4 GEOMETRICAL APPLICATIONS OF DIFFERENTIAL CALCULUS

4.1 Curvature and Radius of Curvature

Let P be any point on a given curve and Q be a neighbouring point. Let Arc $AP = S$, and arc $PQ = \delta s$. Let the tangents at P and Q make angles ψ and $\psi + \sigma\psi$ with the x - axis, so that the angle between the tangents at P and $Q = \delta\psi$. In moving from P to Q through a distance δs, the tangent has turned through the angle $\delta\psi$. This is called total bending or total curvature of the arc PQ. The average curvature of arc $PQ = \frac{\delta\psi}{\delta S}$. The limiting value of average curvature when Q approaches P (i.e. $\delta s \to 0$) is defined as the curvature of the curve at P.

This curvature K (at P) $\rho = \frac{d\psi}{ds}$

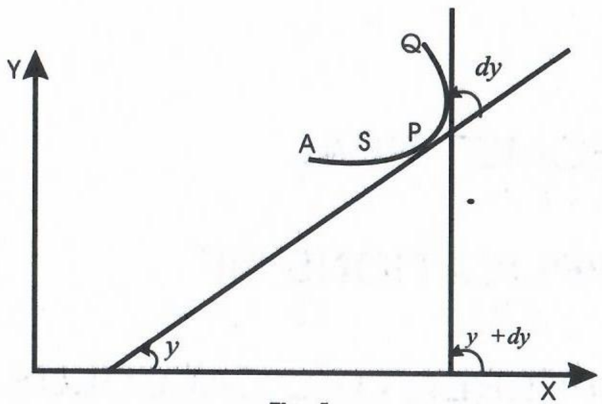

Fig. 1

4.1.1 Radius of Curvature

The reciprocal of the curvature of a curve at any point P is called the radius of curvature at P and is denoted by ρ, so that $\frac{ds}{d\psi}$. To find ρ:

(i). For Cartesian curve $y = f(x)$

$$\rho = \frac{(1+y_1^2)^{3/2}}{y_2}$$

We know that $tan\psi = \frac{dy}{dx} = y_1$ (or) $\psi = tan^{-1}(y_1)$

Differentiating both sides w.r.t.x,

$$\begin{aligned}
\frac{d\psi}{dx} &= \frac{1}{(1+y_1^2)}\frac{d(y_1)}{dx} = \frac{y_2}{(1+y_1^2)} \\
\rho &= \frac{dS}{d\psi} = \frac{dS}{dx}.\frac{dx}{d\psi} \\
&= \sqrt{(1+y_1^2)}\frac{(1+y_1^2)}{y_2} \quad as \quad \frac{dS}{dx} = \sqrt{(1+y_1^2)}
\end{aligned}$$

$$\rho = \frac{(1+y_1^2)^{3/2}}{y_2} \tag{4.1}$$

(ii). For parametric equations $x = f(t), y = \varphi(t)$

Let $x = f(t), y = \varphi(t)$. Differentiate w.r.to t, we have

$$\frac{dx}{dt} = \frac{df}{dt} \Rightarrow x' = f' \text{ and } \frac{dy}{dt} = \frac{d\varphi}{dt} \Rightarrow y' = (\varphi)'$$

$$\begin{aligned}
y_1 &= \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{y'}{x'} \\
y_2 &= \frac{dy_1}{dx} = \frac{d}{dt}\left(\frac{y'}{x'}\right)\frac{dt}{dx} = \frac{x'y'' - y'x''}{(x')^2}.\frac{1}{x'}
\end{aligned}$$

Substituting these values of y_1 and y_2 in (1.1)

$$\rho = \frac{\left[1 + \left(\frac{y'}{x'}\right)^2\right]^{3/2}}{\left[\frac{x'y'' - y'x''}{(x')^3}\right]} = \frac{(x'^2 + y'^2)^{3/2}}{x'y'' - y'x''}$$

Note 1: Curvature of a straight line is zero.
Note 2: Curvature of a circle is the reciprocal of its radius.
Note 3: To calculate ρ when dy/dx becomes infinite, we can use the formula

$$\rho = \frac{\left[1 + \left(\frac{dx}{dy}\right)^2\right]^{3/2}}{\frac{d^2x}{dy^2}}$$

Example 4.1. *Find the radius of curvature at the point* $\left(\frac{1}{4}, \frac{1}{4}\right)$ *on the curve* $\sqrt{x} + \sqrt{y} = 1$.

Solution: Let $\sqrt{x} + \sqrt{y} = 1$. Differentiating with respect to x, we have

$$\frac{1}{2\sqrt{x}} + \frac{1}{2\sqrt{y}}\frac{dy}{dx} = 0 \Rightarrow \frac{1}{\sqrt{x}} + \frac{1}{\sqrt{y}}\frac{dy}{dx} = 0$$

$$\Rightarrow \frac{1}{\sqrt{y}}\frac{dy}{dx} = -\frac{1}{\sqrt{x}} \Rightarrow \frac{dy}{dx} = -\frac{\sqrt{y}}{\sqrt{x}}$$

Therefore $y_1 = -\dfrac{\sqrt{y}}{\sqrt{x}}$

Again differentiating with respect to x, we have

$$\begin{aligned}
\frac{d^2y}{dx^2} &= -\left[\sqrt{x}\frac{1}{2\sqrt{y}} - \sqrt{y}\frac{1}{2\sqrt{x}}\right] \div x \\
&= -\frac{\left[\sqrt{x}\frac{1}{2\sqrt{y}}\left(-\frac{\sqrt{y}}{\sqrt{x}}\right) - \frac{\sqrt{y}}{2\sqrt{x}}\right]}{x}
\end{aligned}$$

$$\Rightarrow \quad \frac{d^2y}{dx^2} = \frac{\frac{1}{2} + \frac{\sqrt{y}}{2\sqrt{x}}}{x} = \frac{\sqrt{x} + \sqrt{y}}{2x\sqrt{x}}$$

At the point $\left(\frac{1}{4}, \frac{1}{4}\right)$, $\dfrac{dy}{dx} = \dfrac{-\sqrt{\frac{1}{4}}}{\frac{1}{4}} = -1$

and $\quad \dfrac{d^2y}{dx^2} = \dfrac{\sqrt{\frac{1}{4}} + \sqrt{\frac{1}{4}}}{2.\frac{1}{4}\sqrt{\frac{1}{4}}} = \dfrac{\frac{1}{2} + \frac{1}{2}}{2.\frac{1}{4}.\frac{1}{2}} = 4$

So $\rho = \dfrac{(1 + y_1^2)^{3/2}}{y_2} = \dfrac{\left[1 + (-1)^2\right]^{3/2}}{4} = \dfrac{2^{3/2}}{4} \Rightarrow \quad \rho = \dfrac{1}{\sqrt{2}}$

Example 4.2. *Show that at any point P on the rectangular hyperbola* $xy = c^2, \rho = \dfrac{r^3}{2c^2}$ *where r is the distance of P from the centre of the curve.*

Solution: Given $xy = c^2$. Differentiating with respect to x, we have

$$x\frac{dy}{dx} + y = 0 \Rightarrow \frac{dy}{dx} = \frac{-y}{x} \tag{4.2}$$

Differentiating again with respect to x, we have

$$\frac{d^2y}{dx^2} = -\left[\frac{\frac{xdy}{dx} - y}{x^2}\right] = -\left[\frac{x(\frac{-y}{x}) - y}{x^2}\right] = \frac{2y}{x^2} \tag{4.3}$$

$$\rho = \frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{3/2}}{\frac{d^2y}{dx^2}} = \frac{\left(1+\frac{y^2}{x^2}\right)^{3/2}}{\frac{2y}{x^3}}$$

$$= \frac{(x^2+y^2)^{3/2}}{2xy} = \frac{r^3}{2c^2}$$

where $r = (x^2+y^2)^{3/2}$ = distance of P from the center $(0,0)$.

Example 4.3. *Find the radius of curvature at $x = 1$ on $y = \frac{logx}{x}$.*

Solution: Given $xy = c^2$. Differentiating with respect to x, we have

$$\frac{dy}{dx} = \frac{x.\frac{d}{dx} - (logx)\frac{d}{dx}(x)}{x^2} = \frac{x.\frac{1}{x} - logx}{x^2}$$

$$= \frac{1 - logx}{x^2}$$

$$\frac{d^2y}{dx^2} = \frac{x^2.\frac{d}{dx}(1 - logx) - (1 - logx)(2x)}{x^4}$$

$$= \frac{x^2.(\frac{-1}{x}) - 2x(1 - logx)}{x^4}$$

$$= \frac{-x - 2x(1 - logx)}{x^4}$$

At $x = 1, \frac{dy}{dx} = \frac{1 - logx}{x^2} = \frac{1}{1} = 1(since\log 1 = 0)$

$$\frac{d^2y}{dx^2} = \frac{-1 - 2(1 - log1)}{1^4} = -3$$

$$\rho = \frac{(1+y_1^2)^{3/2}}{y_2} = \frac{(1+1^2)^{3/2}}{-3} = \frac{-(2)^{3/2}}{3}$$

$$|\rho| = \frac{2\sqrt{2}}{3}$$

Example 4.4. *Find the radius of curvature at the point $\left(\frac{3a}{2}, \frac{3a}{2}\right)$ of the curve $x^3 + y^3 = 3axy$.*

Solution: Differentiating with respect to x, we get

$$3x^2 + 3y^2\frac{dy}{dx} = 3a\left(y + x\frac{dy}{dx}\right)$$

$$\Rightarrow \quad (y^2 - ax)\frac{dy}{dx} = ay - x^2 \tag{4.4}$$

$$\Rightarrow \frac{dy}{dx} \quad \text{at} \quad \left(\frac{3a}{2}, \frac{3a}{2}\right) = -1$$

Again differentiating (4.4), we get

$$\left(2y\frac{dy}{dx} - a\right)\frac{dy}{dx} + (y^2 - ax)\frac{d^2y}{dx^2} = a\frac{dy}{dx} - 2x$$

$$\Rightarrow \quad \frac{d^2y}{dx^2} \quad \text{at} \quad \left(\frac{3a}{2}, \frac{3a}{2}\right) = -\frac{32}{3a}$$

$$\text{Hence} \quad \rho = \left(\frac{3a}{2}, \frac{3a}{2}\right) = \frac{\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{3/2}}{\frac{d^2y}{dx^2}}$$

$$= \frac{\left[1 + (-1)^2\right]^{3/2}}{\frac{-32}{3a}} = \frac{3a}{8\sqrt{2}}$$

Example 4.5. *Show that the radius of curvature at any point of the cycloid $x = a(\theta + sin\theta), y = a(1 - cos\theta)$ is $4a\cos(\frac{\theta}{2})$ [June 2011].*

Solution: Given $x = a(\theta + sin\theta), y = a(1 - cos\theta)$. We have

$\frac{dx}{d\theta} = a(1 + cos\theta), \frac{dy}{d\theta} = a\sin\theta$ so that

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}}$$

$$= \frac{a\sin\theta}{a(1+\cos\theta)}$$

$$= \frac{2sin(\frac{\theta}{2})cos(\frac{\theta}{2})}{cos^2(\frac{\theta}{2})}$$

$$= \tan(\frac{\theta}{2})\frac{d^2y}{dx^2}$$

$$= \frac{d}{d\theta}\left(tan(\frac{\theta}{2})\right)\frac{d\theta}{dx}$$

$$= \frac{1}{2}sec^2\frac{\theta}{2}\frac{1}{2a}\cos^2\frac{\theta}{2}$$

$$= \frac{1}{2}sec^2(\frac{\theta}{2})\frac{1}{2acos^2(\frac{\theta}{2}})$$

$$= \frac{1}{4a}sec^4(\frac{\theta}{2})$$

$$\rho = \frac{\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{3/2}}{\frac{d^2y}{dx^2}} = \frac{4a\left(1 + tan^2\frac{\theta}{2}\right)^{3/2}}{sec^4\frac{\theta}{2}}$$

$$\Rightarrow \quad \rho = 4a\left(\sec^2(\tfrac{\theta}{2})\right)^{3/2}.\cos^4(\tfrac{\theta}{2}) \Rightarrow \rho = 4a\cos(\tfrac{\theta}{2})$$

Example 4.6. *Find the radius of curvature at any point on the curve $x = a(\cos t + t\sin t), y = a(\sin t - t\cos t)$.*

Solution: Let $x = a(\cos t + t\sin t), y = a(\sin t - t\cos t)$

$$x' = \frac{dx}{dt} = a(-\sin t + t\sin t + \sin t) = at\cos t$$

$$y' = \frac{dy}{dt} = a(\cos t + t\sin t - \cos t) = at\sin t$$

$$x'' = \frac{d^2x}{dt^2} = \frac{d}{dt}(\frac{dx}{dt}) = \frac{d}{dt}(at\cos t)\ x'' = a(\cos t - t\sin t)$$

and $y'' = a(t\cos t + \sin t)$ Consider $x'y'' - y'x''$

$$\Rightarrow (at\cos t)(at\cos t + a\sin t) - (at\sin t)(a\cos t - at\sin t)$$

$$= a^2t^2\cos^2 t + a^2t\cos t\sin t - a^2t\sin t\cos t + a^2t^2\sin^2 t$$

$$= a^2t^2$$

Now $\rho = \dfrac{(x'^2 + y'^2)^{3/2}}{x'y'' - y'x''} = \dfrac{(a^2t^2\cos^2 t + a^2t^2\sin^2 t)^{3/2}}{a^2t^2}$

$$\Rightarrow \quad \rho = \frac{(a^2t^2)^{3/2}}{a^2t^2} \Rightarrow \rho = at$$

Example 4.7. *Show that the radius of curvature at an end of the major axis of the ellipse* $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ *is equal to the semi-latus rectum.*

Solution: Given

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \tag{4.5}$$

Differentiating (4.5) with respect to x, we get

$$\frac{2x}{a^2} + \frac{2y}{b^2}\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = \frac{\frac{-x}{a^2}}{\frac{y}{b^2}} = -\frac{xb^2}{ya^2} \tag{4.6}$$

One end of major axis can be taken as $(a, 0)$.

Now $\left(\dfrac{dy}{dx}\right)_{(a,0)} = \infty$

We consider

$$\rho = \frac{\left[1 + \left(\frac{dx}{dy}\right)^2\right]^{3/2}}{\frac{d^2x}{dy^2}} \tag{4.7}$$

At $(a, 0)$ we observe $\dfrac{dx}{dy} = 0$ using (4.6) Consider

$$\frac{dx}{dy} = -\frac{ya^2}{xb^2} \tag{4.8}$$

Differentiating with respect to y, we get,

$$\frac{d^2x}{dy^2} = -\frac{a^2}{b^2}\left(\frac{x.1 - y\frac{dx}{dy}}{x^2}\right)$$

$$Rightarrow \left(\frac{d^2x}{dy^2}\right)_{(a,0)} = -\frac{a^2}{b^2}\cdot\frac{a}{a^2} = -\frac{a}{b^2}$$

From (4.7), $\rho = \dfrac{(1+0^2)^{3/2}}{\left(-\frac{a}{b^2}\right)}$

$|\rho| = \dfrac{b^2}{a}$ = Semi-latus rectum. [since for an ellipse $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$, the length of latus rectum $= \dfrac{2b^2}{a}$]

Example 4.8. *Find 'ρ' at any point $P(at^2, 2at)$ on the parabola $y^2 = 4ax$, Prove that if S is its focus, then ρ^2 varies as sp^3.*

Solution: On the parabola $y^2 = 4ax$, we have $x = at^2, y = 2at$.

Now $\dfrac{dx}{dt} = 2at, \dfrac{dy}{dt} = 2a \Rightarrow y_1 = \dfrac{dy}{dx} = \dfrac{\frac{dy}{dt}}{\frac{dx}{dt}} \Rightarrow \dfrac{dy}{dx} = \dfrac{2a}{2at}$

(or) $\dfrac{dy}{dx} = \dfrac{1}{t}$ and $\dfrac{d^2y}{d^2x} = \dfrac{d}{dt}\left(\dfrac{dy}{dx}\right)\dfrac{dt}{dx} = \dfrac{\frac{d}{dt}\left(\frac{1}{t}\right)}{\frac{dx}{dt}} = \dfrac{\left(-\frac{1}{t^2}\right)}{2at}$

$\Rightarrow y_2 = \dfrac{d^2y}{dx^2} = -\dfrac{1}{2at^3}$

$$rho = \frac{(1+y_1^2)^{3/2}}{y_2} = \frac{\left[1+\left(\frac{1}{t}\right)^2\right]^{3/2}}{\left(-\frac{1}{2at^3}\right)} = \frac{(t^2+1)^{3/2}}{t^3}(-2at^3)$$

$|\rho| = 2a(t^2+1)^{3/2}$. Now $\rho^2 = 4a^2(t^2+1)^3$. For the parabola $y^2 = 4ax$, the focus is $S(a,0)$ any point on the parabola is $P(at^2, 2at)$.

$\Rightarrow$ distance $SP = \sqrt{(at^2-a)^2+(2at-0)^2}$

$$= \sqrt{a^2(t^2-1)^2+4a^2t^2}$$
$$= \sqrt{a^2(t^2-1)^2+4t^2}$$
$$= \sqrt{t^4+1+2t^2}$$

$= a[(t^2+1)^2]^{1/2}$ (or) distance $SP = a(t^2+1)$

$$SP^3 = a^3(t^2+1)^3 \qquad (4.9)$$

$(1) \div (2) \Rightarrow \dfrac{\rho^2}{SP^3} = \dfrac{4}{a}$ = constant (or) ρ^2 varies as SP^3

Example 4.9. *Show that the radius of curvature at the point 'θ' on the curve $x = 3a\cos\theta - a\cos 3\theta, y = 3a\sin\theta - a\sin 3\theta$ is $3a\sin\theta$.*

Solution: $x = 3a\cos\theta - a\cos^3\theta$ Differentiating with respect to 'θ'.

$$\frac{dx}{d\theta} = -3asin\theta + asin3\theta$$
$$= 3a(sin3\theta - sin\theta)$$

$y = 3a\sin\theta - a\sin 3\theta$ Differentiating with respect to 'θ'.

$$\frac{dy}{d\theta} = 3acos\theta - 3acos3\theta$$
$$= 3a(cos\theta - cos3\theta)$$

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{3a(\cos\theta - \cos 3\theta)}{3a(\sin 3\theta - \sin\theta)}$$

$$\text{(or) } \frac{dy}{dx} = \frac{2\sin\left(\frac{3\theta+\theta}{2}\right) sin\left(\frac{3\theta-\theta}{2}\right)}{2\cos\left(\frac{3\theta+\theta}{2}\right)\sin\left(\frac{3\theta-\theta}{2}\right)}$$

$= \tan 2\theta$ [since $\cos C - \cos D = 2\sin\frac{C+D}{2}\sin\frac{D-C}{2}$,
$\sin C - \sin D = 2\cos\frac{C+D}{2}\sin\frac{C-D}{2}$]

$$\text{Now } \frac{d^2y}{dx^2} = \frac{d}{d\theta}\left(\frac{dy}{dx}\right)\frac{d\theta}{dx}$$

$$= \frac{\frac{d}{d\theta}(\tan 2\theta)}{3a(\sin 3\theta - \sin\theta)}$$

$$= \frac{sec^2\theta(2\theta)\times 2}{3a(2\cos 2\theta\sin\theta)}$$

$$= \frac{1}{3a}\sec^3 2\theta . \csc\theta$$

$$\rho = \frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{3/2}}{\frac{d^2y}{dx^2}} = \frac{\left[1+(tan2\theta)^2\right]^{3/2}}{\frac{1}{3a}sec^3 2\theta .cosec\theta}$$

$$= 3a\frac{sec^3 2\theta}{sec^3 2\theta}\frac{1}{cosec\theta} = 3asin\theta$$

Example 4.10. *Find the radius of curvature at the point* $(acos^3\theta, asin^3\theta)$

on the curve $x^{2/3} + y^{2/3} = a^{2/3}$.

Solution: The parametric equation of the given curve are
$x = a\cos^3\theta$ and $y = a\sin^3\theta$.
Differentiating with respect to $'\theta'$.

$$x' = \frac{dx}{d\theta} = 3acos^2\theta(-sin\theta)$$

$$y' = \frac{dy}{d\theta} = 3asin^2\theta(cos\theta)$$

$$x'' = \frac{d^2x}{d\theta^2} = -3a(cos^3\theta - 2cos\theta sin^2\theta)$$

$$y'' = \frac{d^2y}{d\theta^2} = 3a(2sin\theta cos^2\theta - sin^3\theta)$$

$$\rho = \frac{(x'^2+y'^2)^{3/2}}{x'y''-y'x''}$$

$$= \frac{(9a^2\cos^4\theta\sin^2\theta + 9a^2\sin^4\theta\cos^2\theta)^{3/2}}{9a^2(-\cos^2\theta\sin\theta(2\sin\theta\cos^2\theta - \sin^3\theta) + \sin^2\cos\theta(\cos^3\theta - 2\cos\theta\sin}$$

$$\Rightarrow \rho = \frac{27a^3\sin^3\theta\cos^3\theta(\cos^2\theta+\sin^2\theta)^{3/2}}{9a^2\sin^2\theta\cos^2\theta[-(2\cos^2\theta - \sin^2\theta) + (\cos^2\theta - 2\sin^2\theta)]}$$

$$= \frac{3a\sin\theta\cos\theta}{-(\cos^2\theta+\sin^2\theta)}$$

$$\Rightarrow |\rho| = 3a\sin\theta\cos\theta$$

Example 4.11. *Find the radius of curvature for the curve* $y =$

$c\cosh\frac{x}{c}$ *at the point where the curve cross the y-axis.*

Solution: Put $x = 0$ in $y = c\cosh\frac{x}{c}$ as the curve cuts y - axis at $x = 0$. Thus the point is $(0, c)$.

Now $y = c\cosh\frac{x}{c}$

$$y_1 = \frac{dy}{dx} = c\sinh\frac{x}{c}.\frac{1}{c} \text{ (or) } y_1 = \sinh\frac{x}{c}$$

and $y_2 = \frac{d^2y}{dx^2} = \frac{1}{c}c\cosh\frac{x}{c}$

Now $\rho = \frac{(1+y_1^2)^{3/2}}{y_2} = \frac{\left(1+\sinh^2\frac{x}{c}\right)^{3/2}}{\frac{1}{c}c\cosh\frac{x}{c}}$

$\Rightarrow \rho$ at $(0, c) = c$

Example 4.12. *Find the radius of curvature at any point $P(a\cos\theta, b\sin\theta)$ on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.*

Solution: Let $x = a\cos\theta$, $y = b\sin\theta$ then $x' = -a\sin\theta, y' = b\cos\theta$

$x'' = -a\cos\theta, y'' = -b\sin\theta.$

$$\rho = \frac{(x'^2 + y'^2)^{3/2}}{x'y'' - y'x''}$$

$$= \frac{(a^2\sin^2\theta + b^2\cos^2\theta)^{3/2}}{(-a\sin\theta)(-b\sin\theta) - (b\cos\theta)(-a\cos\theta)}$$

$$= \frac{(a^2\sin^2\theta + b^2\cos^2\theta)^{3/2}}{ab(\sin^2\theta + \cos^2\theta)}$$

$$= \frac{(a^2\sin^2\theta + b^2\cos^2\theta)^{3/2}}{ab}$$

Example 4.13. *Find the radius of curvature of the curve $y = x^2(x-3)$ at the points where the tangent is parallel to the x-axis.*

Solution: The equation is $y = x^2(x-3)$ Differentiating w.r.to x, we have $\frac{dy}{dx} = 3x^2 - 6x$ and $\frac{d^2y}{dx^2} = 6x - 6$

At the point where the tangent is parallel to the x - axis, $\frac{dy}{dx} = 0$
$\Rightarrow 3x^2 - 6x = 0; 3x(x-2) = 0$ (or) $x = 0, 2$

when $x = 0, y = 0$ and when $x = 2, y = 4(2-3) = -4$ So the points are $(0, 0)$ and $(2, -4)$

At $(0, 0)$, $\frac{d^2y}{dx^2} = -6$ and $(2, -4)$, $\frac{d^2y}{dx^2} = 6$

$$\rho \text{ at } (0,0) = \frac{(1+y_1^2)^{3/2}}{y_2} = \frac{(1+0)^{3/2}}{-6} = -\frac{1}{6}$$

$$\rho \text{ at } (2,-4) = \frac{(1+y_1^2)^{3/2}}{y_2} = \frac{(1+0)^{3/2}}{6} = \frac{1}{6}$$

Hence $|\rho| = \frac{1}{6}$ at the points where the tangent is parallel to the x - axis.

4.1.2 Formula for ρ in Polar Coordinates:

Let $r = f(\theta)$, then $\rho = \frac{(r^2+r_1^2)^{3/2}}{r^2+2r_1^2-rr_2}$ where

$r_1 = \frac{dr}{d\theta}$ and $r_2 = \frac{d^2r}{d\theta^2}$.

Example 4.14. *Find the radius of curvature at any point (r, θ) on the equiangular spiral $r = ae^{\theta\cot\alpha}$*

Solution: Given $r = ae^{\theta\cot\alpha}$ Taking logarithms,

$$\log r = \log a + \theta\cot\alpha \tag{4.10}$$

Differentiating (1) w.r.to $'\theta'$, we have

$$\frac{1}{r}\frac{dr}{d\theta} = \cot\alpha \Rightarrow r_1 = r\cot\alpha \tag{4.11}$$

Differentiating (2) w.r.to $'\theta'$, $r_2 = r_1\cot\alpha$ (since $\cot\alpha$ is a constant) $= r\cot^2\alpha$ using (4.11)

Now $\rho = \frac{(r^2+r_1^2)^{3/2}}{r^2+2r_1^2-rr_2}$

$= \frac{(r^2+r^2\cot^2\alpha)^{3/2}}{r^2+2r^2\cot^2\alpha-r^2\cot^2\alpha}$

$= \frac{(r^2+r^2\cot^2\alpha)^{3/2}}{r^2+r^2\cot^2\alpha}$

$= [r^2(1+\cot^2\alpha)]^{1/2}$

$= (r^2+r^2\cot^2\alpha)^{1/2}$

$= r\csc\alpha$
$\Rightarrow \rho = r\csc\alpha$

Example 4.15. *Find the radius of curvature at any point (r, θ) for the curve $r = a\cos\theta$.*

Solution: Given $r = a\cos\theta$. Differentiating w.r.to θ, we get $r_1 = \frac{dr}{d\theta} = -a\sin\theta$ and $r_2 = \frac{d^2r}{d\theta^2} = -a\cos\theta$ Now

$$\rho = \frac{(r^2+r_1^2)^{3/2}}{r^2+2r_1^2-rr_2} = \frac{(a^2\cos^2\theta+a^2\sin^2\theta)^{3/2}}{a^2\cos^2\theta+2a^2\sin^2\theta+a^2\cos^2\theta}$$

$$= \frac{a^3}{2a^2} \Rightarrow \rho = \frac{a}{2}$$

Example 4.16. *Show that the radius of curvature of the curve $r^n = a^n\cos n\theta$ is $\frac{a^n r^{-n+1}}{n+1}$.*

Hence prove that the radius of curvature of the curve $r^2 = a^2\cos 2\theta$

is $\frac{a^2}{3r}$

Solution: Given
$$r^n = a^n \cos n\theta \tag{4.12}$$

Taking logarithms of both sides of (1),

we get $n \log r = n \log a + \log \cos n\theta$. Differentiating (1) with respect to $'\theta'$,

we get $\frac{n}{r}\frac{dr}{d\theta} = \frac{n \sin n\theta}{\cos n\theta}$

$$\Rightarrow r_1 = \frac{dr}{d\theta} = -r \tan n\theta \tag{4.13}$$

Differentiating (4.13) with respect to $'\theta'$,

we get $r_2 = \frac{d^2r}{d\theta^2} = -[rn \sec^2 n\theta + \tan n\theta r_1]$

$= -[rn \sec^2 n\theta - r \tan^2 n\theta]$ [by (4.13)]

$$= r \tan^2 n\theta - rn \sec^2 n\theta \tag{4.14}$$

$$\rho = \frac{(r^2 + r_1^2)^{3/2}}{r^2 + 2r_1^2 - rr_2}$$

$$= \frac{(r^2 + r^2 \tan^2 n\theta)^{3/2}}{r^2 + 2r^2 \tan^2 n\theta - r(r \tan^2 n\theta - rn \sec^2 n\theta)}$$

$$= \frac{(r^2 sec^2 n\theta)^{3/2}}{r^2 n \sec^2 n\theta + r^2 + r^2 \tan^2 n\theta}$$

$$= \frac{r^3 \sec^3 n\theta}{r^2 \sec^2 n\theta(n+1)}$$

$$= \frac{r \sec n\theta}{n+1} = \frac{r}{(n+1)\cos n\theta}$$

$$= \frac{r}{\left(n+1)(\frac{r^n}{a^n}\right)}$$

using (4.13)
$$= \frac{a^n r}{(n+1)r^n} = \frac{a^n r^{1-n}}{n+1} \tag{4.15}$$

putting $n = 2$, curve (1) is $r^2 = a^2 cos2\theta$

From (4), $\rho = \frac{a^2 r^{-1}}{3} = \frac{a^2}{3r}$ **Note:** When $n = -2$, we get the rectangular hyperbola $rcos\theta = a^2; \rho = -\frac{a^2}{3r}$

Example 4.17. *Find the radius of curvature of the curve $r = a(1 + \cos\theta)$ at the point $\theta = \frac{\pi}{2}$.*

Solution: Given $r = a(1 + \cos\theta) \Rightarrow r_1 = -a \sin\theta$ and $r_2 =$

$-a\cos\theta$

$$\rho = \frac{(r^2+r_1^2)^{3/2}}{r^2+2r_1^2-rr_2}$$

$$= \frac{[a^2(1+\cos\theta)^2+a^2\sin^2\theta]^{3/2}}{a^2(1+\cos\theta)^2+2a^2\sin^2\theta+a^2\cos\theta(1+\cos\theta)}$$

$$= \frac{a^3[2(1+\cos\theta)]^{3/2}}{a^2[3(1+\cos\theta)]}$$

$$= \frac{2\sqrt{2}}{3}a(1+\cos\theta)^{1/2} = \frac{4a}{3}\cos\frac{\theta}{2}$$

Hence $(\rho)_{\theta=\frac{\pi}{2}} = \frac{4a}{3}\cos\frac{\pi}{4} = \frac{2\sqrt{2}}{3}a$

4.2 Centre of Curvature and Circle of Curvature

Let P be a point (x, y) on the curve $y = f(x)$ and $(\overline{x}, \overline{y})$ be the coordinates of C, the centre of curvature corresponding to P.

We know that C lies on the normal at P such that $CP = \rho$ (on the Inward drawn normal to the curve at P, cut off a length PC = radius of curvature of the curve at P (namely ρ). The point C is called the centre of curvature at P for the curve). Let ψ be the angle made by the tangent at ρ) with OX. Draw CK, PN perpendiculars to OX and PM perpendicular to CK.

Then $\angle PCM = \psi$. Now $\overline{x} = CK = ON - KN = ON - MP$

$$= x - \rho\sin\psi \qquad (4.16)$$

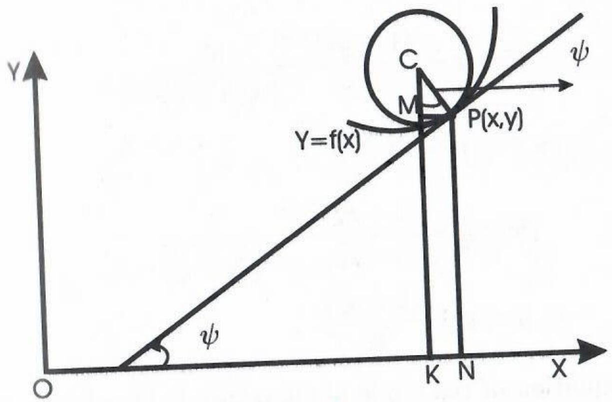

Fig. 2

Also $\quad \overline{y} = KC = KM + MC$

$$= NP + CP\cos\psi = y + \rho\sin\psi \qquad (4.17)$$

We know that $\rho = \frac{(1+y_1^2)^{3/2}}{y_2}$ and $\tan\psi = \frac{dy}{dx} = y_1$

Let $\sin\psi = \frac{\sin\psi}{\cos\psi}\cos\psi = \frac{\tan\psi}{\sec\psi} = \frac{y_1}{\sqrt{1+y_1^2}}$

and $\cos\psi = \frac{1}{\sec\psi} = \frac{1}{\sqrt{1+\tan^2\psi}}$

$$= \frac{1}{\sqrt{1+y_1^2}}$$

Hence (4.16) and (4.17) can be written as

$$\overline{x} = x - \frac{(1+y_1^2)^{3/2}}{y_2} \cdot \frac{y_1}{\sqrt{1+y_1^2}}$$

$$\Rightarrow \overline{x} = x - \frac{y_1(1+y_1^2)}{y_2}$$

$$\overline{y} = y + \frac{(1+y_1^2)^{3/2}}{y_2} \cdot \frac{1}{\sqrt{1+y_1^2}}$$

$$\Rightarrow \overline{y} = y + \frac{(1+y_1^2)}{y_2}$$

Note: The equation of the circle of curvature is $(x-\overline{x})^2 + (y - \overline{y})^2 = \rho^2$ at the point (x, y).

Example 4.18. *Find the coordinates of centre of curvature of the curve $y = x^3 - 6x^2 + 3x + 1$ at $(1, -1)$.*

Solution: Let $(\overline{x}, \overline{y})$ be the centre of curvature corresponding to $(1, -1) \Rightarrow \overline{x} = x - \frac{y_1(1+y_1^2)}{y_2}; \overline{y} = y + \frac{(1+y_1^2)}{y_2}$ Given $y = x^3 - 6x^2 + 3x + 1 \Rightarrow y_1 = \frac{dy}{dx} = 3x^2 - 12x + 3$ and $y_2 = \frac{d^2y}{dx^2} = 6x - 12$
At $(1, -1)$, $y_1 = 3 - 12 + 3 = -6$ and $y_2 = -6$

$$\overline{x} = 1 - \left(\frac{-6}{-6}\right)(1 + 36) = 1 - 1 - 36 = -36$$

$$\overline{y} = -1 + \frac{(1+36)}{-6} = -1 - \frac{1}{6} - 6 = -\frac{43}{6}$$

$\Rightarrow$ Centre of curvature $\left(-36, -\frac{43}{6}\right)$

Example 4.19. *Prove that if the centre of curvature of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ at one end of the minor axis lies at the other hand, then the eccentricity of the ellipse is $\frac{1}{\sqrt{2}}$*

Solution: The ellipse is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \tag{4.18}$$

and BB' is the minor axis. B is $(0, b)$ and B' is $(0, -b)$.

Differentiating (1) w.r.t. x,

$$\frac{2x}{a^2} + \frac{2y}{b^2}\frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = -\frac{b^2x}{a^2y} \tag{4.19}$$

Differentiating (4.18) w.r.t. y, we have

$$\frac{d^2y}{dx^2} = -\frac{b^2}{a^2}\left(\frac{y.1 - x\frac{dy}{dx}}{y^2}\right) \tag{4.20}$$

At $B(0, b)$,

$$\frac{dy}{dx} = 0 \tag{4.21}$$

And

$$\frac{d^2y}{dx^2} = -\frac{b}{a^2} \tag{4.22}$$

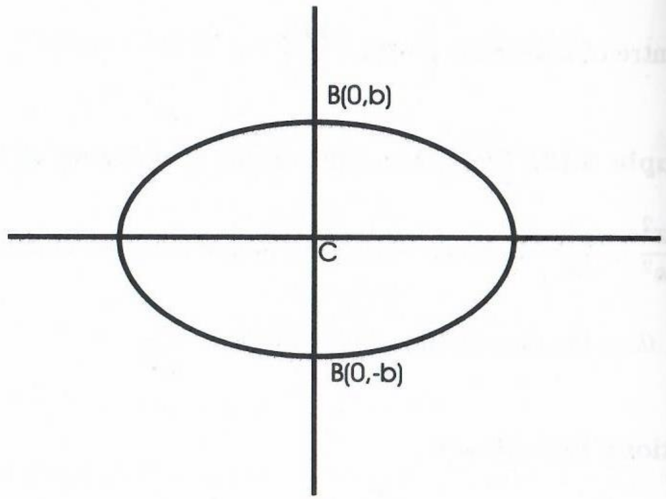

Fig. 3

Let $(\overline{x}, \overline{y})$ be the centre of curvature at $(0, b)$ $\overline{x} = x - \dfrac{y_1(1+y_1^2)}{y_2}$ and $\overline{y} = y + \dfrac{(1+y_1^2)}{y_2}$ Substituting the values of y_1 and y_2 from (4), (5), we get $\overline{x} = 0, \overline{y} = b + \dfrac{1+0}{\left(-\dfrac{b}{a^2}\right)} = b - \dfrac{a^2}{b}$ The centre of curvature is $\left(0, b - \dfrac{a^2}{b}\right)$ and this given to be the point $(0, -b)$, the other end B' of the minor axis. Hence $b - \dfrac{a^2}{b} = -b$ (i.e.)

$$2b^2 = a^2 \; e^2 = \frac{a^2 - b^2}{a^2} = \frac{a^2 - \dfrac{a^2}{2}}{a^2} = \frac{a^2}{2a^2} \Rightarrow e = \frac{1}{\sqrt{2}}$$

Example 4.20. *Find the equation of the circle of curvature of the parabola $y^2 = 12x$ at the point $(3, 6)$.[June 2011]*

Solution: The circle of curvature is $(x - \overline{x})^2 + (y - \overline{y})^2 = \rho^2$ where

$$\rho = \frac{(1+y_1^2)^{3/2}}{y_2} \tag{4.23}$$

Also $\overline{x} = x - \dfrac{y_1(1+y_1^2)}{y_2}$ and $\overline{y} = y + \dfrac{(1+y_1^2)}{y_2}$ Given $y^2 = 12x$

Differentiating with respect to x, $2y\dfrac{dy}{dx} = 12 \Rightarrow \dfrac{dy}{dx} = \dfrac{6}{y} \Rightarrow$

$(y_1)_(3,6) = \dfrac{6}{6} = 1$ Also $\dfrac{d^2y}{dx^2} = -\dfrac{6}{y^2}\dfrac{dy}{dx}$

$\Rightarrow (y_2)_{(3,6)} = \dfrac{-6}{36} \times 1 = -\dfrac{1}{6}$

$\rho = \dfrac{(1+y_1^2)^{3/2}}{y_2} = \dfrac{(1+1)^{3/2}}{-\dfrac{1}{6}} \Rightarrow |\rho| = 6 \times 2^{3/2} = 12\sqrt{2}$

Now $\overline{x} = 3 - \left(\dfrac{1}{-\dfrac{1}{6}}(1+1)\right) = 3 + 12 = 15$ and $\overline{y} = 6 + \left(\dfrac{1+1}{-\dfrac{1}{6}}\right) = -6$ Eqn. (1) $\Rightarrow (x-15)^2 + (y+6)^2 = 288$

$\Rightarrow x^2 + y^2 - 30x + 12y - 27 = 0$ is the circle of curvature.

Example 4.21. *Find the equation of the circle of curvature of* *the curve* $\sqrt{x}+\sqrt{y}=\sqrt{a}$ *at* $\left(\frac{a}{4},\frac{a}{4}\right)$ ***[Dec 2007]***

Solution: Given

$$\sqrt{x}+\sqrt{y}=\sqrt{a} \tag{4.24}$$

Differentiating (4.24) with respect to x,

$$\frac{1}{2\sqrt{x}}+\frac{1}{2\sqrt{y}}\frac{dy}{dx}=0 \Rightarrow \frac{dy}{dx}=-\sqrt{\frac{y}{x}} \tag{4.25}$$

Hence $(y_1)_{\left(\frac{a}{4},\frac{a}{4}\right)} = -1$ Differentiating (4.25) with respect to x, we get $\frac{d^2y}{dx^2} = -\left[\frac{\sqrt{x}\frac{1}{2\sqrt{y}}\frac{dy}{dx}-\sqrt{y}\frac{1}{2\sqrt{x}}}{x}\right] \Rightarrow (y_2)_{\left(\frac{a}{4},\frac{a}{4}\right)} = -\left[\frac{\left(\frac{1}{2}\times(-1)\right)-\frac{1}{2}}{\frac{a}{4}}\right] = -\left((-1)\frac{a}{4}\right) \Rightarrow (y_2) = \frac{4}{a}$ Suppose $(\overline{x},\overline{y})$ denotes the coordinates of centre of curvature at $\left(\frac{a}{4},\frac{a}{4}\right)$, then from $\overline{x}=x-\frac{y_1(1+y_1^2)}{y_2}$ and $\overline{y}=y+\frac{(1+y_1^2)}{y_2}$. We get

$$\overline{x}=\frac{a}{4}-\left(\frac{-1}{\frac{a}{4}}(1+1)\right)=\frac{a}{4}+\frac{2a}{4}\Rightarrow\overline{x}=\frac{3a}{4}$$

$$\overline{y}=\frac{a}{4}+\frac{(1+1)}{\frac{a}{4}}=\frac{a}{4}+\frac{2a}{4}\Rightarrow\overline{y}=\frac{3a}{4}$$

Now radius of curvature $\rho = \frac{(1+y_1^2)^{3/2}}{y_2}$ $\rho = \frac{(1+1)^{3/2}}{\frac{4}{a}} =$

$\frac{a}{4}.2\sqrt{2}$ (i.e.) $\rho = \frac{a}{2}\sqrt{2} \Rightarrow \rho^2 = \frac{a^2}{4}(2) \Rightarrow \rho^2 = \frac{a^2}{2}$ Hence equation to circle of curvature is obtained from the formula $(x-\overline{x})^2+(y-\overline{y})^2=\rho^2. \Rightarrow \left(x-\frac{3a}{4}\right)^2+\left(y-\frac{3a}{4}\right)^2=\frac{a^2}{2}$

EXERCISE

1. Define curvature and radius of curvature.
2. Find ρ for $y=e^x$ at $x=0$.
3. Find ρ for $y=\log\sin x$ at $x=\frac{\pi}{2}$.
4. Write the formula for radius of curvature in parametric coordinates.
5. Write the formula for radius of curvature in polar - coordinates.
6. Write the equation of circle of curvature.
7. What is the curvature of a circle at any point on it?
8. What is the curvature of a straight line?
9. Find ρ for $x=t^2, y=t$ at $t=1$.
10. Find ρ if $r=e^\theta$.
11. Show that for the rectangular hyperbola $xy=c^2, \rho=$

$\frac{(x^2+y^2)^{3/2}}{2c^2}$.

12. Show that the radius of curvature at $(a, 0)$ on the curve $y^2 = a^2(a-x)$ is $\frac{a}{2}$.

13. If $'\rho'$ be the radius of curvature at any point P on the parabola $y^2 = 4ax$ and S be its focus, then show that ρ^2 varies as $(SP)^3$.

14. Prove that the radius of curvature at any point of the astroid $x^{2/3} + y^{2/3} = a^{2/3}$, is three times the length of the perpendicular from the origin to the tangent at that point.

15. Find the radius of curvature at any point (0, c) of the catenary $y = c\cosh\frac{x}{c}$. [Ans: $\frac{y^2}{c}$]

16. Show that the radius of curvature at any point of the cardioid $r = a(1-\cos\theta$ varies as $\sqrt{r}$.

17. Find the radius of curvature at the point (r, θ) on the curve $r^n = a^n \cos n\theta$. [Ans: $\frac{a^n}{(n+1)r^{n-1}}$]

18. Find the radius of curvature at $'t'$ on the curve $x = e^t cost, y = e^t \sin t$. [Ans: $\sqrt{2}e^t$]

19. Find the coordinates of the real points on the curve $y^2 = 2x(3-x^2)$ the tangents at which are parallel to the x - axis. Show that the radius of curvature at each of these points

1/3. [Ans: $(1,2),(1,-2)$]

20. Find the radius of curvature of the parabola $y^2 = 4x$ at the vertex. [Ans: $\frac{1}{2}$]

21. Find the radius of curvature at $x = \frac{\pi}{2}$ on the curve $y = 4\sin x - \sin 2x$ [Ans: $\frac{5^{3/2}}{4}$ Numerically]

22. Find ρ when $x = a\log(\sec\theta + \tan\theta), y = a\sec\theta$. [Ans: $a\sec^2\theta$]

23. Find ρ when $x = 6t^2 - 3t^3, y = 8t^3$. [Ans: $6t(1+t^2)^2$]

24. Find the radius of curvature at $(a, 0)$ on the curve $xy^2 = a^3 - x^3$ [Ans: 3a/2]

25. Show that at the points of intersection of the curves $r = a\theta$ and $r = \frac{a}{\theta}$, their curvatures are in the ration 3:1.

26. Find the centre of curvature of the curve $y = x^3 - 6x^2 + 3x + 1$ at the point $(1, -1)$. [Ans: $(-36, -43/6)$]

27. Find the equation of the circle of curvature of the parabola $y^2 = 4ax$ at the positive end of the latus rectum. [Ans: $x^2 + y^2 - 10ax + 4ay - 3a^2$]

28. Find the equation of the circle of curvature of the rectangular hyperbola $xy = 12$ at the point $(3, 4)$. [Ans: $\left(x - \frac{43}{6}\right)^2 +$

$\left(y-\frac{57}{8}\right)^2=\left(\frac{125^2}{24}\right)$]

29. Find the equation of the circle of curvature of the curve $x^3+y^3=3axy$ at the point $\left(\frac{3a}{2},\frac{3a}{2}\right)$. [Ans: $\left(x-\frac{21a}{16}\right)^2+\left(y-\frac{21a}{16}\right)^2=\frac{9a^2}{128}$]

30. Show that the line joining any point 't' on the cycloid $x=a(t+\sin t), y=a(1-\cos t)$ and its centre of curvature is bisected by the line y = 2a.

4.3 Involutes and Evolutes

Let C be the centre of curvature corresponding to a point P of a given curve. As P moves along the curve, C will trace out a locus which is called evolute of the given curve. If a curve β is the evolute of a curve α, then α is said to be an involute of β. To get the equation to the evolute of the curve $y=f(x)$, we proceed follows. Let $(\overline{x},\overline{y})$ be the centre of curvature corresponding to a point (x,y) on the curve. Then $\overline{x}=x-\frac{y_1(1+y_1^2)}{y_2}$ and $\overline{y}=y+\frac{(1+y_1^2)}{y_2}$, where $y_1=\frac{dy}{dx}$ and $y_2=\frac{d^2y}{dx^2}$. Also $y=f(x)$ is the given curve. By eliminating (x,y) from these three equations, we obtain a relation between $(\overline{x},\overline{y})$ which is the equation of the

evolute. The elimination will be simple, if the coordinates (x, y) of the given curve are taken parametrically.

Example 4.22. *Obtain the equation of the evolute of the curve* $x=a(\cos\theta+\theta\sin\theta), y=a(\sin\theta-\theta\cos\theta)$.

Solution: Given $x=a(\cos\theta+\theta\sin\theta)$

$$\Rightarrow \frac{dx}{d\theta}=a(-\sin\theta+\theta\cos\theta+\sin\theta)$$
$$\Rightarrow \frac{dx}{d\theta}=a\theta\cos\theta$$

and $y=a(sin\theta-\theta\cos\theta)$

$$\Rightarrow \frac{dy}{d\theta}=a(\cos\theta-\cos\theta+\theta\sin\theta)$$
$$\Rightarrow \frac{dy}{d\theta}=a\theta\sin\theta$$

Now $y_1=\frac{dy}{dx}=\frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}}=\frac{a\theta\sin\theta}{a\theta\cos\theta}\Rightarrow y_1=\tan\theta$

Also $y_2=\frac{d^2y}{dx^2}=\frac{d}{d\theta}\left(\frac{dy}{dx}\right)\left(\frac{d\theta}{dx}\right)=\frac{d}{d\theta}(\tan\theta)\frac{1}{a\theta\cos\theta}=\sec^2\theta\frac{1}{a\theta\cos\theta}$

$\Rightarrow y_2=\frac{1}{a\theta}\sec^3\theta$ Let $(\overline{x},\overline{y})$ be the corresponding centre of curvature.

$$\begin{aligned}
\overline{x} &= x - \frac{y_1(1+y_1^2)}{y_2} \\
&= a(\cos\theta + \theta\sin\theta) - \frac{\tan\theta}{\frac{1}{a\theta}\sec^3\theta}(1+\tan^2\theta) \\
&= a\cos\theta + a\theta\sin\theta - \frac{a\theta\tan\theta}{\sec^3\theta}(\sec^2\theta) \\
&= a\cos\theta + a\theta\sin\theta - \frac{a\theta\sin\theta}{\cos\theta} \\
\overline{x} &= a\cos\theta \\
\overline{y} &= y + \frac{(1+y_1^2)}{y_2} \\
&= a\sin\theta + a\theta cos\theta + \frac{1}{\frac{1}{a\theta}\sec^3\theta}(1+\tan^2\theta) \\
&= a\sin\theta - a\theta\cos\theta + a\theta\frac{\sec^2\theta}{\sec^3\theta} \\
\overline{y} &= a\sin\theta
\end{aligned}$$

We have $\frac{\overline{x}}{a} = \cos\theta; \frac{\overline{y}}{a} = \sin\theta$ Squaring and adding $\left(\frac{\overline{x}}{a}\right)^2 + \left(\frac{\overline{y}}{a}\right)^2 = cos^2\theta + sin^2\theta = 1$ is locus of $(\overline{x},\overline{y})$. (i.e.) Evolute is $x^2+y^2=a^2$

Example 4.23. *Find the evolute of the parabola $y^2 = 4ax$*

Solution: The coordinates of any point on $y^2 = 4ax$ can be taken as $x = at^2$ and $y = 2at$. (i.e.) $\frac{dx}{dt} = 2at$ and $\frac{dy}{dt} = 2a$ Then

$\frac{dy}{dx} = \frac{2a}{2at} = \frac{1}{t} = y_1$

$$\begin{aligned}
y_2 &= \frac{d^2y}{dx^2} = \frac{d}{dt}\left(\frac{dy}{dx}\right)\left(\frac{dt}{dx}\right) \\
&= \frac{d}{dt}\left(\frac{1}{t}\right)\left(\frac{dt}{dx}\right) \\
&= -\frac{1}{t^2}.\frac{1}{2at} \\
y_2 &= -\frac{1}{2at^3}
\end{aligned}$$

Let $(\overline{x},\overline{y})$ be the corresponding centre of curvature. Then $\overline{x} = x - \frac{y_1(1+y_1^2)}{y_2} = at^2 - \frac{\left(\frac{1}{t}\right)}{-\frac{1}{2at^3}}\left(1+\frac{1}{t^2}\right) = at^2 + 2at^2\left(\frac{t^2+1}{t^2}\right) = 3at^2 + 2a$

$$\Rightarrow \overline{x} = 3at^2 + 2a \tag{4.26}$$

Also $\overline{y} = y + \frac{(1+y_1^2)}{y_2} = 2at + \frac{\left(1+\frac{1}{t^2}\right)}{-\frac{1}{2at^3}} = 2at - 2at(1+t^2)$

$$\Rightarrow \overline{y} = -2at^3 \tag{4.27}$$

Eliminate $'t'$ from (4.26) and (4.27) From (1),

$$\frac{\overline{x}-2a}{3a} = t^2 \tag{4.28}$$

From (2),

$$-\frac{\overline{y}}{2a} = t^3 \tag{4.29}$$

Now $t^6 = \dfrac{(\overline{x}-2a)^3}{27a^3}$ (from (4.28)) And $t^6 = \dfrac{(\overline{y})^2}{4a^2}$ (from (4.29))
$\Rightarrow \dfrac{(\overline{x}-2a)^3}{27a^3} = \dfrac{(\overline{y})^2}{4a^2} \Rightarrow 27a(\overline{y})^2 = 4(\overline{x}-2a)^3$ Now locus of $(\overline{x},\overline{y})$ is $27ay^2 = 4(x-2a)^3$ which is the required evolute.

Example 4.24. *Find the evolute of the hyperbola* $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$

Solution: The parametric coordinates of any point on the hyperbola are $x = a\sec\theta$ and $y = b\tan\theta$. $\Rightarrow \dfrac{dx}{d\theta} = a\sec\theta\tan\theta$; $\dfrac{dy}{d\theta} = b\sec^2\theta$ Now $y_1 = \dfrac{dy}{dx} = \dfrac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \dfrac{b\sec\theta}{a\tan\theta} = \dfrac{b}{a\sin\theta}$ and $y_2 =$

$\dfrac{d^2y}{dx^2} = \dfrac{d}{d\theta}\left(\dfrac{dy}{dx}\right)\left(\dfrac{d\theta}{dx}\right) = -\dfrac{b}{a\sin^2\theta}\cos\theta\dfrac{d\theta}{dx} = -\dfrac{b\cos\theta}{a\sin^2\theta}\cdot\dfrac{\cos^2\theta}{a\sin\theta}$
$\Rightarrow y_2 = -\dfrac{b\cos^3\theta}{a^2\sin^3\theta}$ Let $(\overline{x},\overline{y})$ be the corresponding centre of curvature. $\overline{x} = x - \dfrac{y_1(1+y_1^2)}{y_2} = a\sec\theta + \dfrac{b}{a\sin\theta}\cdot\dfrac{a^2\sin^3\theta}{b\cos^3\theta}\left(1 + \dfrac{b^2}{a^2\sin^2\theta}\right) = \dfrac{a}{\cos\theta} + \dfrac{1}{a\cos^3\theta}(a^2\sin^2\theta + b^2) = \dfrac{a^2\cos^2\theta + a^2\sin^2\theta + b^2}{a\cos^3\theta}$

$$\Rightarrow \overline{x} = \frac{a^2+b^2}{a\cos^3\theta} \tag{4.30}$$

$\overline{y} = y + \dfrac{1+y_1^2}{y_2}$ $\overline{y} = b\tan\theta - \dfrac{a^3\sin^3\theta}{b\cos^3\theta}\left(1 + \dfrac{b^2}{a^2\sin^2\theta}\right) = \dfrac{b\sin\theta}{\cos\theta} - \dfrac{\sin\theta}{b\cos^3\theta}(a^2\sin^2\theta + b^2) = \dfrac{\sin\theta}{b\cos^3\theta}(b^2\cos^2\theta - a^2\sin^2\theta - b^2)$

$$= -\frac{(a^2+b^2)}{b}\tan^3\theta \tag{4.31}$$

From (1), $\sec^3\theta = \dfrac{a\overline{x}}{a^2+b^2}$ and From (2), $\tan^3\theta = -\dfrac{b\overline{y}}{a^2+b^2}$ We now use $\sec^2\theta - \tan^2\theta = 1 \Rightarrow \left(\dfrac{a\overline{x}}{a^2+b^2}\right)^{2/3} - \left(\dfrac{b\overline{y}}{a^2+b^2}\right)^{2/3} = 1$
Now locus of $(\overline{x},\overline{y})$, gives the required evolute of the hyperbola and it is given by $(ax)^{2/3} - (by)^{2/3} = (a^2+b^2)^{2/3}$.

Example 4.25. *Show that the evolute of the cycloid* $x = a(\theta - \sin\theta)$, $y = a(1-\cos\theta)$ *is another cycloid.*

Solution: Given $x = a(\theta - \sin\theta) \Rightarrow \dfrac{dx}{d\theta} = a(1-\cos\theta)$ and $y = a(1-\cos\theta) \Rightarrow \dfrac{dy}{d\theta} = a\sin\theta$ Now $y_1 = \dfrac{dy}{dx} = \dfrac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \dfrac{a\sin\theta}{a(1-\cos\theta)}$

$= \dfrac{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{2\sin^2\frac{\theta}{2}} \Rightarrow y_1 = \cot\dfrac{\theta}{2}$ and $y_2 = \dfrac{d^2y}{dx^2} = \dfrac{d}{d\theta}\left(\dfrac{dy}{dx}\right)\left(\dfrac{d\theta}{dx}\right)$
$= -\dfrac{1}{2}\csc^2\dfrac{\theta}{2}\cdot\dfrac{d\theta}{dx} = -\dfrac{1}{2}\dfrac{1}{\sin^2\frac{\theta}{2}}\cdot\dfrac{1}{2a\sin^2\frac{\theta}{2}} \Rightarrow y_2 = -\dfrac{1}{4a\sin^4\frac{\theta}{2}}$

$$\begin{aligned}\overline{x} &= x - \frac{y_1(1+y_1^2)}{y_2}\\ &= a(\theta - \sin\theta) + \cot\frac{\theta}{2}\csc^2\frac{\theta}{2}4a\sin^4\frac{\theta}{2}\\ &= a(\theta - \sin\theta) + 4a\sin\frac{\theta}{2}\cos\frac{\theta}{2}\\ &= a(\theta - \sin\theta) + 2a\sin\theta\\ &= a(\theta + \sin\theta)\end{aligned}$$

$$\Rightarrow \overline{x} = a(\theta - \sin\theta) + 2a\sin\theta = a(\theta + \sin\theta) \quad (4.32)$$

and $\overline{y} = y + \dfrac{1+y_1^2}{y_2} = a(1 - cos\theta) - \csc^2\dfrac{\theta}{2}4asin^4\dfrac{\theta}{2} = a(1 - cos\theta) - 2a(1 - cos\theta)$

$$\Rightarrow \overline{y} = -a(1 - cos\theta) \quad (4.33)$$

Now locus of $(\overline{x}, \overline{y})$, which is $x = a(\theta - \sin\theta)$, $y = a(1 - \cos\theta)$ represents another cycloid.

Example 4.26. *Find the evolute of the curve* $x^{2/3} + y^{2/3} = a^{2/3}$.

Solution: The parametric equations are $x = a\cos^3\theta$ and $y = a\sin^3\theta$. $\Rightarrow \dfrac{dx}{d\theta} = -3a\cos^2\theta\sin\theta$ and $\dfrac{dy}{d\theta} = 3a\sin^2\theta\cos\theta$ Now

$y_1 = \dfrac{dy}{dx} = \dfrac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \dfrac{3a\sin^2\theta\cos\theta}{-3a\cos^2\theta\sin\theta} \Rightarrow y_1 = -\tan\theta$ and

$$\begin{aligned} y_2 = \frac{d^2y}{dx^2} &= \frac{d}{d\theta}\left(\frac{dy}{dx}\right)\left(\frac{d\theta}{dx}\right) \\ &= -\sec^2\theta\frac{d\theta}{dx} \\ &= -\sec^2\theta\left(\frac{1}{-3a\cos^2\theta\sin\theta}\right) \\ &= \frac{1}{3a\cos^4\theta\sin\theta} \end{aligned}$$

$$\begin{aligned} \overline{x} &= x - \frac{y_1(1+y_1^2)}{y_2} \\ &= a\cos^3\theta + \frac{\tan\theta(1+\tan\theta)}{\left(\frac{1}{3a\cos^4\theta\sin\theta}\right)} \\ &= a\cos^3\theta + \frac{\sin\theta}{\cos\theta}.\frac{1}{\cos^2\theta}3a\cos^4\theta\sin\theta \\ &= a\cos^3\theta + 3a\cos\theta\sin^2\theta \end{aligned} \quad (4.34)$$

$$\begin{aligned} \overline{y} = y + \frac{1+y_1^2}{y_2} &= a\sin^3\theta + \sec^2\theta.3a\cos^4\theta\sin\theta \\ &= a\sin^3\theta + 3a\sin\theta\cos^2\theta \end{aligned} \quad (4.35)$$

Now $\overline{x} + \overline{y} = a(\cos^3\theta + \sin^3\theta) + 3a\cos\theta\sin^2\theta + 3a\sin\theta\cos^2\theta$

$$\overline{x} + \overline{y} = a(\cos\theta + \sin\theta)^3 \quad (4.36)$$

$\overline{x} - \overline{y} = a(\cos^3\theta + 3\cos\theta\sin^2\theta - 3\sin\theta\cos^2\theta - \sin^3\theta)$

$$\overline{x} - \overline{y} = a(\cos\theta - \sin\theta)^3 \quad (4.37)$$

Now $\left(\dfrac{\overline{x}+\overline{y}}{a}\right)^{2/3} + \left(\dfrac{\overline{x}-\overline{y}}{a}\right)^{2/3} = cos\theta + sin\theta)^2 + cos\theta - sin\theta)^2$
$(\overline{x} + \overline{y})^{2/3} + (\overline{x} - \overline{y})^{2/3} = 2a^{2/3} \Rightarrow$ The equation of the evolute is $(x + y)^{2/3} - (x + y)^{2/3} = 2a^{2/3}$

Example 4.27. *Prove that the evolute of the tractrix* $x = a(cost + log\tan\frac{t}{2})$, $y = a\sin$ *is the catenary* $y = a\cosh\frac{x}{a}$

Solution:

$$x = a\left(\cos t + \log\tan\frac{t}{2}\right) \quad (4.38)$$

Differentiating (4.38) w.r.t. $'t'$, we get

$$\frac{dx}{dt} = a\left(-\sin t + \frac{1}{\tan\frac{t}{2}}\sec^2\frac{t}{2}.\frac{1}{2}\right)$$

$$= a\left(-\sin t + \frac{\cos\frac{t}{2}}{\sin\frac{t}{2}}\frac{1}{\cos^2\frac{t}{2}}\frac{1}{2}\right)$$

$$= a\left(-\sin t + \frac{1}{2\sin\frac{t}{2}\cos\frac{t}{2}}\right)$$

$$= a\left(-\sin t + \frac{1}{\sin t}\right) = \frac{a(1-\sin^2 t)}{\sin t}$$

$$\frac{dx}{dt} = \frac{a\cos^2 t}{\sin t} \tag{4.39}$$

Also

$$y = a\sin t \tag{4.40}$$

$$\Rightarrow \frac{dy}{dt} = a\cos t \tag{4.41}$$

Now $y_1 = \dfrac{dy}{dx} = \dfrac{\frac{dy}{dt}}{\frac{dx}{dt}} = a\cos t \div \dfrac{a\cos^2 t}{\sin t}$

$$\Rightarrow y_1 = \tan t \tag{4.42}$$

$$y_2 = \frac{d}{dx}(y_1)$$
$$= \frac{d}{dx}(\tan t)$$
$$= \frac{d}{dt}(\tan t)\frac{dt}{dx}$$
$$= sec^2 t.\frac{\sin t}{a\cos^2 t}$$

$$= \frac{\sin t}{a\cos^4 t} \tag{4.43}$$

Let $(\overline{x}, \overline{y})$ be the corresponding centre of curvature.

$$\overline{x} = x - \frac{y_1(1+y_1^2)}{y_2}$$
$$= a(\cos t + \log\tan\frac{t}{2}) - \frac{\tan t}{\frac{\sin t}{a\cos^4 t}}(1+\tan^2 t)$$
$$= a(\cos t + \log\tan\frac{t}{2}) - \frac{\tan t}{\sin t}a\cos^4 t\sec^2 t$$
$$= a(\cos t + \log\tan\frac{t}{2}) - a\cos t$$

$$\overline{x} = a\log\tan\frac{t}{2} \tag{4.44}$$

$$\begin{aligned}
\overline{y} &= y + \frac{1+y_1^2}{y_2} \\
&= a\sin t + \frac{1+\tan^2 t}{\dfrac{\sin t}{a\cos^4 t}} \\
&= a\sin t + \frac{a\cos^4 t\sec^2 t}{\sin t} \\
&= a\sin t + \frac{a\cos^2 t}{\sin t} \\
&= \frac{a(\sin^2 + \cos^2 t)}{\sin t}
\end{aligned}$$

$$\overline{y} = \frac{a}{\sin t} \tag{4.45}$$

To get the evolute, eliminate $'t'$ between (4.44) and (4.45). From (4.44), $\log\tan\frac{t}{2} = \frac{\overline{x}}{a}$ and so

$$\tan\frac{t}{2} = e^{\frac{\overline{x}}{a}} \tag{4.46}$$

Equation (4.45) can be expressed interms of $\tan\frac{t}{2}$ From (4.45),

$$\overline{y} = \frac{a}{\left(\dfrac{2\tan\frac{t}{2}}{1+2\tan^2\frac{t}{2}}\right)} = \frac{a\left(1+\tan^2\frac{t}{2}\right)}{2\tan\frac{t}{2}} = a\left(\frac{1+e^{\frac{2\overline{x}}{a}}}{2e^{\frac{\overline{x}}{a}}}\right)$$

substituting from (4.46) $= \frac{a}{2}\left(\frac{1}{e^{\frac{\overline{x}}{a}}} + e^{\frac{\overline{x}}{a}}\right) = \frac{a}{2}\left(e^{\frac{\overline{x}}{a}} + e^{\frac{\overline{x}}{a}}\right)$ $\overline{y} = a\cosh\frac{\overline{x}}{a}$ Locus of $(\overline{x},\overline{y})$ is $y = a\cosh\frac{x}{a}$ which is the well known catenary.

4.4 Envelopes

The equation $xcos\alpha + ysin\alpha = 1 \cdots (1)$ represents a straight line for a given value of α. If different values are given to α, we get different straight lines. All these straight lines are said to constitute a family of straight lines. In general, the curves corresponding to the equation $f(x,y,\alpha) = 0$ for different values of α, constitute a family of curves and α is called the parameter of the family.

The envelope of a family of curves is the curve which touches each member of the family. For example, we know that all the straight lines of the family (1) touch the circle $x^2 + y^2 = 1 \cdots (2)$. (i.e) The envelope of the family of lines (1) is the circle (2), which may also seen as the locus of the ultimate points of intersection of the consecutive members of the family of lines (1). This leads to the following: **Definition:** If $f(x,y,\alpha) = 0$ and $f(x,y,\alpha+\delta\alpha) = 0$ be two consequtive members of a family of curves, then the locus of their ultimate points of intersection is called the envelope of that family.

4.4.1 Rule to find the envelope of the family of curves $f(x,y,\alpha) = 0$:

Eliminate α from $f(x,y,\alpha) = 0$ and $\frac{\partial f(x,y,\alpha)}{\partial\alpha} = 0$ **Remarks:**

1. Let the equation of the family of curves be $A\alpha^2+B\alpha+C=0\cdots(1)$, which is quadratic in α. Then $B^2-4AC=0$ is the equation of the envelope of the family of (1).

2. Evolute of a curve is the envelope of the normals of that curve.

Example 4.28. *Find the envelope of the family of straight lines $y=mx\pm\sqrt{a^2m^2+b^2}$.*

Solution: The given equation is $y=mx\pm\sqrt{a^2m^2+b^2}$ Squaring both sides, we get $(y-mx)^2=a^2m^2+b^2$ (i.e.) $m^2(x^2-a^2)-2mxy+(y^2-b^2)=0$ This is a quadratic in m. So the envelope is $B^2-4AC=0$ (i.e) discriminant $=0$ Hence the envelope is $(-2xy)^2-4(x^2-a^2)(y^2-b^2)=0$ (i.e.) $x^2y^2-(x^2-a^2)(y^2-b^2)=0$ $x^2y^2-(x^2y^2-b^2x^2-a^2y^2+a^2b^2)=0$ (i.e.) $b^2x^2+a^2y^2=a^2b^2$ Dividing by a^2b^2, $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$, which is the required envelope.

Example 4.29. *Find the envelope of the straight line $x\cos\alpha+y\sin\alpha=a\sin\alpha\cos\alpha$, α being the parameter.*

Solution: The given equation is $x\cos\alpha+y\sin\alpha=a\sin\alpha\cos\alpha$ Dividing by $\sin\alpha\cos\alpha$, we have

$$x\csc\alpha+y\sec\alpha=a \tag{4.47}$$

Partially differentiating (1), w.r.t. α, we have $-x\csc\alpha\cot\alpha+y\sec\alpha\tan\alpha=0$ (i.e.) $\dfrac{x\cos\alpha}{\sin^2\alpha}+\dfrac{y\sin\alpha}{\cos^2\alpha}=0$ (or) $y\sin^3\alpha=$

$x\cos^3\alpha\Rightarrow\tan^3\alpha=\dfrac{x}{y}$ (or)

$$\tan\alpha=\frac{x^{1/3}}{y^{1/3}} \tag{4.48}$$

Eliminate 'α' from (4.47) and (4.48)

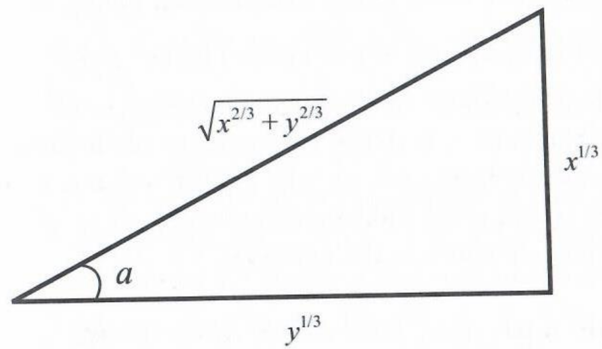

Fig. 4

From the figure, $\csc\alpha=\dfrac{\sqrt{x^{2/3}+y^{2/3}}}{x^{1/3}}$ and $\sec\alpha=\dfrac{\sqrt{x^{2/3}+y^{2/3}}}{y^{1/3}}$

Putting these values in (1), we get $x\dfrac{\sqrt{x^{2/3}+y^{2/3}}}{x^{1/3}}+y\dfrac{\sqrt{x^{2/3}+y^{2/3}}}{y^{1/3}}=a$ (or) $(x^{2/3}+y^{2/3})\sqrt{x^{2/3}+y^{2/3}}=a$ (i.e.) $(x^{2/3}+y^{2/3})^{3/2}=a$ (or) $x^{2/3}+y^{2/3}=a^{2/3}$ is the envelope.

Example 4.30. *Find its envelope of the straight line $\dfrac{x}{a}+\dfrac{y}{b}=1$ where a and b are connected by the relation $a+b=c$, c being a constant.*

Solution: The given equation is

$$\frac{x}{a}+\frac{y}{b}=1 \tag{4.49}$$

where

$$a+b=c \tag{4.50}$$

From (4.50), $b=c-a$ and putting this in (4.49), we have $\frac{x}{a}+\frac{y}{c-a}=1$ (i.e.) $x(c-a)+ay=a(c-a)$ (or) $a^2+a(y-x-c)+cx=0$ This is a quadratic in $'a'$. So the envelope is $B^2-4AC=0$ (i.e) discriminant $=0$ Hence the envelope of the family is $(y-x-c)^2-4cx=0$ (or) $y-x-c=\pm2\sqrt{cx}$ (or) $y=x+c\pm2\sqrt{cx}$ (or) $y=(\sqrt{c}\pm\sqrt{x})^2$ Taking square root, $\sqrt{y}=\sqrt{c}-\sqrt{x}$ (or) $\sqrt{x}+\sqrt{y}=\sqrt{c}$ which is the envelope.

Example 4.31. *Find the envelope of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, where a and b are connected by the relation $a^2+b^2=c^2$, c being a constant.*

Solution: The given equation is

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \tag{4.51}$$

where

$$a^2+b^2=c^2 \tag{4.52}$$

From (2), $b^2=c^2-a^2$ and putting this in (1), we get $\frac{x^2}{a^2}+\frac{y^2}{c^2-a^2}=1$ (i.e) $x^2(c^2-a^2)+a^2y^2=a^2(c^2-a^2)$ (or) $a^4+a^2(y^2-x^2-c^2)+c^2x^2=0$ Put $a^2=\lambda\Rightarrow\lambda^2+\lambda(y^2-x^2-c^2)+c^2x^2=0$

which is a quadratic in λ. So the envelope is $B^2-4AC=0$ (i.e) discriminate $=0$ Hence the envelope of the family is $(y^2-x^2-c^2)^2-4c^2x^2=0$ (i.e.) $(y^2-x^2-c^2+2cx)(y^2-x^2-c^2-2cx)=0$ (i.e.) $y^2=x^2+c^2-2cx$ (or) $y^2=x^2+c^2+2cx$ (i.e.) $(x-c)^2=y^2$ (or)$(x+c)^2=y^2$ Taking square root, $x-c=\pm y$ (or)$x+c=\pm y$ (i.e.) $x\pm c=\pm y$ which can also be written as $x\pm y=\pm c$

Example 4.32. *Find the envelope of the straight lines represented by the equation $x\cos\alpha+y\sin\alpha=a\sec\alpha$ (α is the parameter).*

Solution: The equation to the given family is $x\cos\alpha+y\sin\alpha=a\sec\alpha$ Dividing throughout by $\cos\alpha$, we get $x+y\tan\alpha=\frac{a\sec\alpha}{\cos\alpha}=a\sec^2\alpha=a(1+\tan^2\alpha)$ (i.e) $a\tan^2\alpha-y\tan\alpha+(a-x)=0$ This is a quadratic in $\tan\alpha$ So the envelope is $B^2-4AC=0$ (i.e) discriminate $=0$ Hence the envelope of the family is $(-y)^2-4a(a-x)=0$ (i.e) $y^2-4a(a-x)=0$

Example 4.33. *Find the envelope of the family of lines $y=mx-2am-am^3$*

Solution: We have

$$f(x,y,m)=y-mx-am^3 \tag{4.53}$$

Hence $f_m(x,y,m)=-x-3am^3=0$ (or) $m^2=-\frac{x}{3a}$ From (1),$y=mx+am^3=m(x+am^2)$ Squaring, $y^2=m^2(x+am^2)^2$ Putting $m^2=-\frac{x}{3a}$ in the above step, $y^2=-\frac{x}{3a}\left(x-\frac{ax}{3a}\right)^2$

(i.e) $y^2 = -\frac{x}{3a}\left(\frac{3ax - ax}{3a}\right)^2$ (or) $y^2 = -\frac{x}{3a}\left(\frac{2ax}{3a}\right)^2$ (or) $y^2 = -\frac{x}{3a}\cdot\frac{4x^2}{9}$ (or) $27ay^2 + 4x^3 = 0$ is the required envelope.

Example 4.34. *Find the envelope of the family of straight lines $\frac{x}{a} + \frac{y}{b} = 1$, where a and b are connected by the relation $ab = c^2$, c being a constant.*

Solution: We have

$$\frac{x}{a} + \frac{y}{b} = 1 \tag{4.54}$$

and

$$ab = c^2 \tag{4.55}$$

Differentiating (4.54) and (4.55) with respect to the parameter b regarding $'a'$ as a function of b, we obtain $-\frac{x}{a^2}\frac{da}{db} - \frac{y}{b^2} = 0$ (i.e.)

$$\frac{x}{a^2}\frac{da}{db} + \frac{y}{b^2} = 0 \tag{4.56}$$

From $ab = c^2$, we have $b\frac{da}{db} + a = 0$ (or) $\frac{da}{db} = -\frac{a}{b}$ Using this value in (4.56) $\frac{x}{a^2}\left(-\frac{a}{b}\right) + \frac{y}{b^2} = 0$ (or) $\frac{x}{ab} = \frac{y}{b^2}$ (or) $\frac{x}{a} = \frac{y}{b} = k$ (say) $\Rightarrow \frac{x}{a} + \frac{y}{b} = 2k = 1$ (or) $k = \frac{1}{2} \Rightarrow a = 2x, b = 2y$ using this in (2), $4xy = c^2$ is the required envelope.

Example 4.35. *Find the envelope of the family of lines $x\cos^3\alpha + y\sin^3\alpha = a$, the parameter being α.*

Solution:

$$x\cos^3\alpha + y\sin^3\alpha = a \tag{4.57}$$

Differentiating (4.57) with respect to $'\alpha'$, we get $-3x\cos^2\alpha\sin\alpha + 3y\sin^2\alpha\cos\alpha = 0$ (or) $x\cos\alpha - y\sin\alpha = 0$ (or) $\tan\alpha = \frac{x}{y}$ $\Rightarrow \sin\alpha = \frac{x}{\sqrt{x^2+y^2}}$ and $\cos\alpha = \frac{y}{\sqrt{x^2+y^2}}$ Substituting these values of $\cos\alpha$ and $\sin\alpha$ in (4.57), we get $x.\frac{y^3}{(x^2+y^2)^{3/2}} + y.\frac{x^3}{(x^2+y^2)^{3/2}} = a$ (or) $xy^3 + yx^3 = a(x^2+y^2)^{3/2}$ (or) $xy(x^2+y^2) = a(x^2+y^2)^{3/2}$ (i.e.) $xy = a(x^2+y^2)^{1/2}$ Squaring both sides we have $x^2y^2 = a^2(x^2+y^2)$, which is the required envelope.

Example 4.36. *Find the envelope of a system of concentric and coaxial ellipses of constant area.*

Solution: Let the equation of the ellipse be

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \tag{4.58}$$

where a and b are such that $\pi ab = k$ (or)

$$ab = \frac{k}{\pi} = c^2 \tag{4.59}$$

Differentiating (4.58) and (4.59) w.r.t $'b'$ regarding $'a'$ as a function of $'b'$, we get

$$-\frac{2x^2}{a^3}\frac{da}{db} - \frac{2y^2}{b^3} = 0 \tag{4.60}$$

and

$$a + b\frac{da}{db} = 0 \tag{4.61}$$

Equating the values of $\frac{da}{db}$ from (4.60) and (4.61) $\frac{y^2a^3}{x^2b^3} = \frac{a}{b}$ (or)

$$\frac{x^2}{a^2} = \frac{y^2}{b^2} \quad (4.62)$$

From (4.58) and (4.62), $\frac{x^2}{a^2} = \frac{1}{2}, \frac{y^2}{b^2} = \frac{1}{2}$ (or) $a = \sqrt{2}x$ and $b = \sqrt{2}y$ Substituting these values in (2), the envelope is $2xy = c^2$

Example 4.37. *Find the envelope of the family of parabolas given by* $y = x\tan\alpha - \frac{gx^2}{2u^2\cos^2\alpha}$*, where 'α' is the parameter.*

Solution: $y = x\tan\alpha - \frac{gx^2}{2u^2\cos^2\alpha}$ Putting $t = \tan\alpha$, we get $\frac{gx^2}{2u^2}(1+t^2) - xt + y = 0$ (i.e.) $gx^2t^2 - 2u^2xt + (gx^2 + 2u^2y) = 0$
If we treat 't' as the parameter, we see that this equation is the quadratic equation in the parameter. ⇒ The envelope is given by discriminant $B^2 - 4AC = 0 \Rightarrow 4u^4x^2 - 4gx^2(gx^2 + 2u^2y) = 0$
(or) $g^2x^2 + 2u^2gy - u^4 = 0$ (or) $x^2 = -\frac{2u^2}{g}\left(y - \frac{u^2}{g}\right)$

EXERCISE

1. Define evolute and involute.

2. Define envelope of a family of curves.

3. Find the envelope of the family of lines $\frac{x}{t} + yt = 2c$, 't' being the parameter.

4. Find the envelope of the lines $\frac{x}{a}\cos\theta + \frac{y}{b}\sin\theta = 1$, 'θ' being the parameter.

5. If the centre of curvature of a curve at a variable point 't' on it is $(2a + 3at^2, -2at^3)$, find the evolute of the curve.

6. Find the envelope of the family of lines $x\cos\alpha + y\sin\alpha = \rho$, 'α' being the parameter.

7. Find the envelope of the family of lines $y = mx + \frac{a}{m}$, m being the parameter.

8. Find the envelope of the family of lines $\frac{x}{a}\sec\theta - \frac{y}{b}\tan\theta = 1$, 'θ' being the parameter.

9. Find the evolute of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ [Ans: $(ax)^{2/3} - (by)^{2/3} = (a^2 + b^2)^{2/3}$]

10. Find the evolute of the parabola $x^2 = 4ay$ [Ans: $27ax^2 = 4(y - 2a)^3$]

11. Find the evolute of the rectangular hyperbola $xy = c^2$ [Ans: $(x + y)^{2/3} - (x - y)^{2/3} = (4c)^{2/3}$]

12. Show that evolute of the cycloid $x = a(\theta + \sin\theta), y = a(1 - \cos\theta)$ is another cycloid, given by $x = a(\theta - \sin\theta), y - 2a = a(1 + \cos\theta)$.

13. Find the evolute of the parabola $y^2 = 4ax$, considering it

as envelope of its normals. [Ans: $27ay^2 = 4(x-2a)^3$]

14. Find the envelope of the family of straight lines $y\cos\theta$ - $x\sin\theta = a\cos 2\theta$, '$\theta$' being the parameter. [Ans: $(x+y)^{2/3}$ - $(x-y)^{2/3} = 2a^{2/3}$]

15. Find the envelope of the system $\frac{x}{l} + \frac{y}{m} = 1$, where l and m are connected by the relation $\frac{l}{a} + \frac{m}{b} = 1$ (l and m are the parameters) [Ans: $\sqrt{\frac{x}{a}} + \sqrt{\frac{y}{b}} = 1$]

16. Find the envelope of $\frac{x}{a} + \frac{y}{b} = 1$ when $a^2 + b^2 = c^2$ [Ans: $x^{2/3} + y^{2/3} = c^{2/3}$] [June 2011]

5 THREE DIMENSIONAL ANALYTICAL GEOMETRY

5.1 Introduction

Let XOX', YOY', ZOZ' be any three mutually perpendicular lines through a point O. The point O is called the origin and the lines XOX', YOY' and ZOZ' are called rectangular co-ordinate axes or simply co-ordinate axes. The positive directions of these axes are indicated by arrow heads. The lines XOX', YOY', ZOZ' are simply called x-axis, y-axis and z-axis respectively. The three planes XOY, YOZ and ZOX which are mutually perpendicular are called the co-ordinate planes.

The position of a point P in space is determined by its distances from a fixed point (origin) measured parallel to OX, OY and OZ.

Let x, y, z be the distances of the point P from O measured parallel to OX, OY and OZ respectively. Since OX, OY and OZ are mutually perpendicular, the distances of P from O measured parallel to the axes are nothing but the perpendicular distances