will contain only r terms.

CANONICAL FORM OF SUM OF THE SQUARES FORM USING ORTHOGONAL TRANSFORMATION

Real symmetric matrix A can be reduced to a diagonal form $M'AM = D$ ...(1)

where M is the normalised orthogonal *modal matrix* of A and D is its *spectoral matrix*.

Let the orthogonal transformation be

$$X = MY$$

$$Q = X'AX = (MY)'\,A\,(MY) = (Y'M')\,A\,(MY) = Y'\,(M'AM)\,Y$$

$$= Y'DY \qquad [\because\ M'AM = D]$$

$$= Y'\ \text{Diag.}\ (\lambda_1\ \lambda_2 \ldots \lambda_n)\ Y$$

$$= \begin{bmatrix} y_1 & y_2 \ldots . y_n \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 \ldots . 0 \\ 0 & \lambda_2 \ldots . 0 \\ \vdots & \vdots \quad\quad \vdots \\ 0 & 0 \ldots . \lambda_n \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} = \begin{bmatrix} \lambda_1 y_1 & \lambda_2 y_2 \ldots . \lambda_n y_n \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$$

$= \lambda_1 y_1^2 + \lambda_2 y_2^2 + \ldots . + \lambda_n y_n^2$, which is called canonical form.

Now, we have seen that quadratic form $X'AX$ can be reduced to the sum of the squares by the transformation $X = PI$ where P is the normalised modal matrix of A.

Canonical form. B is a diagonal matrix, then the transformed quadratic is a sum of square terms, known as canonical form.

Index. The number of positive terms in canonical form of a quadratic form is known as index (s) of the form.

Rank of form. Rank (r) of matrix B (or A) is called the rank of the form.

Signature of quadratic form. The difference of positive terms (s) and negative terms (r–s) is known as the signature of quadratic form.

$$\text{Signature} = s - (r - s) = s - r + s = 2s - r$$

CLASSIFICATION OF DEFINITENESS OF A QUADRATIC FORM A

Let Q be $X'AX$ and variables $(x_1, x_2, x_3 \dots x_n)$,

$$\text{Rank } (A) = r,$$
$$\text{Index} = s$$

1. **Positive definite**

   If rank and index are equal *i.e.*, $r = n$, $s = n$ or if all the eigen values of A are positive.

2. **Negative definite**

   If index = 0, *i.e.*, $r = n$, $s = 0$ or if all the eigen values of A are negative.

3. **Positive semi-definite**

   If rank and index are equal but less than n, *i.e.*, $s = r < n$ $[\,|A| = 0]$

   or all eigen values of A are positive at least one eigen value is zero.

4. **Negative semi-definite**

   If index is zero, *i.e.*, $s = 0$, $r < n$ $[\,|A| = 0]$

   or all eigen values of A are negative and at least one eigen value is zero.

5. **Indefinite**

   If some eigen values are positive and some eigen values are negative.

Example-1

Prove that the Q.F $8x_1^2 + 7x_2^2 + 3x_3^2 - 12x_1x_2 + 4x_1x_3 - 8x_2x_3$ ***is positive semi definite.***

Solution

The matrix of the Q.F is

$$A = \begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$$

$$D_1 = |8| = 8$$

$$D_2 = \begin{vmatrix} 8 & -6 \\ -6 & 7 \end{vmatrix} = 56 - 36 = 20$$

$$D_3 = \begin{vmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{vmatrix} = 8\,(21 - 16) + 6\,(-18 + 8) + 2\,(24 - 14) = 0$$

Here $D_1 > 0$, $D_2 > 0$, $D_3 = 0$.

ie $D_n \geq 0$ for all n.

$\therefore$ The given quadratic form is positive semi-definite.

Example-2

Reduce the quadratic form

$$8x_1^2 + 7x_2^2 + 3x_3^2 - 12x_1 x_2 - 8x_2 x_3 + 4x_3 x_1$$

to the canonical form through an orthogonal transformation

Solution

The matrix of the quadratic form is

$$A = \begin{pmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{pmatrix}$$

The characteristic equation of A is $a_0 + a_1 \lambda + a_2 \lambda^2 + a_3 \lambda^3 = 0$

$$a_0 = |A| = 8\,(21 - 16) + 6\,(-18 + 8) + 2\,(24 - 14) = 0$$

$$a_1 = -\left[\begin{vmatrix} 7 & -4 \\ -4 & 3 \end{vmatrix} + \begin{vmatrix} 8 & 2 \\ 2 & 3 \end{vmatrix} + \begin{vmatrix} 8 & -6 \\ -6 & 7 \end{vmatrix} \right] = -45$$

$$a_2 = (-1)^2 [8 + 7 + 3] = 18$$

$$a_3 = (-1)^3 = -1$$

ie $-45\lambda + 18\lambda^2 - \lambda^3 = 0$

ie $\lambda(\lambda^2 - 18\lambda + 45) = 0$

$$\lambda(\lambda - 3)(\lambda - 15) = 0$$

$$\lambda = 0, 3, 15$$

$\therefore$ The eigen values of A are 0, 3, 15.

Let $X = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ be the eigen vector corresponding to the root λ, then we have $(A - \lambda I)X = 0$

ie., $\begin{pmatrix} 8-\lambda & -6 & 2 \\ -6 & 7-\lambda & -4 \\ 2 & -4 & 3-\lambda \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

Case (i) When $\lambda = 0$

$$\begin{pmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Solving, we get $\dfrac{x}{5} = \dfrac{y}{10} = \dfrac{z}{10}$

$$X_1 = \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$$

Case (ii) When $\lambda = 3$

$$\begin{pmatrix} 5 & -6 & 2 \\ -6 & 4 & -4 \\ 2 & -4 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Solving, we get $\dfrac{x}{-16} = \dfrac{y}{-8} = \dfrac{z}{16}$

$$X_2 = \begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix}$$

Case (iii) When $\lambda = 15$

$$\begin{pmatrix} -7 & -6 & 2 \\ -6 & -8 & -4 \\ 2 & -4 & -12 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\frac{x}{80} = \frac{y}{-80} = \frac{z}{40}$$

$$X_3 = \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}$$

Now clearly $X_1X_2^T = X_2X_3^T = X_3X_1^T = 0$

The modal matrix is $M = \begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{pmatrix}$

The normalised modal matrix is

$$N = \begin{pmatrix} \frac{1}{3} & \frac{2}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{1}{3} & \frac{-2}{3} \\ \frac{2}{3} & \frac{-2}{3} & \frac{1}{3} \end{pmatrix}$$

$$N \cdot N^T = \frac{1}{9}\begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{pmatrix}\begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$\therefore N$ is orthogonal matrix

Hence $N^T AN = D\,(0, 3, 15)$

$$Q = Y^T\,(N^T AN)\,Y = (y_1\ y_2\ y_3)\begin{pmatrix} 0 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 15 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$$

$\therefore$ Canonical form is $3y_2^2 + 15y_3^2$

Example-3

Reduce the quadratic form $2x_1^2+6x_2^2+2x_3^2+8x_1x_3$ to canonical form by orthogonal reduction.

Solution

The matrix of quadratic form is

$$A = \begin{array}{c|ccc} & x_1 & x_2 & x_3 \\ \hline x_1 & 2 & 0 & 4 \\ x_2 & 0 & 6 & 0 \\ x_3 & 4 & 0 & 2 \end{array}$$

The characteristic equation of A is $a_0 + a_1\lambda + a_2\lambda^2 + a_3\lambda^3 = 0$

$$a_0 = |A| = 2(12-0)+0+4(0-24) = -72$$

$$a_1 = -\left[\begin{vmatrix} 6 & 0 \\ 0 & 2 \end{vmatrix} + \begin{vmatrix} 2 & 4 \\ 4 & 2 \end{vmatrix} + \begin{vmatrix} 2 & 0 \\ 0 & 6 \end{vmatrix}\right] = -12$$

$$a_2 = (-1)^2[2+6+2] = 10$$

$$a_3 = (-1)^3 = -1$$

$$-72-12\lambda+10\lambda^2-\lambda^3=0$$

ie $\quad \lambda^3-10\lambda^2+12\lambda+72=0$

$\lambda=-2, \Rightarrow -8-40-24+72=0$

$\therefore \lambda=-2$ is one of the root.

$$\begin{array}{r|rrrr} -2 & 1 & -10 & 12 & 72 \\ & 0 & -2 & 24 & -72 \\ \hline & 1 & -12 & 36 & 0 \end{array}$$

$$\lambda^2 - 12\lambda + 36 = 0$$

$$(\lambda - 6)(\lambda - 6) = 0$$

$$\lambda = 6, 6$$

$\therefore$ The eigen values of A are $-2, 6, 6$

Let $X = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ be the eigen vector corresponding to the eigen value λ, then we have $(A - \lambda I)X = 0$

ie $\begin{pmatrix} 2-\lambda & 0 & 4 \\ 0 & 6-\lambda & 0 \\ 4 & 0 & 2-\lambda \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

Case (i) When $\lambda = 2$

$$\begin{pmatrix} 4 & 0 & 4 \\ 0 & 8 & 0 \\ 4 & 0 & 4 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Solving,
$$\frac{x}{32} = \frac{y}{0} = \frac{z}{-32}$$

$$X_1 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$

Case (ii) When $\lambda = 6$

$$\begin{pmatrix} -4 & 0 & 4 \\ 0 & 0 & 0 \\ 4 & 0 & 4 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow x - z = 0$$

$$X_2 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

Let $X_3 = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$ be the third eigen vector which is mutually orthogonal with X_1 and X_2

$$X_3 \perp X_1 \Rightarrow a - c = 0$$
$$X_3 \perp X_2 \Rightarrow a + c = 0$$

Solving $\frac{a}{0} = \frac{b}{-2} = \frac{c}{0}$

$$X_3 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

Clearly $X_1 X_2^T = X_2 X_3^T = X_3 X_1^T = 0$

The Modal matrix $M = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & 1 & 0 \end{pmatrix}$

The Normalised modal matrix $N = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \\ \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \end{pmatrix}$

$$N^T = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \end{pmatrix}$$

Clearly $NN^T = I \Rightarrow N$ is orthogonal.

$$N^T AN = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 2 & 0 & 4 \\ 0 & 6 & 0 \\ 4 & 0 & 2 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} -\frac{2}{\sqrt{2}} & \frac{6}{\sqrt{2}} & 0 \\ 0 & 0 & 6 \\ \frac{2}{\sqrt{2}} & \frac{6}{\sqrt{2}} & 0 \end{pmatrix}$$

$$= \begin{pmatrix} -2 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 6 \end{pmatrix} = D(-2, 6, 6)$$

$N^T AN = D(-2, 6, 6)$

$$Q = Y^T (N^T A N) Y$$

$$= (y_1 \; y_2 \; y_3) \begin{pmatrix} -2 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 6 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$$

$$= (-2y_1 \; 6y_2 \; 6y_3) \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$$

ie $-2y_1^2 + 6y_2^2 + 6y_3^2$

Therefore the quadratic form is indefinite in nature, since canonical form contains both positive and negative terms.