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1 MATRICES

1.1 Introduction

The word Eigen value comes from the German Eigenwert which means proper or characteristic value. Eigen values and eigen vectors are very important concept in linear algebra and differential equations.

The main application comes from the wave equations that are associated with transmitting power through transmission lines. These equations relate the voltages and currents on the sending and receiving ends of a transmission line.

Oil companies frequently use eigen value analysis to explore land for oil. Oil, dirt, and other substances all give rise to linear systems which have different eigen values, so eigen value analysis can give a good indication of where oil reserves are located.

Eigen values can also be used to test for cracks or deformities in a solid. Car designers analyze eigen values in order to damp out the noise so that the occupants have a quiet ride. Eigen value analysis is also used in the design of car stereo systems so that the sounds are directed correctly for the listening pleasure of the passengers and driver.

Eigen values are used in structural analysis to calculate buckling margins of safety. If you have a certain amount of axial and shear stress in a panel you can find the eigen value solution. The eigen value is the ratio of the buckling stress to the total stress.

Eigenvalues and Eigenvectors are also used in many engineering and science applications such as Control theory, vibration analysis, electric circuits etc. Many applications use eigenvalues and vectors to transform a given matrix into a diagonal matrix.

1.2 Eigen Values and Eigen Vectors of a given Square Matrix

Definition 1.1. *Let A be any given non-zero square matrix. If there exists a scalar λ and non-zero column matrix X such that $Ax = \lambda X$, then the scalar λ is called an Eigen value or Characteristic value or Latent value of the matrix A and X is called the corresponding Eigen vector or Characteristic vector of A.*

Determination of λ and X
Let $AX = \lambda X \Rightarrow AX - \lambda IX = O \Rightarrow [A - \lambda I]X = O$. Since $X \neq O$ we have $|A - \lambda I| = 0$. This equation, which determines λ is called the characteristic equation of A. The corresponding eigen vectors X can now be determined by considering the ho-

mogeneous system $[A - \lambda I]X = O$.

1.2.1 Properties of Eigen Values

Property 1.1. *Prove that every square matrix and its transpose have the same eigen values.*

Proof. Let $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ and $A^T = \begin{bmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \end{bmatrix}$

Characteristic equation of A is $|A - \lambda I|X = 0$

$$(i.e.) \begin{vmatrix} a_{11} - \lambda I & a_{12} & a_{13} \\ a_{21} & a_{22} - \lambda I & a_{23} \\ a_{31} & a_{32} & a_{33} - \lambda I \end{vmatrix} = 0$$

Characteristic equation of A^T is $|A^T - \lambda I|X = 0$. Since $|A - \lambda I| = |A^T - \lambda I| = 0$, the characteristic equations are the same for both A and A^T. Hence every square matrix and its transpose have the same eigen values.

Property 1.2. *If $\lambda_1, \lambda_2, \lambda_3, ...\lambda_n$ are the eigen values of the matrix A, then $k\lambda_1, k\lambda_2, k\lambda_3, ...k\lambda_n$ are the eigen values of kA.*

Proof.

Let λ be an eigen value of the matrix A, then $AX = \lambda X$ (1.1)

where X is an eigen vector and $X \neq 0$.
Pre multiply (1.1) by A^{-1}, we have $A^{-1}AX = A\lambda X$
$\Rightarrow IX = \lambda(A^{-1}X) \Rightarrow IX = \lambda(A^{-1}X)(A^{-1}A = I)$.

$$\Rightarrow X = \lambda A^{-1}X \Rightarrow A^{-1}X = \frac{1}{\lambda}X \tag{1.2}$$

From equations (1.1) and (1.2), we get $\frac{1}{\lambda}$ is an eigen value of A^{-1}.

Property 1.3. *If $\lambda_1, \lambda_2, \lambda_3, ...\lambda_n$ are the eigen values of the matrix A, then prove that $\lambda_1^2, \lambda_2^2, \lambda_3^2, ...\lambda_n^2$ are the eigen values of A^2*

Proof.

Let λ be an eigen value of the matrix A, then $AX = \lambda X$ (1.3)

Pre multiply (1.3) by A, we have $A(AX) = A(\lambda X$

$$\Rightarrow A^2X = \lambda(AX) = \lambda(\lambda X) \Rightarrow A^2X = \lambda^2X. \tag{1.4}$$

From equations (1.3) and (1.4), λ^2 is an eigen value of A^2.

Property 1.4. *If $\lambda_1, \lambda_2, \lambda_3, ...\lambda_n$ are the eigen values of the matrix A, then $k\lambda_1, k\lambda_2, k\lambda_3, ...k\lambda_n$ are the eigen values of kA.*

Proof.

Let λ be an eigen value of the matrix A, then $AX = \lambda X$ (1.5)

Multiply both sides of (1.5) by k, we have

$$k(AX) = k(\lambda X \Rightarrow (kA)X = (k\lambda)X \tag{1.6}$$

Comparing (1.5) and (1.6), we have $k\lambda$ is an eigen values of kA.

Property 1.5. *Prove that the eigen values of a real symmetric matrix are all real.*

Proof.

Let λ be an eigen value of the matrix A, then $AX = \lambda X$ (1.7)

Taking conjugate on both sides of (1.7), we have $\overline{AX} = \overline{\lambda X}$, Since A is real, $\overline{A} = A$ and hence $A\overline{X} = \overline{\lambda X}$. Taking transpose on both sides, we have $(A\overline{X})^T = (\overline{\lambda X})^T \Rightarrow \overline{X}^T A^T = \overline{\lambda}^T \overline{X}^T$. Since A is symmetric, $A^T = A$, $\overline{X}^T A = \overline{\lambda X})^T \Rightarrow (\overline{\lambda}^T = \overline{\lambda})$ Post multiply by X, we have $(\overline{X})^T AX = (\overline{\lambda X})^T X \Rightarrow (\overline{X})^T \lambda X = (\overline{\lambda X})^T X \Rightarrow \lambda(\overline{X}^T X) = \overline{\lambda}(\overline{X}^T X) \Rightarrow \lambda = \overline{\lambda}$

Property 1.6. *Show that eigen values of a triangular matrix are just the diagonal elements of the matrix.*

Proof. Let $A = \begin{bmatrix} a_{11} & 0 & 0 \\ a_{21} & a_{22} & 0 \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$

Characteristic equation is $|A - \lambda I| = 0$

$$(i.e.) \begin{vmatrix} a_{11} - \lambda I & 0 & 0 \\ a_{21} & a_{22} - \lambda I & 0 \\ a_{31} & a_{32} & a_{33} - \lambda I \end{vmatrix} = 0$$

. On expansion we get $(a_{11}-\lambda)(a_{22}-\lambda)(a_{33}-\lambda)=0$
$\Rightarrow \lambda = a_{11}, a_{22}, a_{33}$, (which are the diagonal elements of matrix A) are the eigen values.

Property 1.7. *Show that 0 is an eigen value of a matrix A if and only if the matrix is singular.*

Proof. The characteristic equation of the matrix A is given by $|A-\lambda I|=0$. If $\lambda=0$, then from the above it follows that $|A|=0$. (i.e.) Matrix A is singular. We know $\lambda_1, \lambda_2, \cdots \lambda_n = |A|$. A is singular. $\Rightarrow |A|=0 \Rightarrow$ one of $\lambda's$ are zero.

1.2.2 Orthogonal Matrix

If a square matrix satisfies the relation $AA^T = A^TA = I$ then 'A' is said to be an orthogonal matrix. For an orthogonal matrix (i) $A^TA=I$, (ii) $A^{-1}=A^T$, (iii) $|A|=\pm 1$.

Property 1.8. *If A is orthogonal, then prove that (i) A^T is orthogonal (ii) A^{-1} is orthogonal.*

Proof (i). Since A is orthogonal, we have

$$A^TA = I. \tag{1.8}$$

Taking transpose in both sides,

$$(A^TA)^T = I^T \Rightarrow A^T(A^T)^T = I \Rightarrow A^TA = I. \tag{1.9}$$

Comparing (1.8) and (1.9), A^T is orthogonal.
Proof (ii.) $AA^T = I \Rightarrow (AA^T)^{-1} = I^{-1} \Rightarrow (A^T)^{-1}A^{-1} = I \Rightarrow (A^{-1})^TA^{-1} = I$ (i.e.) A^{-1} is orthogonal.

Property 1.9. *If A and B are orthogonal matrices then there product AB is also orthogonal.*

Proof. Since A, B are orthogonal, $A^T = A^{-1}$ and $B^T = B^{-1}$.
We have to prove that $(AB)^T = (AB)^{-1}$
Now $(AB)^T = B^TA^T = B^{-1}A^{-1} = (AB)^{-1}$. Hence proved.

Example 1.1. *Find the eigen values and eigen vectors of the matrix* $A = \begin{bmatrix} 1 & 2 \\ 5 & 4 \end{bmatrix}$

Solution: Characteristic equation is $|A-\lambda I|=0$

$$(i.e.)\begin{vmatrix} 1-\lambda & 2 \\ 5 & 4-\lambda \end{vmatrix} = 0$$

$\Rightarrow \lambda^2 - 5\lambda - 6 = 0$
$\Rightarrow (\lambda+1)(\lambda-6) = 0 \Rightarrow \lambda = -1, 6$ are the eigen values.

To find the eigen vectors:
Let $X_1 = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ be the eigen vector corresponding to the eigen value $\lambda=-1$. Then the equation $|A-\lambda I|X_1 = 0$ becomes

$$\begin{bmatrix} 1-\lambda & 2 \\ 5 & 4-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1+1 & 2 \\ 5 & 4+1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 2 & 2 \\ 5 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 2 & 2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$(R_2 \to R_2 - \frac{5}{2}R_1) \Rightarrow 2x_1 + 2x_2 = 0 \Rightarrow x_1 + x_2 = 0$$

Let $x_2 = k$ then $x_1 = -k$

Therefore the Eigen vectors are $\begin{bmatrix} -k \\ k \end{bmatrix}$ or $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$

When $\lambda = 6$
Let $X_2 = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ be the eigen vector then the equation
$|A - \lambda I|X_2 = 0$ becomes $\begin{bmatrix} 1-6 & 2 \\ 5 & 4-6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$
$\Rightarrow \begin{bmatrix} -5 & 2 \\ 5 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\Rightarrow \begin{bmatrix} -5 & 2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} (R_2b \to R_2 + R_1)$
$\Rightarrow -5x_1 + 2x_2 = 0$

Let $x_2 = k$ then $x_1 = \frac{2}{5}k$

Therefore Eigen vector $X_2 = \begin{bmatrix} \frac{2}{5}k \\ k \end{bmatrix}$ or $\begin{bmatrix} \frac{2}{5} \\ 1 \end{bmatrix}$

Type I: All the roots are distinct.

Example 1.2. *Find the Eigen values and Eigen vectors of*

$$\begin{bmatrix} 2 & -2 & 3 \\ 1 & 1 & 1 \\ 1 & 3 & -1 \end{bmatrix}$$

Solution: Characteristic equation is $|A - \lambda I| = 0$

$$(i.e.) \begin{vmatrix} 2-\lambda & -2 & 3 \\ 1 & 1-\lambda & 1 \\ 1 & 3 & -1-\lambda \end{vmatrix} = 0$$

Instead of evaluating the determinant directly (done in next problem), we can use the formula for its expansion which is as follows:
λ^3- (Sum of diagonal elements of A)λ^2+ Sum of minors of diagonal elements of $A - |A| = 0$
Now $|A| = 2(-4) + 2(-2) + 3(2) = -6$
Characteristic equation is $\lambda^3 + (-2)\lambda^2 + (-4-5+4)\lambda - (-6) = 0 \Rightarrow \lambda^3 - 2\lambda^2 - 5\lambda + 6 = 0$
$(\lambda - 1)$ is a factor (since sum of co-efficients $= 0$). To find other factors, use synthetic division.
$\Rightarrow (\lambda-1)(\lambda^2 - \lambda - 6) = 0$
$\Rightarrow (\lambda-1)(\lambda-3)(\lambda+2) = 0$
$\Rightarrow \lambda = 1, -2, 3$

To find the eigen vectors:
Let $[A - \lambda I]X = 0$

$$\Rightarrow \begin{bmatrix} 2-\lambda & -2 & 3 \\ 1 & 1-\lambda & 1 \\ 1 & 3 & -1-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \qquad (1.10)$$

Case (i): When $\lambda = 1$
Equation (1.10) becomes

$$\begin{bmatrix} 1 & -2 & 3 \\ 1 & 0 & 1 \\ 1 & 3 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Solving by cross multiplication rule, we have

$\frac{x_1}{-2} = -\frac{x_2}{-2} = \frac{x_3}{2} \Rightarrow \frac{x_1}{-1} = -\frac{x_2}{-1} = \frac{x_3}{1}$ Therefore Eigen vector

$$X_1 = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$$

Case (ii): When $\lambda = -2$ Equation (1.10) becomes

$$\begin{bmatrix} 4 & -2 & 3 \\ 1 & 3 & 1 \\ 1 & 3 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Solving by cross multiplication rule, we have

$$\Rightarrow \frac{x_1}{-11} = -\frac{x_2}{1} = \frac{x_3}{14} \Rightarrow \frac{x_1}{-1} = -\frac{x_2}{-1} = \frac{x_3}{1}$$

$$\Rightarrow X_2 = \begin{bmatrix} -11 \\ -1 \\ 14 \end{bmatrix}$$

Case (iii): When $\lambda = 3$ Equation (1.10) becomes

$$\begin{bmatrix} -1 & -2 & 3 \\ 1 & -2 & 1 \\ 1 & 3 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Solving by cross multiplication rule, we have

$$\Rightarrow \frac{x_1}{4} = -\frac{x_2}{-4} = \frac{x_3}{4} \Rightarrow \frac{x_1}{1} = -\frac{x_2}{-1} = \frac{x_3}{1} \Rightarrow X_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Example 1.3. *Find the eigen values and eigen vectors of*

$$\begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$$

Solution. Characteristic equation of A is $|A - \lambda I| = 0$

$$\Rightarrow \begin{vmatrix} 8-\lambda & -6 & 2 \\ -6 & 7-\lambda & -4 \\ 2 & -4 & 3-\lambda \end{vmatrix} = 0$$

$\Rightarrow (8-\lambda)[(7-\lambda)(3-\lambda)-16]+6[-6(3-\lambda)+8]+2[24-2(7-\lambda)] = 0$

$\Rightarrow \lambda^3 - 18\lambda^2 + 45\lambda = 0$

$\Rightarrow \lambda(\lambda-3)(\lambda-15) = 0$

$\Rightarrow \lambda = 0, 3, 15$ which are the eigen values of A.

To find the corresponding eigen vectors:

Let $[A - \lambda I]X = 0$, where $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

$$\Rightarrow \begin{bmatrix} 8-\lambda & -6 & 2 \\ -6 & 7-\lambda & -4 \\ 2 & -4 & 3-\lambda \end{bmatrix} = 0 \qquad (1.11)$$

Case (i): When $\lambda = 0$
Equation (1.11) becomes

$$\begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Solving by cross multiplication rule, we have

$$\Rightarrow \frac{x_1}{21-16} = -\frac{x_2}{-18+8} = \frac{x_3}{24-14}$$

Therefore Eigen vector $X_1 \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$

Case (ii): When $\lambda = 3$
Equation (1.11) becomes

$$\begin{bmatrix} 5 & -6 & 2 \\ -6 & 4 & -4 \\ 2 & -4 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Solving by cross multiplication rule, we have

$$\Rightarrow \frac{x_1}{2} = \frac{x_2}{1} = \frac{x_3}{-2}$$

Therefore Eigen vector $X_2 = \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}$

Case (iii): When $\lambda = 15$

Equation (1) becomes

$$\begin{bmatrix} -7 & -6 & 2 \\ -6 & -8 & -4 \\ 2 & -4 & -12 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Solving by cross multiplication rule, we have

$$\Rightarrow \frac{x_1}{2} = \frac{x_2}{-2} = \frac{x_3}{1}$$

Therefore Eigen vector $X_3 = \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}$

Example 1.4. *Find the eigen values and eigen vectors of*

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

Solution: Characteristic equation is $|A - \lambda I| = 0$

$$\Rightarrow \begin{vmatrix} 1-\lambda & 0 & 0 \\ 0 & 1-\lambda & 1 \\ 0 & 1 & 1-\lambda \end{vmatrix} = 0$$

$\Rightarrow (1-\lambda)(\lambda^2 - 2\lambda) = 0$

$\Rightarrow \lambda(1-\lambda)(\lambda-2) = 0$

$\Rightarrow \lambda = 0, 1, 2$ are the eigen values of A.

To find the corresponding eigen vectors:

Let $[A - \lambda I]X = 0$, where $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

$$\Rightarrow \begin{bmatrix} 1-\lambda & 0 & 0 \\ 0 & 1-\lambda & 1 \\ 0 & 1 & 1-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \qquad (1.12)$$

Case (i): When $\lambda = 0$
Equation (1.12) becomes

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$x_1 + 0x_2 + 0x_3 = 0$

$0x_1 + x_2 + x_3 = 0$

(i.e.) $x_1 = 0, x_2 = -x_3$

Let $x_3 = k$, we get $x_1 = 0, x_2 = --k, x_3 = k$

Therefore Eigen vector X_1 for $\lambda = 0$ is $\begin{bmatrix} 0 \\ -k \\ k \end{bmatrix}$ or $\begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$

Case (ii): When $\lambda = 1$

Equation (1.12) becomes

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\Rightarrow 0x_1 + 0x_2 + 0x_3 = 0$
$0x_1 + 0x_2 + x_3 = 0$
$0x_1 + x_2 + 0x_3 = 0$
(i.e.,) $x_2 = 0, x_3 = 0$. Let x_1 takes any value of k.

Therefore Eigen vector corresponding to $\lambda = 1$ is $X_2 = \begin{bmatrix} k \\ 0 \\ 0 \end{bmatrix}$

or $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

Case (iii): When $\lambda = 2$

Equation (1.12) becomes $\begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 1 \\ 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$\Rightarrow -x_1 + 0x_2 + 0x_3 = 0$
$0x_1 - 0x_2 + x_3 = 0$
$0x_1 + x_2 - x_3 = 0$
$x_1 = 0, x_2 = k, x_3 = k$

Therefore Eigen vector corresponding to $\lambda = 2$ is $X_3 = \begin{bmatrix} 0 \\ k \\ k \end{bmatrix}$

or $\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$

Type II: Repeated eigen values and A is non-symmetric

Example 1.5. *Find the eigen values and eigen vectors for*

$$\begin{bmatrix} 2 & 1 & 1 \\ 2 & 3 & 2 \\ 3 & 3 & 4 \end{bmatrix}$$

Solution. Characteristic equation of A in λ is $|A - \lambda I| = 0$

$$\Rightarrow \begin{vmatrix} 2-\lambda & 1 & 1 \\ 2 & 3-\lambda & 2 \\ 3 & 3 & 4-\lambda \end{vmatrix} = 0$$

$\Rightarrow \lambda^3 - 9\lambda^2 + 15\lambda - 7 = 0$
$\Rightarrow (\lambda - 1)(\lambda - 7)(\lambda - 1) = 0$
$\Rightarrow \lambda = 1, 1, 7$ are the eigen values of A.

To find the eigen vectors:

Let $[A - \lambda I]X = 0$, where $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

$$\Rightarrow \begin{bmatrix} 2-\lambda & 1 & 1 \\ 2 & 3-\lambda & 2 \\ 3 & 3 & 4-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad (1.13)$$

Case (i): When $\lambda = 7$

Equation (1.13) becomes $\begin{bmatrix} -5 & 1 & 1 \\ 2 & -4 & 2 \\ 3 & 3 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

Solving by cross multiplication rule, we have

$\Rightarrow \frac{x_1}{6} = -\frac{x_2}{-12} = \frac{x_3}{18}$
$\Rightarrow \frac{x_1}{1} = \frac{x_2}{2} = \frac{x_3}{3}$

Eigen vector for $\lambda = 7$ is $X_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$

Case (ii): When $\lambda = 1$

Equation (1.13) becomes $\begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & 2 \\ 3 & 3 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$\Rightarrow x_1 + x_2 + x_3 = 0$
Let us assume $x_1 = 0$
$\Rightarrow x_2 = 1, x_3 = -1$

Eigen vector for $\lambda = 1$ is $X_2 = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$

Case (iii): When $\lambda = 1$ (same matrix equation)
$x_1 + x_2 + x_3 = 0$
Now let $x_2 = 0 \Rightarrow x_1 = 1$ and $x_3 = -1$

Therefore Eigen vector for $\lambda = 1$ is $X_3 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$

Example 1.6. *Find the characteristic roots and characteristic vectors of the matrix* $\begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}$

Solution: Characteristic equation is $|A - \lambda I| = 0$

$$\Rightarrow \begin{vmatrix} -2-\lambda & 2 & -3 \\ 2 & 1-\lambda & -6 \\ -1 & -2 & 0-\lambda \end{vmatrix} = 0$$

$\Rightarrow \lambda^3 - \lambda^2 - 21\lambda - 45 = 0$
$\Rightarrow -(\lambda + 3)(\lambda + 3)(\lambda - 5) = 0$
$\Rightarrow \lambda = 5, -3, -3$ which are the eigen values of the matrix A.

To find eigen vectors:

Let $[A - \lambda I]X = 0$, where $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

$$\Rightarrow \begin{bmatrix} -2-\lambda & 2 & -3 \\ 2 & 1-\lambda & -6 \\ -1 & -2 & 0-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \qquad (1.14)$$

Case (i): When $\lambda = 5$ equation (1.14) becomes
$\begin{bmatrix} -7 & 2 & -3 \\ 2 & -4 & -6 \\ -1 & -2 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

Solving by cross multiplication rule, we have

$$\frac{x_1}{20-12} = -\frac{x_2}{-10-6} = \frac{x_3}{-4-4}$$

$$\Rightarrow \frac{x_1}{8} = -\frac{x_2}{-16} = \frac{x_3}{-8}$$

$$\Rightarrow \frac{x_1}{1} = \frac{x_2}{2} = \frac{x_3}{-1}$$

Therefore Eigen vector for $\lambda = 5$ is $X_1 = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$

Case (ii): When $\lambda = -3$

Equation 1.14) becomes $\begin{bmatrix} 1 & 2 & -3 \\ 2 & 4 & -6 \\ -1 & -2 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$\Rightarrow x_1 + 2x_2 - 3x_3 = 0$
Let $x_2 = 0 \Rightarrow x_1 = -3x_3$
Again $x_3 = 1 \Rightarrow x_1 = -3$

Therefore Eigen vector for $\lambda = -3$ is $X_2 = \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}$

Again let $x_3 = 0 \Rightarrow x_1 + 2x_2 = 0$
Again $x_2 = 1 \Rightarrow x_1 = -2$

Therefore Eigen vector for $\lambda = -3$ is $X_3 = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$

Example 1.7. *Find the characteristic roots and vectors of the*

matrix
$$\begin{bmatrix} 6 & -6 & 5 \\ 14 & -13 & 10 \\ 7 & -6 & 4 \end{bmatrix}$$

Solution: Characteristic equation is $|A - \lambda I| = 0$

$$\Rightarrow \begin{vmatrix} 6-\lambda & -6 & 5 \\ 14 & -13-\lambda & 10 \\ 7 & -6 & 4-\lambda \end{vmatrix} = 0$$

$\Rightarrow \lambda^3 + 3\lambda^2 + 3\lambda + 1 = 0$
$\Rightarrow (\lambda + 1)^3 = 0$
$\Rightarrow \lambda = -1, -1, -1$ which are the eigen values

To find eigen vectors:

Let $[A - \lambda I]X = 0$, where $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

$$\Rightarrow \begin{bmatrix} 6-\lambda & -6 & 5 \\ 14 & -13-\lambda & 10 \\ 7 & -6 & 4-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \qquad (1.15)$$

Case (i): When $\lambda = -1$
We get $7x_1 - 6x_2 + 5x_3 = 0, 14x_1 - 12x_2 + 10x_3 = 0$ and
$7x_1 - 6x_2 + 5x_3 = 0$

These three equations are linearly independent since determinant of the coefficients is zero. Number of eigen vectors can be obtained by giving arbitrary value to any two of the quantities x_1, x_2 and x_3. We get from the first equation, $7x_1 = 6x_2 - 5x_3$ When $x_1 = 0$, the second equation becomes $6x_2 = 5x_3$ Again $x_2 = 1 \Rightarrow x_3 = \frac{6}{5}$. Therefore Eigen vector $X_1 = \begin{bmatrix} 0 \\ 1 \\ 6/5 \end{bmatrix}$

When $x_2 = 0$, equation (2) becomes $7x_1 = -5x_3$
When $x_1 = 1 \Rightarrow x_3 = \frac{-7}{5}$

Therefore Eigen vector $X_2 = \begin{bmatrix} 1 \\ 0 \\ -7/5 \end{bmatrix}$

When $x_3 = 0$, equation (2) becomes $7x_1 = 6x_2$
When $x_1 = 1 \Rightarrow x_2 = \frac{7}{6}$

Therefore Eigen vector $X_3 = \begin{bmatrix} 1 \\ 7/6 \\ 0 \end{bmatrix}$

Type III: Matrix A is symmetric and eigen values are repeated

Definition 1.2. *Let X_1 and X_2 be two column matrices. Then*

X_1 and X_2 are orthogonal if $X_1^T X_2 = 0$

$$X_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, X_2 = \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix} \text{ then } X_1^T X_2 = \begin{bmatrix} 1 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix}$$

Note: When the matrix A is symmetric, its eigen vectors are orthogonal.

Example 1.8. *Find the eigen values and eigen vectors for*

$$A = \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$$

Solution: Characteristic equation is $|A - \lambda I| = 0$

$$\Rightarrow \begin{vmatrix} 6-\lambda & -2 & 2 \\ -2 & 3-\lambda & -1 \\ 2 & -1 & 3-\lambda \end{vmatrix} = 0$$

$\Rightarrow \lambda^3 - 12\lambda^2 + 36\lambda - 32 = 0$
$\Rightarrow (\lambda - 2)(\lambda^2 - 10\lambda + 16) = 0$
$\Rightarrow (\lambda - 2)(\lambda - 2)(\lambda - 8) = 0$
$\Rightarrow \lambda = 8, 2, 2$ are the eigen values of A.

To find the eigen vectors:

Let $[A-\lambda I]X=0$, where $X=\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

$$\Rightarrow \begin{bmatrix} 6-\lambda & -2 & 2 \\ -2 & 3-\lambda & -1 \\ 2 & -1 & 3-\lambda \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \qquad (1.16)$$

Case (i): When $\lambda=8$

Equation (1.16) becomes $\begin{bmatrix} -2 & -2 & 2 \\ -2 & -5 & -1 \\ 2 & -1 & -5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

Solving by cross multiplication rule, we have

$\frac{x_1}{12}=-\frac{x_2}{6}=\frac{x_3}{6}$

Eigen vector is $X_1=\begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$

Case (ii): When $\lambda=2$

Equation (1.16) becomes $\begin{bmatrix} 4 & -2 & 2 \\ -2 & 1 & -1 \\ 2 & -1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

Let $x_1=0 \Rightarrow x_2=1, x_3=1$

Therefore Eigen vector is $X_2=\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$

Case (iii): Since A is symmetric, eigen vectors X_1, X_2, X_3 are orthogonal.

Let $X_2=\begin{bmatrix} a \\ b \\ c \end{bmatrix}$ $[X_1, X_3$ are orthogonal $X_1^T X_3=0]$

$$2a-b+c=0 \qquad (1.17)$$

$[X_2, X_3$ are orthogonal $X_2^T X_3=0]$

$$\Rightarrow 0a+b+c=0 \qquad (1.18)$$

Solving (1.17) and (1.21) using cross multiplication rule, we have

$$\frac{a}{-2}=-\frac{b}{2}=\frac{c}{2} \Rightarrow \frac{a}{1}=\frac{b}{1}=\frac{c}{-1}$$

$X_3=\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$

Example 1.9. *Find the eigen values and eigen vectors of the matrix*

$$\begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$$

Solution: Characteristic equation is $|A-\lambda I|=0$

$$\Rightarrow \begin{vmatrix} 2-\lambda & -1 & 1 \\ -1 & 2-\lambda & -1 \\ 1 & -1 & 2-\lambda \end{vmatrix}=0$$

$\lambda^3 - 6\lambda^2 + 9\lambda - 4 = 0$
$(\lambda - 1)(\lambda^2 - 5\lambda + 4) = 0$
$(\lambda - 1)(\lambda - 1)(\lambda - 4)$
$\lambda = 1, 1, 4$ are the eigen values.

To find the eigen vectors:

Let $[A - \lambda I]X = 0$, where $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

$$\Rightarrow \begin{bmatrix} 2-\lambda & -1 & 1 \\ -1 & 2-\lambda & -1 \\ 1 & -1 & 2-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \qquad (1.19)$$

Case (i): When $\lambda = 4$

Equation (1.19) becomes $\begin{bmatrix} -2 & -1 & 1 \\ -1 & -2 & -1 \\ 1 & -1 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

Using cross multiplication rule we get $X_1 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$ for $\lambda = 4$

Case (ii): When $\lambda = 1$
Equation (1.19) $\Rightarrow x_1 - x_2 + x_3 = 0$
Let $x_3 = 0 \Rightarrow x_1 = x_2$
Let $x_1 = 1 = x_2$

Eigen vector is $X_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ for $\lambda = 1$

Case (iii): Since A is symmetric.

Let us choose $X_3 = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ such that $X_1^T X_3 = 0]$ and $X_2^T X_3 = 0]$

$\Rightarrow a - b + c = 0$ and $a + b + 0c = 0$

$$\frac{a}{0-1} = \frac{b}{0-1} = \frac{c}{1+1}$$

Therefore Eigen vector $X_3 = \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix}$

Example 1.10. *Verify that the eigen vectors of a real symmetric matrix* $A = \begin{bmatrix} 3 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{bmatrix}$ *are orthogonal in pairs.*

Solution: Characteristic equation is $|A - \lambda I| = 0$

$$\Rightarrow \begin{vmatrix} 3-\lambda & -1 & 1 \\ -1 & 5-\lambda & -1 \\ 1 & -1 & 3-\lambda \end{vmatrix} = 0$$

$\lambda^3 - 11\lambda^2 + 9\lambda + 36\lambda - 36 = 0$
$\lambda = 3, 6, 2$ are the eigen values of A.

To find the eigen vectors:

Let $[A - \lambda I]X = 0$, where $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

$$\Rightarrow \begin{bmatrix} 3-\lambda & -1 & 1 \\ -1 & 5-\lambda & -1 \\ 1 & -1 & 3-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \qquad (1.20)$$

Case (i): When $\lambda = 2$

Equation (1.20) becomes $\begin{bmatrix} 1 & -1 & 1 \\ -1 & 3 & -1 \\ 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$x_1 - x_2 + x_3 = 0$

$\Rightarrow -x_1 + 3x_2 - x_3 = 0$

$x_1 - x_2 + x_3 = 0$

Taking the first two equations and using cross multiplication rule, we get

$\frac{x_1}{-2} = -\frac{x_2}{0} = \frac{x_3}{2}$

Therefore Eigen vector $X_1 = \begin{bmatrix} -2 \\ 0 \\ 2 \end{bmatrix}$

Example 1.11. *Find the sum and product of the eigen values of the matrix* $\begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}$

Solution: Sum of the eigen values of the matrix = sum of the leading diagonal elements of the matrix = trace of the matrix $= -2 + 1 + 0 = -1$. Product of the eigen values of the matrix $= |A|$.

$$\begin{vmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{vmatrix} = -2(0-12) - 2(0-6) - 3(-4+1)$$

$$= 45$$

Example 1.12. *Two of the eigen values of* $A = \begin{bmatrix} 3 & -1 & 1 \\ -1 & 5 & -6 \\ -1 & -2 & 0 \end{bmatrix}$ *are 3 and 6. Find the eigen values of* A^{-1}.

Solution: Let $\lambda_1, \lambda_2, \lambda_3$ are the eigen values of A.

Now sum of the eigen values is $\lambda_1 + \lambda_2 + \lambda_3 = 3 + 5 + 3 = 11$

Given $\lambda_1 = 3, \lambda_2 = 6$ $3 + 6 + \lambda_3 = 11 \Rightarrow \lambda_3 = 2$

The eigen values of $A^{-1} = \frac{1}{2}, \frac{1}{3}, \frac{1}{6}$.

Example 1.13. *Find the eigen values of* A^3 *if* $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & -7 \\ 0 & 0 & 3 \end{bmatrix}$

Solution: Eigen values of $A = 1, 2, 3$ (As A is triangular matrix,

Eigen values are main diagonal elements of A)
Eigen values of $A^3 = 1^3, 2^3, 3^3$.

Example 1.14. *Find the eigen values of* A^3 *if* $A = \begin{bmatrix} 2 & -2 & 2 \\ 1 & 1 & 1 \\ 1 & 3 & -1 \end{bmatrix}$ *find the other two eigen values.*

Solution: Let $\lambda_1, \lambda_2, \lambda_3$ are the eigen values of A and $\lambda_1 = 2$.

Sum of the eigen values is $\lambda_1 + \lambda_2 + \lambda_3 = 2 + 1 - 1 = 2$

$\Rightarrow \lambda_1 + \lambda_2 + \lambda_3 = 2 \Rightarrow 2 + \lambda_2 + \lambda_3 = 2 \Rightarrow \lambda_2 + \lambda_3 = 0$

$\Rightarrow \lambda_2 = -\lambda_3$.

Now product of the eigen values are

$$\lambda_1\lambda_2\lambda_3 = \begin{vmatrix} 2 & -2 & 2 \\ 1 & 1 & 1 \\ 1 & 3 & -1 \end{vmatrix} = 2(-1-3) + 2(-1-1) + 2(3-1) = -8$$

$\Rightarrow \lambda_1\lambda_2\lambda_3 = -8 \Rightarrow 2\lambda_2\lambda_3 = -8$

$\Rightarrow \lambda_2\lambda_3 = -4$, since $\lambda_1 = 2$. Since $\lambda_2 = -\lambda_3$,

$\lambda_2\lambda_3 = -4 \Rightarrow (-\lambda_3)(\lambda_3) = -4 \Rightarrow \lambda_3^2 = 4$

$\Rightarrow \lambda_3 = 2 \Rightarrow \lambda_2 = -2$ and $\lambda_3 = 2$

Example 1.15. *If the eigen values of* $A = \begin{bmatrix} 3 & 10 & 5 \\ -2 & -3 & -4 \\ 3 & 5 & 7 \end{bmatrix}$ *are 2, 2, 3, find the eigen values of* A^{-1} *and* A^2.

Solution: Eigen values of A are 2, 2, 3 (given).
Eigen values of A^2 are $2^2, 2^2, 3^2$
Eigen values of A^{-1} are $\frac{1}{2}, \frac{1}{2}, \frac{1}{3}$

Example 1.16. *Find the eigen values of the matrix*

$$A = \begin{bmatrix} 1 & -2 \\ -5 & 4 \end{bmatrix}$$

and $A - 3I$

Solution: Characteristic equation is $|A - \lambda I| = 0$

$$\Rightarrow \begin{vmatrix} 1-\lambda & -2 \\ -5 & 4-\lambda \end{vmatrix} = 0$$

$\Rightarrow \lambda^2 - 5\lambda - 6 = 0$
$\Rightarrow \lambda = -1.6.$
We know that if λ_1, λ_2 are the eigen values of A, then $A - kI$ has the eigen values $\lambda_1 - k, \lambda_2 - k$
The eigen values of (A - 3I) are -1-3 and 6-3. (i. e.) -4 and 3.

Example 1.17. *Two of the eigen values of the matrix* $A = \begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix}$ *are equal to 1 each. Find the eigen values of* A^{-1}.

Solution: Let $\lambda_1, \lambda_2, \lambda_3$ are the eigen values of A.
Then $\lambda_1 + \lambda_2 + \lambda_3 = 7 \Rightarrow 1 + 1 + \lambda_3 = 7$
$\lambda_3 = 5 \Rightarrow \lambda_1 = 1, \lambda_2 = 1, \lambda_3 = 5$
The eigen values of A^{-1} are 1, 1, 1/5.

Example 1.18. *Find the constants a and b such that* $\begin{bmatrix} a & 4 \\ 1 & b \end{bmatrix}$ *matrix has 3 and -2 as its eigen values.*

Solution: Sum of the eigen values $a + b = 1$
Sum of the eigen values = Sum of the leading diagonals

And product of the eigen values = Value of the determinant
(i.e.) $ab - 4 = -6 \Rightarrow ab = -2$
But $b = 1 - a$ $a(1 - a) = -2$
$\Rightarrow a^2 - a + 2 = 0 \Rightarrow a = 2, -1$
$b = -1$ or 2.

Example 1.19. *Find the sum and product of the eigen values of* $A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix}$

Solution: Sum of the eigen values $2 + 2 + 1 = 5$
Product of the eigen values $= |A| = 3$

Example 1.20. *Find the sum of the squares of the eigen values of* $\begin{bmatrix} 1 & 7 & 5 \\ 0 & 2 & 9 \\ 0 & 0 & 5 \end{bmatrix}$

Solution: Since the given matrix is triangular matrix, eigen values are the diagonal elements = 1, 2, 5.
The sum of the squares of the eigen values = 1 + 4 + 25 = 30

Example 1.21. *If 3 and 2 are the eigen values of the matrix*

$$A = \begin{bmatrix} 3 & 10 & 5 \\ -2 & -3 & -4 \\ 3 & 5 & 7 \end{bmatrix}$$ *, find the eigen values of A^{-1} and A^3*

Solution: Let $\lambda_1, \lambda_2, \lambda_3$ are the eigen values of A.
Then $\lambda_1 + \lambda_2 + \lambda_3 = 3 - 3 + 7 = 7 \Rightarrow 2 + 3 + \lambda_3 = 7 \Rightarrow \lambda_3 = 2$
Again eigen values of A^{-1} are $\frac{1}{2}, \frac{1}{2}, \frac{1}{3}$ and eigen values of A^3 are $2^3, 2^3, 3^3$.

Example 1.22. *If 2 and 8 are the eigen values of* $\begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$

then find the third eigen value.

Solution: We have $\lambda_1 + \lambda_2 + \lambda_3 = 6 + 3 + 3 = 12$

$\Rightarrow 2 + 8 + \lambda_3 = 12 \Rightarrow \lambda_3 = 2$
Third eigen value is 2.

Example 1.23. *Two of the eigen values of the 3 × 3 matrix, whose determinant is equal to 4 are -1 and 2. Find the third eigen value.*

Solution: *We have $\lambda_1\lambda_2\lambda_3 = 4$ (Since product of the eigen value is the value of the determinant).*

(i.e.,) $(-1)(2)(\lambda_3) = 4 \Rightarrow -2\lambda_3 = 4 \Rightarrow \lambda_3 = -2.$

Example 1.24. *Given the matrix* $A = \begin{bmatrix} 3 & 10 & 5 \\ -2 & -3 & -4 \\ 3 & 5 & 7 \end{bmatrix}$*, find the eigen values of A^2.*

Solution: Eigen values of the given matrix = -1, -3, 2 [the matrix is triangular]
The eigen values of $A^2 = 1, 9, 4$.

Example 1.25. *Find the inverse of the matrix if* $A = \begin{bmatrix} 1 & -2 \\ -5 & 4 \end{bmatrix}$

Solution: Characteristic equation of A is$|A - \lambda I| = 0$

$\Rightarrow \begin{vmatrix} 1-\lambda & -2 \\ -5 & 4-\lambda \end{vmatrix} = 0$

$\Rightarrow (1-\lambda)(4-\lambda)(-10) = 0 \Rightarrow \lambda^2 - 5\lambda - 6 = 0$

$\Rightarrow \lambda = -1, 6.$

Example 1.26. *Two eigen values of* $A = \begin{bmatrix} 4 & 6 & 6 \\ 1 & 3 & 2 \\ -1 & -5 & -2 \end{bmatrix}$ *are equal and they are double the third. Find the eigen values of* A^2.

Solution: Let the third eigen value be λ.

The remaining eigen values are $2\lambda, 2\lambda$.

$\lambda + 2\lambda + 2\lambda = 4 + 3 - 2 = 5 \Rightarrow 5\lambda = 5 \Rightarrow \lambda = 1.$

The eigen values of $A = 1, 2, 2$. The eigen values of $A^2 = 1, 4, 4.$

EXERCISE

1. Find the sum and product of the eigen values of

$$\begin{bmatrix} 7 & -2 & -2 \\ -2 & 1 & 4 \\ -2 & 4 & 1 \end{bmatrix} \quad [Ans: 9, 81]$$

2. If $\lambda_1, \lambda_2, \lambda_3$ are the eigen values of A then find λ_3 if $\lambda_1 = 3\lambda_2 = 15$ and $A = \begin{bmatrix} 8 & -6 & 2 \\ 6 & 7 & -4 \\ -2 & -4 & 3 \end{bmatrix}$ [Ans: $\lambda_3 = 0$]

3. The product of two eigen value of $A = \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ -2 & -1 & 3 \end{bmatrix}$ is 16 then find third eigen value. [Ans: $\lambda_3 = 2$]

4. Find the sum of squares of the eigen values of $A = \begin{bmatrix} 3 & 2 & 0 \\ 0 & 2 & 6 \\ 0 & 0 & 5 \end{bmatrix}$ [Ans: 9, 4, 25]

5. If 1, 1, 5 are the eigen values of $A = \begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix}$, find the eigen values of 5A.[Ans: 5, 5, 25]

6. Find the eigen values of $adjA$, if $A = \begin{bmatrix} 3 & 2 & 1 \\ 0 & 4 & 2 \\ 0 & 0 & 1 \end{bmatrix}$ [Ans:

$12\left[\frac{1}{3}, \frac{1}{4}, 1\right]$]

7. Prove that $A = \frac{1}{3}\begin{bmatrix} 2 & 2 & 1 \\ -2 & 1 & 2 \\ 1 & -2 & 2 \end{bmatrix}$ is orthogonal.

8. Find the eigen values and eigen vectors of

(i) $A = \begin{bmatrix} 2 & -2 & 2 \\ 1 & 1 & 3 \\ 1 & 3 & -1 \end{bmatrix}$

(ii) $A = \begin{bmatrix} 6 & -6 & 5 \\ 14 & -13 & 10 \\ 7 & -6 & 4 \end{bmatrix}$

(iii) $\begin{bmatrix} 2 & -1 \\ -8 & 4 \end{bmatrix}$

(iv) $\begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 1 & 0 & 2 \end{bmatrix}$

(v) $\begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix}$

(vi) $\begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}$

(vii) $\begin{bmatrix} 1 & 2 & 2 \\ 0 & 2 & 1 \\ -1 & 2 & 2 \end{bmatrix}$

(viii) $\begin{bmatrix} 3 & 1 & 4 \\ 0 & 2 & 6 \\ 0 & 0 & 5 \end{bmatrix}$

(ix) $\begin{bmatrix} -2 & 5 & 4 \\ 5 & 7 & 5 \\ 4 & 5 & -2 \end{bmatrix}$

(x) $\begin{bmatrix} 2 & 0 & 1 \\ 0 & 3 & 0 \\ 1 & 0 & 2 \end{bmatrix}$

Ans: i) $\lambda = -2, 2, 2, X_1 = \begin{bmatrix} -4 \\ -1 \\ 7 \end{bmatrix}, X_2 = X_3 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$

ii) $\lambda = -1, -1, -1,\ X_1 = \begin{bmatrix} -5 \\ 0 \\ 7 \end{bmatrix}, X_2 = \begin{bmatrix} 6 \\ 7 \\ 0 \end{bmatrix},$

$X_3 = \begin{bmatrix} 0 \\ 5 \\ 6 \end{bmatrix}$

iii) $\lambda = 0, 6,\ X_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, X_2 = \begin{bmatrix} 1 \\ -4 \end{bmatrix}$

iv) $\lambda = 1, 2, 3,\ X_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, X_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix},$

$$X_3 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

v) $\lambda = 1, 1, 5$, $X_1 = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}$, $X_2 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$,

$$X_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

vi) $\lambda = -3, -3, 5$, $X_1 = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}$, $X_2 = \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}$,

$$X_3 = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$$

vii) $\lambda = 1, 1, 2$, $X_1 = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$, $X_2 = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$

viii) $\lambda = 3, 2, 5$, $X_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $X_2 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$,

$$X_3 = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$$

ix) $\lambda = -3, -6, 12$, $X_1 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$, $X_2 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$,

$$X_3 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$

x) $\lambda = 1, 3, 3$, $X_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$, $X_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$,

$$X_3 = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$$

1.3 Cayley - Hamilton Theorem

Statement: Every square matrix satisfies its own characteristic equation.

Proof. Let A be a $n \times n$ square matrix.

$$A = \begin{bmatrix} a_{11} & a_{12} & ...a_{1n} \\ a_{21} & a_{22} & ...a_{2n} \\ . & . & . \\ . & . & . \\ . & . & . \\ a_{n1} & a_{n2} & ...a_{nn} \end{bmatrix} \tag{1.21}$$

Characteristic equation of A is$|A - \lambda I| = 0$

$$(i.e.,) \begin{vmatrix} a_{11} - \lambda I & a_{12} & ...a_{1n} \\ a_{21} & a_{22} - \lambda I & ...a_{2n} \\ . & . & . \\ . & . & . \\ . & . & . \\ a_{n1} & a_{n2} & ...a_{nn} - \lambda I \end{vmatrix} = 0 \tag{1.22}$$

On expansion we get a polynomial in λ of degree 'n'.

$$(i.e.) \quad a_0\lambda^n + a_1\lambda^{n-1} + \cdots + a_n = 0 \qquad (1.23)$$

We have to prove that 'A' satisfies equation (1.22).

$$(i.e.) \quad a_0A^n + a_1A^{n-1} + \cdots + a_nI = 0 \qquad (1.24)$$

Let $B = adj[A - \lambda I]$
Since the elements of $adj[A - \lambda I]$ are the co-factors of the elements of (1.22), each element of B will be a polynomial in λ of degree $n-1$ or less.
B can be written as a matrix polynomial.

$$B = B_0 + B_1\lambda + B_2\lambda^2 + \cdots + B_n\lambda^{n-1} \qquad (1.25)$$

where $B_0, B_1, B_2, \cdots B_n$ are the matrices of order 'A'.
We know $A.adjA = |A|I \Rightarrow [A - \lambda I]adj[A - \lambda I] = |A - \lambda I|I$
$\Rightarrow [A - \lambda I][B_0 + B_1\lambda + B_2\lambda^2 + \cdots + B_n\lambda^{n-1}]$

$$= (a_0\lambda^n + a_1\lambda^{n-1} + \cdots + a_n)I.\text{using}[(1.25)and(1.26)] \qquad (1.26)$$

Equating the co-efficients of I, $\lambda, \lambda^2, \lambda^3, \cdots \lambda^n$, we have

Co-efficient of λ^n, $-B_n = a_0I$ (6)

Co-efficient of λ^{n-1}, $AB_n - B_{n-1} = a_1I$ (7)

Co-efficient of λ, $AB_1 - B_0 = a_{n-1}I$ (8)

Constant $AB_0 = a_nI$ (9)

Pre multiplying (6), (7), (8) and (9) by $A^n, A^{n-1}, A^{n-2}, A, I$ respectively and adding $a_0A^n + a_1A^{n-1} + \cdots + a_nI = 0$ which is same as equation

(i. e) The matrix satisfies its own characteristic equation. Hence the theorem is proved.

Example 1.27. *Using Cayley-Hamilton theorem find the inverse of the matrix* $\begin{bmatrix} 2 & 1 \\ 1 & -5 \end{bmatrix}$

Solution: Characteristic equation of A is$|A - \lambda I| = 0$
(i.e.) $\begin{vmatrix} 2-\lambda & 1 \\ 1 & -5-\lambda \end{vmatrix} = 0$
$(2-\lambda)(-5-\lambda) - 1 = 0 \Rightarrow \lambda^2 + 3\lambda - 11 = 0$
By Cayley-Hamilton theorem, $A^2+3A-11I = 0$ (1)
To find A^{-1}:
Pre multiplying equation (1) by A^{-1}, we get

$$\begin{aligned} A^{-1} &= \frac{1}{11}[A + 3I] \\ &= \frac{1}{11}\left\{\begin{bmatrix} 2 & 1 \\ 1 & -5 \end{bmatrix} + 3\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right\} \\ &= \frac{1}{11}\begin{bmatrix} 5 & 1 \\ 1 & -2 \end{bmatrix} \end{aligned}$$

Example 1.28. *Verify Cayley - Hamilton theorem and hence find A^{-1}, where*

$$A = \begin{bmatrix} 8 & -8 & -2 \\ 4 & -3 & -2 \\ 3 & -4 & 1 \end{bmatrix}$$

Solution: Characteristic equation of A is $|A - \lambda I| = 0$

$$(i.e.,) \begin{vmatrix} 8-\lambda & -8 & -2 \\ 4 & -3-\lambda & -2 \\ 3 & -4 & 1-\lambda \end{vmatrix} = 0$$

$\Rightarrow \lambda^3 - 6\lambda^2 + 11\lambda - 6 = 0$. We have to show that

$$A^3 - 6A^2 + 11A - 6I = 0 \qquad (1.27)$$

$$\text{Now} \quad A^2 = \begin{bmatrix} 8 & -8 & -2 \\ 4 & -3 & -2 \\ 3 & -4 & 1 \end{bmatrix} \begin{bmatrix} 8 & -8 & -2 \\ 4 & -3 & -2 \\ 3 & -4 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 26 & -32 & -2 \\ 14 & -15 & -4 \\ 11 & -16 & 3 \end{bmatrix}$$

$$\text{Now} \quad A^3 = \begin{bmatrix} 26 & -32 & -2 \\ 14 & -15 & -4 \\ 11 & -16 & 3 \end{bmatrix} \begin{bmatrix} 8 & -8 & -2 \\ 4 & -3 & -2 \\ 3 & -4 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 74 & -104 & 10 \\ 40 & -51 & -2 \\ 33 & -52 & 13 \end{bmatrix}$$

Equation (1.27) implies

$$A^3 - 6A^2 + 11A - 6I = \begin{bmatrix} 74 & -104 & 10 \\ 40 & -51 & -2 \\ 33 & -52 & 13 \end{bmatrix} - \begin{bmatrix} 156 & -192 & -12 \\ -84 & -90 & -24 \\ 66 & -93 & 18 \end{bmatrix}$$

$$+ \begin{bmatrix} 88 & -88 & -22 \\ 44 & -33 & -22 \\ 33 & -44 & 11 \end{bmatrix} - \begin{bmatrix} 6 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 6 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = 0$$

Hence Cayley - Hamilton theorem is verified.

To find A^{-1}:

Pre multiplying equation (1.27) by A^{-1}, we get

$$A^2 - 6A + 11I - 6A^{-1} = 0 \Rightarrow A^{-1} = \frac{1}{6}[A^2 - 6A + 11I]$$

$$A^{-1} = \frac{1}{6}\left\{\begin{bmatrix} 26 & -32 & -2 \\ 14 & -15 & -4 \\ 11 & -16 & 3 \end{bmatrix} - \begin{bmatrix} 48 & -48 & -12 \\ 24 & -18 & -12 \\ 18 & -24 & 6 \end{bmatrix}\right.$$
$$\left. + \begin{bmatrix} 11 & 0 & 0 \\ 0 & 11 & 0 \\ 0 & 0 & 11 \end{bmatrix}\right\} = \frac{1}{6}\begin{bmatrix} -11 & 16 & 10 \\ -10 & 14 & 8 \\ -7 & 8 & 8 \end{bmatrix}$$

Example 1.29. *Using Cayley - Hamilton theorem, find A^{-1} if exists given* $A = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$

Solution: Characteristic equation of A is $|A - \lambda I| = 0$

$$(i.e.,)\begin{vmatrix} 2-\lambda & -1 & 1 \\ -1 & 2-\lambda & -1 \\ 1 & -1 & 2-\lambda \end{vmatrix} = 0$$

$\Rightarrow \lambda^3 - 6\lambda^2 + 9\lambda - 4 = 0$

Using Cayley - Hamilton theorem, we have

$$A^3 - 6A^2 + 9A - 4I = 0 \qquad (1.28)$$

To find A^{-1}:

Pre multiplying equation (1) by A^{-1}, we have

$A^2 - 6A + 9I - 4A^{-1} = 0$

$$A^{-1} = \frac{1}{4}[A^2 - 6A + 9I] \qquad (2)$$

Now

$$A^2 = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}\begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$$
$$= \begin{bmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{bmatrix}$$

Equation (2) $\Rightarrow$

$$A^{-1} = \frac{1}{4}\left\{\begin{bmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{bmatrix} - \begin{bmatrix} 12 & -6 & 6 \\ 6 & 12 & -6 \\ -6 & -6 & 12 \end{bmatrix}\right.$$
$$\left. + \begin{bmatrix} 9 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 9 \end{bmatrix}\right\} = \frac{1}{4}\begin{bmatrix} 3 & 1 & -1 \\ 1 & 3 & 1 \\ -1 & 1 & 3 \end{bmatrix}$$

Example 1.30. *Verify Cayley - Hamilton theorem for the matrix* $A = \begin{bmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{bmatrix}$. *Use it to find A^{-1} and A^4.*

Solution: Characteristic equation of A is $|A - \lambda I| = 0$

$$(i.e.,)\begin{vmatrix} 1-\lambda & 2 & -2 \\ -1 & 3-\lambda & 0 \\ 0 & -2 & 1-\lambda \end{vmatrix} = 0 \Rightarrow \lambda^3 - 5\lambda^2 + 9\lambda - 1 = 0$$

We have to verify that $A^3 - 5A^2 + 9A - I = 0$ (1)

Now

$$A^2 = \begin{bmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{bmatrix}$$
$$= \begin{bmatrix} -1 & 12 & -4 \\ -4 & 7 & 2 \\ 2 & -8 & 1 \end{bmatrix}$$

$$A^3 = \begin{bmatrix} -1 & 12 & -4 \\ -4 & 7 & 2 \\ 2 & -8 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{bmatrix}$$
$$= \begin{bmatrix} -13 & 42 & -2 \\ -11 & 9 & 10 \\ 10 & -22 & -3 \end{bmatrix}$$

Equation (1) $\Rightarrow$

$$A^3 - 5A^2 + 9A - I = \begin{bmatrix} -13 & 42 & -2 \\ -11 & 9 & 10 \\ 10 & -22 & -3 \end{bmatrix}\begin{bmatrix} -5 & 60 & -20 \\ -20 & 35 & 10 \\ 10 & -40 & 5 \end{bmatrix}$$
$$+ \begin{bmatrix} 9 & 18 & -18 \\ -9 & 27 & 0 \\ 0 & -18 & 9 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$
$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = 0$$

Cayley - Hamilton theorem is verified.

To find A^{-1}:

Pre multiplying equation (1) by A^{-1}, we have

$$A^2 - 5A + 9I - A^{-1} = 0$$

$$A^{-1} = A^2 - 5A + 9I$$
$$= \begin{bmatrix} -1 & 12 & -4 \\ -4 & 7 & 2 \\ 2 & -8 & 1 \end{bmatrix} - \begin{bmatrix} 5 & 10 & -10 \\ -5 & 15 & 0 \\ 0 & -10 & 5 \end{bmatrix} + \begin{bmatrix} 9 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 9 \end{bmatrix}$$
$$= \begin{bmatrix} 3 & 2 & 6 \\ 1 & 1 & 2 \\ 2 & 2 & 5 \end{bmatrix}$$

To find A^4:

Multiply equation (1) by A, we get $A^4 - 5A^3 + 9A^2 - A = 0$

$$A^4 = 5A^3 - 9A^2 + A$$

$$A^4 = \begin{bmatrix} -65 & 210 & -10 \\ -55 & 45 & 50 \\ 50 & -110 & -15 \end{bmatrix} - \begin{bmatrix} -9 & 108 & -36 \\ -36 & 63 & 18 \\ 18 & -72 & 9 \end{bmatrix}$$
$$+ \begin{bmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{bmatrix} = \begin{bmatrix} -55 & 104 & 24 \\ -20 & -15 & 32 \\ 32 & -42 & -23 \end{bmatrix}$$

Example 1.31. *If* $A = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}$, *find the value of* $A^5 + 5A^4 - 6A^3 + 2A^2 - 4A + 7I$, *using Cayley - Hamilton theorem.*

Solution: Characteristic equation of A is $|A - \lambda I| = 0$

(i.e.) $\begin{bmatrix} 1-\lambda & 4 \\ 2 & 3-\lambda \end{bmatrix} = 0 \Rightarrow \lambda^2 - 4\lambda - 5 = 0$

Using Cayley - Hamilton theorem, we have

$A^2 - 4A - 5I = 0 \Rightarrow A^2 = 4A + 5I$

$$\begin{aligned} A^3 &= 4A^2 + 5A = 4(4A+5I) + 5A \\ &= 21A + 20I \\ A^4 &= 21A^2 + 20A = 2(4A+5I) + 20A \\ &= 104A + 105I \\ A^5 &= 104A^2 + 105A = 104(4A+5I) + 105A \\ &= 521A + 520I \end{aligned}$$

$A^5 + 5A^4 - 6A^3 + 2A^2 - 4A + 7I = (521A + 520I)$
$+ 5(104A + 105I) - 6(21A + 20I) + 2(4A + 5I) - 4A + 7I$

$$\begin{aligned} &= 919A + 942I \\ &= 919\begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix} + 942\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1861 & 3676 \\ 1838 & 3699 \end{bmatrix} \end{aligned}$$

Example 1.32. *Given* $A = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}$, *find* A^{-1} *using Cayley - Hamilton theorem.*

Solution: Characteristic equation of A is $|A - \lambda I| = 0$

(i.e.) $\begin{vmatrix} 1-\lambda & 2 \\ 4 & 3-\lambda \end{vmatrix} = 0 \Rightarrow \lambda^2 - 4\lambda - 5 = 0$

Using Cayley - Hamilton theorem, we have

$A^2 - 4A - 5I = 0$ (1)

To find A^{-1}:

Multiply both sides of equation (1) by A^{-1}, we have

$5A^{-1} = A - 4I = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix} - 4\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -3 & 2 \\ 4 & -1 \end{bmatrix}$

$A^{-1} = \frac{1}{5}\begin{bmatrix} -3 & 2 \\ 4 & -1 \end{bmatrix}$

Example 1.33. *Verify Cayley - Hamilton theorem for the matrix*

$A = \begin{bmatrix} 1 & 3 & 7 \\ 4 & 2 & 3 \\ 1 & 2 & 1 \end{bmatrix}$ *and hence find* A^{-1}

Solution: Characteristic equation of A is $|A - \lambda I| = 0$

$$(i.e.) \begin{vmatrix} 1-\lambda & 3 & 7 \\ 4 & 2-\lambda & 3 \\ 1 & 2 & 1-\lambda \end{vmatrix} = 0$$

$\Rightarrow \lambda^3 - 4\lambda^2 - 20\lambda - 35 = 0$

We have to prove that by Cayley - Hamilton theorem,

$$A^3 - 4A^2 - 20A - 35I = 0 \tag{1.29}$$

Now $A^2 = \begin{bmatrix} 1 & 3 & 7 \\ 4 & 2 & 3 \\ 1 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 3 & 7 \\ 4 & 2 & 3 \\ 1 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 20 & 23 & 23 \\ 15 & 22 & 37 \\ 10 & 9 & 14 \end{bmatrix}$

$$A^3 = \begin{bmatrix} 1 & 3 & 7 \\ 4 & 2 & 3 \\ 1 & 2 & 1 \end{bmatrix} \begin{bmatrix} 20 & 23 & 23 \\ 15 & 22 & 37 \\ 10 & 9 & 14 \end{bmatrix} = \begin{bmatrix} 135 & 152 & 232 \\ 140 & 163 & 208 \\ 60 & 76 & 111 \end{bmatrix}$$

Equation (1.29) becomes

$$A^3 - 4A^2 - 20A - 35I = \begin{bmatrix} 135 & 152 & 232 \\ 140 & 163 & 208 \\ 60 & 76 & 111 \end{bmatrix} + \begin{bmatrix} -80 & -92 & -9 \\ -60 & -88 & -4 \\ -40 & -36 & -5 \end{bmatrix}$$

$$+ \begin{bmatrix} -20 & -60 & -40 \\ -80 & -40 & -60 \\ -20 & -40 & -20 \end{bmatrix} + \begin{bmatrix} -35 & 0 & 0 \\ 0 & -35 & 0 \\ 0 & 0 & -35 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Hence Cayley - Hamilton theorem is verified.

To find A^{-1}:

Multiply both sides of equation (1.29) by A^{-1}, we have
$A^2 - 4A - 20 - 35A^{-1} = 0$
$\Rightarrow A^{-1} = \frac{1}{35}(A^2 - 4A - 20)$

$$A^{-1} = \frac{1}{35}\left\{ \begin{bmatrix} 20 & 23 & 23 \\ 15 & 22 & 37 \\ 10 & 9 & 14 \end{bmatrix} + \begin{bmatrix} -4 & -12 & -28 \\ -16 & -8 & -12 \\ -4 & -8 & -4 \end{bmatrix} \right.$$

$$\left. + \begin{bmatrix} -20 & 0 & 0 \\ 0 & -20 & 0 \\ 0 & 0 & -20 \end{bmatrix} \right\}$$

Example 1.34. *Given the matrix* $A = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 2 & 0 \\ 1 & 1 & 2 \end{bmatrix}$, *find the value of* $A^8 - 5A^7 + 7A^6 - 3A^5 + A^4 - 5A^3 - 8A^2 + 2A - I$.

Solution: Characteristic equation of A is $|A - \lambda I| = 0$

$$(i.e.) \begin{vmatrix} 2-\lambda & 1 & 1 \\ 0 & 2-\lambda & 0 \\ 1 & 1 & 2-\lambda \end{vmatrix} = 0$$

$\Rightarrow \lambda^3 - 5\lambda^2 + 7\lambda - 3 = 0$

By Cayley - Hamilton theorem, we have

$$A^3 - 5A^2 + 7A - 3I = 0 \tag{1.30}$$

Now

$$A^2 = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 2 & 0 \\ 1 & 1 & 2 \end{bmatrix} \begin{bmatrix} 2 & 1 & 1 \\ 0 & 2 & 0 \\ 1 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 5 & 4 & 4 \\ 0 & 1 & 0 \\ 4 & 4 & 5 \end{bmatrix}$$

and

$$A^3 = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 2 & 0 \\ 1 & 1 & 2 \end{bmatrix} \begin{bmatrix} 5 & 4 & 4 \\ 0 & 1 & 0 \\ 4 & 4 & 5 \end{bmatrix} = \begin{bmatrix} 14 & 13 & 13 \\ 0 & 1 & 0 \\ 13 & 13 & 14 \end{bmatrix}$$

$$A^8 - 5A^7 + 7A^6 - 3A^5 + A^4 - 5A^3 - 8A^2 + 2A - I$$

$$\begin{aligned} &= A^5(A^3 - 5A^2 + 7A - 3I) \\ &+ A(A^3 - 5A^2 + 7A - 3I) \\ &\quad -15A^2 + 5A - I(using(1)) \\ &= -15\begin{bmatrix} 5 & 4 & 4 \\ 0 & 1 & 0 \\ 4 & 4 & 5 \end{bmatrix} + 5\begin{bmatrix} 2 & 1 & 1 \\ 0 & 2 & 0 \\ 1 & 1 & 2 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\ &= -11\begin{bmatrix} 6 & 5 & 5 \\ 0 & 1 & 0 \\ 5 & 5 & 6 \end{bmatrix} \end{aligned}$$

Example 1.35. *Verify Cayley - Hamilton theorem for*

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & 4 \\ 3 & 1 & -1 \end{bmatrix}$$ *and hence find A^{-1} and A^4.*

Solution: Characteristic equation of A is $|A - \lambda I| = 0$

$$\Rightarrow \begin{vmatrix} 1-\lambda & 2 & 3 \\ 2 & -1-\lambda & 4 \\ 3 & 1 & -1-\lambda \end{vmatrix} = 0$$

$$\Rightarrow \lambda^3 + \lambda^2 - 18\lambda - 40 = 0$$

We have to show that $A^3 + A^2 - 18A - 40I = 0$ (1)

Now $A^2 = \begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & 4 \\ 3 & 1 & -1 \end{bmatrix}\begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & 4 \\ 3 & 1 & -1 \end{bmatrix} = \begin{bmatrix} 14 & 3 & 8 \\ 12 & 9 & -2 \\ 2 & 4 & 14 \end{bmatrix}$

and

$$A^3 = \begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & 4 \\ 3 & 1 & -1 \end{bmatrix}\begin{bmatrix} 14 & 3 & 8 \\ 12 & 9 & -2 \\ 2 & 4 & 14 \end{bmatrix} = \begin{bmatrix} 44 & 33 & 46 \\ 24 & 13 & 74 \\ 54 & 14 & 8 \end{bmatrix}$$

$$\begin{aligned} A^3 + A^2 - 18A - 40I &= \begin{bmatrix} 44 & 33 & 46 \\ 24 & 13 & 74 \\ 54 & 14 & 8 \end{bmatrix} + \begin{bmatrix} 14 & 3 & 8 \\ 12 & 9 & -2 \\ 2 & 4 & 14 \end{bmatrix} \\ &\quad -18\begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & 4 \\ 3 & 1 & -1 \end{bmatrix} - 40\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \end{aligned}$$

Hence Cayley - Hamilton theorem is verified.

To find A^{-1}:
Pre multiplying equation (1) by A^{-1}, we have

$$A^2 + A - 18I - 40A^{-1} = 0 \Rightarrow A^{-1} = \frac{1}{40}[A^2 + A - 18I]$$

$$A^{-1} = \frac{1}{40}\left\{\begin{bmatrix} 14 & 3 & 8 \\ 12 & 9 & -2 \\ 2 & 4 & 14 \end{bmatrix} + \begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & 4 \\ 3 & 1 & -1 \end{bmatrix} - \begin{bmatrix} 18 & 0 & 0 \\ 0 & 18 & 0 \\ 0 & 0 & 18 \end{bmatrix}\right\} = \frac{1}{40}\begin{bmatrix} -3 & -4 & 11 \\ 14 & -10 & 2 \\ 5 & 5 & -5 \end{bmatrix}$$

To find A^4:

Multiply equation (1) by A, we have

$$A^4 = -A^3 - 18A^2 - 40A = \begin{bmatrix} 248 & 101 & 218 \\ 272 & 109 & 50 \\ 204 & 98 & 204 \end{bmatrix}$$

Example 1.36. *Find the characteristic equation of the matrix* $A = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix}$ *and hence find the matrix* $A^8 - 5A^7 + 7A^6 - 3A^5 + A^4 - 5A^3 + 8A^2 - 2A + I$

Solution: Characteristic equation of A is $|A - \lambda I| = 0$

(i.e.) $\begin{vmatrix} 2-\lambda & 1 & 1 \\ 0 & 2-\lambda & 0 \\ 1 & 1 & 2-\lambda \end{vmatrix} = 0 \Rightarrow \lambda^3 - 5\lambda^2 + 7\lambda - 3 = 0$

By Cayley - Hamilton theorem, we have

$A^3 - 5A^2 + 7A - 3I = 0$

Now $A^8 - 5A^7 + 7A^6 - 3A^5 + A^4 - 5A^3 + 8A^2 - 2A + I$

$$\begin{aligned} &= A^5(A^3 - 5A^2 + 7A - 3I) + A(A^3 - 5A^2 + 8A - 2I) + I \\ &= A^5(0) + A(A^3 - 5A^2 + 7A - 3I + A + I) + I \\ &= 0 + A(0 + A + I) + I \\ &= A^2 + A + I \\ &= \begin{bmatrix} 8 & 5 & 5 \\ 0 & 3 & 0 \\ 5 & 5 & 8 \end{bmatrix} \end{aligned}$$

EXERCISE

1. Verify Cayley - Hamilton theorem for $A = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}$
2. Verify Cayley-Hamilton theorem for $A = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$ and hence find A^{-1}.
3. Verify Cayley - Hamilton theorem for $A = \begin{bmatrix} 1 & 3 & 7 \\ 4 & 2 & 3 \\ 1 & 2 & 1 \end{bmatrix}$ and hence find A^{-1}.
4. Find A^{-1} and A^4 for the matrix $A = \begin{bmatrix} 1 & 0 & -1 \\ 3 & 4 & 5 \\ 0 & -6 & -7 \end{bmatrix}$

5. Verify Cayley - Hamilton theorem for i) $A = \begin{bmatrix} 1 & 0 & 3 \\ 2 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix}$,

ii) $A = \begin{bmatrix} 1 & 2 & -2 \\ 1 & 1 & 1 \\ 1 & 3 & -1 \end{bmatrix}$ and hence find A^{-1}.

6. Verify Cayley - Hamilton theorem for $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & -3 \\ 2 & -1 & 3 \end{bmatrix}$ and hence find A^{-1} and A^4.

7. If $A = \begin{bmatrix} 1 & 2 & -2 \\ 1 & 1 & 1 \\ 1 & 3 & -1 \end{bmatrix}$ find A^{-1} using Cayley - Hamilton theorem.

1.4 Diagonolisation of a Matrix by Orthogonal Transformation

Let A be a symmetric matrix. Form a matrix N whose columns are normalized eigen vectors of A. Since N is orthogonal we use $N^{-1} = N^T$

Again $D = N^T AN$ which is called orthogonal reduction or orthogonal transformation.

Note: Since A is a symmetric matrix, orthogonal transformation is possible only for real symmetric matrices.

Example 1.37. *Diagonolize the matrix* $A = \begin{bmatrix} 3 & 1 & 1 \\ 1 & 3 & -1 \\ 1 & -1 & 3 \end{bmatrix}$ *by means of orthogonal transformation.*

Solution: Characteristic equation of A is $|A - \lambda I| = 0$

(i.e.) $\begin{vmatrix} 3-\lambda & 1 & 1 \\ 1 & 3-\lambda & -1 \\ 1 & -1 & 3-\lambda \end{vmatrix} = 0 \Rightarrow \lambda^3 - 9\lambda^2 + 24\lambda - 16 = 0$

$\Rightarrow \lambda = 1, 4, 4$ are the eigen values.

To find eigen vectors:

Let $|A - \lambda I|X = 0$

$\Rightarrow \begin{bmatrix} 3-\lambda & 1 & 1 \\ 1 & 3-\lambda & -1 \\ 1 & -1 & 3-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

Case (i): When $\lambda = 1$

Equation (1) becomes

$$\begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Solve by using cross multiplication method,

we have $\frac{x_1}{3} = -\frac{x_2}{3} = \frac{x_3}{-3}$

Eigen vector for $\lambda = 1$ is $X_1 = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$

Case (ii): When $\lambda = 4$, equation (1) becomes

$$\begin{bmatrix} -1 & 1 & 1 \\ 1 & -1 & -1 \\ 1 & -1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow x_1 - x_2 - x_3 = 0$$

Let $x_2 = 1, x_3 = 0$ we get $x_1 = 1$

Eigen vector for $\lambda = 4$ is $X_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$

Case (iii): To find the third eigen vector since the matrix is symmetric.

Let us choose $X_3 = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ such that $X_1^T X_3 = 0$ and $X_2^T X_3 = 0$

$\Rightarrow \begin{bmatrix} -1 & 1 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = 0$ and $\begin{bmatrix} 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = 0$

$\Rightarrow a + b + c = 0$ and $a + b + 0c = 0$

$\Rightarrow \dfrac{a}{-1} = -\dfrac{b}{-1} = \dfrac{c}{-2}$

Therefore Eigen vector $X_3 = \begin{bmatrix} -1 \\ 1 \\ -2 \end{bmatrix}$

Normalized modal matrix $N = \begin{bmatrix} -1/\sqrt{3} & 1/\sqrt{2} & -1/\sqrt{6} \\ 1/\sqrt{3} & 1/\sqrt{2} & 1/\sqrt{6} \\ 1/\sqrt{3} & 0 & -2/\sqrt{6} \end{bmatrix}$

Now

$$\begin{aligned} D &= N^T A N \\ &= \begin{bmatrix} -1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \\ 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ -1/\sqrt{6} & 1/\sqrt{6} & -2/\sqrt{6} \end{bmatrix} \begin{bmatrix} 3 & 1 & 1 \\ 1 & 3 & -1 \\ 1 & -1 & 3 \end{bmatrix} \\ &\times \begin{bmatrix} -1/\sqrt{3} & 1/\sqrt{2} & -1/\sqrt{6} \\ 1/\sqrt{3} & 1/\sqrt{2} & 1/\sqrt{6} \\ 1/\sqrt{3} & 0 & -2/\sqrt{6} \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix} \end{aligned}$$

Example 1.38. *Reduce the matrix* $A = \begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$ *to a diagonal form using orthogonal transformation.*

Solution: Characteristic equation of A is $|A - \lambda I| = 0$

(i.e.) $\begin{vmatrix} 8-\lambda & -6 & 2 \\ -6 & 7-\lambda & -4 \\ 2 & -4 & 3-\lambda \end{vmatrix} = 0$

$\Rightarrow \lambda^3 - 18\lambda^2 + 45\lambda = 0$

$\Rightarrow \lambda = 0, 3, 15$

Case (i): When $\lambda = 0$, eigen vector $X_1 = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$

and normalized eigen vector $X_1 = \begin{bmatrix} 1/3 \\ 1/3 \\ 1/3 \end{bmatrix}$

Case (ii): When $\lambda = 3$, eigen vector $X_2 = \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}$

and normalized eigen vector $X_2 = \begin{bmatrix} 2/3 \\ 1/3 \\ -2/3 \end{bmatrix}$

Case (iii): When $\lambda = 15$, eigen vector $X_3 = \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}$

and normalized eigen vector $X_3 = \begin{bmatrix} 2/3 \\ -2/3 \\ 1/3 \end{bmatrix}$

Normalized modal matrix $N = \begin{bmatrix} 1/3 & 2/3 & 2/3 \\ 1/3 & 1/3 & -2/3 \\ 1/3 & -2/3 & 1/3 \end{bmatrix}$

$$N^T = \begin{bmatrix} 1/3 & 1/3 & 1/3 \\ 2/3 & 1/3 & 2/3 \\ 2/3 & -2/3 & 1/3 \end{bmatrix}$$

$$\begin{aligned} D &= N^T A N \\ &= \begin{bmatrix} 1/3 & 1/3 & 1/3 \\ 2/3 & 1/3 & 2/3 \\ 2/3 & -2/3 & 1/3 \end{bmatrix} \begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix} \\ &\times \begin{bmatrix} 1/3 & 2/3 & 2/3 \\ 1/3 & 1/3 & -2/3 \\ 1/3 & -2/3 & 1/3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 15 \end{bmatrix} \end{aligned}$$

Example 1.39. *Reduce the matrix* $A = \begin{bmatrix} 10 & -2 & -5 \\ -2 & 2 & 3 \\ -5 & 3 & 5 \end{bmatrix}$ *to*

Diagonal form by an orthogonal transformation.

Solution: Characteristic equation of A is $|A - \lambda I| = 0$

(i.e.) $\begin{vmatrix} 10-\lambda & -2 & -5 \\ -2 & 2-\lambda & 3 \\ -5 & 3 & 5-\lambda \end{vmatrix} = 0 \Rightarrow \lambda^3 - 17\lambda^2 + 42\lambda = 0$

$\Rightarrow \lambda = 0, 3, 14$

Find eigen vectors as in the previous example.

We get eigen vector X_1 for $\lambda = 0$ is $X_1 = \begin{bmatrix} 1 \\ -5 \\ 4 \end{bmatrix}$

Therefore Eigen vector X_2 for $\lambda = 3$ is $X_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

Therefore Eigen vector X_3 for $\lambda = 14$ is $X_3 = \begin{bmatrix} -3 \\ 1 \\ 2 \end{bmatrix}$

Now the normalized model matrix is

$$N = \begin{bmatrix} 1/\sqrt{42} & 1/\sqrt{3} & -3/\sqrt{14} \\ -5/\sqrt{42} & 1/\sqrt{3} & 1/\sqrt{14} \\ 4/\sqrt{42} & 1/\sqrt{3} & 2/\sqrt{14} \end{bmatrix}$$

$$\begin{aligned} D &= N^T A N \\ &= \begin{bmatrix} 1/\sqrt{42} & -5/\sqrt{42} & 4/\sqrt{42} \\ 1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \\ -3/\sqrt{14} & 1/\sqrt{14} & 2/\sqrt{14} \end{bmatrix} \begin{bmatrix} 10 & -2 & -5 \\ -2 & 2 & 3 \\ -5 & 3 & 5 \end{bmatrix} \\ &\quad \begin{bmatrix} 1/\sqrt{42} & 1/\sqrt{3} & -3/\sqrt{14} \\ -5/\sqrt{42} & 1/\sqrt{3} & 1/\sqrt{14} \\ 4/\sqrt{42} & 1/\sqrt{3} & 2/\sqrt{14} \end{bmatrix} \end{aligned}$$

$$\Rightarrow \quad D = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 14 \end{bmatrix}$$

Example 1.40. *Diagonolize* $\begin{bmatrix} 2 & 1 & -1 \\ 1 & 1 & -2 \\ -1 & -2 & 1 \end{bmatrix}$ *by means of orthogonal transformation.*

Solution: Let $A = \begin{bmatrix} 2 & 1 & -1 \\ 1 & 1 & -2 \\ -1 & -2 & 1 \end{bmatrix}$

Characteristic equation of A is $|A - \lambda I| = 0$

(i.e.) $\begin{vmatrix} 2-\lambda & 1 & -1 \\ 1 & 1-\lambda & -2 \\ -1 & -2 & 1-\lambda \end{vmatrix} = 0 \Rightarrow \lambda^3 - 4\lambda^2 - \lambda + 4 = 0$

(i.e.) $\lambda = -1, 1, 4$

To find the eigen vectors:

Let $|A - \lambda I|X = 0$

$$\begin{bmatrix} 2-\lambda & 1 & -1 \\ 1 & 1-\lambda & -2 \\ -1 & -2 & 1-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Case (i): When $\lambda = -1$, eigen vector $X_1 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$

Case (ii): When $\lambda = 1$, eigen vector $X_2 = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$

Case (iii): When $\lambda = 4$, eigen vector $X_3 = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$

Normalized modal matrix $N = \begin{bmatrix} 0 & 2/\sqrt{6} & 1/\sqrt{3} \\ 1/\sqrt{2} & -1/\sqrt{6} & 1/\sqrt{3} \\ 1/\sqrt{2} & 1/\sqrt{6} & 1/\sqrt{3} \end{bmatrix}$

$$\begin{aligned} D &= N^T A N \\ &= \begin{bmatrix} 0 & 1/\sqrt{2} & 1/\sqrt{2} \\ 2/\sqrt{6} & -1/\sqrt{6} & 1/\sqrt{6} \\ 1/\sqrt{3} & 1/\sqrt{3} & -1/\sqrt{3} \end{bmatrix} \begin{bmatrix} 2 & 1 & -1 \\ 1 & 1 & -2 \\ -1 & -2 & 1 \end{bmatrix} \\ & \quad \begin{bmatrix} 0 & 2/\sqrt{6} & 1/\sqrt{3} \\ 1/\sqrt{2} & -1/\sqrt{6} & 1/\sqrt{3} \\ 1/\sqrt{2} & 1/\sqrt{6} & 1/\sqrt{3} \end{bmatrix} \\ &= \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 4 \end{bmatrix} \end{aligned}$$

Example 1.41. *Diagonolize the matrix* $\begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$ *using orthogonal transformation.*

Solution: Characteristic equation of A is $|A - \lambda I| = 0$

(i.e.) $\begin{vmatrix} 2-\lambda & -1 & 1 \\ -1 & 2-\lambda & -1 \\ 1 & -1 & 2-\lambda \end{vmatrix} = 0 \Rightarrow \lambda^3 - 6\lambda^2 + 9\lambda - 4 = 0$

(i.e.) $\lambda = 1, 1, 4$

Case (i): When $\lambda = 1$, eigen vector $X_1 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$

and $X_2 = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$

Case (ii): When $\lambda = 4$, eigen vector $X_3 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$

Normalized model matrix $N = \begin{bmatrix} 0 & 2/\sqrt{6} & 1/\sqrt{3} \\ 1/\sqrt{2} & 1/\sqrt{6} & -1/\sqrt{3} \\ 1/\sqrt{2} & -1/\sqrt{6} & 1/\sqrt{3} \end{bmatrix}$

$N^T = \begin{bmatrix} 0 & 1/\sqrt{2} & 1/\sqrt{2} \\ 2/\sqrt{6} & 1/\sqrt{6} & -1/\sqrt{6} \\ 1/\sqrt{3} & -1/\sqrt{3} & 1/\sqrt{3} \end{bmatrix}$

Now

$$\begin{aligned} D &= N^T A N \\ &= \begin{bmatrix} 0 & 1/\sqrt{2} & 1/\sqrt{2} \\ 2/\sqrt{6} & 1/\sqrt{6} & -1/\sqrt{6} \\ 1/\sqrt{3} & -1/\sqrt{3} & 1/\sqrt{3} \end{bmatrix} \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} \\ &\quad \begin{bmatrix} 0 & 2/\sqrt{6} & 1/\sqrt{3} \\ 1/\sqrt{2} & 1/\sqrt{6} & -1/\sqrt{3} \\ 1/\sqrt{2} & -1/\sqrt{6} & 1/\sqrt{3} \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 4 \end{bmatrix} \end{aligned}$$

EXERCISE

1. Diagonolize the matrix $A = \begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix}$

   Ans: $D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 5 \end{bmatrix}$

2. Diagonolize the matrix $A = \begin{bmatrix} 2 & 0 & 4 \\ 0 & 6 & 0 \\ 4 & 0 & 2 \end{bmatrix}$

   Ans: $D = \begin{bmatrix} -2 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 6 \end{bmatrix}$

3. Diagonolize the matrix $A = \begin{bmatrix} 3 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 3 \end{bmatrix}$

   Ans: $D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{bmatrix}$

1.5 Quadratic Forms

Definition 1.3. *A homogeneous expression of second degree in any number of unknowns is called a quadratic form (Q.F).*

E. g: $Q = x_1^2 - 2x_2^2 + 4x_3^2 - 3x_1x_2 + 4x_2x_3 + 6x_1x_3$ is a Q.F in three unknowns x_1, x_2, x_3.
Consider a simple Q.F: $Q = ax^2 + 2hxy + by^2$
We regard Q as a matrix of order $|x|$.

$$(i.e.)Q = [ax^2 + 2hxy + by^2]_{|x|}$$
$$= \begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} ax+by \\ hx+by \end{bmatrix}$$
$$Q = \begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} a & h \\ h & b \end{bmatrix} \begin{bmatrix} x & y \end{bmatrix}$$

Let $X = \begin{bmatrix} x \\ y \end{bmatrix}$, then $X' = \begin{bmatrix} x & y \end{bmatrix}$ and let $A = \begin{bmatrix} a & h \\ h & b \end{bmatrix}$ then $Q = X'AX$.
Thus we observe that a Q.F can be written as a product of three matrices X', A, X where (i) X is a column matrix of unknowns or variables.
(ii) X' is a transpose of X.
(iii) A is a symmetric matrix of order 2 (which is equal to number of unknowns in Q.F).
Symmetric matrix A is called as matrix of the quadratic form.

Note:
(i) Diagonal entries of A are the co-efficients of square terms in Q.
(ii) Non-diagonal entries of A are half the co-efficients of product terms in Q.

Example 1.42. *Write the Q.F as product of matrices $Q = x_1^2 - 2x_2^2 + 3x_3^2 - 4x_1x_2 + 5x_2x_3 + 6x_1x_3$*

Solution: The required form is

$$Q = \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} \begin{bmatrix} 1 & -2 & 3 \\ -2 & -2 & 5/2 \\ 3 & 5/2 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Example 1.43. *Write the Q.F where $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 9 \\ 3 & 9 & 3 \end{bmatrix}$*

Solution:
$Q = x_1^2 + 4x_2^2 - 3x_3^2 + 4x_1x_2 + 18x_2x_3 + 6x_1x_3$

Example 1.44. *Write the Q.F where* $\begin{bmatrix} 3 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & -4 \end{bmatrix}$

x **Solution:** $Q = 3x_1^2 + 9x_2^2 - 4x_3^2$

Note: The transformed Q.F is called as canonical form.

Index: The number of positive terms in the canonical form is called as index of the form and denoted by p.

Signature: The difference between positive terms (p) and negative terms (r-p) where r is the rank of the matrix in the canonical form is called as signature of the quadratic form.

Signature $= p - (r - p) = 2p - r$

Some definitions:

i A quadratic form $Q = X'AX$ in $'n'$ variables is called positive definite if $r = p = n(here\ |A| \neq 0)$

ii A quadratic form $Q = X'AX$ is called as positive semi-definite if $r = p < n(here\ |A| \neq 0)$

iii A quadratic form is negative definite if $p = 0$ and $r = n$

iv A quadratic form $Q = X'AX$ is negative semi-definite if $p = 0, r < n$

v A quadratic form is indefinite in other cases.

1.6 Reduction of Quadratic forms into Canonical forms using Orthogonal Transformation

i) First we write $Q = X'AX$

ii) (ii) We find eigen values $\lambda_1, \lambda_2, \lambda_3$ and eigen vectors X_1, X_2, X_3 of A noting that A is symmetric.

iii) We normalize eigen vectors and call them as $\overline{X_1}, \overline{X_2}, \overline{X_3}$ and form a matrix $P = \left[\ \overline{X_1}\ \ \overline{X_2}\ \ \overline{X_3}\ \right]$

iv) Find $P^T AP = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$ and using the transformation $X = PY$, we have $Q = X'AX = Y'(P'AP)Y \Rightarrow Q = \lambda_1 y_1^2 + \lambda_2 y_2^2 + \lambda_3 y_3^2$

Thus using the transformation X = PY we have reduced Q to a sum of squares in which coefficient of square terms are eigen values of A. Since P is orthogonal.
$X = PY$ is called an orthogonal transformation.
Determine the nature of the quadratic forms without reducing

them to canonical forms.
Method is explained below:

Let A to a square matrix of the quadratic form is $\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$

Let $D_1 = |a_{11}|$; $D_2 = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$; $D_3 = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$

i) The Q.F is positive definite if D_1, D_2, D_3 are all positive.

ii) The Q.F is negative definite if $D_1, D_3, \cdots$ are negative and $D_2, D_4, \cdots$ are all positive in $(-1)^n D_n > 0$, for all n.

iii) The Q.F is positive semi definite if $D_1, D_2, \cdots \geq 0$ and atleast one D_1 is equal to zero.

iv) The Q.F is negative semi definite if $(-1)^n D_n \geq 0$ and atleast one $D_i = 0$

v) In all the other cases it is indefinite.

Example 1.45. *Determine the nature of the following quadratic forms without reducing then to canonical form.*

i) $6x^2 + 3y^2 + 14z^2 + 4yz + 18xz + 4xy$

ii) $x_1^2 + 3x_2^2 + 6x_3^2 + 2x_1x_2 + 2x_2x_3 + 4x_3x_1$

iii) $2x_1x_2 + 2x_2x_3 - 2x_3x_1$

Solution: i) $A = \begin{bmatrix} 6 & 2 & 9 \\ 2 & 3 & 2 \\ 9 & 2 & 14 \end{bmatrix}$

Now $D_1 = 6$; $D_2 = \begin{vmatrix} 6 & 2 \\ 2 & 3 \end{vmatrix} > 0$ Now $D_1 = 6$; $D_3 = |A| = -ve$
(i.e.) The Q.F is indefinite.

ii) $A = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 3 & 1 \\ 2 & 1 & 6 \end{bmatrix}$

Now $D_1 = 1$; $D_2 = \begin{vmatrix} 1 & 1 \\ 1 & 3 \end{vmatrix} = 2$; $D_3 = |A| = 3$
Here D_1, D_2, D_3 are all positive.
(i.e.) The Q.F is positive definite.
iii) Now $D_1 = 0$

$D_2 = \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} = -1$

$D_3 = |A| = -2$
(i.e.) The Q.F is negative semi definite.

Example 1.46. *Reduce the quadratic form* $Q = 3x_1^2 + 5x_2^2 + 3x_3^2 - 2x_1x_2 - 2x_2x_3 + 2x_3x_1$ *to a diagonal canonical form and* *hence find its nature, rank, index and signature.*

Solution: Given $Q = \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} \begin{bmatrix} 3 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

Now, we find the eigen values and eigen vectors of

$$A = \begin{bmatrix} 3 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{bmatrix}$$

Characteristic equation of A is $|A - \lambda I| = 0$

$$\Rightarrow \begin{vmatrix} 3-\lambda & -1 & 1 \\ -1 & 5-\lambda & -1 \\ 1 & -1 & 3-\lambda \end{vmatrix} = 0$$

$\lambda^3 - 11\lambda^2 + 36\lambda - 36 = 0$

(i.e.) $\lambda_1 = 2, \lambda_2 = 3, \lambda_3 = 6$

To find the eigen vectors:

Let $[A - \lambda I]X = 0$

$$\begin{bmatrix} 3-\lambda & -1 & 1 \\ -1 & 5-\lambda & -1 \\ 1 & -1 & 3-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \qquad (1.31)$$

Case(i): When $\lambda_1 = 2$ equation (1.31) becomes

$$\begin{bmatrix} 1 & -1 & 1 \\ -1 & 3 & -1 \\ 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

(i.e.) $\frac{x_1}{2} = -\frac{x_2}{0} = \frac{x_3}{-2}$

Therefore Eigen vector $X_1 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$

Case(ii): When $\lambda_2 = 3$ equation (1.31) becomes

$$\begin{bmatrix} 0 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

(i.e.) $\frac{x_1}{-1} = -\frac{x_2}{1} = \frac{x_3}{-1}$

Therefore Eigen vector $X_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

Case(iii): When $\lambda_3 = 6$ equation (1.31) becomes

$$\begin{bmatrix} -3 & -1 & 1 \\ -1 & -1 & -1 \\ 1 & -1 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

(i.e.) $\frac{x_1}{2} = -\frac{x_2}{4} = \frac{x_3}{2}$

Therefore Eigen vector $X_3 = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$

Now normalized eigen vectors are

$$\overline{X_1} = \begin{bmatrix} -1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{bmatrix}, \overline{X_2} = \begin{bmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix}, \overline{X_3} = \begin{bmatrix} 1/\sqrt{6} \\ -2/\sqrt{6} \\ 1/\sqrt{6} \end{bmatrix}$$

$$P = \begin{bmatrix} \overline{X_1} & \overline{X_2} & \overline{X_3} \end{bmatrix} \begin{bmatrix} -1/\sqrt{2} & 1/\sqrt{3} & 1/\sqrt{6} \\ 0 & 1/\sqrt{3} & -2/\sqrt{6} \\ 1/\sqrt{2} & 1/\sqrt{3} & 1/\sqrt{6} \end{bmatrix}$$

Now P is orthogonal.

Let $X = PY$ be the orthogonal transformation then

$$Q = X'AX = Y'(P'AP)Y \tag{2}$$

$$P'AP = \begin{bmatrix} -1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \\ 1/\sqrt{6} & -2/\sqrt{6} & 1/\sqrt{6} \end{bmatrix} \begin{bmatrix} -3 & -1 & 1 \\ -1 & -1 & -1 \\ 1 & -1 & -3 \end{bmatrix}$$

$$\times \begin{bmatrix} -1/\sqrt{2} & 1/\sqrt{3} & 1/\sqrt{6} \\ 0 & 1/\sqrt{3} & -2/\sqrt{6} \\ 1/\sqrt{2} & 1/\sqrt{3} & 1/\sqrt{6} \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 6 \end{bmatrix}$$

Equation (2) gives $Q = 2y_1^2 + 3y_2^2 + 6y_3^2$
Hence rank $r = 3$
Index $p = 3$
Signature $= 2p - r = 3$
The quadratic form is positive definite, since all the eigen values are positive.

Example 1.47. *Reduce the quadratic form $x_1^2 + 2x_2^2 + x_3^2 - 2x_1x_2 + 2x_2x_3$ to a canonical form using orthogonal transformation and hence show that it is positive semi definite. Give also a non-zero set of values x_1, x_2, x_3 which makes this quadratic form zero.*

Solution: Given $Q = x_1^2 + 2x_2^2 + x_3^2 - 2x_1x_2 + 2x_2x_3$

Let $A = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & 1 \\ 0 & 1 & 1 \end{bmatrix} \Rightarrow \lambda^3 - 4\lambda^2 + 3\lambda = 0$

(i.e.) $\lambda_1 = 0, \lambda_2 = 3, \lambda_3 = 1$

To find the eigen vectors:

Let $|A - \lambda I|X = 0$

$$\Rightarrow \begin{bmatrix} 1-\lambda & -1 & 0 \\ -1 & 2-\lambda & 1 \\ 0 & 1 & 1-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \tag{1}$$

Case(i): When $\lambda_1 = 0$ equation (1) becomes

$$\begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & 1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

(i.e.) $\dfrac{x_1}{1} = -\dfrac{x_2}{-1} = \dfrac{x_3}{-1}$

Therefore Eigen vector $X_1 = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$

Case(ii): When $\lambda_2 = 3$ equation (1) becomes

$$\begin{bmatrix} -2 & -1 & 0 \\ -1 & -1 & 1 \\ 0 & 1 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

(i.e.) $\dfrac{x_1}{1} = -\dfrac{x_2}{2} = \dfrac{x_3}{-1}$

Therefore Eigen vector $X_2 = \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}$

Case(iii): When $\lambda_3 = 1$ equation (1) becomes

$$\begin{bmatrix} 0 & -1 & 0 \\ -1 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

(i.e.) $\frac{x_1}{-1} = -\frac{x_2}{0} = \frac{x_3}{-1}$

Therefore Eigen vector $X_3 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$

Normalized modal matrix $P = \begin{bmatrix} 1/\sqrt{3} & -1/\sqrt{6} & 1/\sqrt{2} \\ 1/\sqrt{3} & 2/\sqrt{6} & 0 \\ 1/\sqrt{3} & 1/\sqrt{6} & 1/\sqrt{2} \end{bmatrix}$

Now

$$P'AP = \begin{bmatrix} 1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \\ -1/\sqrt{6} & 2/\sqrt{6} & 1/\sqrt{6} \\ 1/\sqrt{2} & 0 & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$
$$\times \begin{bmatrix} 1/\sqrt{3} & -1/\sqrt{6} & 1/\sqrt{2} \\ 1/\sqrt{3} & 2/\sqrt{6} & 0 \\ 1/\sqrt{3} & 1/\sqrt{6} & 1/\sqrt{2} \end{bmatrix}$$
$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$Q = Y'(P'AP)Y$

$= \begin{pmatrix} y_1 & y_2 & y_3 \end{pmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$

$= 0y_1^2 + 3y_2^2 + y_3^2$ which is the canonical form and the form is positive semi definite.

Again $X = PY$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{3} & -1/\sqrt{6} & 1/\sqrt{2} \\ 1/\sqrt{3} & 2/\sqrt{6} & 0 \\ 1/\sqrt{3} & 1/\sqrt{6} & 1/\sqrt{2} \end{bmatrix}$$
$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{3}}y_1 & -\frac{1}{\sqrt{6}}y_2 & \frac{1}{\sqrt{2}}y_3 \\ \frac{1}{\sqrt{3}}y_1 & \frac{2}{\sqrt{6}}y_2 & 0y_3 \\ \frac{1}{\sqrt{3}}y_1 & \frac{1}{\sqrt{6}}y_2 & \frac{1}{\sqrt{2}}y_3 \end{bmatrix}$$
$$x_1 = \frac{1}{\sqrt{3}}y_1 - \frac{1}{\sqrt{6}}y_2 + \frac{1}{\sqrt{6}}y_2\frac{1}{\sqrt{2}}y_3$$
$$x_2 = \frac{1}{\sqrt{3}}y_1 + \frac{2}{\sqrt{6}}y_2 + 0y_3$$
$$x_3 = \frac{1}{\sqrt{3}}y_1 + \frac{1}{\sqrt{6}}y_2 + \frac{1}{\sqrt{2}}y_3$$

By assuming y_2 and y_3 as 0 and y_1 as arbitrary value, the quadratic form will become zero.
Substituting $y_1 = \sqrt{3}, y_2 = 0, y_3 = 0$ in the above equation, we get
$x_1 = 1-0+0 = 1, x_2 = 1+0+0 = 1$ and $x_3 = -1+0+0 = -1$

Example 1.48. *Reduce the quadratic form* $x_1^2 + 5x_2^2 + x_3^2 + 2x_1x_2 + 2x_2x_3 + 6x_3x_1$ *to a canonical form using orthogonal transformation.*

Solution: Let the matrix the quadratic form $A = \begin{bmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{bmatrix}$

Characteristic equation of A is $|A - \lambda I| = 0$

(i.e.) $\begin{vmatrix} 1-\lambda & 1 & 3 \\ 1 & 5-\lambda & 1 \\ 3 & 1 & 1-\lambda \end{vmatrix} = 0$

$\Rightarrow \lambda^3 - 7\lambda^2 + 36 = 0$

$\Rightarrow \lambda = -2, 3, 6$ are the eigen values.

To find eigen vectors:

Let $|A - \lambda I|X = 0$

$$\begin{bmatrix} 1-\lambda & 1 & 3 \\ 1 & 5-\lambda & 1 \\ 3 & 1 & 1-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \qquad (1)$$

Case(i): When $\lambda_1 = -2$ equation (1) becomes

$$\begin{bmatrix} 3 & 1 & 3 \\ 1 & 7 & 1 \\ 3 & 1 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\Rightarrow \dfrac{x_1}{21-1} = -\dfrac{x_2}{3-3} = \dfrac{x_3}{1-21}$

Eigen vector $X_1 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$

Case(ii): When $\lambda_1 = 3$ equation (1) becomes

$$\begin{bmatrix} -2 & 1 & 3 \\ 1 & 2 & 1 \\ 3 & 1 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\Rightarrow \dfrac{x_1}{-5} = -\dfrac{x_2}{-5} = \dfrac{x_3}{-5}$

Eigen vector $X_2 = \begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix}$

Case(iii): When $\lambda_1 = 6$ equation (1) becomes

$$\begin{bmatrix} -5 & 1 & 3 \\ 1 & -1 & 1 \\ 3 & 1 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\Rightarrow \dfrac{x_1}{4} = -\dfrac{x_2}{-8} = \dfrac{x_3}{4}$

Eigen vector $X_1 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$

Normalized modal matrix is

$$P = \begin{bmatrix} -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} \\ 0 & \frac{1}{\sqrt{3}} & \frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} \end{bmatrix}$$

and

$$P^T = \begin{bmatrix} -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{6}} & \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{6}} \end{bmatrix}$$

$$P^TAP = \begin{bmatrix} -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{6}} & \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{6}} \end{bmatrix} \begin{bmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{bmatrix}$$

$$\times \begin{bmatrix} -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} \\ 0 & \frac{1}{\sqrt{3}} & \frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} \end{bmatrix}$$

$$= \begin{bmatrix} -2 & 0 & 0 \\ 0 & 3 & 1 \\ 0 & 0 & 6 \end{bmatrix}$$

Hence $(P^TAP)X = \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} \begin{bmatrix} -2 & 0 & 0 \\ 0 & 3 & 1 \\ 0 & 0 & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

$= -2x_1^2 + 3x_2^2 + 6x_3^2$ which is the canonical form.

Example 1.49. *Reduce the quadratic form* $2x_1^2 + 5x_2^2 + 3x_3^2 +$ $4x_1x_2$ *to a canonical form and hence find rank, index and signature of the quadratic form.*

Solution: Let the matrix the quadratic form $A = \begin{bmatrix} 2 & 2 & 0 \\ 2 & 5 & 0 \\ 0 & 0 & 3 \end{bmatrix}$

Characteristic equation of A is $|A - \lambda I| = 0$

(i.e.) $\begin{vmatrix} 2-\lambda & 2 & 0 \\ 2 & 5-\lambda & 0 \\ 0 & 0 & 3-\lambda \end{vmatrix} = 0$

$\Rightarrow \lambda^3 - 10\lambda^2 + 27\lambda - 18 = 0$

(i.e.) $\lambda_1 = 1, \lambda_2 = 3, \lambda_3 = 6$

To find the eigen vectors:

Let $|A - \lambda I|X = 0$

$$\Rightarrow \begin{bmatrix} 2-\lambda & 2 & 0 \\ 2 & 5-\lambda & 0 \\ 0 & 0 & 3-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0 \qquad (1)$$

Case(i): When $\lambda_1 = 1$ equation (1) becomes

$\begin{bmatrix} 1 & 2 & 0 \\ 2 & 4 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$\Rightarrow \frac{x_1}{-2} = \frac{x_2}{1} = \frac{x_3}{0}$

Eigen vector $X_1 = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$

Case(ii): When $\lambda_2 = 3$, $X_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

Case(iii): When $\lambda_3 = 6$, $X_3 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$

Normalised modal matrix

$$P = \begin{bmatrix} -2/\sqrt{5} & 0 & 1/\sqrt{5} \\ 2/\sqrt{5} & 0 & 1/\sqrt{5} \\ 0 & 1 & 0 \end{bmatrix}$$

$$\begin{aligned} P'AP &= \begin{bmatrix} -2/\sqrt{5} & 1/\sqrt{5} & 0 \\ 0 & 0 & 1 \\ 1/\sqrt{5} & 2/\sqrt{5} & 0 \end{bmatrix} \begin{bmatrix} 2 & 2 & 0 \\ 2 & 5 & 0 \\ 0 & 0 & 3 \end{bmatrix} \\ &= \times \begin{bmatrix} -2/\sqrt{5} & 0 & 1/\sqrt{5} \\ 2/\sqrt{5} & 0 & 1/\sqrt{5} \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 6 \end{bmatrix} \end{aligned}$$

$$Q = Y'(P'AP)Y = \begin{pmatrix} y_1 & y_2 & y_3 \end{pmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 6 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

$= y_1^2 + 3y_2^2 + 6y_3^2$ which is the required canonical form.
Rank of the matrix $r = 3$
Index $p = 3$
Signature $= 3$
Matrix of the quadratic form is positive definite.

Example 1.50. *Reduce the quadratic form $x_1^2 + 2x_2x_3$ into a canonical form by means of an orthogonal transformation. Determine its nature.*

Solution: Let $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$

Characteristic equation of A is $|A - \lambda I| = 0$

$$\Rightarrow \begin{vmatrix} 1-\lambda & 0 & 0 \\ 0 & 0-\lambda & 1 \\ 0 & 1 & 0-\lambda \end{vmatrix}$$

$\lambda^3 - \lambda^2 - \lambda + 1 = 0$
$\lambda = 1, 1$ and -1 are the eigen values.

To find the eigen vectors:
$|A - \lambda I|X = 0$

$$\Rightarrow \begin{bmatrix} 1-\lambda & 0 & 0 \\ 0 & 0-\lambda & 1 \\ 0 & 1 & 0-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0 \qquad (1)$$

Case(i): When $\lambda_1 = -1$ equation (1) gives
$2x_1 = 0 \Rightarrow x_1 = 0$
$x_1 + x_2 = 0$
$x_2 + x_3 = 0 \Rightarrow x_2 = -x_3$
Put $x_3 = 1$, we get $x_2 = -1$

Therefore eigen vector X_1 for $\lambda_1 = -1$ is $\begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$

Case(ii): When $\lambda_1 = 1$ equation (1) gives
$0x_1 = 0 \Rightarrow x_1$ takes any value.
$-x_2 + x_3 = 0$
$x_2 - x_3 = 0 \Rightarrow x_2 = x_3$
Put $x_2 = 1$, we get $x_3 = 1$
And let $x_1 = -1$

Therefore eigen vector $X_2 = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$

Let $X_3 = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ be third eigen vector which is orthogonal to X_2.
(i.e.) $-x+y+z=0$
Again $0x-y+z=0$
$\Rightarrow y=z$
Therefore $x=2z$
Let $z=1$, we get $y=1$ and $x=2$

Therefore eigen vector $X_2 = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$

The normalized modal matrix is

$$P = \begin{bmatrix} 0 & -\frac{1}{\sqrt{3}} & \frac{2}{\sqrt{6}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} \end{bmatrix}$$

Now $X'AX = Y'(P^TAP)Y =$

$$\begin{pmatrix} y_1 & y_2 & y_3 \end{pmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

$= y_1^2 + y_2^2 - y_3^2$ which is the required canonical form of the given quadratic form. Again one eigen value is negative and two

eigen values are positive therefore the given quadratic form is indefinite.

Example 1.51. *Write down one quadratic form, whose associated matrix is*

$$\begin{bmatrix} 3 & 1 & 1 \\ 1 & 0 & 2 \\ 1 & 2 & 0 \end{bmatrix}$$

and reduce it to its canonical form.

Solution: Quadratic form is $3x_1^2 + 2x_1x_2 + 4x_2x_3 + 2x_3x_1$
Characteristic equation of A is $|A - \lambda I| = 0$

$$\Rightarrow \begin{vmatrix} 3-\lambda & 1 & 1 \\ 1 & 0-\lambda & 2 \\ 1 & 2 & 0-\lambda \end{vmatrix} = 0$$

(i.e.) $\lambda^3 - 3\lambda^2 - 6\lambda + 8 = 0$
$\Rightarrow \lambda = 1, -2, 4$
To find the eigen vectors:
Let $|A - \lambda I|X = 0$

$$\Rightarrow \begin{bmatrix} 3-\lambda & 1 & 1 \\ 1 & 0-\lambda & 2 \\ 1 & 2 & 0-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0 \qquad (1)$$

Case(i): When $\lambda_1 = 1$ equation (1) becomes

$$\Rightarrow \begin{bmatrix} 2 & 1 & 1 \\ 1 & -1 & 2 \\ 1 & 2 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$$\frac{x_1}{1-4} = -\frac{x_2}{-1-2} = \frac{x_3}{2+1}$$

Therefore Eigen vector $X_1 = \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}$

Case(ii): When $\lambda_1 = -2$ equation (1) becomes

$$\Rightarrow \begin{bmatrix} 5 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

ThereforeEigen vector $X_2 = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$

Case(iii): When $\lambda_1 = 4$ equation (1) becomes

$$\Rightarrow \begin{bmatrix} -1 & 1 & 1 \\ 1 & -4 & 2 \\ 1 & 2 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

Therefore Eigen vector $X_3 = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$ Normalised modal matrix

$$P = \begin{bmatrix} \frac{1}{\sqrt{3}} & 0 & \frac{2}{\sqrt{6}} \\ -\frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} \end{bmatrix}$$

$$\begin{aligned} P^T AP &= \begin{bmatrix} \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} \\ 0 & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} \end{bmatrix} \begin{bmatrix} 3 & 1 & 1 \\ 1 & 0 & 2 \\ 1 & 2 & 0 \end{bmatrix} \\ &= \times \begin{bmatrix} \frac{1}{\sqrt{3}} & 0 & \frac{2}{\sqrt{6}} \\ -\frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 4 \end{bmatrix} \\ &= y_1^2 - 2y_2^2 + 4y_3^2 \end{aligned}$$

Example 1.52. *Reduce the quadratic form $2x_1x_2+2x_2x_3+2x_3x_1$ into the canonical form.*

Solution: Let $A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$

Characteristic equation of A is $|A - \lambda I| = 0$
(i.e.) $\lambda^3 - -2\lambda - 2 = 0$
(i.e.) $\lambda_1 = 2, \lambda_2 = -1, \lambda_3 = -1$

To find the eigen vectors:
Let $|A - \lambda I|X = 0$

$$\Rightarrow \begin{bmatrix} -\lambda & 1 & 1 \\ 1 & -\lambda & 1 \\ 1 & 1 & -\lambda \end{bmatrix} \tag{1.32}$$

Case(i): When $\lambda_1 = 2$ equation (1.32) becomes

$$\Rightarrow \begin{bmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$\Rightarrow \frac{x_1}{3} = -\frac{x_2}{-3} = \frac{x_3}{3}$.

Therefore Eigen vector $X_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$.

Case(ii): When $\lambda_1 = -1$ equation (1.32) becomes

$$\Rightarrow \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

(i.e.) $x_1 + x_2 + x_3 = 0$ Let $x_2 = 0 \Rightarrow x_1 = -x_3$

Therefore Eigen vector $X_2 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$

To find the eigen vector X_3 :

Let $X_3 = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ such that $X_3^T X_1 = 0$

(i.e.) $\begin{bmatrix} a & b & c \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = 0$

(i.e.) $a + b + c = 0$ $\quad$ (i)

Again $X_3^T X_2 = 0 \Rightarrow a - b = 0$ $\quad$ (ii)

From equations (i) and (ii)
$\frac{a}{-1} = \frac{-b}{-2} = \frac{c}{-1}$

Therefore Eigen vector $X_3 = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$

Normalized modal matrix

$$P = \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & 0 & \frac{-2}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \end{bmatrix}$$

$$P^TAP = \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{6}} & \frac{-2}{\sqrt{6}} & \frac{1}{\sqrt{6}} \end{bmatrix} \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

$$\times \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & 0 & \frac{-2}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

$$Q.F = \begin{pmatrix} y_1 & y_2 & y_3 \end{pmatrix} \begin{bmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

$$= 2y_1^2 - y_2^2 - y_3^2$$

which is the canonical form.

Example 1.53. *Reduce the quadratic form $6x_1^2 + 3x_2^2 + 3x_3^2 -$ $4x_1x_2 - 2x_2x_3 + 4x_3x_1$ into the canonical form.*

Solution: Let $A = \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$

Characteristic equation of A is $|A - \lambda I| = 0$

$$\Rightarrow \begin{vmatrix} 6-\lambda & -2 & 2 \\ -2 & 3-\lambda & -1 \\ 2 & -1 & 3-\lambda \end{vmatrix} = 0$$

$\Rightarrow \lambda = 2, 2, 8$

To find the eigen vectors:

Let $|A - \lambda I|X = 0$

Case(i): When $\lambda = 8$, eigen vector $X_1 = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$

Case(ii): When $\lambda = 2$, eigen vector $X_2 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$

Case(iii): When $\lambda = 2$, eigen vector $X_3 = \begin{bmatrix} 2 \\ -1 \\ -5 \end{bmatrix}$

Normalised modal matrix $P = \begin{bmatrix} \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{30}} \\ -\frac{1}{\sqrt{6}} & \frac{2}{\sqrt{5}} & -\frac{1}{\sqrt{30}} \\ \frac{1}{\sqrt{6}} & 0 & -\frac{5}{\sqrt{30}} \end{bmatrix}$

Now $P^TAP = \begin{bmatrix} 8 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$

Canonical form is $8y_1^2 + 2y_2^2 + 2y_3^2$

EXERCISE

1. Reduce the quadratic form $3x_1^2 - 3x_2^2 - 5x_3^2 - 2x_1x_2 - 6x_2x_3 + 6x_3x_1$ to a canonical form using orthogonal transformation and find rank, index and signature. [Ans: $\lambda = -1, -8, 4, r = 3$, index $= 1$, signature $= 1$]

2. Reduce the quadratic form $8x_1^2 + 7x_2^2 + 3x_3^2 - 12x_1x_2 - 8x_2x_3 + 4x_3x_1$ to a canonical form and state its nature. [Ans: $\lambda = 15, 0, 3, (1, 2, 2)^{\gamma}, (2, 1, -2)^{\gamma}, (2, -2, 1)^{\gamma}$ positive semi definite]

3. Reduce the following to a canonical forms by orthogonal transformation:
(i) $Q = 7x_1^2 + 10x_2^2 + 7x_3^2 - 4x_1x_2 + 4x_2x_3 + 2x_3x_1$ (ii) $Q = x_1^2 + 4x_2^2 + 9x_3^2 + 4x_1x_2 + 12x_2x_3 + 6x_3x_1$ (iii) $Q = 7x_1^2 - 8x_2^2 - 8x_3^2 + 8x_1x_2 - 2x_2x_3 - 6x_3x_1$ [Ans: i) $2y_1^2 + 4y_2^2 + 6y_3^2$ ii) $14y_1^2$ iii) $9(y_1^2 - y_2^2 + y_3^2)$]

4. Find the nature of Q.F without reducing to canonical form i) $x_1^2 + 2x_1x_2 + x_2^2$ ii) $2x_1^2 + x_2^2 - 3x_3^2 + 12x_1x_2 - 8x_2x_3 - 4x_3x_1$

2 FUNCTIONS OF SEVERAL VARIABLES

2.1 Introduction

This chapter is devoted to a study of functions depending on more than one independent variable. A real function $z = f(x, y)$ of two independent variables x and y, can be thought to represent a surface in the three-dimensional space referred to a set of co-ordinate axes X, Y, Z.

A simple example of a function of two independent variables x and y is $z = xy$, which represents the area of a rectangle whose sides are x and y.

Continuity of a function of two variables: