18MAB102T
Advanced Calculus and Complex Analysis
Unit II -Vector Calculus

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Scalar and Vector Fields:

- A physical quantity expressible as a continuous function and which can assume one or more definite values at each point of a region of space, is called point function in the region and the region concerned is called a field.
- Point functions are classified as scalar point function and vector point function according as the nature of the quantity concerned is a scalar or a vector.
- At each point P of the field if the function denoted by f(P) is a scalar, it is known as scalar point function while if $\vec{f}(P)$ is a vector, then the function $\vec{f}(P)$ is called a vector point function. The concerned field is called a scalar field or a vector field respectively.

Example of Scalar Fields:

- The temperature distribution in a medium, the gravitational potential of a system of masses and the electrostatic potential of a system of charges.

Example of Vector Fields:

- The velocity of a moving particle, the electrostatic, the magneto static and gravitational fields.

Vector Differential Operator DEL(∇):

$$\nabla = \vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}$$

Gradient:

Let $\phi(x, y, z)$ defines a differentiable scalar field. (i.e) ϕ is differentiable at each point (x, y, z) is a certain region of space. Then the gradient of ϕ denoted by $\nabla\phi$ (or) grad ϕ is defined by

$$\nabla\phi = \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right)\phi = \vec{i}\frac{\partial\phi}{\partial x} + \vec{j}\frac{\partial\phi}{\partial y} + \vec{k}\frac{\partial\phi}{\partial z} = \sum \vec{i}\frac{\partial\phi}{\partial x}$$

Divergence :

If $\vec{F}(x, y, z)$ is defined and differentiable vector point function at each point (x, y, z) is a certain region of space, then the divergence of $\vec{F}$ denoted by $\nabla.\vec{F}$ (or) div$\vec{F}$ is defined by

$$\text{div}\vec{F} = \nabla.\vec{F} = \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right)\vec{F} = \sum \vec{i}.\frac{\partial \vec{F}}{\partial x}$$

If $\vec{F} = F_1\vec{i} + F_2\vec{j} + F_3\vec{k}$, then $\text{div}\vec{F} = \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right)\left(F_1\vec{i} + F_2\vec{j} + F_3\vec{k}\right)$

$$\text{div}\vec{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$$

Solenoidal :

If $\vec{F}$ is a vector such that $\nabla.\vec{F} = 0$ for all points is a given region, then it is said to be a solenoidal vector in that region.

Curl :

If $\vec{F}(x, y, z)$ is a differentiable vector point function in a certain region of space, then the curl or rotation of $\vec{F}$ denoted by $\nabla \times \vec{F}$ (or) curl $\vec{F}$ (or) rot $\vec{F}$ is defined by

$$\nabla \times \vec{F} = curl\ \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix}$$

Irrotational :

If $\vec{F}$ is vector such that $\nabla \times \vec{F} = 0$ for all points in the region, then it is called an irrotational vector (or) Lamellar vector in that region.

Directional derivation : $\frac{\nabla\phi.\vec{a}}{|\vec{a}|}$

Unit normal vector : $\hat{n} = \frac{\nabla\phi}{|\nabla\phi|}$

Angle between the surfaces :

$$\cos\theta = \frac{\nabla\phi_1.\nabla\phi_2}{|\nabla\phi_1||\nabla\phi_2|}$$

Problem: 1

If $\phi = xyz$, find $\nabla\phi$ at $(1, 2, 3)$

Solution:

$$\nabla\phi = \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right)(xyz)$$

$$= \vec{i}\frac{\partial}{\partial x}(xyz) + \vec{j}\frac{\partial}{\partial y}(xyz) + \vec{k}\frac{\partial}{\partial z}(xyz)$$

$$= \vec{i}\, yz + \vec{j}\, xz + \vec{k}\, xy$$

$$\nabla\phi = yz\,\vec{i} + xz\,\vec{j} + xy\,\vec{k}\mathrm{v}$$

$$\nabla\phi_{(1,2,3)} = 6\vec{i} + 3\vec{j} + 2\vec{k}.$$

Problem: 2

Prove that $\nabla(r^n) = nr^{n-2}\vec{r}$

Solution:

$$\nabla(r^n) = \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right)(r^n)$$

$$= \vec{i}\frac{\partial}{\partial x}(r^n) + \vec{j}\frac{\partial}{\partial y}(r^n) + \vec{k}\frac{\partial}{\partial z}(r^n)$$

$$= \vec{i}\, nr^{n-1}\frac{\partial r}{\partial x} + \vec{j}\, nr^{n-1}\frac{\partial r}{\partial y} + \vec{k}\, nr^{n-1}\frac{\partial r}{\partial z}$$

$$\nabla(r^n) = nr^{n-1}\left(\vec{i}\frac{\partial r}{\partial x} + \vec{j}\frac{\partial r}{\partial y} + \vec{k}\frac{\partial r}{\partial z}\right) \qquad (1)$$

$$\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$$

$$\vec{r}.\vec{r} = r^2 = x^2 + y^2 + z^2$$

$2r\dfrac{\partial r}{\partial x} = 2x$	$2r\dfrac{\partial r}{\partial y} = 2y$	$2r\dfrac{\partial r}{\partial z} = 2z$
$\dfrac{\partial r}{\partial x} = \dfrac{x}{r}$	$\dfrac{\partial r}{\partial y} = \dfrac{y}{r}$	$\dfrac{\partial r}{\partial z} = \dfrac{z}{r}$

(2)

Sub (2) in (1),

$$\nabla(r^n) = nr^{n-1}\left(\vec{i}\frac{x}{r} + \vec{j}\frac{y}{r} + \vec{k}\frac{z}{r}\right)$$

$$\nabla(r^n) = nr^{n-2}\vec{r}.$$

Problem: 3

Find the directional derivative of $\phi = x^2yz + 4xz^2 + xyz$ at $(1, 2, 3)$ in the direction of $2\vec{\imath} + \vec{\jmath} - \vec{k}$.

Solution:

Directional derivation $= \frac{\nabla\phi.\vec{a}}{|\vec{a}|}$

Given $\phi = x^2yz + 4xz^2 + xyz$

$$\nabla\phi = (2xyz + 4z^2 + yz)\vec{\imath} + (x^2z + xz)\vec{\jmath} + (x^2 + 8xz + xy)\vec{k}$$

$$\nabla\phi_{(1,2,3)} = 54\vec{\imath} + 6\vec{\jmath} + 28\vec{k}$$

Let $\vec{a} = 2\vec{i} + \vec{j} - \vec{k}$

$$|\vec{a}| = \sqrt{2^2 + 1^2 + (-1)^2}$$

$$|\vec{a}| = \sqrt{6}$$

$$\text{Directional derivation} = \frac{\nabla\phi.\vec{a}}{|\vec{a}|}$$

$$\text{Directional derivation} = \frac{\left(54\vec{i}+6\vec{j}+28\vec{k}\right).\left(2\vec{i}+\vec{j}-\vec{k}\right)}{\sqrt{6}}$$

$$\text{Directional derivation} = \frac{86}{\sqrt{6}}$$

Problem: 4

Find a unit normal to the surface $x^2 + 2xz^2 = 8$ at the point $(1, 0, 2)$.

Solution:

Let $\phi = x^2 + 2xz^2 - 8$

$$\nabla\phi = (2xy + 2x^2)\vec{i} + x^2\vec{j} + 4xz\vec{k}$$

$$\nabla\phi_{(1,0,2)} = 8\vec{i} + \vec{j} + 8\vec{k}$$

$$|\nabla\phi| = \sqrt{8^2 + 1^2 + 8^2} = \sqrt{129}$$

Unit normal= $\hat{n} = \frac{\nabla\phi}{|\nabla\phi|} = \frac{8\vec{i}+\vec{j}+8\vec{k}}{\sqrt{129}}$.

Problem: 5

Find the angle between the surfaces $z = x^2 + y^2 - 3$ and $x^2 + y^2 + z^2 = 9$ at (2, -1, 2).

Solution:

Given $\phi_1 = x^2 + y^2 - 2 - 3$

$$\nabla\phi_1 = 2x\vec{i} + 2y\vec{j} - \vec{k}$$

$$\nabla\phi_1\,(2,-1,2) = 4\vec{i} - 2\vec{j} - \vec{k}$$

$$|\nabla\phi_1| = \sqrt{21}$$

$$\phi_2 = x^2 + y^2 + z^2 - 9$$

$$\nabla\phi_2 = 2x\vec{\imath} + 2y\vec{\jmath} + 2z\vec{k}$$

$$\nabla\phi_2\,(2,-1,2) = 4\vec{\imath} - 2\vec{\jmath} + 4\vec{k}$$

$$|\nabla\phi_2| = 6$$

$$\cos\theta = \frac{\nabla\phi_1 . \nabla\phi_2}{(\nabla\phi_1)(\nabla\phi_2)}$$

$$= \frac{(4\vec{\imath} - 2\vec{\jmath} - \vec{k}).(4\vec{\imath} - 2\vec{\jmath} + 4\vec{k})}{(\sqrt{21})\,(6)}$$

$$\cos\theta = \frac{8}{3\sqrt{21}}$$

$$\theta = \cos^{-1}\frac{8}{3\sqrt{21}}$$

Problem: 6

If $\nabla\phi = \left(yz\vec{i} + zx\vec{j} + xy\vec{k}\right)$, find ϕ.

Solution:

$$\nabla\phi = \left(yz\vec{i} + zx\vec{j} + xy\vec{k}\right)$$

$$\vec{i}\frac{\partial\phi}{\partial x} + \vec{j}\frac{\partial\phi}{\partial y} + \vec{k}\frac{\partial\phi}{\partial z} = \left(yz\vec{i} + zx\vec{j} + xy\vec{k}\right)$$

$$\frac{\partial\phi}{\partial x} = yz$$

function not involving x.

$$\frac{\partial \phi}{\partial y} = zx$$

$\phi = xyz + a$, function not involving y.

$$\frac{\partial \phi}{\partial z} = xy$$

$\phi = xyz + a$, function not involving z.

From the last three statements,

we conclude

$\phi = xyz + a$ is a constant.

Problem: 7

If $\vec{F} = x^2\vec{\imath} + y^2\vec{\jmath} + z^2\vec{k}$, then find $\nabla . \vec{F}$ and $\nabla \times \vec{F}$.

Solution:

$$\nabla . \vec{F} = \left(\vec{\imath}\frac{\partial}{\partial x} + \vec{\jmath}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right) . \left(x^2\vec{\imath} + y^2\vec{\jmath} + z^2\vec{k}\right)$$

$$= \frac{\partial}{\partial x}(x^2) + \frac{\partial}{\partial y}(y^2) + \frac{\partial}{\partial z}(z^2)$$

$$\nabla . \vec{F} = 2\mathrm{x} + 2\mathrm{y} + 2\mathrm{z}$$

$$\nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ x^2 & y^2 & z^2 \end{vmatrix}$$

$$= \vec{i}\left[\frac{\partial}{\partial y}(z^2) - \frac{\partial}{\partial z}(y^2)\right] - \vec{j}\left[\frac{\partial}{\partial x}(z^2) - \frac{\partial}{\partial z}(x^2)\right] + \vec{k}\left[\frac{\partial}{\partial x}(y^2) - \frac{\partial}{\partial y}(x^2)\right]$$

$$= \vec{i}[0] - \vec{j}[0] + \vec{k}[0]$$

$$\nabla \times \vec{F} = 0.$$

Problem: 8

Prove that the vector $\vec{F} = z\vec{i} + x\vec{j} + y\vec{k}$ is solenoidal.

Solution:

$$\nabla \cdot \vec{F} = \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right) \cdot \left(z\vec{i} + x\vec{j} + y\vec{k}\right)$$

$$= \frac{\partial}{\partial x}(z) + \frac{\partial}{\partial y}(x) + \frac{\partial}{\partial z}(y)$$

$$\nabla \cdot \vec{F} = 0$$

$\therefore \vec{F}$ is solenoidal.

Problem: 9

If $\vec{F} = (x + 3y)\vec{i} + (y - 2z)\vec{j} + (x + \lambda z)\vec{k}$ is solenoidal, find the value of λ.

Solution:

$$\nabla . \vec{F} = 0$$

$$\frac{\partial}{\partial x}(x + 3y) + \frac{\partial}{\partial y}(y - 2z) + \frac{\partial}{\partial z}(x + \lambda z) = 0$$

$$1 + 1 + \lambda = 0$$

$$\lambda = -2.$$

Problem: 10

Show that $\vec{F} = \left(yz\vec{i} + zx\vec{j} + xy\vec{k}\right)$ is irrotational.

Solution:

$$\nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ yz & zx & xy \end{vmatrix}$$

$$= \vec{i}\left[\frac{\partial}{\partial y}(xy) - \frac{\partial}{\partial z}(xz)\right] - \vec{j}\left[\frac{\partial}{\partial x}(xy) - \frac{\partial}{\partial z}(yz)\right] + \vec{k}\left[\frac{\partial}{\partial x}(zx) - \frac{\partial}{\partial y}(yz)\right]$$

$$\nabla \times \vec{F} = 0$$

$\therefore \vec{F}$ is irrotational.

Laplace operator :

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

Problem: 11

Prove that $\nabla^2 \mathrm{r}^n = n(n+1)\mathrm{r}^{n-2}$ where $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$ and $r = |\vec{r}|$ and deduce $\nabla^2\left(\frac{1}{r}\right)$.

Solution:

$$r = |\vec{r}| = \sqrt{x^2 + y^2 + z^2}$$

$$\mathrm{r}^2 = x^2 + y^2 + z^2$$

$$\frac{\partial r}{\partial x} = \frac{x}{r}; \quad \frac{\partial r}{\partial y} = \frac{y}{r}; \quad \frac{\partial r}{\partial z} = \frac{z}{r}$$

$$\nabla^2 \mathrm{r}^n = \sum \frac{\partial^2}{\partial x^2}(\mathrm{r}^n) = \sum \frac{\partial}{\partial x}\left[\mathrm{nr}^{n-1}\frac{\partial r}{\partial x}\right]$$

$$= \sum \frac{\partial}{\partial x}\left[\mathrm{nr}^{n-1}\frac{x}{r}\right] = \sum \frac{\partial}{\partial x}\left[\mathrm{nr}^{n-2}x\right]$$

$$= \sum n\left[\left((\mathrm{n}-2)\mathrm{r}^{n-3}\frac{\partial r}{\partial x}\right)x + \mathrm{r}^{n-2}\right]$$

$$= \sum n\left[\left((\mathrm{n}-2)\mathrm{r}^{n-3}\frac{x}{r}\right)x + \mathrm{r}^{n-2}\right]$$

$$= \sum n[(\mathrm{x}^2(\mathrm{n}-2)\mathrm{r}^{n-4}) + \mathrm{r}^{n-2}]$$

$$= \sum[(n(\mathrm{n}-2)\mathrm{r}^{n-4}\mathrm{x}^2) + n\mathrm{r}^{n-2}]$$

$$= n(n-2)\mathrm{r}^{n-4}(x^2 + y^2 + z^2) + 3\mathrm{nr}^{n-2}$$

$$= n(n-2)\mathrm{r}^{n-4}r^2 + 3\mathrm{nr}^{n-2}$$

$$= n(n-2)\mathrm{r}^{n-2} + 3\mathrm{nr}^{n-2}$$

$$= n\mathrm{r}^{n-2}[n-2+3]$$

$$\nabla^2(\mathrm{r}^n) = n(n+1)\mathrm{r}^{n-2}.$$

Line Integral

Problem: 12

Find the work done in moving a particle in the force field $\vec{F} = 3x^2\vec{i} + (2xz - y)\vec{j} - z\vec{k}$ from $t = 0$ to $t = 1$ along the cone $x = 2t^2, y = t, z = 4t^3$.

Solution:

Work done $= \int_c \vec{F}.\overrightarrow{dr}$

$$\vec{F} = 3x^2\vec{i} + (2xz - y)\vec{j} - z\vec{k}$$

$$\overrightarrow{dx} = dx\,\vec{i} + dy\,\vec{j} + dz\,\vec{k}$$

$$\vec{F}.\overrightarrow{dr} = 3x^2dx + (2xz - y)dy - zdz$$

$x = 2t^2$ | $y = t$ | $z = 4t^3$

$dx = 4t\, dt$ | $dy = dt$ | $dz = 12t^2\, dt$

$$\vec{F}.\overrightarrow{dr} = 48\, t^5\, dt + (16\, t^5 - t)dt - 48\, t^5\, dt$$

$$\int_c \vec{F}.\overrightarrow{dr} = \int_0^1 (16\, t^5 - t)dt$$

$$= \left[16\, \frac{t^6}{6} - \frac{t^2}{2}\right]_0^1$$

$$= \frac{16}{6} - \frac{1}{2}$$

$$= \frac{13}{6}$$

Surface Integrals

$$\iint_S \vec{F}.\hat{n}\ ds \ = \ \iint_R \frac{\vec{F}.\hat{n}}{|\vec{n}.\hat{k}|}\ ds\ dy$$

Problem: 13

Evaluate $\iint_S \vec{F}.\hat{n}\ ds$ where $\vec{F} = z\,\vec{i} + x\,\vec{j} - \ y^2 z\,\vec{k}$ and S is the surface of the cylinder $x^2 + \ y^2 \ = 1$ included in the first octant between the planes $z = 0$ and $z = 2$.

Solution :

$$\vec{F} = z\,\vec{i} + x\,\vec{j} - \ y^2 z\,\vec{k}$$

$$\varphi = \ x^2 + \ y^2 - 1$$

$$|\nabla\varphi| = \sqrt{4x^2 + 4y^2} = 2$$

$$\hat{n} = \frac{\nabla\varphi}{|\nabla\varphi|}$$

$$= \frac{2x\,\vec{i} + 2y\,\vec{j}}{2}$$

$$\hat{n} = x\,\vec{i} + y\,\vec{j}$$

$$\vec{F}.\hat{n} = \left(z\,\vec{i} + x\,\vec{j} - y^2 z\,\vec{k}\right).(x\,\vec{i} + y\,\vec{j}) = xz + xy$$

$$\iint_S \vec{F}.\hat{n}\;ds = \iint_R \frac{\vec{F}.\hat{n}}{|\vec{n}.\hat{i}|}\,dy\,dz$$

Where R is the projection of S on yz plane.

$$= \iint_R (xz + xy)\ \frac{dy\,dz}{x}$$

$$= \iint_R (z + y)\ dy\,dz$$

$$= \int_0^2 \int_0^1 (z + y)\ dy\,dz$$

$$= \int_0^2 \left[zy + \frac{y^2}{2}\right]_0^1\ dz$$

$$= \int_0^2 (z + \tfrac{1}{2})\ dz$$

$$= \left[\frac{z^2}{2} + \frac{z}{2}\right]_0^2 = 3.$$

Volume Integrals

Problem: 14

If $\vec{F} = (2x^2 - 3x)\vec{i} - 2xy\vec{j} - 4x\vec{k}$. Evaluate $\iiint_v \nabla \times \vec{F} dv$ where v is the region bounded by $x = 0, y = 0, z = 0$ and $2x + 2y + z = 4$.

Solution:

$$\nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ 2x^2 - 3z & -2xy & -4x \end{vmatrix}$$

$$\nabla \times \vec{F} = \vec{j} - 2y\vec{k}$$

$$\iiint_v \nabla \times \vec{F}\, dv = \int_0^2 \int_0^{2-x} \int_0^{4-2x-2y} \left(\vec{j} - 2y\vec{k}\right) dzdydx$$

$$= \int_0^2 \int_0^{2-x} \left[z\vec{j} - 2yz\vec{k}\right]_0^{4-2x-2y} dydx$$

$$= \int_0^2 \int_0^{2-x} \left[(4 - 2x - 2y)\vec{j} - 2y(4 - 2x - 2y)\vec{k}\right] dydx$$

$$= \int_0^2 \left[\left(4y - 2xy - \frac{2y^2}{2}\right)\vec{j} - \left(4y^2 - 2xy^2 - \frac{4y^3}{3}\right)\vec{k}\right]_0^{2-x} dx$$

$$= \int_0^2 \left\{[4(2-x) - 2x(2-x) - (2-x)^2]\vec{j} - \left[4(2-x)^2 - 2x(2-x)^2 - \frac{4}{3}(2-x)^3\right]\vec{k}\right\} dx$$

$$= \int_0^2 \left[(4 - 4x + x^2)\vec{\imath} - \frac{\vec{k}}{3}(16 - 24x + 12x^2 - 2x^3)\right] dx$$

$$\iiint_v \nabla \times \vec{F}\, dv = \left[4x - 2x^2 + \frac{x^3}{3}\right]_0^2 \vec{\imath} - \frac{\vec{k}}{3}\left[16x - 12x^2 + 4x^3 - \frac{x^4}{2}\right]_0^2$$

$$= \left(8 - 8 + \frac{8}{3}\right)\vec{\imath} - \frac{\vec{k}}{3}(32 - 48 + 32 - 8)$$

$$= \frac{8}{3}\left(\vec{j} - \vec{k}\right).$$

Thank You

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18MAB102T- ADVANCED CALCULUS AND COMPLEX ANALYSIS; Unit II (Part-3) - Green's, Stoke's and Gauss Divergence theorem

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Outline

1. Green's theorem
2. Stoke's theorem
3. Gauss divergence theorem

Statement (Green's theorem):

Let C be a positively oriented, piecewise smooth, simple, closed curve and let R be the region enclosed by the curve C in the xy-plane. If $P(x, y)$ and $Q(x, y)$ have continuous first order partial derivatives on R, then

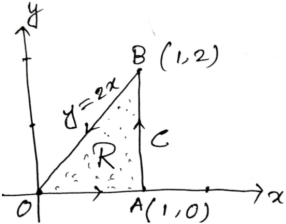
$$\oint_C Pdx + Qdy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dxdy.$$

Applications of Green's theorem

Example 1:

Use Green's theorem to evaluate $\oint_C xydx + x^2y^3dy$, where C is the triangle with vertices $(0,0), (1,0), (1,2)$ with positive orientation.

Solution: Let $P = xy, Q = x^2y^3$ and the positive orientation curve C is as shown in the figure.



Applications of Green's theorem

Using Green's theorem,

$$\oint_C xydx + x^2y^3dy = \oint_C Pdx + Qdy$$

$$= \iint\limits_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dxdy = \iint\limits_R (2xy^3 - x)dxdy$$

$$= \int_0^1 \int_0^{2x} (2xy^3 - x)dydx = \int_0^1 \left[\frac{xy^4}{2} - xy\right]_0^{2x} dx$$

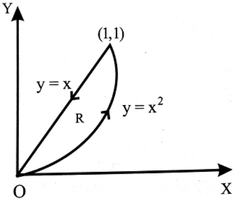
$$= \int_0^1 (8x^5 - 2x^2)dx = \left[\frac{4x^6}{3} - \frac{2x^3}{3}\right]_0^1 = \frac{2}{3}.$$

Applications of Green's theorem

Example 2:

Verify Green's theorem in the plane for $\oint_C [(xy + y^2)dx + x^2 dy]$, where C is the closed curve of the region bounded by $y = x$ and $y = x^2$.

Solution: Let $P = xy + y^2$, $Q = x^2$ and the positive orientation curve C is as shown in the figure. The curves $y = x$ and $y = x^2$ intersect at $(0, 0)$ and $(1, 1)$.



Applications of Green's theorem

Using Green's theorem,

$$\oint_C [(xy + y^2)dx + x^2 dy] = \oint_C Pdx + Qdy$$

$$= \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dxdy = \iint_R (2x - x - 2y) dxdy$$

$$= \iint_R (x - 2y) dxdy = \int_0^1 \int_{y=x^2}^{x} (x - 2y) dydx$$

$$= \int_0^1 \left[xy - y^2 \right]_{y=x^2}^{x} dx = \int_0^1 (x^4 - x^3) dx$$

$$= \left[\frac{x^5}{5} - \frac{x^4}{4} \right]_0^1 = -\frac{1}{20}.$$

Applications of Green's theorem

Now let us evaluate the line integral along C. Along $y = x^2, dy = 2xdx$ and the line integral equals

$$\int_0^1 [(x(x^2) + x^4)dx + x^2(2x)dx] = \int_0^1 (3x^3 + x^4)dx$$
$$= \left[\frac{3x^4}{4} + \frac{x^5}{5}\right]_0^1 = \frac{19}{20}.$$

Along $y = x, dy = dx$ and the line integral equals

$$\int_1^0 [(x(x) + x^2)dx + x^2dx] = \int_1^0 (3x^2)dx = \left[\frac{3x^3}{3}\right]_1^0 = -1.$$

Therefore, the required line integral $= \frac{19}{20} - 1 = -\frac{1}{20}$. Hence the theorem is verified.

Statement (Stoke's theorem):

Let S be a smooth surface that is bounded by a simple closed, smooth boundary curve C with positive orientation and $\overrightarrow{F} = F_1\overrightarrow{i} + F_2\overrightarrow{j} + F_3\overrightarrow{k}$ be any vector function having continuous first order partial derivatives, then

$$\oint_C \overrightarrow{F}.\overrightarrow{dr} = \iint_S curl\overrightarrow{F}.\widehat{n}ds,$$

where $\widehat{n}$ is the outward normal unit vector at any point of S.

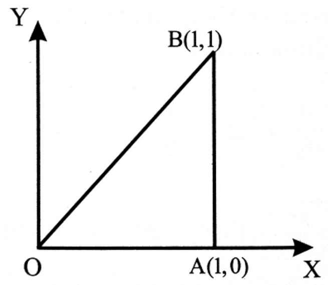
Applications of Stoke's theorem

Example 1:

Use Stoke's theorem to evaluate $\oint_C \overrightarrow{F}.\overrightarrow{dr}$, where $\overrightarrow{F} = y^2\overrightarrow{i} + x^2\overrightarrow{j} - (x+z)\overrightarrow{k}$ and C is the boundary of the triangle with vertices $(0,0,0), (1,0,0), (1,1,0)$ with positive orientation.

Solution: We note that z-coordinate of each vertex of the triangle is 0. Therefore, the triangle lies in the xy-plane. So $\widehat{n} = \overrightarrow{k}$ and the positive orientation curve C is as shown in the figure.

Applications of Stoke's theorem



Let $F_1 = y^2, F_2 = x^2, F_3 = -(x+z)$ and we have

$$curl\,\overrightarrow{F} = \begin{vmatrix} \overrightarrow{i} & \overrightarrow{j} & \overrightarrow{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2 & x^2 & -(x+z) \end{vmatrix} = 0\,\overrightarrow{i} + \overrightarrow{j} + 2(x-y)\,\overrightarrow{k}$$

Applications of Stoke's theorem

and $curl\,\overrightarrow{F}.\widehat{n} = [\overrightarrow{j} + 2(x-y)\overrightarrow{k}].\overrightarrow{k} = 2(x-y)$.
The equation of the line OB is $y = x$. Using Stoke's theorem,

$$\oint_C \overrightarrow{F}.\overrightarrow{dr} = \iint_S curl\,\overrightarrow{F}.\widehat{n}ds = \int_0^1 \int_{y=0}^{x} 2(x-y)dxdy$$

$$= 2\int_0^1 \left[xy - \frac{y^2}{2}\right]_0^x dx = 2\int_0^1 \frac{x^2}{2}dx = \frac{1}{3}.$$

Applications of Stoke's theorem

Example 2:

Verify Stoke's theorem for $\overrightarrow{F} = (2x - y)\overrightarrow{i} - yz^2\overrightarrow{j} - y^2z\overrightarrow{k}$ over the upper half surface S of the sphere $x^2 + y^2 + z^2 = 1$ bounded by its projection on the xy-plane and C is its boundary.

Solution: The boundary C of S is a circle in the xy-plane of radius unity and centre at origin. Let $x = \cos t, y = \sin t$, $z = 0, 0 \leq t \leq 2\pi$ are parametric equations of C.

Applications of Stoke's theorem

Now

$$\oint_C \overrightarrow{F}.\overrightarrow{dr}$$
$$= \oint_C [(2x-y)\overrightarrow{i} - yz^2\overrightarrow{j} - y^2z\overrightarrow{k}].[dx\overrightarrow{i} + dy\overrightarrow{j} + dz\overrightarrow{k}]$$
$$= \oint_C (2x-y)dx - yz^2dy - y^2zdz = \oint_C (2x-y)dx$$
$$= -\int_0^{2\pi} (2\cos t - \sin t)\sin t dt = \pi. \quad (1)$$

Applications of Stoke's theorem

Also $\hat{n} = \overrightarrow{k}$, $ds = dxdy$,

$$curl\,\overrightarrow{F} = \begin{vmatrix} \overrightarrow{i} & \overrightarrow{j} & \overrightarrow{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2x-y & -yz^2 & -y^2z \end{vmatrix} = \overrightarrow{k}$$

and $curl\,\overrightarrow{F}.\hat{n} = \overrightarrow{k}.\overrightarrow{k} = 1.$
Using Stoke's theorem,

$$\iint_S curl\,\overrightarrow{F}.\hat{n}ds = \iint_S dxdy = \pi, \qquad (2)$$

where $\pi(1)^2$ is the area of the circle C.
Hence from (1) and (2), the theorem is verified.

Statement (Gauss divergence theorem):

If V is the volume bounded by a closed surface S and $\overrightarrow{F}$ is a vector point function with continuous derivatives in V, then

$$\iint_S curl\overrightarrow{F}.\widehat{n}ds = \iiint_V div\overrightarrow{F}dV,$$

where $\widehat{n}$ is the outward normal unit vector at any point of S.

Applications of Gauss divergence theorem

Example 1:

Use Gauss divergence theorem to evaluate $\iint\limits_S [(x^3 - yz)dydz - 2x^2ydzdx + zdxdy]$ over the surface S of a cube bounded by the coordinate planes and the plane $x = y = z = a$.

Solution: Let $F_1 = x^3 - yz, F_2 = -2x^2y, F_3 = z$. Using Gauss divergence theorem,

$$\iint\limits_S curl\,\overrightarrow{F}.\hat{n}ds = \iiint\limits_V div\,\overrightarrow{F}\,dV$$

$$= \iiint\limits_V \left(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}\right) dxdydz$$

Applications of Gauss divergence theorem

$$= \int_{x=0}^{a}\int_{y=0}^{a}\int_{z=0}^{a}(x^2+1)dxdydz = \int_{z=0}^{a}\int_{y=0}^{a}\left[\frac{x^3}{3}+x\right]_{x=0}^{a} dydz$$

$$= \left[\frac{a^3}{3}+a\right]\int_{z=0}^{a}\int_{y=0}^{a} dydz = a\left[\frac{a^3}{3}+a\right]\int_{z=0}^{a} dz = a^2\left[\frac{a^3}{3}+a\right].$$

Applications of Gauss divergence theorem

Example 2:

Use Gauss divergence theorem to evaluate $\iint\limits_S [(x+z)dydz + (y+z)dzdx + (x+y)dxdy]$ over the surface S of the sphere $x^2 + y^2 + z^2 = 4$.

Solution: Let $F_1 = x + z, F_2 = y + z, F_3 = x + y$. Using Gauss divergence theorem,

$$\iint\limits_S curl\,\overrightarrow{F}.\widehat{n}ds = \iiint\limits_V div\,\overrightarrow{F}\,dV$$

$$= \iiint\limits_V \left(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}\right) dV$$

Applications of Gauss divergence theorem

$$= \iiint_V 2dV = 2\iiint_V dV = 2V,$$

where V is the volume of the sphere $x^2 + y^2 + z^2 = 2^2$ ($\because$ the volume of a sphere of radius r is $\frac{4}{3}\pi r^3$).

$$= 2\left[\frac{4}{3}\pi(2)^3\right] = \frac{64}{3}\pi.$$