SEQUENCES AND SERIES

1
CHAPTER

quences

Definition. A set of numbers, a_1, a_2 a_3, ..., a_n, ... such that to each positive integer n, there rresponds a number a_n of the set, is called a *sequence* and it is denoted by $\{a_n\}$.

In a *sequence,* the elements are arranged in a definite order while in a *set* order of the elements ay be in any order. Thus, a sequence and the set of natural numbers N have one to one correspondence.

amples.

1. If $a_n = \dfrac{1}{n}$, the sequence is $1, \dfrac{1}{2}, \dfrac{1}{3}, \ldots \dfrac{1}{n}, \ldots$
2. If $a_n = n^3$, the sequence is $1^3, 2^3, 3^3, \ldots. n^3, \ldots$
3. If $a_n = k$, the sequence is $k, k, k, \ldots$
4. If $a_n = (-1)^n$, then the sequence is $-1, 1, -1, 1, \ldots (-1)^n, \ldots$
5. If $a_n = 1$ for n even
   $= -1$ for n odd, then the sequence is
   $-1, 1, -1, 1, \ldots$ which is oscillating.

Limit of a sequence or Convergence of a sequence. Let $\{a_n\}$ be a sequence and l, a real mber. We say that $\{a_n\}$ *tends to the limit 1* or *converges to l* if given any positive number, ε, wever small it may be, there exists a positive integer N (depending upon ε) such that

$$|a_n - l| < \varepsilon \text{ for all } n \geq N$$

e write $\lim\limits_{n\to\infty} a_n = l$

Here l is the *limit* of the sequence $\{a_n\}$

Note. $|a_n - l| < \varepsilon$ implies $l - \varepsilon < a_n < l + \varepsilon$

Bounded Sequence. A sequence $\{a_n\}$ is said to be *bounded above* if there exists a real number such that $a_n \leq M$ for all n.

If such a number *M does not exist,* the $\{a_n\}$ is *unbounded above.*

A sequence $\{a_n\}$ is said to be *bounded below* if there exists a real number m such that $a_n \geq m$ for n. If such a 'm' does not exist, then the sequence $\{a_n\}$ is *unbounded below.*

Definition. A sequence $\{a_n\}$ is said to be *bounded* if it is bounded above and bounded below. other words, there exists two real numbers m and M such that $m \leq a_n \leq M$ for all n ($\in N$).

Divergent Sequence. A sequence $\{a_n\}$ is said to *diverge to* $+\infty$ if given any positive number however large it may be, there exists a positive integer N such that

$$a_n > M \text{ for all } n \geq N$$

We write $\lim\limits_{n\to\infty} a_n = +\infty$

A sequence $\{a_n\}$ is said to *diverge to* $-\infty$ if given any positive number M, however large it may there exists a positive integer N such that

$$a_n < -M \text{ for all } n \geq N$$

We write $\lim_{n \to \infty} a_n = -\infty$

Monotonic Sequences. A sequence $\{a_n\}$ is said to be *monotonically increasing* if $a_n \le a_{n+1}$ for all n.

That is if $a_1 \le a_2 \le a_3 \le a_4 \dots \le a_n \le a_{n+1} \le \dots$

A sequence $\{a_n\}$ is said to be *monotonically decreasing* if $a_n \ge a_{n+1}$ for all n.

That is, if $a_1 \ge a_2 \ge a_3 \ge \dots \ge a_n \ge a_{n+1} \ge \dots$ A sequence which is *either* monotonically increasing or decreasing is called a *monotonic* sequence.

Oscillatory Sequences : A sequence $\{a_n\}$ is said to be *oscillatory* (or oscillating) if it does not converge and does not diverge to $+\infty$ or $-\infty$

A sequence is said to oscillate *finitely* if it is bounded and is an *oscillatory* sequence.

A sequence is said to oscillate *infinitely* if it is not bounded and is an oscillatory sequence.

Theorems. The following theorems can be assumed (without proof).

1. $\lim (a_n \pm b_n) = \lim a_n \pm \lim b_n$
2. $\lim (a_n \cdot b_n) = (\lim a_n)(\lim b_n)$
3. $\lim \left(\dfrac{1}{a_n}\right) = \dfrac{1}{\lim a_n}$

where $\{a_n\}$, $\{b_n\}$ are convergent sequences.

4. **If $\{a_n\}$ and $\{b_n\}$ are convergent sequences, then $\{a_n \pm b_n\}$ is also a convergent sequence.**
5. **A sequence $\{a_n\}$ cannot converge to two distinct limits.**
6. **If $\{a_n\}$ converges to 'a' and $\{b_n\}$ converges to b then $\{a_n b_n\}$ converges to ab.**
7. **If $\{a_n\}$ converges to 'a' and $\{b_n\}$ converges to b ($\ne a$), then $\left\{\dfrac{a_n}{b_n}\right\}$ converges to $\dfrac{a}{b}$.**
8. **A monotonic increasing sequence which is bounded above converges.**
9. **A monotonic increasing sequence which is not bounded above diverges to $+\infty$.**
10. **A monotonic decreasing sequence which is bounded below converges.**
11. **A monotonic decreasing sequence which is unbounded below diverges to $-\infty$.**
12. **If a sequence $\{a_n\}$ of positive terms converges to l then l is ≥ 0.**
13. **If $|x| < 1$ then $\lim_{n \to \infty} x^n = 0$**
14. **If $x > 1$ then $\lim_{n \to \infty} x^n = \infty$**
15. **A monotonic sequence always tends to a limit, finite or infinite.**

Example 1. *Show that the sequence (i)* $\left\{\dfrac{1}{n}\right\}$ *(ii)* $\left\{\dfrac{n+1}{2n+7}\right\}$

(iii) $\{a_n\}$ *where* $a_n = 3 + \dfrac{(-1)^n}{n}$ *are convergent.*

Solution. (*i*) $a_n = \dfrac{1}{n}$ and $a_n \to 0$ as $n \to \infty$.

We will apply the definition of convergence and find the existence of N.

For a given positive number ε,

$$\left|\frac{1}{n} - 0\right| < \varepsilon \text{ for all } n \ge N.$$

i.e., $\dfrac{1}{n} < \varepsilon$ for all $n \ge N$

i.e., for $n > \dfrac{1}{\varepsilon}$, this is possible.

Therefore select N, a positive integer, such that $N > \dfrac{1}{\varepsilon}$.

Then for all $n \ge N$, $\left|\dfrac{1}{n} - 0\right| < \varepsilon$.

$\therefore \left\{\dfrac{1}{n}\right\}$ converges to zero.

(*ii*) Let $$a_n = \frac{n+1}{2n+7} = \frac{1+\frac{1}{n}}{2+\frac{7}{n}}$$

This tends to $\dfrac{1}{2}$ as $n \to \infty$

Let us find the existence of N.

$$|a_n - l| = \left|\frac{n+1}{2n+7} - \frac{1}{2}\right| = \left|\frac{-5}{2(2n+7)}\right|$$

$$\left|a_n - \frac{1}{2}\right| < \varepsilon \text{ implies } \frac{5}{2(2n+7)} < \varepsilon$$

i.e., $$2n + 7 > \frac{5}{2\varepsilon}$$

i.e., $$n > \frac{5}{4\varepsilon} - \frac{7}{2}$$

$\therefore$ For all $n \ge N$ where $N > \dfrac{5}{4\varepsilon} - \dfrac{7}{2}$, $\left|a_n - \dfrac{1}{2}\right| < \varepsilon$

Hence $\{a_n\} \to \dfrac{1}{2}$ and it is convergent.

(*iii*) $|a_n - 3| = \left|3 + \dfrac{(-1)^n}{n} - 3\right| = \left|\dfrac{1}{n}\right|$ since, here $l = 3$

$|a_n - 3| < \varepsilon$ implies $\dfrac{1}{n} < \varepsilon$ *i.e.,* $n > \dfrac{1}{\varepsilon}$

We find a positive integer $N > \dfrac{1}{\varepsilon}$ such that for all $n \ge N$, the sequence is convergent.

Example 2. *The sequences* $\{n^2\}$, $\{-n^2\}$, $\{3n+2\}$, $\{n\}$ *are examples of divergent sequence since N cannot be found out such that* $\{a_n - l\} < \varepsilon$ *for* $n \ge N$. $\{(-1)^n\}$, $\{4(-1)^n\}$ *are examples of oscillating sequences.*

Example 3. *The sequence* $\{n\}$, $\{n^2\}$, $\{2n+7\}$ *are examples of monotonically increasing sequences which are unbounded above.*

The sequences $\left\{\frac{1}{n}\right\}, \left\{\frac{1}{2n+3}\right\}$ *are examples of monotonically decreasing sequences which are bounded below.*

Solution. Let $a_n = n$; then $a_{n+1} - a_n = (n+1) - n = 1$

i.e., $a_{n+1} - a_n = 1 > 0$

$a_{n+1} > a_n$ for all n.

Hence $\{a_n\}$ is monotonically increasing.

Since no M can be found out such that $n < M$ the sequence is not bounded above.

Similarly all the other problems can be worked out.

Example 4. *Prove* $\left\{\frac{n+1}{2n+3}\right\}$ *is monotonically increasing.*

Solution.
$$a_{n+1} - a_n = \frac{(n+1)+1}{2(n+1)+3} - \frac{n+1}{2n+3} = \frac{1}{(2n+5)(2n+3)} > 0 \text{ for all } n$$

$\therefore$ $a_{n+1} > a_n$ for all n.

$\therefore$ $\{a_n\}$ is monotonically increasing $\left(\text{bounded above by } \frac{1}{2}\right)$

EXERCISE 1

1. Show that the following sequences $\{a_n\}$ is convergent where a_n is given by

((*i*) $\frac{n+1}{n}$ (*ii*) $x^{\frac{1}{n}}$ where $x > 0$ (*iii*) $\frac{2n+3}{3n+7}$ (*iv*) $\frac{1}{n(n+1)}$

(*v*) $n \tan\left(\frac{1}{n}\right)$ (*iv*) $\left(1+\frac{1}{n}\right)^n$ (*vii*) $\frac{1}{n+1} + \frac{1}{n+2} + \ldots \frac{1}{2n}$

(*viii*) $\frac{1}{n^2+1}$ (*ix*) $\frac{an^2+bn+c}{\alpha n^2+\beta n+\gamma}$

2. Show that the following sequences $\{a_n\}$ where a_n is given below are divergent or oscillating

(*i*) $\frac{1+n^2}{n}$ (*ii*) $n + \frac{(-1)^n}{n^2}$ (*iii*) $3 + (-1)^n$ (*iv*) $n\,(-1)^n$

(*v*) $(-1)^n$ (*vi*) $1 + \frac{1}{2} + \frac{1}{3} + \ldots \frac{1}{n}$ (*vii*) $n^3 + n$

3. Prove that the sequence $\{a_n\}$ where $a_n = \frac{1}{1!} + \frac{1}{2!} + \ldots + \frac{1}{n!}$ is convergent.

4. Prove that the sequence $\{a_n\}$ where $a_n = \frac{a+nb}{c+nd}$ where a, b, c, d are positive is descending or ascending according as $ad \gtrless bc$.

5. Prove $\left\{\frac{n}{n+1}\right\}$ is monotonic increasing and bounded above. Hence what is your conclusion ?

6. If $a_n = 3 + \frac{1}{2^n}$, find the least positive integer m such that $|a_n - 3| < 10^{-6}$ for all $n > m$. Find the limit of $\{a_n\}$.

(**Ans.** $m = 20$)

Cauchy's General Principle of Convergence

A necessary and sufficient condition for the existence of a limit to the sequence $\{a_n\}$ *is that, given any positive value* ε*, however small it may be, there exists a positive integer* m*, such that* $|a_{n+p} - a_n| < \varepsilon$ *for all* $n > m$ *and for all integral values of* $p \geq 0$.

INFINITE SERIES

If $\{a_n\}$ is a sequence, then

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \ldots \text{ to } \infty \text{ is called an infinite series.}$$

Convergence or divergence of a series. Let

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \ldots + a_n + \ldots \text{ to } \infty \text{ be a given series.}$$

Define $s_n = a_1 + a_2 + \ldots + a_n$.

That is s_n is the sum of the first n terms of the series $\sum a_n$.

The convergence or divergence of the series $\sum a_n$ is defined in terms of the covergence or divergence of the sequence $\{s_n\}$.

That is, $\sum a_n$ is said to converge if sequence $\{s_n\}$ converges.

$\sum a_n$ converges or diverges according as $\{s_n\}$ converges or diverges.

If $\{s_n\}$ diverges to ∞, then $\sum a_n$ diverges to ∞.

If $\{s_n\}$ diverges to $-\infty$, then $\sum a_n$ diverges to $-\infty$.

If $\{s_n\}$ oscillates finitely or infinitely, then we say $\sum a_n$ oscillates finitely or infinitely.

If $\{s_n\}$ converges to l, then we say the series $\sum a_n$ converges to l. That is, $\sum a_n \to l$.

Geometric series. The geometric series

$$1 + r + r^2 + \ldots + r^n + \ldots \text{ to } \infty = \sum_{n=0}^{\infty} r^n$$

(*i*) converges to $\frac{1}{1-r}$ if $|r| < 1$ (*ii*) diverges if $r \geq 1$

(*iii*) oscillates finitely if $r = -1$ (*iv*) oscillates infinitely if $r < -1$

Theorems 1. If $\sum u_n$ is convergent then $\sum ku_n$ is convergent where k is a constant.

1. If $\sum u_n$ and $\sum v_n$ are two convergent series then $\sum (u_n + v_n)$ is convergent and its sum is the sum of the two series.

SERIES OF POSITIVE TERMS

A series whose terms are all positive is called a series of positive terms.

Theorem 1. *A series of positive terms either converges or diverges. It cannot oscillate.*

2. *If* $\sum u_n$ *is convergent then* $\lim\limits_{n\to\infty} u_n = 0$. *This is a necessary condition but not a sufficient condition.*

That is, if $\lim\limits_{n\to\infty} u_n = 0$, then $\sum u_n$ need not converge. *(Converse is not true)*

For example, in the series

$$1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n} + \ldots, u_n = \frac{1}{n} \text{ and } \lim_{n\to\infty} \frac{1}{n} = 0;$$

but still $\sum u_n = \sum \frac{1}{n}$ is not convergent; it diverges although $\frac{1}{n} \to 0$ as $n \to \infty$.

Note. *If the nth term of a positive termed series tends to zero as* $n \to \infty$*, we cannot say any thing; whereas if the nth term of the series does not tend to zero as* $n \to \infty$ *we can say that the series is not convergent (therefore, divergent).*

TESTS. (proofs of tests are assumed)

Auxiliary series test or Harmonic series test. The series

$$\sum \frac{1}{n^p} = \frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + \dots + \frac{1}{n^p} + \dots \text{ to } \infty \text{ is}$$

(*i*) convergent if $p > 1$ and (*ii*) divergent if $p \le 1$

COMPARISON TEST, (different forms)

1. *If $\sum u_n$ and $\sum v_n$ are two series of positive terms and the second series $\sum v_r$ is convergent and if $u_n \le kv_n$ for all n (k constant) then $\sum u_n$ is also convergent and it converges to the value which is less or equal to k times the convergent value of $\sum v_n$.*

Useful forms

2. *If $\sum u_n$ and $\sum v_n$ are two positive termed series and if $\sum v_n$ is convergent and $\frac{u_n}{v_n}$ tends to a limit other then zero as n tends to infinity then $\sum u_n$ is also convergent.*

3. *If $\sum u_n$ and $\sum v_n$ are two series of positive terms and if $\sum v_n$ is divergent and $\frac{u_n}{v_n}$ tends to a limit other than zero as n tends to infinity, then $\sum u_n$ is also divergent.*

4. If $\sum u_n$ and $\sum v_n$ are two positive termed series and if $u_n \le kv_n$ for all n and if $\sum u_n$ is divergent then $\sum v_n$ is divergent.

Note. $\sum \frac{1}{n}$ is divergent while $\sum \frac{1}{n^2}$ is convergent.

5. Cauchy's general principal of convergence of an infinite series.

A necessary and sufficient condition for the convergence of the series $\sum u_n$ is that, given any positive number ε, however small it may be, there exists a positive integer m such that $|u_{n+1} + u_{n+2} + \dots + u_{n+p}| < \varepsilon$ for all $n \ge m$ and for all integers $p \ge 0$.

Example 1. *Test the convergence of the series :*

(*i*) $\sum \frac{1}{\sqrt{n^2+1}}$ (*ii*) $\sum \frac{1}{\sqrt{n+1}}$ (*iii*) $\sum \frac{1}{(n+1)(2n+1)}$

(*iv*) $\sum (\sqrt{n^2+1} - n)$ (*v*) $\sum (\sqrt{n^4+1} - \sqrt{n^4-1})$

Solution. (*i*) Let $u_n = \frac{1}{\sqrt{n^2+1}}$ Take $v_n = \frac{1}{n}$.

Now
$$\frac{u_n}{v_n} = \frac{\frac{1}{\sqrt{n^2+1}}}{\frac{1}{n}} = \frac{n}{\sqrt{n^2+1}} = \frac{1}{\sqrt{1+\frac{1}{n^2}}}$$

$\frac{u_n}{v_n} \to 1$ as $n \to \infty$. Further $\sum v_n = \sum \frac{1}{n}$ is divergent. Hence, $\sum u_n$ is divergent by comparison test.

(*ii*)
$$u_n = \frac{1}{\sqrt{n+1}}; \text{ take } v_n = \frac{1}{n^{1/2}}$$

$$\frac{u_n}{v_n} = \frac{\sqrt{n}}{\sqrt{n+1}} = \sqrt{\frac{1}{1+\frac{1}{n}}}$$

$\frac{u_n}{v_n} \to 1$ as $n \to \infty$ and $\sum v_n = \sum \frac{1}{n^{1/2}}$ is divergent.

$\therefore \sum u_n$ is divergent.

(*iii*)
$$u_n = \frac{1}{(n+1)(2n+1)} \left(= \frac{1}{\text{second degree in } n} \right)$$

So, take $v_n = \frac{1}{n^2}$

$$\frac{u_n}{v_n} = \frac{n^2}{(n+1)(2n+1)} = \frac{1}{\left(\frac{n+1}{n}\right)\left(\frac{n+1}{n}\right)} = \frac{1}{\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right)} \quad \frac{u_n}{v_n} \to \frac{1}{2} \text{ as } n \to \infty.$$

Also $\sum v_n = \sum \frac{1}{n^2}$ is convergent.

$\therefore \sum \frac{1}{(n+1)(2n+1)}$ is convergent.

(*iv*) Let
$$u_n = \sqrt{n^2+1-n}$$
$$= \frac{(\sqrt{n^2+1}-n)(\sqrt{n^2+1}+n)}{(\sqrt{n^2+1}+n)} = \frac{1}{\sqrt{n^2+1+n}}$$

Therefore, take $v_n = \frac{1}{n}$

$$\frac{u_n}{v_n} = \frac{n}{\sqrt{n^2+1+n}} = \frac{1}{\sqrt{1+\frac{1}{n^2}+1}} \quad \frac{u_n}{v_n} \to \frac{1}{2} \neq 0 \text{ as } n \to \infty$$

Also $\sum v_n$ is divergent. By comparison test, $\sum u_n$ is also divergent.

(*v*) Let
$$u_n = \sqrt{n^4+1} - \sqrt{n^4-1}$$
$$= \frac{(\sqrt{n^4+1} - \sqrt{n^4-1})(\sqrt{n^4+1} + \sqrt{n^4-1})}{(\sqrt{n^4+1} + \sqrt{n^4-1})}$$
$$= \frac{(n^4+1)-(n^4-1)}{\sqrt{n^4+1}+\sqrt{n^4-1}} = \frac{2}{\sqrt{n^4+1}+\sqrt{n^4-1}}$$

Take $v_n = \frac{1}{n^2}$

$$\frac{u_n}{v_n} = \frac{2n^2}{\sqrt{n^4+1}+\sqrt{n^4-1}} = \frac{2}{\sqrt{1+\frac{1}{n^4}}+\sqrt{1-\frac{1}{n^4}}}$$

$$\frac{u_n}{v_n} \to 1 \neq 0 \text{ as } n \to \infty$$

Since $\sum v_n = \sum \frac{1}{n^2}$ is convergent, $\sum u_n$ is also convergent.

Example 2. *Discuss the convergence of* $\sum \frac{1}{(n+a)^p\,(n+b)^q}$

Solution. Take $v_n = \frac{1}{n^{p+q}}$ and use comparison test.

$$\frac{u_n}{v_n} = \frac{n^{p+q}}{(n+a)^p\,(n+b)^q} = \frac{1}{\left(1+\frac{a}{n}\right)^p\left(1+\frac{b}{n}\right)^q}$$

$\frac{u_n}{v_n} \to 1$ as $n \to \infty$

$\therefore \sum u_n$ and $\sum v_n$ converge or diverge together.

$\therefore \sum u_n$ is convergent if $p+q>1$ and divergent if $p+q \le 1$

Example 3. *Discuss the convergence of the series*

(*i*) $\sum \frac{1}{n} \sin\left(\frac{1}{n}\right)$ *and* (*ii*) $\frac{1}{\sqrt{n}} \tan\left(\frac{1}{n}\right)$

Solution. (*i*) Let $u_n = \frac{1}{n}\sin\left(\frac{1}{n}\right)$; take $v_n = \frac{1}{n^2}$

$$\frac{u_n}{v_n} = \frac{\sin\left(\frac{1}{n}\right)}{\left(\frac{1}{n}\right)}$$

$$\lim_{n\to\infty} \frac{u_n}{v_n} = \lim_{\frac{1}{n}\to 0} \frac{\sin\left(\frac{1}{n}\right)}{\left(\frac{1}{n}\right)} = 1 \neq 0$$

$\therefore \sum u_n$ is convergent since $\sum v_n = \sum \frac{1}{n^2}$ is convergent.

(*ii*) Let $u_n = \frac{1}{\sqrt{n}} \tan\left(\frac{1}{n}\right)$; $v_n = \frac{1}{n^{3/2}}$

$$\lim_{n\to\infty} \frac{u_n}{v_n} = \lim_{\frac{1}{n}\to 0} \frac{\tan\left(\frac{1}{n}\right)}{\frac{1}{n}} = 1 \neq 0$$

$\therefore \sum u_n$ is convergent since $\sum v_n$ is convergent.

Example 4. *Show that the series* $\frac{1}{1.2.3} + \frac{3}{2.3.4} + \frac{5}{3.4.5} + \ldots$ *to* ∞ *is convergent.*

Solution. u_n = nth term of the series

$$= \frac{2n-1}{n(n+1)(n+2)}$$

Since the right hand side is of the form $\frac{\text{first degree}}{\text{third degree}}$

Take $v_n = \frac{1}{\text{second degree}} = \frac{1}{n^2}$

Now $\frac{u_n}{v_n} = \frac{(2n-1).n^2}{n(n+1)(n+2)} = \frac{(2n-1)n}{(n+1)(n+2)} = \frac{\left(2-\frac{1}{n}\right)}{\left(1+\frac{1}{n}\right)\left(1+\frac{2}{n}\right)}$

$$\lim_{n\to\infty} \frac{u_n}{v_n} = 2 \neq 0$$

$\therefore \sum u_n$ is convergent since $\sum v_n = \sum \frac{1}{n^2}$ is convergent.

Example 5. *If* $\sum u_n$ *and* $\sum v_n$ *are two convergent series of positive terms, then prove*

(*i*) $\sum (u_n + v_n)$ (*ii*) $\sum \sqrt{u_n\, v_n}$ (*iii*) $\sum u_n\, v_n$

(*iv*) $\sum u_n^2, \sum u_n^3 \ldots$ *are also convergent.*

Solution. Let $\sum u_n$ and $\sum v_n$ converge to *a* and *b* i.e., if $U_n = u_1 + u_2 + \ldots + u_n$ and $V_n = v_1 + v_2 + \ldots + v_n$

$\lim_{n\to\infty} U_n = a$ *and* $\lim_{n\to\infty} V_n = b$.

(*i*) Let $s_n = (u_1+v_1) + (u_2+v_2) + \ldots + (u_n+v_n)$

$$= \sum_{n=1}^{n} (u_n + v_n)$$

$$\lim_{n\to\infty} s_n = \lim_{n\to\infty} \sum_{1}^{n} (u_n + v_n)$$

$$= \lim_{n\to\infty} \sum_{1}^{n} u_n + \lim_{n\to\infty} \sum_{1}^{n} v_n = a + b.$$

$\therefore \sum_{1}^{\infty} (u_n + v_n)$ converges.

(*ii*) Since A.M. > G.M.

$$\frac{u_n + v_n}{2} > \sqrt{u_n\, v_n}$$

Since $\sum (u_n + v_n)$ is convergent, $\sum \left(\frac{u_n + v_n}{2}\right)$ is also convergent.

$\therefore \sum_{1}^{\infty} \sqrt{u_n\, v_n}$ is convergent.

(*iii*) $u_n v_n < a v_n$ since $u_n < a$

Since $\sum a v_n = a \sum v_n$ is convergent

$\sum u_n v_n$ is also convergent.

(*iv*) Taking $v_n = u_n$, $\sum u_n^2$ is convergent

Similarly $\sum u_n^3, \sum u_n^4, ...$ are all convergent.

Example 6. *Discuss the convergence of the series* $\sum \frac{1}{x^n + x^{-n}}$ *for* $x > 0$.

Solution. Case 1. Let $x > 1$; hence $\frac{1}{x}$ is positive and less than 1

$\therefore \sum \left(\frac{1}{x}\right)^n$ is convergent since it is a geometric series.

$$x^n + x^{-n} > x^n \qquad (\because x^{-n} > 0)$$

$$\frac{1}{x^n + x^{-n}} < \frac{1}{x^n} \text{ and } \sum \frac{1}{x^n} \text{ is convergent.}$$

$\therefore \sum \frac{1}{x^n + x^{-n}}$ is also convergent for $x > 1$.

Case 2. Let $x = 1$; $\frac{1}{x^n + x^{-n}} = \frac{1}{2}$ and $\sum \left(\frac{1}{2}\right)$ is divergent

$\therefore \sum \frac{1}{x^n + x^{-n}}$ is divergent for $x = 1$

Case 3. Let $0 < x < 1$

$x^n + x^{-n} > x^{-n}$ and $\frac{1}{x^n + x^{-n}} < \frac{1}{x^{-n}} = x^n$ and $\sum x^n$ is convergent

$\therefore \sum \frac{1}{x^n + x^{-n}}$ is convergent for $0 < x < 1$

Example 7. *If* $a_n > 0$ *and (i) if* $\sum a_n^2$ *is convergent, can you say that* $\sum a_n$ *is convergent? (ii) If* $\sum a_n$ *is divergent can you say* $\sum a_n^2$ *is divergent?*

Solution. (*i*) **No.** As a counter example, $\sum \frac{1}{n^2}$ is convergent while $\sum \frac{1}{n}$ is not convergent (it is divergent)

(*ii*) **No.** $\sum \frac{1}{n}$ is divergent while $\sum \frac{1}{n^2}$ is not divergent.

Example 8. *If* $a_n > 0$ *and* $\sum a_n$ *is convergent, prove* $\sum \frac{a_n}{1 + a_n}$ *and* $\sum \frac{a_n}{1 + n^2 a_n}$ *are also convergent.*

Solution. If $a_n > 0$, $1 + a_n > 1$ and $\frac{1}{1 + a_n} < 1$

$\therefore \quad \frac{a_n}{1 + a_n} < a_n$ and $\sum a_n$ is convergent.

$\therefore \sum \frac{a_n}{1 + a_n}$ is also convergent.

Similarly $\frac{a_n}{1 + n^2 a_n} < a_n$ and $\sum a_n$ is convergent.

Hence $\sum \frac{a_n}{1 + n^2 a_n}$ is also convergent.

Example 9. *Discuss the convergence the series*

(*i*) $\sum_{n=1}^{\infty} \frac{\sqrt{n+1} - \sqrt{n}}{n^p}$ (*ii*) $\sum_{n=1}^{\infty} \frac{n^p}{\sqrt{n+1} + \sqrt{n}}$ (*iii*) $\sum_{n=1}^{\infty} \frac{n^p}{(n+1)^{n+1}}$ (*iv*) $\sum_{n=10}^{\infty} \frac{1}{n+5}$

Solution. (*i*) Here, $u_n = \frac{\sqrt{n+1} - \sqrt{n}}{n^p} = \frac{\sqrt{(n+1} - \sqrt{n})(\sqrt{n+1} + \sqrt{n})}{n^p (\sqrt{n+1} + \sqrt{n})} = \frac{1}{n^p (\sqrt{n+1} + \sqrt{n})}$

Take $\qquad v_n = \frac{1}{n^{p+\frac{1}{2}}}$

Now $\lim_{n\to\infty} \frac{u_n}{v_n} = \lim_{n\to\infty} \frac{n^{p+\frac{1}{2}}}{n^p (\sqrt{n+1} + \sqrt{n})} = \lim_{n\to\infty} \frac{n^{1/2}}{\sqrt{n+1} + \sqrt{n}} = \lim_{n\to\infty} \frac{1}{\sqrt{1 + \frac{1}{n}} + 1} = 1/2 \neq 0$

$\sum u_n$ is convergent for $p + \frac{1}{2} > 1$ and divergent for $p + \frac{1}{2} \leq 1$.

Hence, the given series is convergent for $p > \frac{1}{2}$ and divergent for $p + \frac{1}{2} \leq 1$.

(*ii*) Let $u_n = \frac{n^p}{\sqrt{n+1} + \sqrt{n}}$; Take $v_n = \frac{1}{n^{1/2-p}}$

Now, $\lim_{n\to\infty} \frac{u_n}{v_n} = \lim_{n\to\infty} \frac{n^p . n^{1/2-p}}{\sqrt{n+1} + \sqrt{n}}$

$$= \lim_{n\to\infty} \frac{1}{\sqrt{1 + \frac{1}{n}} + 1} = \frac{1}{2}$$

$\sum v_n = \sum \frac{1}{n^{1/2-p}}$ is convergent if $\frac{1}{2} - p > 1$ and divergent if $\frac{1}{2} - p \leq 1$

$\therefore$ The given series is convergent if $p < -\frac{1}{2}$ and divergent if $p \geq -\frac{1}{2}$

(*iii*) Let $u_n = \frac{n^n}{(n+1)^{n+1}}$; Take $v_n = \frac{1}{n}$

Now, $\lim_{n\to\infty} \frac{u_n}{v_n} = \lim_{n\to\infty} \frac{n^{n+1}}{(n+1)^{n+1}} = \lim_{n\to\infty} \left(\frac{n}{n+1}\right)^{n+1}$

$$= \lim_{n\to\infty} \frac{1}{\left(1 + \frac{1}{n}\right)^n} \cdot \frac{1}{1 + \frac{1}{n}} = \frac{1}{e} \neq 0$$

Since $\sum v_n = \sum \frac{1}{n}$ is divergent, the given series $\sum u_n$ is also divergent by comparison test.

(*iv*) **Method 1.**

The series is $\frac{1}{15}, \frac{1}{16}, \frac{1}{17}$ ... to ∞.

This is nothing but the series $\sum_{n=1}^{\infty} \frac{1}{n}$ from which the first few terms are removed. Since $\sum_{n=1}^{\infty} \frac{1}{n}$ is divergent, the given series is also divergent.

Method 2. Take $v_n = \frac{1}{n}$;

Then,
$$\lim_{n\to\infty} \frac{u_n}{v_n} = \lim_{n\to\infty} \frac{n}{n+5} = \lim_{n\to\infty} \frac{1}{1+\frac{5}{n}} = 1 \neq 0$$

$\therefore$ $\sum_{n=1}^{\infty} u_n$ is divergent since $\sum_{n=1}^{\infty} v_n$ is divergent by comparison test.

$\therefore$ $\sum_{n=10}^{\infty} u_n$ is also divergent.

Example 10. *Show that the series* $\sum_{n=1}^{\infty} e^{-n^2}$ *converges.*

Solution. Let $u_n = e^{-n^2} = \frac{1}{e^{n^2}}$

we know, for $x > 0$, $e^x > x$

$$\therefore \quad e^{n^2} > n^2$$

$$\frac{1}{e^{n^2}} < \frac{1}{n^2} \text{ and } \sum \frac{1}{n^2} \text{ is convergent.}$$

$\therefore$ $\sum u_n = \sum \frac{1}{e^{n^2}}$ is convergent by comparison test.

Example 11. *Find the sum to n terms of the series* $\sum \frac{1}{4n^2-1}$. *Hence discuss the convergence of the series.*

Solution. Let $u_n = \frac{1}{4n^2-1} = \frac{1}{(2n-1)(2n+1)} = \frac{1}{2}\left[\frac{1}{2n-1} - \frac{1}{2n+1}\right]$

Sum to n terms of the series $= u_1 + u_2 + \ldots + u_n$

$$= \frac{1}{2}\left[\left(\frac{1}{1}-\frac{1}{3}\right)+\left(\frac{1}{3}-\frac{1}{5}\right)+\left(\frac{1}{5}-\frac{1}{7}\right)+\ldots+\left(\frac{1}{2n-1}-\frac{1}{2n+1}\right)\right] = \frac{1}{2}\left[1-\frac{1}{2n+1}\right]$$

Since $\sum_{n=1}^{\infty} u_n = \lim_{n\to\infty} s_n = \lim_{n\to\infty} \frac{1}{2}\left[1-\frac{1}{2n+1}\right] = \frac{1}{2}$

The series converges and it converges to the value 1/2.

Note. Here, we are using the basic idea of the convergence of a series, by finding s_n and its limit as $n \to \infty$.

Example 12. *Discuss the convergence of* $\frac{1^3}{1^p+2^p} + \frac{2^3}{2^p+3^p} + \frac{3^3}{3^p+4^p} + \ldots$ *to* ∞

Solution. Let $u_n = \frac{n^3}{n^p+(n+1)^p}$ Take $v_n = \frac{1}{n^{p-3}}$

Here
$$\lim_{n\to\infty} \frac{u_n}{v_n} = \lim_{n\to\infty} \frac{n^{p-3}.n^3}{n^p+(n+1)^p} = \lim_{n\to\infty} \frac{n^p}{n^p+(n+1)^p}$$
$$= \lim_{n\to\infty} \frac{1}{1+\left(1+\frac{1}{n}\right)^p} = \frac{1}{2} \neq 0$$

$\sum v_n$ is convergent for $p-3>1$ and divergent for $p-3 \le 1$

$\therefore$ The given series is convergent for $p>4$ and divergent for $p \le 4$, by comparison test.

EXERCISE 2

Test for convergence of the series

1. $\frac{1}{3.4.5} + \frac{2}{4.5.6} + \frac{3}{5.6.7} + \ldots$ to ∞
2. $\frac{1.2}{3.4.5} + \frac{2.3}{4.5.6} + \frac{3.4}{5.6.7} + \ldots$ to ∞
3. $\frac{1}{x} + \frac{1}{x+1} + \frac{1}{x+2} + \ldots$ to ∞, if $x > 0$
4. $\frac{1}{x} + \frac{1}{x+y} + \frac{1}{x+2y} + \frac{1}{x+3y} + \ldots$, if $x, y > 0$
5. $1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \ldots$ to ∞
6. $\frac{1}{1.2} + \frac{1}{3.4} + \frac{1}{5.6} +$ to ...∞
7. $1 + \frac{1+2}{1+2^2} + \frac{1+3}{1+3^2} + \ldots + \frac{1+n}{1+n^2} + \ldots$
8. $\sum \frac{(n+a)(n+b)}{n(n+1)(n+2)}$
9. $\frac{1}{1^2-x} + \frac{1}{2^2-x} + \frac{1}{3^2-x} + \ldots$.
10. $\frac{1}{\sqrt{1^2-x^2}} + \frac{1}{\sqrt{2^2-x^2}} + \frac{1}{\sqrt{3^2-x^2}} + \ldots$

Test for convergence of the series whose *n*th term is

11. $\frac{\sqrt{n+1}}{n+2}$
12. $\frac{1}{1+3n^4}$
13. $\sin^2\left(\frac{1}{n}\right)$
14. $\sqrt[3]{n^3+1} - n$
15. $\frac{n^p}{\sqrt{n+1}-\sqrt{n}}$
16. $\sqrt{n^3+1} - \sqrt{n^3}$
17. $\frac{1}{\sqrt{n+1}-\sqrt{n}}$
18. $\frac{1}{an+b}$
19. $\frac{n+1}{n^p}$
20. $\frac{1}{\sqrt{n(n+1)}}$
21. $\frac{n+1}{n^2}$
22. n^2
23. $\sin\left(\frac{1}{n}\right)$

Answers. C-Convergent; D-divergent.

1. (*C*) 2. to 5. (*D*) 6. (*C*) 7. (*D*) 8. (*D*) 9. (*C*) 10. (*D*)
11. (*D*) 12. to 14. (*C*) 15. (*C*) if $p < -3/2$ and (*D*) if $p \ge -3/2$ 16. (*C*), 17. (*D*),
18. (*D*), 19. (*C*) if $p > 2$ and (*D*) if $p \le 2$ 20. (*D*), 21. (*D*) 22. (*D*) 23. (*D*).

D'ALEMBERT'S RATIO TEST

If $\sum u_n$ *is a series of positive terms and* $\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = l$, *then the series* $\sum u_n$ *is*

(*i*) *convergent if* $l < 1$, *and*

(*ii*) *divergent if* $l > 1$.

Example 1. *Test the convergence of the series*

$$\frac{3}{4}+\frac{3.4}{4.6}+\frac{3.4.5}{4.6.8}+\ldots \text{ to } \infty$$

Solution. Here
$$u_n=\frac{3.4.5\ldots(n+2)}{4.6.8\ldots(2n+2)}$$

$\therefore$
$$u_{n+1}=\frac{3.4.5\ldots(n+2)(n+3)}{4.6.8\ldots(2n+2)(2n+4)}$$

$$\frac{u_{n+1}}{u_n}=\frac{n+3}{2n+4}=\frac{1+\frac{3}{n}}{2+\frac{4}{n}}$$

$$\lim_{n\to\infty}\frac{u_{n+1}}{u_n}=\frac{1}{2}<1$$

$\therefore$ By ratio test, the given series is convergent.

Example 2. *Test the convergence or divergence of*

$$1+\frac{a+1}{b+1}+\frac{(a+1)(2a+1)}{(b+1)(2b+1)}+\frac{(a+1)(2a+1)(3a+1)}{(b+1)(2b+1)(3b+1)}+\ldots \text{ to } \infty$$

Solution. Here $u_n=\dfrac{(a+1)(2a+1)(3a+1)\ldots[(n-1)a+1]}{(b+1)(2b+1)(3b+1)\ldots[(n-1)b+1]}$

$\therefore$
$$\frac{u_{n+1}}{u_n}=\frac{na+1}{nb+1}=\frac{a+\frac{1}{n}}{b+\frac{1}{n}} \quad \lim_{n\to\infty}\frac{u_{n+1}}{u_n}=\frac{a}{b}$$

If a, b are positive, the given series become positive termed series.

$\therefore$ by Ratio rest,

If $\dfrac{a}{b}<1$, *i.e.*, $a<b$ and $a, b>0$, the given series is convergent.

If $\dfrac{a}{b}>1$, *i.e.*, $a>b>0$, the series is divergent.

If $a=b$, then $u_n=1$. Hence the series is $1+1+1+\ldots$ which is divergent.

$\therefore$ If $a=b$, the given series diverges.

Example 3. *Test for convergency the series* $\displaystyle\sum_{n=1}^{\infty}\frac{n^3+1}{2^n+1}$

Solution. Here
$$u_n=\frac{n^3+1}{2^n+1};\ u_{n+1}=\frac{(n+1)^3+1}{2^{n+1}+1}$$

$\therefore$
$$\frac{u_{n+1}}{u_n}=\frac{(n+1)^3+1}{n^3+1}\cdot\frac{2^n+1}{2^{n+1}+1}=\frac{\left(1+\frac{1}{n}\right)^3+\frac{1}{n^3}}{1+\frac{1}{n^3}}\cdot\frac{1+\frac{1}{2^n}}{2+\frac{1}{2^n}}$$

$$\therefore \lim_{n\to\infty}\frac{u_{n+1}}{u_n}=1\times\frac{1}{2}=\frac{1}{2}<1$$

$\therefore$ The seriesis convergent.

Example 4. *Test for convergency the series* $\sum\frac{n!}{n^n}$.

Solution. Here $u_n=\dfrac{n!}{n^n}$

$\therefore$
$$\frac{u_{n+1}}{u_n}=\frac{(n+1)!}{(n+1)^{n+1}}\cdot\frac{n^n}{n!}=\frac{n+1}{(n+1)^n\cdot(n+1)}\cdot n^n=\frac{1}{\left(\frac{n+1}{n}\right)^n}=\frac{1}{\left(1+\frac{1}{n}\right)^n}$$

$$\lim_{n\to\infty}\frac{u_{n+1}}{u_n}\to\frac{1}{e}<1$$

$\therefore$ The series is convergent.

Example 5. *Test for convergency the series* $\sum\dfrac{x^n}{1+x^n}$, $x>0$.

Solution. Here,
$$\frac{u_{n+1}}{u_n}=\frac{x^{n+1}}{1+x^{n+1}}\cdot\frac{1+x^n}{x^n}=\frac{1+x^n}{1+x^{n+1}}\cdot x$$

If $x<1$ and $x>0$, $x^n\to 0$ as $n\to\infty$

$\therefore \displaystyle\lim_{n\to\infty}\frac{u_{n+1}}{u_n}=x$ since $x^n, x^{n+1}\to 0$ as $n\to\infty$

$\therefore \sum u_n$ is convergent if $0<x<1$.

If $x>1$, the series is divergent.

When $x=1$, the sercies is $\sum u_n=\sum\dfrac{1^n}{1+1^n}=\sum\dfrac{1}{2}$

$\therefore \sum u_n$ is divergent when $x=1$.

Example 6. *Test for convergence of (i)* $\sum\dfrac{x^n}{n!}$

(ii) $\displaystyle\sum_{1}^{\infty}\sqrt{\frac{n}{n+1}}\cdot x^n$, $\quad x>0$

Solution. (*i*) Let $u_n=\dfrac{x^n}{n!}$

$\therefore$
$$\frac{u_{n+1}}{u_n}=\frac{x^{n+1}}{(n+1)!}\cdot\frac{n!}{x^n}=\frac{x}{n+1}\to 0 \text{ as } n\to\infty$$

$$\lim_{n\to\infty}\frac{u_{n+1}}{u_n}=0<1.$$

$\therefore \sum u_n$ is convergent for all real x.

(*ii*) Let
$$u_n=\sqrt{\frac{n}{n+1}}\,x^n;\ u_{n+1}=\sqrt{\frac{n+1}{n+2}}\cdot x^{n+1}$$

$\therefore$
$$\frac{u_{n+1}}{u_n}=\sqrt{\frac{n+1}{n+2}\cdot\frac{n+1}{n}}\cdot x=\sqrt{\frac{\left(1+\frac{1}{n}\right)^2}{\left(1+\frac{2}{n}\right)}}\cdot x$$

$$\lim_{n\to\infty} \frac{u_n+1}{u_n} = x$$

If $0 < x < 1$, the series is convergent.
If $x > 1$ the series is divergent.
If $x = 1$, D' Alembert's test fails.

when $$x = 1,\ u_n = \sqrt{\frac{n}{n+1}} = \sqrt{\frac{1}{\left(1+\frac{1}{n}\right)}}$$

$\lim\limits_{n\to\infty} u_n = 1$ which does not tend to zero.

$\therefore$ The series is divergent.

$\therefore$ The series is convergent if $0 < x < 1$ and divergent if $x \ge 1$.

Example 7. *Examine the convergence of the series*

(*i*) $\frac{x}{1} + \frac{1}{2}\cdot\frac{x^2}{3} + \frac{1.3}{2.4}\frac{x^3}{5} + \frac{1.3.5}{2.4.6}\frac{x^4}{7} + \ldots \text{ to } \infty, x > 0$

(*ii*) $\frac{1}{2} + \frac{4}{9}x + \frac{9}{28}x^2 + \ldots + \frac{n^2}{1+n^3}x^{n-1} \ldots \text{ to } \infty, x > 0$

Solution. Let $$u_n = \frac{1.3.5.7\ldots(2n-3)}{2.4.6\ldots(2n-2)}\cdot\frac{x^n}{2n-1}$$

$$u_{n+1} = \frac{1.3.5\ldots(2n-3)(2n-1)}{2.4.6\ldots(2n-2)(2n)}\cdot\frac{x^{n+1}}{2n+1}$$

$$\frac{u_{n+1}}{u_n} = \frac{2n-1}{2n}\cdot\frac{2n-1}{2n+1}\cdot x = \frac{\left(1-\frac{1}{2n}\right)^2}{\left(1+\frac{1}{2n}\right)}\cdot x$$

$$\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = x$$

$\therefore$ The series is convergent if $0 < x < 1$

The series diverges if $x > 1$

If $x = 1$, $\lim\limits_{n\to\infty}\frac{u_{n+1}}{u_n} = 1$ and hence the ratio test fails, In this case we can use Raabe's test.

(*ii*) Let $$u_n = \frac{n^2}{1+n^3}x^{n-1};\ u_{n+1} = \frac{(n+1)^2}{1+(n+1)^3}\cdot x^n$$

$$\frac{u_{n+1}}{u_n} = \frac{(n+1)^2}{n^2}\cdot\frac{1+n^3}{1+(n+1)^3}\cdot x$$

$$= \left(1+\frac{1}{n}\right)^2 \frac{\left(1+\frac{1}{n^3}\right)}{\frac{1}{n^3}+\left(1+\frac{1}{n}\right)^3}\cdot x \to x \text{ as } n\to\infty$$

$\therefore$ If $0 < x < 1$, then $\sum u_n$ converges and if $x > 1$ the series diverges.

When $x = 1$, $u_n = \frac{n^2}{1+n^3}$

We will use comparison test. Take $v_n = \frac{1}{n}$

$$\frac{u_n}{v_n} = \frac{n^3}{1+n^3} = \frac{1}{1+\frac{1}{n^3}} \to 1 \text{ as } n\to\infty.$$

$\therefore$ $\sum u_n$ and $\sum v_n$ converge or diverge together.

Since $\sum v_n = \sum\frac{1}{n}$ is divergent, $\sum u_n$ also diverges.

RAABE'S TEST

The positive termed series $\sum u_n$ *is convergent or divergent according as*

$$\lim_{n\to\infty}\left\{n\left(\frac{u_n}{u_{n+1}}-1\right)\right\} > 1 \text{ or } < 1.$$

Note : If D' Alembert's test fails, use Raabe's test.

Example 1. *Test for convergence of the series.*

(*i*) $\frac{2}{3.4} + \frac{2.4}{3.5.6} + \frac{2.4.6}{3.5.7.8} + \frac{2.4.6.8}{3.5.7.9.10} + \ldots \text{ to } \infty$

(*ii*) $\frac{3}{4}\cdot\frac{x}{5} + \frac{3.6}{4.7}\cdot\frac{x^2}{8} + \frac{3.6.9}{4.7.10}\frac{x^3}{11} + \ldots \text{ to } \infty, x > 0$

Solution. (*i*) $$u_n = \frac{2.4.6.8\ldots(2n)}{3.5.7\ldots(2n+1)}\cdot\frac{1}{2n+2}$$

$$u_{n+1} = \frac{2.4.6.8\ldots(2n)(2n+2)}{3.5.7\ldots(2n+3)}\cdot\frac{1}{2n+4}$$

$$\frac{u_{n+1}}{u_n} = \frac{2n+2}{2n+3}\cdot\frac{2n+2}{2n+4} = \frac{\left(1+\frac{1}{n}\right)^2}{\left(1+\frac{3}{2n}\right)\left(1+\frac{4}{2n}\right)}$$

$$\lim_{n\to\infty}\frac{u_{n+1}}{u_n} = 1$$

Hence ratio test fails.

$$n\left(\frac{u_n}{u_{n+1}}-1\right) = n\left[\frac{(2n+3)(2n+4)}{(2n+2)^2}-1\right] = \frac{n(6n+8)}{(2n+2)^2} \to \frac{3}{2} \text{ as } n\to\infty$$

$\therefore$ **By Raabe's test,** $\sum u_n$ is convergent.

(ii) Let $\quad u_n = \dfrac{3.6.9...(3n)}{4.7.10...(3n+1)} \cdot \dfrac{x^n}{(3n+2)}$

$$\frac{u_{n+1}}{u_n} = \frac{(3n+3)}{(3n+4)} \cdot \frac{3n+2}{3n+5} \cdot x \to x \text{ as } n \to \infty$$

$\therefore \sum u_n$ is convergent if $0 < x < 1$ and divergent if $x > 1$

If $x = 1$, the test fails.

$\therefore$ When $x = 1$,

$$n\left[\frac{u_n}{u_{n+1}} - 1\right] = n\left[\frac{(3n+4)(3n+5)}{(3n+3)(3n+2)} - 1\right] = n\left[\frac{12n+14}{(3n+3)(3n+2)}\right]$$

$$= \frac{\left(12 + \frac{14}{n}\right)}{\left(3 + \frac{3}{n}\right)\left(3 + \frac{2}{n}\right)} \to \frac{12}{9} \text{ as } n \to \infty$$

$$\lim_{n \to \infty} n\left(\frac{u_n}{u_{n+1}} - 1\right) = \frac{12}{9} > 1$$

$\therefore \sum u_n$ is convergent when $x = 1$.

Example 2. *Test for convergence of the series*

(i) $\sum \dfrac{1.3.5...(2n-1)}{2.4.6...(2n)} \cdot \dfrac{1}{n}$

(ii) $1 + \dfrac{a}{b} + \dfrac{a(a+1)}{b(b+1)} + \dfrac{a(a+1)(a+2)}{b(b+1)(b+2)} + ...$ *to* ∞, *if* $a, b > 0$

(iii) $\sum \dfrac{1.3.5...(2n-1)}{2.4.6...(2n)} . x^n$.

Solution. (i) Let $\quad u_n = \dfrac{1.3.5...(2n-1)}{2.4.6...(2n)} \cdot \dfrac{1}{n}$

$$u_{n+1} = \frac{1.3.5...(2n+1)}{2.4.6...(2n)(2n+2)} \cdot \frac{1}{n+1}$$

$$\frac{u_{n+1}}{u_n} = \frac{2n+1}{2n+2} \cdot \frac{n}{n+1} = \frac{\left(1 + \frac{1}{2n}\right)}{\left(1 + \frac{2}{2n}\right)} \cdot \frac{1}{\left(1 + \frac{1}{n}\right)}$$

$\lim\limits_{n \to \infty} \dfrac{u_{n+1}}{u_n} = 1$. Therefore, D'Alembert's ratio test fails.

We will try Raabe's test.

$$n\left(\frac{u_n}{u_{n+1}} - 1\right) = n\left[\frac{2n+2}{2n+1} \cdot \frac{n+1}{n} - 1\right] = \frac{3n+2}{2n+1} = \frac{3 + \frac{2}{n}}{2 + \frac{1}{n}} \to \frac{3}{2} (> 1) \text{ as } n \to \infty$$

$\therefore$ By Raabe's test, the series is convergent.

(ii) $\quad u_n = \dfrac{a(a+1)(a+2)...(a+n-1)}{b(b+1)(b+2)...(b+n-1)}$

$$u_{n+1} = \frac{a(a+1)...(a+n)}{b(b+1)...(b+n)}$$

$$\therefore \quad \frac{u_{n+1}}{u_n} = \frac{a+n}{b+n} = \frac{1 + \frac{a}{n}}{1 + \frac{b}{n}} \to 1 \text{ as } n \to \infty$$

$\therefore$ Ratio test fails.

We will apply Raabe's test

$$n\left(\frac{u_n}{u_{n+1}} - 1\right) = n\left(\frac{n+b}{n+a} - 1\right) = \frac{n(b-a)}{(n+a)} \to b - a \text{ as } n \to \infty$$

By Raabe's test, $\sum u_n$ is convergent if $b - a > 1$ and divergent if $b - a < 1$

If $b - a = 1$, Raabe's test also fails.

In this case $\quad u_n = \dfrac{a(a+1)(a+2)...(a+n-1)}{(a+1)(a+2)...(a+n)} = \dfrac{a}{a+n}$

Taking $\quad v_n = \dfrac{1}{n}, \dfrac{u_n}{v_n} = \dfrac{an}{n+a} = \dfrac{a}{1 + \frac{a}{n}} \to a$ if $n \to \infty$

If $a \neq 0$, $\sum u_n$ is divergent since $\sum v_n$ is divergent by comparison test.

(iii) $\quad u_n = \dfrac{1.3.5...(2n-1)}{2.4.6...(2n)} x^n$

$$\frac{u_{n+1}}{u_n} = \frac{2n+1}{(2n+2)} . x = \frac{1 + \frac{1}{2n}}{1 + \frac{2}{2n}} . x \to x \text{ as } n \to \infty$$

$\therefore \sum u_n$ is convergent if $0 < x < 1$ and divergent if $x > 1$.

If $x = 1$, D'Alembert's ratio test fails. Apply Raabe's test

$$n\left(\frac{u_n}{u_{n+1}} - 1\right) = \frac{n}{2n+1} \to \frac{1}{2} \text{ as } n \to \infty$$

$\therefore$ when $x = 1$, $\sum u_n$ is divergent.

Example 3. *Prove that the series*

$1 + \dfrac{1}{2} \cdot \dfrac{a}{b} + \dfrac{1.3}{2.4} \dfrac{a(a+1)}{b(b+1)} + \dfrac{1.3.5}{2.4.6} \cdot \dfrac{a(a+1)(a+2)}{b(b+1)(b+2)} + ...$ *to* ∞ *is convergent if* $a > 0, b > 0$ *and*

$b > a + \dfrac{1}{2}$

Solution. Let $\quad u_n = \dfrac{1.3.5...(2n-3)}{2.4.6...(2n-2)} \cdot \dfrac{a(a+1)...(a+n-2)}{b(b+1)...(b+n-2)}$

$$\therefore \quad \frac{u_{n+1}}{u_n} = \frac{(2n-1)(a+n-2)}{2n(b+n-1)} = \left(1-\frac{1}{2n}\right)\frac{\left(1+\frac{a-1}{n}\right)}{\left(1+\frac{b-1}{n}\right)} \to 1 \text{ as } n\to\infty.$$

Since $a, b > 0$, it is a positive termed series.

Ratio test fails.

Using Raabe's test,

$$n\left(\frac{u_n}{u_{n+1}}-1\right) = n\left[\frac{2n(b+n-1)}{(2n-1)(a+n-1)}-1\right] = n\left[\frac{2n(b-a)+a+n-1}{(2n-1)(a+n-1)}\right]$$

$$= \frac{(2b-2a+1)+\frac{a-1}{n}}{\left(2-\frac{1}{n}\right)\left(1+\frac{a-1}{n}\right)} \to b-a+\frac{1}{2} \text{ as } n\to\infty$$

By Raabe's test, $\sum u_n$ is convergent if $b-a+\frac{1}{2}>1$ *i.e.* if $b>a+\frac{1}{2}$

Example 4. *Discuss the convergence of* $\sum \frac{(n!)^2}{(2n)!}x^n$.

Solution. Let $\quad u_n = \frac{(n!)^2}{(2n)!}x^n;\ u_{n+1} = \frac{[(n+1)!]^2}{(2n+2)!}x^{n+1}$

$$\frac{u_{n+1}}{u_n} = \frac{(n+1)^2}{(2n+1)(2n+2)^2}.x = \frac{n+1}{2(2n+1)}x$$

$$\lim_{n\to\infty}\frac{u_{n+1}}{u_n} = \lim_{n\to\infty}\frac{1+\frac{1}{n}}{2\left(2+\frac{1}{n}\right)}.x = \frac{x}{4}$$

Hence the series is convergent if $0<x<4$ and divergent if $x>4$.

If $x = 4$. D' Alembert's ratio test fails.

$$\lim_{n\to\infty} n\left(\frac{u_n}{u_{n+1}}-1\right) = \lim_{n\to\infty} n\left[\frac{2(2n+1)}{4(n+1)}-1\right] = \lim_{n\to\infty}\frac{-n}{2(n+1)} = -\ 1/2 < 1$$

$\therefore$ when $x = 4$, the series is divergent.

Example 5. *Discuss the convergency or otherwise of the series whose nth term is* $(x>0)$

(*i*) $\frac{1}{1+nx^n}$ (*ii*) $\frac{x^n}{1+x^n}$ (*iii*) $\frac{x^n}{1+n^2x^{2n}}$ (*iv*) $\frac{x^n}{1+x^{2n}}$

Solution. (*i*) Let $\quad u_n = \frac{1}{1+nx^n};\ \therefore u_{n+1} = \frac{1}{1+(n+1)x^{n+1}}$

$$\frac{u_{n+1}}{u_n} = \frac{1+nx^n}{1+(n+1)x^{n+1}} = \frac{\frac{1}{nx^n}+1}{\frac{1}{nx^n}+\left(1+\frac{1}{n}\right)x}$$

If $x>1$, $\lim_{n\to\infty}\frac{u_{n+1}}{u_n} = \frac{1}{x}<1$

The series is convergent for $x>1$. When $x = 1$, the series $\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+...$ to ∞ which is divergent.

For $0<x<1$.

$$\frac{1}{1+x}+\frac{1}{1+2x^2}+\frac{1}{1+3x^3}+...>\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+...$$

Since $\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+...$ is divergent, the given series is also divergent when $0<x<1$.

(*ii*) Let $u_n = \frac{x^n}{1+x^n}$

$$\therefore \quad \frac{u_{n+1}}{u_n} = \frac{1+x^n}{1+x^{n+1}}.x$$

If $0<x<1$, $x^n\to 0$ as $n\to\infty$

$\therefore \lim_{n\to\infty}\frac{u_{n+1}}{u_n} = x<1 \qquad \therefore \sum u_n$ is convergent.

If $x>1$, $\lim_{n\to\infty} u_n = \lim_{n\to\infty}\frac{1}{1+\frac{1}{x^n}} = 1\neq 0$

$\therefore$ The *n*th term does not tend to zero as $n\to\infty$

$\therefore \sum u_n$ is divergent.

when $\quad x = 1, u_n = \frac{1}{2}\ \therefore \sum u_n = \sum\frac{1}{2}$ is divergent.

(*iii*) Let $\quad u_n = \frac{x^n}{1+n^2x^{2n}};\ u_{n+1} = \frac{x^{n+1}}{1+(n+1)^2x^{2n+2}}$

$$\frac{u_{n+1}}{u_n} = \frac{1+n^2x^{2n}}{1+(n+1)^2x^{2n+2}}.x$$

If $0<x<1$, n^2x^{2n}, $(n+1)^2x^{2n+2}$ all tend to zero as $n\to\infty$

$\therefore \lim_{n\to\infty}\frac{u_{n+1}}{u_n} = x<1\ \therefore \sum u_n$ is convergent if $0<x<1$

when $x>1$, $\frac{u_{n+1}}{u_n} = \frac{\frac{1}{n^2x^{2n}}+1}{\frac{1}{n^2x^{2n}}+\left(1+\frac{1}{n}\right)^2x^2}.x \to \frac{1}{x^2}.x = \frac{1}{x}<1$

$\therefore \sum u_n$ is convergent if $x>1$.

If $\quad x = 1, u_n = \frac{1}{1+n^2}$ and $\sum u_n$ is convergent.

$\therefore$ The given series is convergent for all real x.

(*iv*) Let $u_n = \frac{x^n}{1+x^{2n}};\ u_{n+1} = \frac{x^{n+1}}{1+x^{2n+2}}$

∴ $\frac{u_{n+1}}{u_n} = \frac{1+x^{2n}}{1+x^{2n+2}}\cdot x$

If $0<x<1$, $\lim_{n\to\infty}\frac{u_{n+1}}{u_n} = x<1$ ∴ $\sum u_n$ is convergent.

If $x>1$, $\frac{u_{n+1}}{u_n} = \frac{\frac{1}{x^{2n}}+1}{\frac{1}{x^{2n}}+x^2}\cdot x \to \frac{1}{x}<1$ since $\frac{1}{x^{2n}}\to 0$ as $n\to\infty$

∴ $\sum u_n$ is convergent if $x>1$

When $x=1$, $u_n=\frac{1}{2}$; The series is $\frac{1}{2}+\frac{1}{2}+\frac{1}{2}+\ldots$ which is divergent.

∴ $\sum u_n$ is convergent if $x \gtrless 1$ and it is divergent if $x=1$

Example 6. *Test the convergence of (i)* $\sum_{n=1}^{\infty}\frac{n^p}{n!}\ (p>0)$

(*i*) $\sum_{n=1}^{\infty}\frac{x^n}{n}$ (*ii*) $\sum_{n=1}^{\infty}\frac{3^n\, n!}{n^n}$ (*iii*) $\sum\frac{n!}{3.5.7\ldots(2n+1)}$

Solution. (*i*) Let $u_n = \frac{n^p}{n!}$

Then $\frac{u_{n+1}}{u_n} = \frac{(n+1)^p}{(n+1)!}\cdot\frac{n!}{n^p} = \left(\frac{n+1}{n}\right)^p\cdot\frac{1}{n+1} = \left(1+\frac{1}{n}\right)^p\cdot\frac{1}{n+1}$

$\lim_{n\to\infty}\frac{u_{n+1}}{u_n} = 0 < 1$

∴ The given series is convergent, by Ratio test.

(*ii*) Let $u_n = \frac{x^n}{n}$

Then $\lim_{n\to\infty}\frac{u_{n+1}}{u_n} = \lim_{n\to\infty}\frac{x^{n+1}}{n+1}\cdot\frac{n}{x^n} = \lim_{n\to\infty}\frac{x}{\left(1+\frac{1}{n}\right)} = x$

∴ The series is convergent if $0<x<1$ and diverges for $x>1$. If $x=1$, then $u_n = \frac{1}{n}$ and hence $\sum u_n$ is divergent.

(*iii*) Let $u_n = \frac{3^n\, n!}{n^n}$

$\frac{u_{n+1}}{u_n} = \frac{3^{n+1}\,(n+1)!}{(n+1)^{n+1}}\cdot\frac{n^n}{3^n\, n!} = \frac{3(n+1)}{\left(\frac{n+1}{n}\right)^n}\cdot\frac{1}{n+1}$

$\lim_{n\to\infty}\frac{u_{n+1}}{u_n} = \lim_{n\to\infty}\frac{3}{\left(1+\frac{1}{n}\right)^n} = \frac{3}{e} > 1$

∴ $\sum u_n$ is divergent by Ratio test.

(*iv*) Let $u_n = \frac{n!}{3.5.7\ldots(2n+1)}$

Then $\frac{u_{n+1}}{u_n} = \frac{(n+1)!}{3.5.7\ldots(2n+3)}\cdot\frac{3.5.7\ldots(2n+1)}{n!} = \frac{n+1}{2n+3}$

$\lim_{n\to\infty}\frac{u_{n+1}}{u_n} = \frac{1}{2} < 1$ ∴ $\sum u_n$ is convergent.

Example 7. *Test for the convergence of (i)* $\sum_{1}^{\infty}\frac{3n^2+1}{4^n}$

(*ii*) $\sum_{1}^{\infty}\left(\frac{1}{2^n}+\frac{1}{3^n}\right)$ (*iii*) $\sum\frac{1}{n!}$ (*iii*) $\frac{1}{3}+\frac{1.2}{3.5}+\frac{1.2.3}{3.5.7}+\ldots$ *to* ∞

Solution. (*i*) Let $u_n = \frac{3n^2+1}{4^n}$

$\frac{u_{n+1}}{u_n} = \frac{3(n+1)^2+1}{4^{n+1}}\cdot\frac{4^n}{3n^2+1} = \frac{3n^2+6n+4}{3n^2+1}\cdot\frac{1}{4}$

$\lim_{n\to\infty}\frac{u_{n+1}}{u_n} = \lim_{n\to\infty}\frac{3+\frac{6}{n}+\frac{4}{n^2}}{3+\frac{1}{n^2}}\cdot\frac{1}{4} = \frac{1}{4} < 1$

∴ $\sum u_n$ is convergent by Ratio test.

(*ii*) Let $u_n = \frac{1}{2^n}+\frac{1}{3^n} = \frac{3^n+2^n}{6^n}$

$\frac{u_{n+1}}{u_n} = \frac{3^{n+1}+2^{n+1}}{6^{n+1}}\cdot\frac{6^n}{3^n+2^n} = \frac{1}{6}\cdot\frac{3^{n+1}+2^{n+1}}{3^n+2^n}$

$= \frac{1}{6}\cdot\frac{3^{n+1}}{3^n}\left(\frac{1+\left(\frac{2}{3}\right)^{n+1}}{1+\left(\frac{2}{3}\right)^n}\right)$

$\lim_{n\to\infty}\frac{u_{n+1}}{u_n} = \frac{1}{2} < 1$ since $\left(\frac{2}{3}\right)^n \to 0$ as $n\to\infty$

Hence, by Ratio test, the given series converges.

(*iii*) Let $u_n = \frac{1}{n!}$; Here $u_n > 0$

$\frac{u_{n+1}}{u_n} = \frac{n!}{(n+1)!} = \frac{1}{n+1}$

$$\lim_{n\to\infty}\frac{u_{n+1}}{u_n}=0<1$$

$\therefore$ The given series is convergent.

(*iv*) Here, $u_n=\dfrac{n!}{3.5.7...(2n+1)}$

$$\frac{u_{n+1}}{u_n}=\frac{(n+1)}{(2n+3)}=\frac{\left(1+\frac{1}{n}\right)}{2+\frac{3}{n}}$$

$\displaystyle\lim_{n\to\infty}\frac{u_{n+1}}{u_n}=\frac{1}{2}<1$; By Ratio test, the series converges.

EXERCISE 3

Test for convergency and divergency of the following series. (Take $x>0$)

1. $\sum\dfrac{1}{1+x^n}$ **2.** $\sum n^{100}x^n$ **3.** $\sum(\sqrt{n^2+1}-n)\,x^n$ **4.** $\sum\dfrac{\sqrt{n}}{n^2+3}.x^n$

5. $\sum\dfrac{x^n}{\sqrt{n}}$ **6.** $\sum\dfrac{a^nx^n}{n^2+1}.a>0$ **7.** $\sum n!x^n$ **8.** $\displaystyle\sum_1^\infty\frac{x^n}{n(n+1)}$

9. $\displaystyle\sum_{n=1}^\infty\frac{x^{n-1}}{(2n-1)^p}$ **10.** $\dfrac{1.x}{2.4}+\dfrac{1.3}{2.4.6}x^2+...+\dfrac{1.3.5...(2n-1)}{2.4.6...(2n+2)}x^n+...$

11. $1+\dfrac{2^2}{3^2}+\dfrac{2^2.4^2}{3^2\,5^2}+\dfrac{2^2.4^2.6^2}{3^2.5^2.7^2}+...\text{to }\infty$ **12.** $\sum\sqrt{\dfrac{2^n-1}{3^n-1}}$ **13.** $\sum\dfrac{n^2}{n!}$

14. $\sum\dfrac{n!}{n^n}$ **15.** $\sum\dfrac{x^n}{1+n^2}$ **16.** $\sum\sqrt{\dfrac{n}{n^2+1}}.x^n$ **17.** $\sum\dfrac{n^2}{3^n}$

18. $\sum\dfrac{n}{1+2^n}$ **19.** $\sum\dfrac{n^3+k}{2^n+k}$ **20.** $\sum\dfrac{n^2-1}{n^2+1}x^n$ **21.** $3x+5x^2+7x^3+...$ to ∞

22. $\dfrac{1}{1+\sqrt{2}}+\dfrac{2}{1+2\sqrt{3}}+\dfrac{3}{1+3\sqrt{4}}+...$ to ∞ **23.** $\sum\dfrac{10^n}{n!}$ **24.** $\sum\dfrac{3^{n-1}}{4^n+1}$

25. $\sum\dfrac{1.3.5...(2n-1)}{4.6.8...(2n+2)}$ **26.** $\sum\dfrac{x^n}{n^2+1}$ **27.** $\dfrac{1}{2}+\dfrac{1.6}{2.6}+\dfrac{1.6.11}{2.6.10}+...\text{to}\infty$

ANSWERS

1. Convergent if $x>1$ and divergent if $0<x\le1$ **2, 3.** Convergent if $0<x<1$ and divergent if $x\ge1$
4. Convergent if $0<x\le1$ and divergent if $x>1$ **5.** Convergent if $0<x<1$ and divergent if $x\ge1$
6. Convergent if $0<x\le\dfrac{1}{a}$ and divergent if $x>\dfrac{1}{a}$
7. Divergent for all real $x(x\ne0)$, and convergent for $x=0$
8. Convergent for $0<x\le1$ and divergent for $x>1$
9. Convergent for $x>1$, for all p divergent for $x>1$ for all p

For $x=1$, convergent if $p>1$. divergent if $p\le1$
10. Convergent if $0<x\le1$, divergent if $x>1$ **11.** Divergent. **12, 13, 14** Convergent
15. Convergent for $x\le1$, divergent for $x>1$ **16.** Convergent for $x<1$, divergent if $x\ge1$
17. 18, 19. Convergent. **20, 21.** Convergent if $x<1$, divergent if $x\ge1$
22. Divergent. **23, 24, 25.** Convergent **26.** Convergent if $0<x\le1$ **27.** Convergent

CAUCHY'S ROOT TEST

If $\displaystyle\sum_{n=1}^\infty u_n$ is a series of positive terms, then the series is convergent or divergent according as $\displaystyle\lim_{n\to\infty}u_n^{1/n}$ is less than 1 or greater than 1.

Example 1. *Test the convergent of the series*

(*i*) $\sum\dfrac{1}{(\log n)^n}$ (*ii*) $\sum\dfrac{(n+1)(n+2)...(n+n)}{n^n}$

(*iii*) $\sum\dfrac{1}{\left(1+\frac{1}{n}\right)^{n^2}}$ (*iv*) $\sum\dfrac{x^n}{(n+1)^n},x>0$

(*v*) $\sum\dfrac{\{r(n+1)\}^n}{n^{n+1}}$ *where r is a constant.*

Solution. (*i*) Let $u_n=\dfrac{1}{(\log n)^n}$; since the power contains n we will use Root test.

$$u_n^{1/n}=\left\{\frac{1}{(\log n)^n}\right\}^{1/n}=\frac{1}{\log n}$$

$\displaystyle\lim_{n\to\infty}u_n^{1/n}\to0$ which is less than 1 as $n\to\infty$

By Cauchy's Root test, the series converges.

(*ii*) Let
$$u_n=\frac{(n+1)(n+2)...(n+n)}{n^n}$$

Let
$$k=\lim_{n\to\infty}u_n^{\frac{1}{n}}=\lim_{n\to\infty}\left\{\frac{n+1}{n}.\frac{n+2}{n}...\frac{n+n}{n}\right\}^{1/n}$$
$$=\lim\left\{\left(1+\frac{1}{n}\right)\left(1+\frac{2}{n}\right)...\left(1+\frac{n}{n}\right)\right\}^{1/n}$$

Taking logrithm

$$\log k=\lim_{n\to\infty}\frac{1}{n}\left\{\sum_{r=1}^{n}\log\left(1+\frac{r}{n}\right)\right\}$$
$$=\int_0^1\log(1+x)\,dx \text{ using integration as a summation}$$
$$=[x\log(1+x)]_0^1-\int_0^1\frac{x}{x+1}dx=2\log2-1=\log\frac{4}{e}$$

$\therefore$ $k=\dfrac{4}{e}>1$ since $2<e<3$

$\therefore \quad \sum u_n$ is divergent.

(*iii*) Let $u_n = \dfrac{1}{\left(1+\dfrac{1}{n}\right)^{n^2}}$

$$u_n^{1/n} = \left\{\frac{1}{\left(1+\frac{1}{n}\right)^{n^2}}\right\}^{1/n} = \frac{1}{\left(1+\frac{1}{n}\right)^{n}}$$

$$\lim_{n\to\infty} u_n^{1/n} = \lim_{n\to\infty} \frac{1}{\left(1+\frac{1}{n}\right)^{n}} = \frac{1}{e} < 1$$

$\therefore$ The given series is convergent.

(*iv*) Let $\quad u_n = \dfrac{x^n}{(n+1)^n};\ u_n^{1/n} = \dfrac{x}{n+1}$

$\lim_{n\to\infty} u_n^{1/n} = 0$ for all real x.

Since this limit is < 1, the series is convergent.

(*v*) Let $u_n = \dfrac{\{r(n+1)\}^n}{n^{n+1}}$

$$\therefore \quad u_n^{1/n} = \frac{r(n+1)}{n^{\left(1+\frac{1}{n}\right)}} = \frac{r.\left(1+\frac{1}{n}\right)}{n^{1/n}}$$

$\lim_{n\to\infty} u_n^{1/n} = r$ since $\lim_{n\to\infty} n^{1/n} = 1$

$\therefore \sum u_n$ is convergent if $r < 1$ and divergent if $r > 1$ when $r = 1$, Root test fails. Therefore go to basic idea.

When $\quad r = 1,\ u_n = \left(1+\dfrac{1}{n}\right)^n . \dfrac{1}{n}$

Take $v_n = \dfrac{1}{n}$ and use comparison test.

$$\frac{u_n}{v_n} = \left(1+\frac{1}{n}\right)^n \to e \text{ as } n \to \infty$$

$\therefore \sum u_n$ and $\sum v_n$ converge or diverge together. But $\sum v_n$ is divergent.

$\therefore \sum u_n$ is divergent when $r = 1$.

Example 2. *Test the convergence of (i)* $\sum \dfrac{n^3}{3^n}$ *(ii)* $\sum_{n=1}^{\infty} e^{\sqrt{n}} x^n$ *(iii)* $\sum_{n=1}^{\infty} \left(\dfrac{n+a}{n+b}\right)^n x^n$

Solution. (*i*) Let $\quad u_n = \dfrac{n^3}{3^n};\ u_n^{\frac{1}{n}} = \left(\dfrac{n^3}{3^n}\right)^{1/n} = \dfrac{(n^{1/n})^3}{3}$

$$\lim_{n\to\infty} u_n^{\frac{1}{n}} = \frac{1}{3} < 1 \text{ since } \lim_{n\to\infty} n^{1/n} = 1$$

By Root test, $\sum u_n$ converges.

(*ii*) Let $\quad u_n = e^{\sqrt{n}} x^n$

$$u_n^{\frac{1}{n}} = (e^{\sqrt{n}} x^n)^{1/n} = e^{1/\sqrt{n}}.x$$

$$\lim_{n\to\infty} u_n^{\frac{1}{n}} = x$$

$\therefore$ By Root test, $\sum u_n$ converges if $0 < x < 1$ and diverges if $x > 1$

If $\quad x = 1,\ u_n = e^{\sqrt{n}}$ $\lim_{n\to\infty} u_n = \infty \therefore \sum u_n$ diverges if $x = 1$

(*iii*) Let $\quad u_n = \left(\dfrac{n+a}{n+b}\right)^n . x^n$

$$\therefore \quad u_n^{\frac{1}{n}} = \frac{n+a}{n+b}.x = \frac{\left(1+\frac{a}{n}\right)}{1+\frac{1}{n}}.x \quad \therefore \lim_{n\to\infty} u_n^{\frac{1}{n}} = x$$

$\therefore \sum u_n$ converges if $0 < x < 1$ and diverges if $x > 1$ by Root test.

If $\quad x = 1,\ u_n = \left(\dfrac{n+a}{n+b}\right)^n = \dfrac{\left(1+\frac{a}{n}\right)^n}{\left(a+\frac{b}{n}\right)^n}$

$$\lim_{n\to\infty} u_n = \frac{\lim_{n\to\infty}\left(1+\frac{a}{n}\right)^n}{\lim_{n\to\infty}\left(1+\frac{b}{n}\right)^n} = \frac{e^a}{e^b} = e^{a-b}$$

$\lim_{n\to\infty} u_n$ does not tend to zero.

$\therefore \quad \sum u_n$ diverges if $x = 1$

Example 3. *Test the convergence of* $\sum_{n=1}^{\infty} \dfrac{n^k}{a^n}$ *and hence discuss the convergence of* $\sum_{n=1}^{\infty} \dfrac{n^k+b}{a^n+c}$

Solution. Let $\quad u_n = \dfrac{n^k}{a^n};\ u_n^{\frac{1}{n}} = \left(\dfrac{n^k}{a^n}\right)^{1/n} = \dfrac{(n^{1/n})^k}{a}$

$$\lim_{n\to\infty} u_n^{\frac{1}{n}} = \frac{1}{a} \lim_{n\to\infty} (n^{1/n})^k = \frac{1}{a}.1^k = \frac{1}{a}$$

$\therefore$ If $\dfrac{1}{a} < 1$, $\sum \dfrac{n^k}{a^n}$ converges.

That is, $\sum \dfrac{n^k}{a^n}$ converges if $a > 1$ and diverges if $a < 1$

If $a = 1$, $\sum u_n = \sum n^k = \sum \frac{1}{n^{-k}}$

which converges if $-k > 1$ and diverges if $-k \le 1$

Taking $\quad u_n = \frac{n^k}{a^n}$ and $v_n = \frac{n^k + b}{a^n + c}$

$$\frac{v_n}{u_n} = \frac{n^k + b}{a^n + c} \cdot \frac{a^n}{n^k} = \left(\frac{n^k + b}{n^k}\right)\left(\frac{a^n}{a^n + c}\right) = \left(1 + \frac{b}{n^k}\right)\left(\frac{1}{1 + \frac{c}{a^n}}\right)$$

$$\lim_{n\to\infty} \frac{v_n}{u_n} = 1 \text{ since } n^k \to \infty \text{ if } k > 0 \text{ and } a^n \to \infty \text{ if } a > 1$$

$\sum v_n$ and $\sum u_n$ converge or diverge together.

$\therefore \sum v_n$ converges if $a > 1$ and $k > 0$

If $0 < a < 1$, $a^n \to 0$ and then $\lim\limits_{n\to\infty} \frac{v_n}{u_n} = 0$

Hence comparison test fails.

Example 4. *Test for convergence of* $\sum \frac{n!}{n^n}$

Solution. Let $u_n = \frac{n!}{n^n}$

$$k = \lim_{n\to\infty} u_n^{\frac{1}{n}} = \lim_{n\to\infty} \left\{\frac{n!}{n^n}\right\}^{1/n} = \lim_{n\to\infty} \left\{\frac{1}{n}\cdot\frac{2}{n}\cdots\frac{n}{n}\right\}^{1/n}$$

$$\log k = \lim_{n\to\infty} \frac{1}{n}\left\{\sum_{r=1}^{n} \log \frac{x}{n}\right\} = \int_0^1 \log x \, dx$$

by using the concept that integration is the summation

$$= (x \log x)_0^1 - \int_0^1 dx = -1$$

$$\therefore \qquad k = e^{-1} = \frac{1}{e} < 1$$

Hence $\sum u_n$ is convergent.

EXERCISE 4

Test for convergence of the series below:

1. $\sum \frac{x^n}{n^n}, x > 0$ 2. $\sum \left(1 + \frac{1}{\sqrt{n}}\right)^{n^{-3/2}}$ 3. $(\sqrt[n]{n} - 1)^n$ 4. $a + b + a^2 + b^2 + a^3 + b^3 + \ldots$ to ∞

5. $\sum \left(\frac{n}{2n+1}\right)^n$ 6. $\sum \left(\frac{n}{n^2+1}\right)^n$ 7. $\sum \frac{2^n n!}{n^n}$ 8. $\sum \frac{3^n n!}{n^n}$

9. $\sum e^{-\sqrt{n}} x^n$ 10. $\sum_{n=1}^{\infty} \frac{n^3 + 100}{2^n + 10}$

ANSWERS

1. 2, 3, Convergent 4. Convergent if $0 < a < 1$ and $0 < b < 1$ and divergent if $a \ge 1$ or $b \ge 1$
5. 6, 7, Convergent 8. Divergent 9. Converges if $0 < x < 1$ and diverges if $x > 1$
10. Convergent

INTEGRAL TEST

If $\sum f(n) = f(1) + f(2) + \ldots + f(n) + \ldots$ is a series of positive terms and $f(n)$ decreases as n increases, then $\sum f(n)$ converges or diverges according as the integral

$\int_1^{\infty} f(x)\, dx$ is finite or infinite

Example 1. *By using integral test, show that the harmonic series* $\sum \frac{1}{n^p}$ *converges for* $p > 1$ *and diverges for* $p \le 1$

Solution. Here, $f(n) = \frac{1}{n^p}$ decrease as n increases. For $p \ne 1$, $\int_1^{\infty} \frac{1}{x^p} dx = \mathop{\mathrm{Lt}}\limits_{N\to\infty} \int_1^N \frac{dx}{x^p}$

$$= \mathop{\mathrm{Lt}}_{N\to\infty} \left(\frac{x^{1-p}}{1-p}\right)_1^N = \mathop{\mathrm{Lt}}_{N\to\infty} \frac{N^{1-p} - 1}{1-p} = \frac{1}{p-1} \text{ if } p > 1$$

and $\to \infty$ if $p < 1$

Here $\frac{1}{p-1}$ is finite. If $p = 1$, $\int_1^{\infty} \frac{1}{x^p} dx = \int_1^{\infty} \frac{1}{x} dx = (\log x)_1^{\infty} \to \infty$

Hence, $\sum \frac{1}{n^p} = \sum f(n)$ converges if $p > 1$ and diverges if $p \le 1$.

Example 2. Using integral test, show that the series $\frac{1}{1.3} + \frac{2}{3.5} + \frac{3}{5.7} + \ldots$, to ∞, diverges

Solution. $\qquad n$th term $= \frac{n}{(2n-1)(2n+1)}$

Let $\qquad f(n) = \frac{n}{(2n-1)(2n+1)}$

$$f(n) - f(n+1) = \frac{n}{(2n-1)(2n+1)} - \frac{n+1}{(2n+1)(2n+3)}$$

$$= \frac{1}{2n+1}\left[\frac{n}{2n-1} - \frac{n+1}{2n+3}\right] = \frac{1}{(2n+1)} \cdot \frac{2n+1}{(2n-1)(2n+3)}$$

$$= \frac{1}{(2n-1)(2n+3)} = +\text{ve}$$

$\therefore f(n+1) < f(n)$

That is, $f(n)$ decreases as n increases.

$$\int_1^{\infty} f(x)\, dx = \int_1^{\infty} \frac{x}{(2x-1)(2x+1)} dx = \int_1^{\infty} \frac{1}{4}\left[\frac{1}{2x-1} + \frac{1}{2x+1}\right] dx$$

$$= \mathop{\mathrm{Lt}}_{N\to\infty} = \int_1^{N} \frac{1}{4}\left[\frac{1}{2x-1} + \frac{1}{2x+1}\right] dx$$

$$= \underset{N\to\infty}{\text{Lt}} \frac{1}{8}\left[\log(2x-1)+\log(2x+1)\right]_1^N$$

$$= \underset{N\to\infty}{\text{Lt}} \frac{1}{8}\left[\log(4x^2-1)\right]_1^N$$

$$= \underset{N\to\infty}{\text{Lt}} \frac{1}{8}\left[\log(4N^2-1)-\log 3\right] \to \infty$$

$\therefore$ Given series diverges by integral test.

Note : This could be done easily by comparison test

EXERCISE 5

Using integral test, test for convergence or divergence of the following series. (Given under Exercise 2) whose nth term is

1. $\dfrac{n}{(n+2)(n+3)(n+4)}$ 2. $\dfrac{n(n+1)}{(n+2)(n+3)(n+4)}$ 3. $\dfrac{1}{x+n}$ if $x>0$ 4. $\dfrac{1}{x+ny}$ if $x, y>0$

5. $\dfrac{1}{2n-1}$ 6. $\dfrac{1}{(2n-1)(2n)}$

ANSWERS

1. 6. Convergent; 2 to 5 divergent.

Note : All these problems can be easily done by comparison list.

ALTERNATING SERIES

Hitherto, we were dealing with infinite series whose terms are all positive. In practice, this need not be the case. There are series whose terms are positive and negative.

Alternating series. A series in which the terms are alternately positive and negative is called an *alternating series.*

Absolute convergence of series. The series $\sum u_n$ containing positive and negative terms is said to be *absolutely convergent,* if the series formed by the numerical values of the terms of $\sum u_n$ is convergent. That is, $\sum u_n$ is absolutely convergent if $\sum |u_n|$ is convergent.

Conditionally convergent. The series $\sum u_n$ containing positive and negative terms is said to be *conditionally convergent* or *semi convergent,* if

(*i*) $\sum u_n$ is convergent while

(*ii*) $\sum |u_n|$ is not convergent. (That is, divergent)

Example $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2} = \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2}$ ... *is absolutely convergent since.*

$$\sum \left|\frac{(-1)^{n-1}}{n^2}\right| = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots \text{ is convergent.}$$

While,

$$\sum u_n = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots \text{ is convergent whereas}$$

$$\sum |u_n| = 1 + \frac{1}{2} + \frac{1}{3} + \ldots \text{ is divergent and hence}$$

$1 - \frac{1}{2} + \frac{1}{3} - \ldots$ *is conditionally convergent.*

Theorems. (without proof)

1. An absolutely convergent series is convergent.

2. If the terms of an absolutely convergent series are rearranged, the series remains convergent and its sum is unaltered.

3. In a conditionally convergent series, a rearrangement of terms *may alter the sum of the series.*

4. If $\sum u_n$ is an alternating series and if $\displaystyle\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = l < 1$, the series $\sum u_n$ is absolutely convergent and hence it is convergent while if $l > 1$, the series $\sum u_n$ is not convergent.

Example 1. *Prove that the exponential series* $1 + \frac{x}{1!} + \frac{x^2}{2!} + \ldots + \frac{x^n}{n!} + \ldots$ *to* ∞ *is absolutely convergent and hence convergent for all values of x.*

Solution. Let
$$u_n = \frac{x^{n-1}}{(n-1)!};\ u_{n+1} = \frac{x^n}{n!}$$

$$\frac{u_{n+1}}{u_n} = \frac{x^n}{n!} \times \frac{(n-1)!}{x^{n-1}} = \frac{x}{n}$$

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\frac{|x|}{n} = 0 < 1 \quad \text{for all } x.$$

Hence the series is absolutely convergent and hence convergence for all x. (real)

Example 2. *Discuss the convergence of the Binomial series.*

Solution. The Binomial series is

$$1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \ldots \text{ to } \infty$$

Here
$$u_r = \frac{n(n-1)(n-2)\ldots(n-r+2)}{(r-1)!}x^{r-1}$$

$$u_{r+1} = \frac{n(n-1)(n-2)\ldots(n-r+1)}{r!}x^r$$

$$\frac{u_{r+1}}{u_r} = \frac{n-r+1}{r}.x = \left(\frac{n+1}{r} - 1\right)x$$

$$\left|\frac{u_{r+1}}{u_r}\right| = \left|\frac{n+1}{r} - 1\right||x|$$

$$\lim_{r\to\infty}\left|\frac{u_{r+1}}{u_r}\right| = |x| . \lim_{r\to\infty}\left|\frac{n+1}{r} - 1\right| = |x|$$

$\therefore$ The series is absolutely convergent if $|x| < 1$ and hence convergent if $|x| < 1$

If $|x| > 1$, the series $\sum |u_n|$ is divergent.

Example 3. *Discuss the convergence of the logarithmic series.*

Solution. The logarithmic series is

$$x - \frac{x^2}{2} + \frac{x^3}{3} - \dots + (-1)^{n-1}\frac{x^n}{n} + \dots \text{ to } \infty$$

$$u_n = (-1)^{n-1}\frac{x^n}{n}$$

$$\frac{u_{n+1}}{u_n} = -\frac{n}{n+1}x = \frac{-x}{\left(1+\frac{1}{n}\right)}$$

$\lim\limits_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = |x|$. $\therefore$ If $|x| < 1$, $\sum u_n$ is absolutely convergent and hence convergent. If $|x| > 1$, u_n does not tend to zero as $n \to \infty$ hence the series is not convergent.

If $x = 1$, the series is $\sum \frac{(-1)^{n-1}}{n}$ which is convergent by alternating series test since

$$\frac{1}{n} > \frac{1}{n+1} \text{ and } \lim_{n\to\infty}\frac{1}{n} = 0$$

If $x = -1$, the series becomes $\sum\left(-\frac{1}{n}\right)$ which is divergent. Hence the logarithmic series converges if $-1 < x \le 1$ and diverges if $x \le -1$ and $x > 1$.

LEIBNITZ'S TEST

The alternating series $\sum_{n=1}^{\infty}(-1)^{n-1}u_n$ converges if

(*i*) $\{u_n\}$ is a monotonic decreasing sequence

and (*ii*) $\lim\limits_{n\to\infty} u_n = 0$.

In other words,

The alternating series $u_1 - u_2 + u_3 - u_4 + \dots$ to ∞ converges if

(*i*) Each term is numerically less than its preceding term and

(*ii*) $\lim\limits_{n\to\infty} u_n = 0$.

Note. In the above theorem, if $\lim\limits_{n\to\infty} u_n \ne 0$, then the alternating series $\sum_{n=1}^{\infty}(-1)^{n-1}u_n$ oscillates.

Example 1. *Prove* $\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{\log(n+1)}$ *converges.*

Solution. Let $u_n = \frac{1}{\log(n+1)}$

The given series is an alternating series.

$\log(n+2) > \log(n+1)$ for $n > 0$

$$\frac{1}{\log(n+2)} < \frac{1}{\log(n+1)} \quad u_{n+1} < u_n \text{ for } n > 0$$

$\{u_n\}$ is monotonic decreasing.

$$\lim_{n\to\infty} u_n = \lim_{n\to\infty}\frac{1}{\log(n+1)} = 0$$

Hence by Leibnitz's test, the given series converges.

Example 2. *Test for convergence of the series*

$$(1+1) - \left(1+\frac{1}{2}\right) + \left(1+\frac{1}{3}\right) - \left(1+\frac{1}{4}\right) + \dots \text{ to } \infty$$

Solution. Here, $a_n = 1 + \frac{1}{n}$ where the given series is $\sum_{1}^{\infty}(-1)^{n-1}a_n$

Since $\frac{1}{n+1} < \frac{1}{n}$, $a_{n+1} < a_n$ and

$\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty}\left(1+\frac{1}{n}\right) = 1 \ne 0$; Hence the series given is not convergent.

Example 3. *Bring out the fallacy in the following proof:*

"*We know,* $1 - \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \dots$ to $\infty = \log_e 2$

Also, $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots$ to $\infty = \left(1 + \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots \text{ to } \infty\right) - 2\left(\frac{1}{2} + \frac{1}{4} + \frac{1}{6} + \dots \text{ to } \infty\right)$

$$= \left(1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \dots \text{ to } \infty\right) - \left(1 + \frac{1}{2} + \frac{1}{3} + \dots \text{ to } \infty\right) = 0"$$

Solution. In the above argument, we have done rearrangement of terms which is not permissible. Also, to find the sum of an infinite series, we should get s_n = sum to n terms and then get limit s_n as $n \to \infty$. This is not done here in the above proof. Hence the fallacy.

Example 4. *Discuss the convergence of*

(*i*) $\sum_{n=1}^{\infty}(-1)^n(\sqrt{n+1} - \sqrt{n})$ (*ii*) $\sum_{n=1}^{\infty}(-1)^{n-1}\cdot\frac{n}{3n-2}$

Solution. (*i*) The series is evidently alternating series

$$u_n = \sqrt{n+1} - \sqrt{n} = \frac{(\sqrt{n+1}-\sqrt{n})(\sqrt{n+1}+\sqrt{n})}{\sqrt{n+1}+\sqrt{n}} = \frac{1}{\sqrt{n+1}+\sqrt{n}}$$

$$u_{n+1} = \frac{1}{\sqrt{n+2}+\sqrt{n+1}}$$

Since $(\sqrt{n+2}+\sqrt{n+1}) > (\sqrt{n+1}+\sqrt{n})$, evidently $u_{n+1} < u_n$ for all n *i.e.*, $\{u_n\}$ is a monotonic decreasing sequence

$$\lim_{n\to\infty} u_n = \lim_{n\to\infty}\frac{1}{\sqrt{n+1}+\sqrt{n}} = 0.$$

$\therefore$ By Leibnitz's test, the given series converges.

(*ii*) Let $u_n = \frac{n}{3n-2}$. The given series is an alternating series.

We have to prove $u_n > u_{n+1}$ for all n.

i.e., $$\frac{n}{3n-2} > \frac{n+1}{3(n+1)-2}$$

i.e., $$\frac{3n-2}{n} < \frac{3(n+1)-2}{n+1}$$

i.e., $$3-\frac{2}{n}<3-\frac{2}{n+1}$$

i.e., $$-\frac{2}{n}<-\frac{2}{n+1}$$

i.e., $$\frac{2}{n}>\frac{2}{n+1}$$

i.e., $n<n+1$ which is true.

Hence $u_n > u_{n+1}$ for all

Further $$\lim_{n\to\infty} u_n = \lim_{n\to\infty}\frac{n}{3n-2}=\frac{1}{3}\neq 0$$

$\therefore$ The given series oscillates.

Note. By testing $\lim_{n\to\infty} u_n = \frac{1}{3}\neq 0,$ we can immediately say that the given series does not converge; the first part need not be proved.

Example 5. *Discuss the convergence of*

(*i*) $\sum(-1)^n\left(1+\frac{1}{n}\right)$ (*ii*) $\sum(-1)^n\sin\left(\frac{1}{n}\right)$ (*iii*) $\frac{1}{1.2}-\frac{1}{3.4}+\frac{1}{5.6}-\ldots$ *to* ∞

Solution. (*i*) Let $u_n = 1+\frac{1}{n}$

$$u_{n+1} = 1+\frac{1}{n+1} \qquad \therefore u_{n+1}<u_n \text{ for all } n$$

and $$\lim_{n\to\infty} 1+\frac{1}{n} = 1\neq 0.$$

Hence the given series does not converge. In fact, it oscillates.

(*ii*) Here $u_n = \sin\left(\frac{1}{n}\right); u_{n+1} = \sin\left(\frac{1}{n+1}\right)$

$$\frac{1}{n+1}<\frac{1}{n} \text{ and } \sin\left(\frac{1}{n+1}\right)<\sin\left(\frac{1}{n}\right)$$

i.e., $u_{n+1}<u_n$ for all n.

Further $\lim_{n\to\infty}\sin\left(\frac{1}{n}\right)=0$

Hence, by alternating series test, it converges.

(*iii*) Here $$u_n = \frac{1}{(2n-1)(2n)}$$

and $$u_{n+1} = \frac{1}{(2n+1)(2n+2)}$$

Since $(2n+1)(2n+2)>(2n-1)(2n)$

$$u_{n+1} = u_{n+1} < u_n$$

and $$\lim_{n\to\infty} u_n = \lim_{n\to\infty}\frac{1}{(2n-1)(2n)}=0$$

Hence, the given series is convergent.

Example 6. *Discuss the convergence of the series*

(*i*) $1-\frac{\sqrt{2}}{2}+\frac{\sqrt{3}}{3}-\frac{\sqrt{4}}{4}+\ldots\infty$ (*ii*) $1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\ldots\infty$

(*iii*) $\sum\frac{(-1)^{n-1}}{n^p}$ *where* $p>0$ (*iv*) $\sum_{n=1}^{\infty}\frac{n(-1)^{n-1}}{2n-1}$

Solution. Consider $1-\frac{\sqrt{2}}{2}+\frac{\sqrt{3}}{3}-\frac{\sqrt{4}}{4}+\ldots$to ∞.

(*i*) In alternating series, we try to use Leibnitz's test.

Points to check

P1. The terms are alternately positive and negative

P2. $$u_n = \frac{\sqrt{n}}{n}=\frac{1}{\sqrt{n}}$$

$$u_{n+1} = \frac{1}{\sqrt{n+1}}$$

Evidently $u_{n+1}<u_n$ since $\frac{1}{\sqrt{n+1}}<\frac{1}{\sqrt{n}}$ for all n

P3. Further $\lim_{n\to\infty} u_n = \lim_{n\to\infty}\frac{1}{\sqrt{n}}=0$

All the conditions of the Leibnitz's test are satisfied.

Hence the given series is convergent.

However, it is not absolutely convergent (why ?)

(*ii*) Consider $1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\ldots$ to ∞

Here $$u_n = \frac{1}{n} \text{ and } u_{n+1} = \frac{1}{n+1}$$

P1. The series is alternately positive and negative.

P2. $u_{n+1}<u_n$ since $\frac{1}{n+1}<\frac{1}{n}$ for all n.

P3. $\lim_{n\to\infty} u_n = \lim_{n\to\infty}\frac{1}{n}=0.$

Hence, by Leibnitz's test, the given series is convergent.

Note. This series is semi-convergent or conditionally convergent but not absolutely convergent. (why ?)

(*iii*) $$u_n = \frac{1}{n^p} \text{ and } u_{n+1} = \frac{1}{(n+1)^p}$$

$$\frac{u_{n+1}}{u_n} = \frac{n^p}{(n+1)^p} = \frac{1}{\left(1+\frac{1}{n}\right)^p}<1$$

since $p>0, \left(1+\frac{1}{n}\right)^p>1$

$\therefore u_{n+1}<u_n$

and the series is alternating.

Also $$\lim u_n = \lim_{n\to\infty}\frac{1}{n^p}=0 \text{ if } p>0$$

(*iv*) Here

$$u_n = \frac{n}{2n-1}$$

$$\lim_{n\to\infty} u_n = \lim_{n\to\infty} \frac{1}{2-\frac{1}{n}} = \frac{1}{2} \neq 0$$

∴ The series is not convergent.
In fact, it is oscillatory.

Example 7. *Discuss the convergence of the series* $\frac{x}{1+x} - \frac{x^2}{1+x^2} + \frac{x^3}{1+x^3} - \ldots\infty$, *if* $0 < x < 1$.

Solution. Since x is positive, the series is alternately positive and negative.

Here $\quad u_n = \frac{x^n}{1+x^n}$ and $u_{n+1} = \frac{x^{n+1}}{1+x^{n+1}}$

$$u_n - u_{n+1} = \frac{x^n}{1+x^n} - \frac{x^{n+1}}{1+x^{n+1}} = x^n\left(\frac{1}{1+x^n} - \frac{x}{1+x^{n+1}}\right)$$

$$= \frac{x^n(1-x)}{(1+x^n)(1+x^{n+1})} = \text{a positive value since } 0 < x < 1$$

∴ $u_{n+1} < u_n$ for all n.

Also, $\quad \lim_{n\to\infty} u_n = \lim_{n\to\infty} \frac{x^n}{1+x^n} = 0$

since $\quad x^n \to 0$ as $n \to \infty$

∴ The series is convergent.

EXERCISE 6

Discuss the convergence of the series :

1. $1 - \frac{1}{5} + \frac{1}{9} - \frac{1}{13} + \ldots$ to ∞
2. $1 - \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} - \frac{1}{\sqrt{4}} + \ldots$
3. $1 - \frac{1}{2\sqrt{2}} + \frac{1}{3\sqrt{3}} - \frac{1}{4\sqrt{4}} + \ldots$
4. $\sum \frac{(-1)^n}{n+7}$
5. $\sum \frac{(-1)^{n-1}}{x+n}$
6. $1 - \frac{1}{2!} + \frac{1}{3!} - \frac{1}{4!} + \ldots$ to ∞
7. $\sum (-1)^n(\sqrt{n+1} - \sqrt{n})$
8. $x - \frac{x^2}{\sqrt{2}} + \frac{x^3}{\sqrt{3}} - \frac{x^4}{\sqrt{4}} + \ldots$ to ∞
9. $1 - \frac{x}{1+a} + \frac{x^2}{1+2a} - \ldots + \frac{(-1)^{n-1}x^{n-1}}{1+(n-1)a} + \ldots$
10. $\sum \frac{(-1)^n n}{2n+7}$
11. $\sum \frac{(-1)^n x^n}{\sqrt{n}}$
12. $\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \ldots$ to ∞
13. $1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \ldots$ to ∞
14. $\sum (-1)^n \frac{1+n^2}{1+n^3}$

ANSWERS

1 to 7. Convergent **8., 9.** Convergent if $|x| < 1$ **10.** Not convergent
11. Convergent if $|x| < 1$ **12.** abs. convergent **13, 14.** Convergent

SHORT ANSWER QUESTIONS

1. Define convergence and divergence of a sequence.
2. Define oscillating sequence, bounded sequence.

3. Define monotoric sequence.
4. State Cauchy's general principal of convergence of a sequence
5. What is the necessary condition for $\sum u_n$ to converge ?
6. Give an example to show that $\lim_{n\to\infty} u_n = 0$ is not sufficient for covergence of $\sum u_n$
7. State D'Alembert's ratio test, Comparison test, Cauchy's Root test, Raabe's test, Leibnitz's test
8. Define absolute convergence and conditional convergence of a series. Give one example for each case.
9. Every absolutely convergent series is ....
10. $\sum \frac{(-1)^{n-1}}{n^2}$ is ....
11. A series of positive terms cannot ....
12. .... test is used to prove $\sum \frac{(-1)^{n-1}}{n}$ is convergent.
13. If $u_n \geq k\, v_n$ and $\sum v_n$ is divergent then $\sum u_n$ is ....
14. If $\sum_{n=1}^{\infty} u_n$ is convergent then $\sum_{n=100}^{\infty} u_n$ is ....
15. $\left\{\frac{n}{n+1}\right\}$ is monotonically .... and bounded ....
16. Under what conditions $\sum \frac{1}{n^p}$ is convergent, or divergent ?
17. If $|a_n - l| < \varepsilon$ then a_n lies between ....
18. $\lim_{n\to\infty} n^{\frac{1}{n}}$ is .... **Ans.** 1
19. $\sum_{1}^{\infty} \frac{n^2}{3^n}$ is ...... **Ans.** Convergent

TRUE OR FALSE

20. If $\sum a_n$ is convergent then $a_n \to 0$ as $n \to \infty$ (T)
21. $\sum_{1}^{\infty} \frac{1}{n(n+1)}$ converges to the value 1. (T)
22. If $a_n > 0$ and $\sum a_n$ is convergent then $\sum a_n^2, \sum a_n^3$ are also convergent (T)
23. In general, if $\sum a_n^{\ 2}$ in convergent then $\sum a_n$ is convergent (F)
24. If $\sum a_n$ is divergent then $\sum a_n^2$ is divergent. (F)
25. $\sum_{n=2}^{\infty} \frac{1}{(\log n)^n}$ converges. (T)

Ans. Leibnitz's

26. The convergence of $\sum \frac{(-1)^{n-1}}{n^p}$ is tested by using .... test. **Ans.** $\sum \frac{(-1)^{n-1}}{n}$
27. Give an example of a series which is conditionally convergent.
28. Define: absolutely convergent, semi-convergent, conditionally convergent.
29. State the tests: Ratio test, Cauchy's Root test, Leibnitz's test, Comparison test.
30. Prove $\frac{\sin x}{1^3} - \frac{\sin 2x}{2^3} + \frac{\sin 3x}{3^3} - \ldots$ to ∞ is absolutely convergent.