# Linear Differential Equations with Variable Coefficients

We will now study two types of linear differential equations with variables coefficients which can be reduced to linear differential equations with constant coefficients by suitable substitution.

**a. Cauchy's homogeneous linear equation (Euler type)**

An equation of the form

$$x^n \frac{d^n y}{dx^n} + a_1 x^{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_n y = F(x) \tag{1}$$

where $a_1, a_2, \cdots a_n$ are constants and $F(x)$ is a function of $x$ is called Cauchy's (Euler's) homogeneous linear differential equation. Equation (1) can be transformed to a linear differential equation with constant coefficients by the transformation

$x = e^z$ (or) $z = \log x$ and $\dfrac{dz}{dx} = \dfrac{1}{x}$

Now $\dfrac{dy}{dx} = \dfrac{dy}{dz} = \dfrac{dz}{dx} = \dfrac{dy}{dz}.\dfrac{1}{x} \Rightarrow x\dfrac{dy}{dx} = \dfrac{dy}{dz}$

Hence $xDy = D'y$ where $D = \dfrac{d}{dx}, D' = \dfrac{d}{dz}$ (2)

Also $\dfrac{d^2y}{dx^2} = \dfrac{d}{dx}\left(\dfrac{dy}{dx}\right) = \dfrac{d}{dx}\left(\dfrac{1}{x}\dfrac{dy}{dz}\right)$

$$= \frac{1}{x}\frac{d}{dx}\left(\frac{dy}{dz}\right) - \frac{1}{x^2}\frac{dy}{dz}$$

$$= \frac{1}{x}\frac{d}{dz}\left(\frac{dy}{dz}\right)\frac{dz}{dx} - \frac{1}{x^2}\frac{dy}{dz}$$

$$= \frac{1}{x}\frac{d^2y}{dz^2}\frac{1}{x} - \frac{1}{x^2}\frac{dy}{dz}$$

$$\frac{d^2y}{dx^2} = \frac{1}{x^2}\frac{d^2y}{dz^2} - \frac{1}{x^2}\frac{dy}{dz} = \frac{1}{x^2}\left(\frac{d^2y}{dz^2} - \frac{d^2y}{dx^2}\right)$$

$$\Rightarrow x^2\frac{d^2y}{dx^2} = \frac{d^2y}{dz^2} - \frac{dy}{dz}$$

That is $x^2D^2y = D'^2y - D'y = (D'^2 - D')y$

$$x^2D^2y = D'(D'-1)y \tag{3}$$

$$\text{Similarly, } x^3D^3y = D'(D'-1)(D'-2)y \tag{4}$$

Substituting (2), (3), (4) and so on in (1) we get a linear differential equation with constant coefficients and can be solved by any one of the known method.

Example 1: $Solve: \dfrac{d^2y}{dx^2} + \dfrac{1}{x}\dfrac{dy}{dx} = \dfrac{12\log x}{x^2}$

**Solution:** Given $\dfrac{d^2y}{dx^2} + \dfrac{1}{x}\dfrac{dy}{dx} = \dfrac{12\log x}{x^2}$

Multiplying throughout by $x^2$, we have

$$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} = 12\log x$$

(i.e.) $(x^2D^2 + xD)y = 12\log x$ (5)

Let $x = e^z$ (or) $z = \log x$ so that $xD = D'$, $x^2D^2 = D'(D' - 1)$,

where $D = \dfrac{d}{dx}, D' = \dfrac{d}{dz}$

Now equation (5) becomes $(D(D'-1)+D')y = 12z$ $(D'^2 - D' + D')y = 12z$
$\Rightarrow D'^2 y = 12z$

$$\Rightarrow \frac{d^2y}{dz^2} = 12z$$

Integrating w. r. to $z$, we have

$$\frac{dy}{dz} = 12\frac{z^2}{2} + C_1 \text{ and } y = 6\frac{z^3}{3} + C_1 z + C_2$$

$y = 2z^2 + C_1 z + C_2$. But $z = \log x$, so that
$y = (2\log x)^3 + C_1 z + C_2$ is the required solution.

Example 2: $Solve\text{:}\ x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + y = 4\sin(\log x)$

Solution:

Given $(x^2D^2 + xD + 1)y = 4\sin(\log x)$ (6)

Let $x = e^z$ (or) $z = \log x$ so that $xD = D'$, $x^2D^2 = D'(D' - 1)$,

where $D = \dfrac{d}{dx}, D' = \dfrac{d}{dz}$

Now equation (6)

$$(D(D'-1) + D' + 1)y = 4\sin(z)$$
$$\Rightarrow (D'^2 - D' + D' + 1)y = 4\sin z$$
$$\Rightarrow (D'^2 + 1)y = 4\sin z \qquad (7)$$

(i.e.) $\phi(D')y = F(z)$

We have to solve equation (7). The auxiliary equation is $m^2 + 1 = 0$

$\Rightarrow m^2 = -1 \Rightarrow m = \pm i \Rightarrow m = 0 \pm i = \alpha \pm i\beta \Rightarrow \alpha = 0, \beta = 1$

Roots are imaginary.

$$\Rightarrow C.F. = C_1 \cos z + C_2 \sin z$$

$$\text{Now } P.I. = \frac{1}{\phi(D')} F(z) = \frac{1}{D'^2 + 1} 4 \sin(z)$$

$$= z.\frac{1}{2D} 4 \sin(z) = 2z.\frac{1}{D'} \sin z = 2z.\frac{D'}{D'^2} \sin z$$

$$= 2z.\frac{D'}{-1} \sin z \ (\text{since} \quad D'^2 = -1, D'^2 + 1 = 0, \ (\text{i.e.}) \ Dr = 0)$$

$$= -2z \cos z$$

The complete solution of (7) is $y = C_1 \cos z + C_2 \sin z - 2z \cos z$

Then the required solution is

$y = C_1 \cos(\log x) + C_2 \sin(\log x) - 2(\log x) \cos(\log x)$

## b. Homogeneous Equations of Legendre's Type

An equation of the form

$$(ax+b)^n \frac{d^n y}{dx^n} + p_1 (ax+b)^{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \cdots + p_n y = F(x) \quad (8)$$

where $p_1, p_2, \cdots p_n$ are constants, is known as Legendre linear differential equation.

Equation (8) can be reduced to linear differential equation with constant coefficients by putting $ax + b = e^z$

$$\text{(or)} \;\; z = \log(ax+b) \text{ so that } \frac{dz}{dy} = \frac{a}{ax+b}.$$

Now $\frac{dy}{dx} = \frac{dy}{dz}.\frac{dz}{dx} = \frac{dy}{dz}.\frac{a}{ax+b}$

(i.e.) $(ax+b)\frac{dy}{dx} = a\frac{dy}{dz} \Rightarrow (ax+b)D = aD'$, where $D = \frac{d}{dx}$ and $D' = \frac{d}{dz}$

Similarly $(ax + b)^2D^2 = a^2D'(D' - 1)$,
$(ax + b)^3D^3 = a^3D'(D' - 1)(D' - 2)$ and so on.
Substituting these in (8), we get a linear differential equation with constant coefficients which can be solved by any one of the known methods.

Example 1: *Solve:* $(2x+5)^2\frac{d^2y}{dx^2} - 6(2x+5)\frac{dy}{dx} + 8y = 0$

Solution: $(2x+5)^2\frac{d^2y}{dx^2} - 6(2x+5)\frac{dy}{dx} + 8y = 0$

$$\text{(i.e.) } [(2x+5)^2 D^2 - 6(2x+5)D + 8]y = 0 \qquad (9)$$

Let $(2x+5) = e^z$ (or) $z = \log(2x+5)$ so that $(2x+5)D = 2D'$

$$(2x+5)^2 D^2 = 2^2 D'(D'-1), \text{ where } D = \frac{d}{dx} \text{ and } D' = \frac{d}{dz}$$

Now equation (9) becomes

$[4D'(D'-1) - 6.2D' + 8]y = 0$

$[4D'^2 - 4D' - 12D' + 8]y = 0$

$$\text{(i.e.) } [4D'^2 - 16D' + 8]y = 0 \qquad (10)$$

The auxiliary equation of (10) is $4m^2 - 16m + 8 = 0$

$$\Rightarrow m^2 - 4m + 2 = 0 \Rightarrow m = \frac{4 \pm \sqrt{16-8}}{2}$$

$$= \frac{4 \pm \sqrt{8}}{2} = \frac{4 \pm 2\sqrt{2}}{2} = 2 \pm \sqrt{2}$$

$$\Rightarrow m = 2+\sqrt{2}, m - \sqrt{2}$$

$$\Rightarrow C.F. = C_1 e^{(2+\sqrt{2})z} + C_2 e^{(2-\sqrt{2})z} \text{ and } P.I. = 0$$

The complete solution of (10) is $y = CF + PI$.

$$y = C_1 e^{(2+\sqrt{2})z} + C_2 e^{(2-\sqrt{2})z}$$

$\Rightarrow$ The complete solution of (9) is

$$y = (2x+5)^{(2+\sqrt{2})} + (2x+5)^{(2-\sqrt{2})}$$

Example 2: $Solve{:}\ (1+x)^2 \frac{d^2y}{dx^2} + (1+x)\frac{dy}{dx} + y = 4\cos[\log(1+x)]$

Solution: Given $(1+x)^2 \frac{d^2y}{dx^2} + (1+x)\frac{dy}{dx} + y = 4\cos[\log(1+x)]$

(i.e.) $[(1+x)^2D^2 + (1+x)D + 1]y = 4\cos[\log(1+x)]$ (11)

Let $(1+x) = e^z$ (or) $z = \log(1+x)$ so that $(1+x)D = 2D'$

$$(1+x)^2D^2 = 2^2D'(D'-1), \text{ where } D = \frac{d}{dx} \text{ and } D' = \frac{d}{dz}.$$

Now equation (11) becomes

$[D'(D'-1)+D'+1]y = 4\cos z$

(i.e.) $(D'^2+1)y = 4\cos z$

(i.e.) $\phi(D')y = F(z)$ (12)

The auxiliary equation of (12) is $m^2+1=0$

$\Rightarrow m = \pm i = 0 \pm i = \alpha \pm i\beta \Rightarrow \alpha = 0, \beta = 1$

$\Rightarrow C.F. = C_1 \cos z + C_2 \sin z$

Now $P.I. = \dfrac{1}{\phi(D')}F(z) = \dfrac{1}{D'^2+1}4\cos z$

$= z.\dfrac{1}{2D'}4\cos z = 2z.\dfrac{1}{D'}\cos z = 2z\int \cos z dz = 2z\sin z$

The complete solution of (12) is $y = C_1 \cos z + C_2 \sin z + 2z \sin z$

Hence the complete solution of (11) is

$y = C_1 \cos[\log(1+x)] + C_2 \sin[\log(1+x)] + 2[\log(1+x)]\sin[\log(1+x)]$

## EXERCISE

1. Solve: $x^2\dfrac{d^2y}{dx^2} + x\dfrac{dy}{dx} - 3y = x^2(\log x)$ [Ans: $y = C_1x^3 + \dfrac{C_2}{x} - \dfrac{x^2}{9}(3\log x + 2)$]

2. Solve: $x^2\dfrac{d^2y}{dx^2} - 3x\dfrac{dy}{dx} + y = \dfrac{\sin(\log x)}{x}$ [Ans: $y = x^2(C_1x^{\sqrt{3}} + C_2x^{-\sqrt{3}}) + \dfrac{1}{61x}[5\sin(\log x) + 6\cos(\log x)]$]

3. Solve: $x^2\dfrac{d^2y}{dx^2} - 3x\dfrac{dy}{dx} = x + 11$ [Ans: $y = C_1 + C_2x^4 - \left(\dfrac{x}{3} + \dfrac{11}{4}\log x + \dfrac{11}{16}\right)$]

## METHOD OF VARIATION OF PARAMETERS

This method is very useful for finding the particular integral of a second order linear differential equation whose complementary function is known.

Consider the equation
$$\frac{d^2y}{dx^2} + a_1\frac{dy}{dx} + a_2y = F(x) \tag{13}$$

where $a_1$, $a_2$ are constants, $F(x)$ is a function of $x$. Let the complementary function of (3.24) is $C.F. = C_1f_1 + C_2f_2$

where $C_1$, $C_2$ are constants and $f_1$, $f_2$ are functions of $x$.

Then $PI = Pf_1 + Qf_2$ (14)

Where
$$P = -\int \frac{f_2}{f_1f_2' - f_2f_1'}F(x)dx \tag{15}$$

And
$$Q = \int \frac{f_1}{f_1f_2' - f_2f_1'}F(x)dx \tag{16}$$

Substituting (15) and (16) in (14), we get the PI. Hence the complete solution is $y = CF + PI$.

Example 1: Solve $\frac{d^2y}{dx^2} + y = \sec x$ e method of variation of parameters.

Solution: Given $\frac{d^2y}{dx^2} + y = \sec x$

(i.e.) $(D^2 + 1)y = \sec x$

(i.e.) $\phi(D)y = F(x)$

The auxiliary equation is $m^2 + 1 = 0 \Rightarrow m = \pm i = 0 \pm i$

$\Rightarrow \alpha \pm i\beta = 0 \pm i \Rightarrow \alpha = 0, \beta = 1$

$\Rightarrow C.F. = C_1 \cos x + C_2 \sin x = C_1 f_1 + C_2 f_2$

Here $f_1 = \cos x$, $f_2 = \sin x$ so that $f_1' = -\sin x$, $f_2' = \cos x$

$\Rightarrow f_1 f_2' - f_2 f_1' = \cos^2 x + \sin^2 x = 1$

Let $PI = Pf_1 + Qf_2 = P\cos x + Q\sin x$ where

$$P = -\int \frac{f_2}{f_1 f_2' - f_2 f_1'} F(x)dx$$

$$= -\int \frac{\sin x}{1} \sec x dx = -\int \tan x dx$$

$$= \int \frac{-\sin x}{\cos x} dx = \log(\cos x)$$

$$\text{and } Q = \int \frac{f_1}{f_1 f_2' - f_2 f_1'} F(x)dx$$

$$= \int \frac{\cos x}{1} \sec x dx = \int dx = x$$

$\Rightarrow P.I. = \cos x \log(\cos x) + x \sin x$

Hence the complete solution is $y = CF + PI$.

$y = C_1 \cos x + C_2 \sin x + \cos x \log(\cos x) + x \sin x$

Example 1: Solve $\frac{d^2y}{dx^2} + 4y = 4\tan 2x$ method of variation of parameters.

Solution: Given $\frac{d^2y}{dx^2} + 4y = 4\tan 2x$

(i.e.) $(D^2 + 4)y = 4\tan 2x$

(i.e.) $\phi(D)y = F(x)$

The auxiliary equation is $m^2 + 4 = 0 \Rightarrow m = \pm 2i = 0 \pm 2i$

$\Rightarrow \alpha \pm i\beta = 0 \pm 2i \Rightarrow \alpha = 0, \beta = 2$

$\Rightarrow C.F. = C_1 \cos 2x + C_2 \sin 2x = C_1 f_1 + C_2 f_2$

Here $f_1 = \cos 2x, f_2 = \sin 2x$

so that $f_1' = -2\sin 2x, f_2' = 2\cos 2x$

$\Rightarrow f_1 f_2' - f_2 f_1' = 2\cos^2 2x + 2\sin^2 2x = 2$

Let $PI = Pf_1 + Qf_2$ where

$$P = -\int \frac{f_2}{f_1 f_2' - f_2 f_1'} F(x) dx$$

$$= -\int \frac{\sin 2x}{2} 4 \tan 2x dx$$

$$= -2\int \frac{\sin^2 2x}{\cos 2x} dx = -2\int \frac{(1-\cos^2 2x)}{\cos 2x} dx$$

$$= -2\int \left( \frac{1}{\cos 2x} - \frac{\cos^2 2x}{\cos 2x} \right) dx$$

$$= -2\int (\sec 2x - \cos 2x) dx = -2\int \sec 2x dx + 2\int \cos 2x dx$$

$$= -2\left(\frac{1}{2}\right) \log(\sec 2x + \tan 2x) + 2\left(\frac{\sin 2x}{2}\right)$$

$$= -\log(\sec 2x + \tan 2x) + \sin 2x$$

and $Q = \int \frac{f_1}{f_1 f_2' - f_2 f_1'} F(x) dx$

$$= \int \left( \frac{\cos 2x}{2} \right) 4 \tan 2x dx$$

$$= 2 \int \sin 2x dx = -2 \left( \frac{\cos 2x}{2} \right) = -\cos 2x$$

$\Rightarrow P.I. = -\cos 2x \log(\sec 2x + \tan 2x) + \sin 2x \cos 2x - \sin 2x \cos 2x$

$= -\cos 2x \log(\sec 2x + \tan 2x)$

Hence the complete solution is $y = CF + PI$.

$y = C_1 \cos 2x + C_2 \sin 2x - \cos 2x \log(\sec 2x + \tan 2x)$

## EXERCISE

1. Explain the method of variation of parameters.

2. Solve: $y'' + y = \tan x$ by the method of variation of parameters. **SRM June 2006, Nov 2007** [Ans: $y = C_1 \cos x + C_2 \sin x - \cos x \log(\sec x + \tan x)$]

3. Solve: $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = e^x \log x$ by the method of variation of parameters. [Ans: $y = (C_1 + C_2 x)e^x + \frac{e^x x^2}{4}(2\log x - 3)$]

# Simultaneous Linear Differential Equations with Constant Coefficients

Here we discuss differential equations in which there is one independent variable and two or more dependent variables. Such equations are termed as simultaneous equations. Here we consider only a system of linear differential equations with constant coefficients. We shall discuss the solution of these equations in the same manner as we do in the case of simultaneous linear algebraic equations.

**Example 1:** Solve the simultaneous linear differential equations:

$$\frac{dx}{dt} + 7x - y = 0; \frac{dy}{dt} + 2x + 5y = 0$$

**Solution:** Given $\frac{dx}{dt} + 7x - y = 0; \frac{dy}{dt} + 2x + 5y = 0$

(i.e.) $(D + 7)x - y = 0$ (1)

and $(D + 5)y + 2x = 0$ (2)

Now (2) + (D+5) (1), we have $2x + (D + 5)y = 0$

$(D + 5)(D + 7)x - (D + 5)y = 0$

$(D + 5)(D + 7)x + 2x = 0$

That is $(D^2+12D+35+2)x = 0 \Rightarrow (D^2+12D+37)x = 0$ (3)

The auxiliary equation of (3) is $m^2 + 12m + 37 = 0$

$$\Rightarrow m = \frac{-12 \pm \sqrt{144 - 148}}{2} = \frac{-12 \pm i2}{2} = -6 \pm i = \alpha \pm i\beta$$

$$\Rightarrow C.F. = e^{-6t}(C_1 \cos t + C_2 \sin t) \text{ and } P.I. = 0$$

The complete solution of (3) is

$$x = e^{-6t}(C_1 \cos t + C_2 \sin t)$$

Now $\frac{dx}{dt} + 7x - y = 0$ we have $y = \frac{dx}{dt} + 7x$ (5)

From equation (4), $\frac{dx}{dt} = e^{-6t}(-C_1 \sin t + C_2 \cos t)$

$$-6e^{-6t}(C_1 \cos t + C_2 \sin t)$$
$$y = e^{-6t}(-C_1 \sin t + C_2 \cos t) - 6e^{-6t}(C_1 \cos t + C_2 \sin t)$$
$$+7e^{-6t}(C_1 \cos t + C_2 \sin t)$$
$$= e^{-6t}[(C_2 + C_1) \cos t + (C_2 - C_1) \sin t]$$
$$y = e^{-6t}[C_3 \cos t + C_4 \sin t] \text{ where } C_3 = C_2 + C_1, C_4 = C_2 - C_1$$
$$\Rightarrow x = e^{-6t}(C_1 \cos t + C_2 \sin t) \text{ and } y = e^{-6t}(C_3 \cos t + C_4 \sin t)$$

**Example 1:** Solve: $\frac{dx}{dt} + 2y = \sin 2t; \frac{dy}{dt} - 2x = \cos 2t$

**Solution:** Given $\frac{dx}{dt} + 2y = \sin 2t; \frac{dy}{dt} - 2x = \cos 2t$

(i.e.) $Dx + 2y = \sin 2t$ (6)

and $Dy - 2x = \cos 2t$ (7)

(6) $\times D \Rightarrow D^2x + 2Dy = 2\cos 2t$

(7) $\times 2 \Rightarrow -4x + 2Dy = 2\cos 2t$

$D^2x + 4x = 0$

(i.e.) $(D2 + 4)x = 0$ (8)

The auxiliary equation of (8) is $m^2 + 4 = 0$

$\Rightarrow m^2 = -4 \Rightarrow m = 0 \pm 2i = \alpha \pm i\beta \Rightarrow \alpha = 0, \beta = 2$

$\Rightarrow C.F. = C1 \cos 2t + C2 \sin 2t$

The complete solution of (8) is $x = C1 \cos 2t + C2 \sin 2t$

Now from (6), $2y = \sin 2t - Dx$

$2y = \sin 2t - (-2C_1 \sin 2t - 2C_2 \cos 2t)$

$= \sin 2t + 2C_1 \sin 2t - 2C_2 \cos 2t$

$$\Rightarrow y = \frac{\sin 2t}{2} + C_1 \sin 2t - C_2 \cos 2t$$

$\Rightarrow$ The solutions are

$$x = C_1 \cos 2t + C_2 \sin 2t \text{ and } y = C_1 \sin 2t - C_2 \cos 2t + \frac{\sin 2t}{2}$$

## EXERCISE

1. Solve: $\frac{dx}{dt} + y = e^t; \frac{dy}{dt} = t$ [Ans: $x = C_1 \cos t + C_2 \sin t + \frac{e^t}{2} + t, y = C_1 \sin t - C_2 \cos t + \frac{e^t}{2} - 1$]

2. Solve: $\frac{dx}{dt} + 2x + 3y = 2e^{2t}; \frac{dy}{dt} + 3x + 2y = 0$ [Ans: $x = C_1 e^t + C_2 e^{-5t} + \frac{8}{7}e^{2t}, y = C_2 e^{-5t} - C_1 e^t - \frac{6}{7}e^{2t}$]

3. Solve: $\frac{dx}{dt} + 2y = -\sin t; \frac{dy}{dt} - 2x = \cos t$ [Ans: $x = C_1 \cos t + C_2 \sin t - \frac{\cos 2t}{3}, y = C_1 \sin t - C_2 \cos t + \frac{\sin 2t}{3}$]