

Unit-5 Sequences and Series

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Sequence

Definition:

A set of numbers $a_1, a_2, \dots, a_n, \dots$ such that to each positive integer n , there corresponds a number a_n of the set, is called a sequence and it is denoted by $\{a_n\}$. In other words, a sequence of real numbers is a function s from the set of natural numbers N into the set of real numbers R .

Examples:

- ① If $a_n = \frac{1}{n}$, then the sequence is $1, \frac{1}{2}, \frac{1}{3}, \dots, \frac{1}{n}, \dots$
- ② If $a_n = (-1)^n$, then the sequence is $-1, 1, -1, \dots$
- ③ If $a_n = k$, then the sequence is k, k, \dots

Note:

- 1. A finite sequence has a finite number of terms.
- 2. A sequence which is not finite, is an infinite sequence.
- 3. Example 3 is a constant sequence.

Sequence

Operations on Sequences:

If $\{s_n\}$ and $\{t_n\}$ are sequences then

- 1 Sum sequence is $\{s_n + t_n\} = \{s_n\} + \{t_n\}$.
- 2 Product sequence is $\{s_n t_n\} = \{s_n\} \cdot \{t_n\}$.
- 3 If $c \in R$, then $c\{s_n\} = \{cs_n\}$.
- 4 $\{\frac{1}{s_n}\}$ is called the reciprocal of $\{s_n\}$.
- 5 $\{\frac{s_n}{t_n}\}$ is defined as the quotient of sequence $\{s_n\}$ and $\{t_n\}$, $t_n \neq 0$.

Bounded Sequence

A sequence $\{s_n\}$ is said to be bounded if there exist numbers m, M such that $m < a_n < M$ for all $n \in N$. Otherwise it is said to be unbounded.

Example:

1. $\{\frac{1}{n}\}$ which is bounded by 1.
2. $\{2^n\}$ which is unbounded.

Sequence

Monotonic Sequence:

A sequence $\{s_n\}$ is said to be

- (i) Monotonically increasing if $s_{n+1} \geq s_n$ for every n , $s_1 \leq s_2 \leq s_3 \dots \leq s_n \leq s_{n+1} \leq \dots$
- (ii) Monotonically decreasing if $s_{n+1} \leq s_n$ for every n , $s_1 \geq s_2 \geq s_3 \dots \geq s_n \geq s_{n+1} \geq \dots$
- (iii) Monotonic if it is either monotonically increasing or monotonically decreasing.

Example:

- ① $1, 4, 7, 10, \dots$ is a monotonic sequence.
- ② $1, \frac{1}{2}, \frac{1}{3}, \dots$ is monotonic sequence.
- ③ $1, -1, 1, -1, \dots$ is not a monotonic sequence.

Limit of a sequence:

Let $\{s_n\}$ be a sequence. l is said to be limit of the sequence $\{s_n\}$, if to each $\varepsilon > 0$ there exists $m \in \mathbb{Z}^+$ such that $|s_n - l| < \varepsilon$, $\forall n \geq m$. That is $\lim_{n \rightarrow \infty} s_n = l$.

Sequence

Convergent Sequence:

A sequence $\{s_n\}$ is said to be convergent if it has a finite limit. That is $\lim_{n \rightarrow \infty} s_n = l$.

Divergent Sequence:

If $\lim_{n \rightarrow \infty} s_n = \infty$, $\{s_n\}$ is divergent.

Oscillatory Sequence:

If $\lim_{n \rightarrow \infty}$ is not unique (oscillates finitely) or $\pm\infty$ (oscillates infinitely) then $\{s_n\}$ is oscillatory sequence.

Examples:

- ① $\{\frac{1}{n^2}\}$ is a convergent sequence.
- ② $\{n\}$ is a divergent sequence.
- ③ $\{(-1)^n\}$ oscillates finitely.
- ④ $\{(-1)^n n^2\}$ oscillates infinitely.

Sequence

Problems:

Which of the following sequence are convergent?

- (i) $\{\frac{n}{n^2+1}\}$ (ii) $\{(-1)^{n+1}\}$ (iii) $\{\frac{n}{n+1}\}$ (iv) $\{1 + \frac{(-1)^n}{n}\}$ (v) $\{(\frac{1}{2})^n\}$

Solution: (i)

$$\begin{aligned}\lim_{n \rightarrow \infty} s_n &= \lim_{n \rightarrow \infty} \frac{n}{n^2 + 1} \\ &= \lim_{n \rightarrow \infty} \frac{n}{n(n + \frac{1}{n})} \\ &= \lim_{n \rightarrow \infty} \frac{1}{(n + \frac{1}{n})} \\ &= 0\end{aligned}$$

Hence the sequence is convergent.

(ii) $\{(-1)^{n+1}\} = 1, -1, 1, -1, \dots$ is an oscillating sequence.

(iii)

$$\begin{aligned}\lim_{n \rightarrow \infty} s_n &= \lim_{n \rightarrow \infty} \frac{n}{n+1} \\ &= \lim_{n \rightarrow \infty} \frac{n}{n(1 + \frac{1}{n})} \\ &= \lim_{n \rightarrow \infty} \frac{1}{(1 + \frac{1}{n})} \\ &= 1\end{aligned}$$

Hence the sequence is convergent.

(iv)

$$\begin{aligned}\lim_{n \rightarrow \infty} s_n &= \lim_{n \rightarrow \infty} 1 + \frac{(-1)^n}{n} \\ &= 1\end{aligned}$$

Hence the sequence is convergent.

(v) The sequence $\{(\frac{1}{2})^n\}$ is convergent.

Definition:

$u_1, u_2, \dots, u_n \dots$ is an infinite sequence. The expression $u_1 + u_2 + \dots + u_n + \dots$ is called the series. It is denoted by $\sum_{n=1}^{\infty} u_n$.

Note:

1. If the number of terms are finite in a series then the series is called a finite series.
2. If the number of terms are infinite in a series then the series is called an infinite series.

Definition:

The sum of a finite number of terms (the first n -terms) of a series is called the n^{th} partial sum of the series. $S_n = u_1 + u_2 + \dots + u_n = \sum_{n=1}^{\infty} u_n$.

1. If $\lim_{n \rightarrow \infty} S_n = S(\text{finite})$, then the series $\sum_{n=1}^{\infty} u_n$ converges.
2. If $\lim_{n \rightarrow \infty} S_n = \pm\infty$, then the series $\sum_{n=1}^{\infty} u_n$ diverges.
3. If $\lim_{n \rightarrow \infty} S_n$ is more than one limit (or) $\pm\infty$, then $\sum_{n=1}^{\infty} u_n$ is oscillatory (or) non converges.

Problems:

1. Examine the nature of the series $1 + 3 + 5 + 7 + \dots \infty$

Solution:

The n^{th} partial sum is $S_n = 1 + 3 + 5 + 7 + \dots + n$. It is an arithmetic series with $a = 1$, $d = 2$, $S_n = \frac{n}{2}[2a + (n-1)d] = n^2$. It follows that $\lim_{n \rightarrow \infty} S_n = \infty$. Hence the series is divergent.

2. Show that the series $1 + r + r^2 + \dots \infty$ (i) Converges if $|r| < 1$
(ii) Diverges if $r \geq 1$ and (iii) Oscillatory if $r \leq -1$

Solution:

(i) If $|r| < 1$, the n^{th} partial sum is $S_n = 1 + r + r^2 + \dots + r^{n-1} = \frac{1(1 - r^n)}{1 - r}$. $\lim_{n \rightarrow \infty} r^n = 0$, if $|r| < 1$. Thus the series is convergent.

(ii) If $r > 1$, $\lim_{n \rightarrow \infty} r^n = \infty$. If $r = 1$, then $S_n = 1 + 1 + \dots + 1 = n$. Hence $\lim_{n \rightarrow \infty} r^n = \infty$. Hence the series is divergent if $r \geq 1$.

(iii) If $r < -1$, then $\lim_{n \rightarrow \infty} S_n = \begin{cases} \infty, & \text{if } n \text{ is odd} \\ -\infty, & \text{if } n \text{ is even} \end{cases}$

Series

If $r = -1$, then $S_n = 1 - 1 + 1 - 1 + \dots 1$. $S_n = \begin{cases} 1, & \text{if } n \text{ is odd} \\ -1, & \text{if } n \text{ is even} \end{cases}$

Hence the series $1 + r + r^2 + \dots \infty$ is oscillatory if $r \leq -1$

3. Examine the converges of the following series.

(i) $5 - 4 - 1 + 5 - 4 - 1 + \dots \infty$ (ii) $1 + \frac{5}{4} + \frac{6}{4} + \dots \infty$ (iii) $\sum_{n=1}^{\infty} \frac{1}{n(n+2)}$

(iv) $1 + \frac{1}{2} + \frac{1}{2^2} \dots \infty$ (v) $1 + \frac{4}{3} + \left(\frac{4}{3}\right)^2 + \dots \infty$

Solution:

(i) n^{th} partial sum is $S_n = 5$ or 1 or 0 . Hence the series is oscillatory.

(ii) The series is arithmetic series $a = 1$, $d = \frac{1}{4}$,
 $S_n = \frac{n}{2}[2a + (n-1)d] = \frac{n}{8}[7 + n]$. Therefore $\lim_{n \rightarrow \infty} S_n = \infty$. Hence the series is divergent.

(iii) $u_n = \frac{1}{n(n+2)}$. By using partial fractions $\frac{1}{n(n+2)} = \frac{A}{n} + \frac{B}{n+2}$.

Which implies $1 = A(n+2) + Bn$. When $n = 0$, $A = \frac{1}{2}$ and when $n = -2$, $B = \frac{-1}{2}$.

Therefore $u_n = \frac{1}{n(n+2)} = \frac{1}{2n} - \frac{1}{2(n+2)}$. The n^{th} partial sum is $S_n = u_1 + u_2 + \cdots + u_n = \frac{1}{2} - \frac{1}{2(n+2)}$. $\lim_{n \rightarrow \infty} S_n = \frac{1}{2}$. Hence the given series is convergent.

(iv) The n^{th} partial sum is $S_n = 1 + \frac{1}{2} + \left(\frac{1}{2}\right)^2 + \cdots + \left(\frac{1}{2}\right)^{n-1}$.

$S_n = \frac{1 - \frac{1}{2^n}}{\frac{1}{2}} = 2 \left(1 - \frac{1}{2^n}\right)$. $\lim_{n \rightarrow \infty} S_n = 2$. Hence the given series is convergent.

(v) The n^{th} partial sum is $S_n = 1 + \frac{4}{3} + \left(\frac{4}{3}\right)^2 + \cdots + \left(\frac{4}{3}\right)^{n-1}$.

$S_n = \frac{\left(\frac{4}{3}\right)^n - 1}{\frac{1}{3}} = 3 \left(\left(\frac{4}{3}\right)^n - 1\right)$. $\lim_{n \rightarrow \infty} S_n = \infty$. Hence the given series is divergent.

Series of Positive terms

Properties of Series:

1. Convergence of a series remains unchanged by the replacement, inclusion or omission of a finite number of terms.
2. A series remains convergent, divergent or oscillatory when each term of it is multiplied by a fixed number other than zero.
3. A series of positive terms either converges or diverges to $+\infty$. That is omitting the negative terms the sum of first n terms tends to either a finite limit or $+\infty$
4. Every finite series is a convergent series.

Definition:

If all terms after few positive terms in an infinite series are positive, such a series is a positive term series.

Example: $-10-6-1+5+12+20+\dots$

Series of positive terms

Necessary Condition for Convergence:

If a positive term series $\sum_{n=1}^{\infty} u_n$ is convergent, then $\lim_{n \rightarrow \infty} u_n = 0$.

Note:

Converse of the above theorem is not true.

Example:

The series $1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \cdots + \frac{1}{\sqrt{n}} + \cdots$ is divergent even though $\lim_{n \rightarrow \infty} u_n = \lim_{n \rightarrow \infty} \frac{1}{\sqrt{n}} = 0$.

Test for Divergence:

If $\lim_{n \rightarrow \infty} u_n \neq 0$, the series $\sum_{n=1}^{\infty} u_n$ must be divergent.

Comparison Test for Convergence:

If two positive term series $\sum_{n=1}^{\infty} u_n$ and $\sum_{n=1}^{\infty} v_n$ be such that $\sum_{n=1}^{\infty} v_n$ converges and $u_n \leq v_n$ for all values of n , then $\sum_{n=1}^{\infty} u_n$ also converges.

Series of positive terms

Comparison Test for divergence:

If two positive term series $\sum_{n=1}^{\infty} u_n$ and $\sum_{n=1}^{\infty} v_n$ be such that $\sum_{n=1}^{\infty} v_n$ diverges and $u_n \geq v_n$ for all values of n , then $\sum_{n=1}^{\infty} u_n$ also diverges.

Limit Comparison test:

If two positive term series $\sum_{n=1}^{\infty} u_n$ and $\sum_{n=1}^{\infty} v_n$ be such that $\lim_{n \rightarrow \infty} \frac{u_n}{v_n} = \text{non zero finite value}$, then $\sum_{n=1}^{\infty} u_n$ and $\sum_{n=1}^{\infty} v_n$ converges or diverges together.

Auxiliary Series:

(a) p -series:

The harmonic series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ is convergent for $p > 1$ and divergent for $p \leq 1$.

(b) Geometric series:

The geometric series $\sum_{n=1}^{\infty} r^{n-1}$ is convergent if $r < 1$ and divergent if $r \geq 1$.

Problems:

1. Examine the series $\frac{1}{1.3.5} + \frac{2}{3.5.7} + \frac{3}{5.7.9} + \dots$ for convergence.

Solution:

To find u_n :

1,2,3,... is an arithmetic series where $a = 1, d = 1$,

$$t_n = a + (n-1)d = n$$

1,3,5,... is an arithmetic series where $a = 1, d = 2$,

$$t_n = a + (n-1)d = 2n-1$$

3,5,7,... is an arithmetic series where $a = 3, d = 2$,

$$t_n = a + (n-1)d = 2n+1$$

5,7,9,... is an arithmetic series where $a = 5, d = 2$,

$$t_n = a + (n-1)d = 2n+3$$

$$\text{Therefore } u_n = \frac{n}{(2n-1)(2n+1)(2n+3)} = \frac{1}{n^2(2-\frac{1}{n})(2+\frac{1}{n})(2+\frac{3}{n})}.$$

$$\text{Choose } v_n = \frac{1}{n^2}.$$

Problems:

$\lim_{n \rightarrow \infty} \frac{u_n}{v_n} = \lim_{n \rightarrow \infty} \frac{1}{(2 - \frac{1}{n})(2 + \frac{1}{n})(2 + \frac{3}{n})} = \frac{1}{8} \neq 0$. By comparison test, $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is convergent, implies that $\sum_{n=1}^{\infty} \frac{n}{(2n-1)(2n+1)(2n+3)}$ is convergent.

2. Discuss the convergence or divergence of the series

$$\frac{2}{1^p} + \frac{3}{2^p} + \frac{4}{3^p} + \cdots \infty.$$

Solution:

$u_n = \frac{(n+1)}{n^p}$. Choose $v_n = \frac{1}{n^{p-1}}$. We have

$\lim_{n \rightarrow \infty} \frac{u_n}{v_n} = \lim_{n \rightarrow \infty} (1 + \frac{1}{n}) = 1 \neq 0$. The series $\sum_{n=1}^{\infty} \frac{1}{n^{p-1}}$ is convergent if $p > 2$ and it is divergent if $p \leq 2$. Hence the given series $\sum_{n=1}^{\infty} \frac{(n+1)}{n^p}$ is convergent if $p > 2$ and it is divergent if $p \leq 2$.

Problems:

3. Determine whether the following series is convergent or divergent $\frac{\sqrt{2}-1}{3^3-1} + \frac{\sqrt{3}-1}{4^3-1} + \frac{\sqrt{4}-1}{5^3-1} + \dots$.

Solution:

$$u_n = \frac{\sqrt{n+1}-1}{(n+1)^3-1} = \frac{\sqrt{1+\frac{1}{n}} - \frac{1}{\sqrt{n}}}{n^{\frac{5}{2}} \left(\left(1 + \frac{2}{n}\right)^3 - \frac{1}{n^3} \right)}. \text{ Choose } v_n = \frac{1}{n^{\frac{5}{2}}}. \lim_{n \rightarrow \infty} \frac{u_n}{v_n} =$$

$$\lim_{n \rightarrow \infty} \frac{\sqrt{1+\frac{1}{n}} - \frac{1}{\sqrt{n}}}{\left(\left(1 + \frac{2}{n}\right)^3 - \frac{1}{n^3} \right)} = 1 \neq 0. \text{ Since } \sum_{n=1}^{\infty} v_n \text{ is convergent.}$$

Hence by comparison test $\sum_{n=1}^{\infty} u_n$ is convergent.

Problems:

4. Examine the nature of series $\sum_{n=1}^{\infty} \frac{1}{(a+n)^p(b+n)^q}$ where a, b, p, q all positive.

Solution:

$$\begin{aligned} u_n &= \frac{1}{(a+n)^p(b+n)^q} \\ &= \frac{1}{n^{p+q} \left(1 + \frac{a}{n}\right)^p \left(1 + \frac{b}{n}\right)^q} \end{aligned}$$

$$\text{Choose } v_n = \frac{1}{n^{p+q}}$$

$$\lim_{n \rightarrow \infty} \frac{u_n}{v_n} = \lim_{n \rightarrow \infty} \frac{1}{\left(1 + \frac{a}{n}\right)^p \left(1 + \frac{b}{n}\right)^q} = 1$$

Since $\sum_{n=1}^{\infty} \frac{1}{n^{p+q}}$ is convergent if $p+q > 1$ and is divergent if $p+q \leq 1$.

Which implies $\sum_{n=1}^{\infty} \frac{1}{(a+n)^p(b+n)^q}$ is convergent if $p+q > 1$ and is divergent if $p+q \leq 1$.

Problems:

5. Determine whether the following series is convergent or divergent

$$\sum_{n=1}^{\infty} \left(\sqrt{n^4 + 1} - \sqrt{n^4 - 1} \right).$$

Solution:

$$\begin{aligned} u_n &= \left(\sqrt{n^4 + 1} - \sqrt{n^4 - 1} \right) \left(\frac{\sqrt{n^4 + 1} + \sqrt{n^4 - 1}}{\sqrt{n^4 + 1} + \sqrt{n^4 - 1}} \right) \\ &= \frac{(n^4 + 1) - (n^4 - 1)}{\sqrt{n^4 + 1} + \sqrt{n^4 - 1}} = \frac{2}{\sqrt{n^4 + 1} + \sqrt{n^4 - 1}} \\ &= \frac{2}{n^2} \frac{1}{\sqrt{1 + \frac{1}{n^4}} + \sqrt{1 - \frac{1}{n^4}}} \end{aligned}$$

$$\text{Choose } v_n = \frac{1}{n^2}$$

$$\lim_{n \rightarrow \infty} \frac{u_n}{v_n} = \lim_{n \rightarrow \infty} \frac{2}{2} = 1.$$

Since the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is convergent, $\sum_{n=1}^{\infty} \left(\sqrt{n^4 + 1} - \sqrt{n^4 - 1} \right)$ is convergent.

Problems:

6. Determine whether the following series is convergent or divergent

$$\sum_{n=1}^{\infty} \sqrt{\frac{3^n - 1}{2^n + 1}}.$$

Solution:

$$\begin{aligned} u_n &= \sqrt{\frac{3^n - 1}{2^n + 1}} \\ &= \left(\sqrt{\frac{3}{2}} \right)^n \sqrt{\frac{1 - \frac{1}{3^n}}{1 + \frac{1}{2^n}}} \end{aligned}$$

$$\text{Let } v_n = \left(\sqrt{\frac{3}{2}} \right)^n$$

$$\lim_{n \rightarrow \infty} \frac{u_n}{v_n} = \lim_{n \rightarrow \infty} \sqrt{\frac{1 - \frac{1}{3^n}}{1 + \frac{1}{2^n}}} = 1$$

Since $\sum_{n=1}^{\infty} v_n$ is divergent, $\sum_{n=1}^{\infty} \sqrt{\frac{3^n - 1}{2^n + 1}}$ is also divergent.

7. Examine the nature of the series $\sum_{n=1}^{\infty} \sin\left(\frac{1}{n}\right)$.

Solution:

$$u_n = \sin\left(\frac{1}{n}\right), \text{ choose } v_n = \frac{1}{n}. \quad \lim_{n \rightarrow \infty} \frac{u_n}{v_n} = \lim_{n \rightarrow \infty} \frac{\sin\left(\frac{1}{n}\right)}{\frac{1}{n}} = 1.$$

Since $\sum_{n=1}^{\infty} \frac{1}{n}$ is divergent. Hence $\sum_{n=1}^{\infty} \sin\left(\frac{1}{n}\right)$ is also divergent.

Cauchy's Integral Test

If $\sum_{n=1}^{\infty} u_n$ is a series of positive terms and if $u(x) = f(x)$ be such that

- $f(x)$ is continuous in $1 < x < \infty$.
- $f(x)$ decreases as x increases, then the series $\sum_{n=1}^{\infty} u_n$ is convergent or divergent according as the integral $\int_1^{\infty} f(x) dx$ is finite or infinite.

Cauchy's Integral Test:

1. Apply Cauchy's integral test to discuss the nature of the harmonic series (p-series) $\sum_{n=1}^{\infty} \frac{1}{n^p}$.

Solution:

Let $u_x = f(x) = \frac{1}{x^p}$. As x increases $f(x)$ decreases.

$$\begin{aligned}\int_1^{\infty} f(x) dx &= \int_1^{\infty} \frac{1}{x^p} dx \\&= \int_1^{\infty} x^{-p} dx \\&= \left[\frac{x^{-p+1}}{-p+1} \right] \\&= \frac{-1}{p-1} [x^{-p+1}]_1^{\infty} \\&= \begin{cases} \frac{1}{p-1} & \text{if } p > 1 \\ \infty & \text{if } p \leq 1 \end{cases}\end{aligned}$$

Therefore $\sum u_n$ is convergent if $p > 1$ and divergent if $p \leq 1$.

Cauchy's Integral Test:

2. Find the nature of the series $\sum_{n=2}^{\infty} \frac{1}{n(\log n)^p}$.

Solution:

Let $u_x = f(x) = \frac{1}{x(\log x)^p}$. As x increases $f(x)$ decreases.

$$\begin{aligned}\int_2^{\infty} \frac{1}{x(\log x)^p} &= \int_{\log 2}^{\infty} \frac{dt}{t^p} \\ &= \left[\frac{t^{-p+1}}{-p+1} \right] \\ &= \begin{cases} \text{finite,} & \text{if } p = 1 \\ \text{finite,} & \text{if } p > 1 \\ \text{infinite,} & \text{if } p \leq 1 \end{cases}\end{aligned}$$

Therefore $\sum u_n$ is convergent if $p > 1$ and divergent if $p \leq 1$.

Cauchy's Integral Test:

3. Discuss the convergence of the series $\sum_{n=2}^{\infty} \frac{1}{n \log n}$.

Solution:

Let $u_x = f(x) = \frac{1}{x \log x}$. As x increases $f(x)$ decreases.

$$\begin{aligned}\int_2^{\infty} f(x) dx &= \int_2^{\infty} \frac{1}{x(\log x)} dx \\ &= \int_{\log 2}^{\infty} \frac{dt}{t} \\ &= \text{infinite}\end{aligned}$$

Therefore $\sum u_n$ is divergent.

Cauchy's Integral Test:

4. Examine the convergence of the series $1 + \frac{1}{4^{\frac{2}{3}}} + \frac{1}{9^{\frac{2}{3}}} + \frac{1}{16^{\frac{2}{3}}} + \dots$

Solution:

Let $u_x = f(x) = \frac{1}{x^{\frac{4}{3}}}$. As x increases $f(x)$ decreases.

$$\begin{aligned}\int_1^{\infty} f(x) dx &= \int_1^{\infty} \frac{1}{x^{\frac{4}{3}}} dx \\ &= \int_1^{\infty} x^{-\frac{4}{3}} dx \\ &= -3[x^{-\frac{1}{3}}]_1^{\infty} \\ &= 3(\text{finite})\end{aligned}$$

Therefore $\sum u_n$ is convergent.

D'Alembert's Ratio Test:

D'Alembert's Ratio Test

The series $\sum_{n=1}^{\infty} u_n$ of positive terms is convergent if $\lim_{n \rightarrow \infty} \frac{u_{n+1}}{u_n} <$

1 is divergent if $\lim_{n \rightarrow \infty} \frac{u_{n+1}}{u_n} > 1$. If $\lim_{n \rightarrow \infty} \frac{u_{n+1}}{u_n} = 1$, the test fails.

1. Test the convergence of the series $1 + \frac{1+\alpha}{1+\beta} + \frac{(1+\alpha)(1+2\alpha)}{(1+\beta)(1+2\beta)} +$
.....

Solution:

Let $u_n = \frac{(1+\alpha)(1+2\alpha)\dots(1+(n)\alpha)}{(1+\beta)(1+2\beta)\dots(1+(n)\beta)}$ then

$u_{n+1} = \frac{(1+\alpha)(1+2\alpha)\dots(1+(n)\alpha)(1+(n+1)\alpha)}{(1+\beta)(1+2\beta)\dots(1+(n)\beta)(1+(n+1)\beta)}$. It follows that

$\frac{u_{n+1}}{u_n} = \frac{(1+(n+1)\alpha)}{(1+(n+1)\beta)} \Rightarrow \lim_{n \rightarrow \infty} \frac{u_{n+1}}{u_n} = \frac{\alpha}{\beta}$. Thus the series converges if $\frac{\alpha}{\beta} < 1$ and diverges if $\frac{\alpha}{\beta} > 1$. If $\frac{\alpha}{\beta} = 1$, then $\alpha = \beta$.

Therefore the series $\sum_{n=1}^{\infty} u_n = 1 + 1 + 1 + \dots$ is a divergent series.

D'Alembert's Ratio Test:

2. Test the convergence of the series $x + \frac{2^2 x^2}{2!} + \frac{3^3 x^3}{3!} + \dots$

Solution:

Let $u_n = \frac{x^n n^n}{n!}$ then $u_{n+1} = \frac{x^{n+1} (n+1)^{(n+1)}}{(n+1)!}$. It follows that

$\frac{u_{n+1}}{u_n} = x \left(\frac{n+1}{n}\right)^n \Rightarrow \lim_{n \rightarrow \infty} \frac{u_{n+1}}{u_n} = x \lim_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right)^n = xe$. Thus the series converges if $ex < 1$ and diverges if $ex > 1$. If $ex = 1$, then the ratio test fails.

3. Find the nature of the series $\frac{2}{1} + \frac{2.5.8}{1.5.9} + \frac{2.5.8.11}{1.5.9.13} + \dots$

Solution:

$u_n = \frac{2.5.8 \dots (3n-1)}{1.5.9 \dots (4n-3)}$ and $u_{n+1} = \frac{2.5.8 \dots (3n-1)(3n+2)}{1.5.9 \dots (4n-3)(4n+1)}$. Thus

$\frac{u_{n+1}}{u_n} = \frac{3n+2}{4n+1}$. Hence $\lim_{n \rightarrow \infty} \frac{u_{n+1}}{u_n} = \lim_{n \rightarrow \infty} \frac{3 + \frac{2}{n}}{4 + \frac{1}{n}} = \frac{3}{4} < 1$. Hence

the given series is convergent.

Raabe's Test:

Raabe's Test

The positive termed series $\sum_{n=1}^{\infty} u_n$ is convergent or deivergent according as $\lim_{n \rightarrow \infty} n \left(\frac{u_{n+1}}{u_n} \right) - 1 > 1$ or < 1 . If D'Alemberts test fails then use Raabe's test.

1. Test the convergence of the series $\frac{2}{3.4} + \frac{2.4}{3.5.6} + \frac{2.4.6}{3.5.7.8} + \dots$

Solution:

Let $u_n = \frac{2.4.6.8 \dots 2n}{3.5.7 \dots (2n+1)} \frac{1}{2n+2}$ then

$u_{n+1} = \frac{2.4.6.8 \dots 2n.2(n+1)}{3.5.7 \dots (2n+1).(2n+3)} \frac{1}{2n+4}$. It follows that

$\frac{u_{n+1}}{u_n} = \frac{\left(1 + \frac{1}{n}\right)^2}{\left(1 + \frac{3}{2n}\right)\left(1 + \frac{4}{2n}\right)} \Rightarrow \lim_{n \rightarrow \infty} \frac{u_{n+1}}{u_n} = 1$. Hence the ratio test

fails. $n \left(\frac{u_{n+1}}{u_n} \right) - 1 = n \left[\frac{(2n+3)(2n+4)}{(2n+2)^2} - 1 \right] \rightarrow \frac{3}{2}$ as $n \rightarrow \infty$.

Therefore by Raabe's test the given series is convergent.

Raabe's Test:

2. Test the convergence of the series $\sum_{n=1}^{\infty} \frac{1.3.5....(2n-1)}{2.4.6....(2n)} x^n$.

Solution:

$\frac{u_{n+1}}{u_n} = x \left(\frac{2n+1}{2n+2} \right) \Rightarrow \lim_{n \rightarrow \infty} \frac{u_{n+1}}{u_n} = x$. Thus the series converges if $0 < x < 1$ and diverges if $x > 1$. If $x = 1$, then the D'Alemberts ratio test fails. Apply Raabe's test.

$n \left(\frac{u_n}{u_{n+1}} - 1 \right) = \frac{n}{2n+1} \rightarrow \frac{1}{2}$ as $n \rightarrow \infty$. Therefore when $x = 1$ the given series is divergent.

Cauchy's root Test:

Cauchy's root Test

If $\sum_{n=1}^{\infty} u_n$ is a series of positive terms, then the series is convergent or divergent according as $\lim_{n \rightarrow \infty} u_n^{\frac{1}{n}} < 1$ or > 1 .

1. Test the convergence of the series $\sum_{n=1}^{\infty} \frac{1}{(\log n)^n}$.

Solution:

Let $u_n = \frac{1}{(\log n)^n}$. By Cauchy's root test $u_n^{\frac{1}{n}} = \frac{1}{(\log n)^n}^{\frac{1}{n}} = \frac{1}{\log n}$. It follows that $u_n^{\frac{1}{n}} \rightarrow 0$ as $n \rightarrow \infty$. By Cauchy's root test the given series is convergent.

2. Test the convergence of the series $\sum_{n=1}^{\infty} \frac{1}{(1 + \frac{1}{n})^{n^2}}$.

Solution:

Let $u_n = \frac{1}{(1 + \frac{1}{n})^{n^2}}$. By Cauchy's root test $u_n^{\frac{1}{n}} = \frac{1}{(1 + \frac{1}{n})^n}$. It follows that $u_n^{\frac{1}{n}} \rightarrow \frac{1}{e} < 1$ as $n \rightarrow \infty$. By Cauchy's root test the given series is convergent.

Alternating Series-Lebnitz's Test:

Alternating Series

A series in which the terms are alternatively positive or negative is called an alternating series.

Lebnitz's Rule

An alternating series $u_1 - u_2 + u_3 - u_4 + \dots$ converges if $u_n - u_{n-1} < 0$ and $\lim_{n \rightarrow \infty} u_n = 0$. The alternating series is not convergent if one of the condition is satisfied. If $\lim_{n \rightarrow \infty} u_n \neq 0$, then the given series is oscillatory.

1. Discuss the convergence of the series $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots$

Solution:

$u_n = \frac{1}{n}$, $u_{n-1} = \frac{1}{n-1}$. Then $u_n - u_{n-1} = \frac{-1}{n(n-1)} < 0$. $\lim_{n \rightarrow \infty} u_n = 0$.

Therefore by Lebnitz's Rule the given series is convergent.

Alternating Series-Lebnitz's Test:

2. Examine the nature of the series $\sum_{n=2}^{\infty} \frac{(-1)^{n-1} x^n}{n(n-1)}$, $0 < x < 1$.

Solution:

$$u_n = \frac{x^n}{n(n-1)}, u_{n-1} = \frac{x^{n-1}}{(n-2)(n-1)}. u_n - u_{n-1} \text{ is less than zero.}$$

Also $\lim_{n \rightarrow \infty} u_n = \lim_{n \rightarrow \infty} \frac{x^n}{n(n-1)} = 0$. Thus the Lebnitz's Conditions are satisfied. Hence the given series is convergent.

3. Examine the nature of the series $\sum_{n=1}^{\infty} (-1)^{n-1} [\sqrt{n^2+1} - n]$.

Solution:

$$u_n = \sqrt{n^2+1} - n, u_{n-1} = \sqrt{n^2-2n+2} - n + 1. \text{ Also } u_n - u_{n-1} < 0.$$

$$\lim_{n \rightarrow \infty} u_n = \lim_{n \rightarrow \infty} [\sqrt{n^2+1} - n] = \lim_{n \rightarrow \infty} \left[\frac{(\sqrt{n^2+1})^2 - n^2}{\sqrt{n^2+1} + n} \right] = 0.$$

Thus the Lebnitz's Conditions are satisfied. Hence the given series is convergent.

Absolute Convergence and Conditional Convergence:

Absolute Convergence: If the series of arbitrary terms $u_1 + u_2 + \dots + u_n + \dots$ be such that the series $|u_1| + |u_2| + \dots + |u_n| + \dots$ is convergent, then the series $\sum_{n=1}^{\infty} u_n$ is absolutely convergent.

Conditional Convergence: If the series $\sum_{n=1}^{\infty} |u_n|$ is divergent but $\sum_{n=1}^{\infty} u_n$ is convergent, then the series $\sum_{n=1}^{\infty} u_n$ is conditionally convergent.

1. Test the series $1 - \frac{1}{2\sqrt{2}} + \frac{1}{3\sqrt{3}} - \frac{1}{4\sqrt{4}} + \dots$ for (i) Absolute Convergence (ii) Conditional Convergence.

Solution:

(i) $\sum_{n=1}^{\infty} u_n = 1 - \frac{1}{2\sqrt{2}} + \frac{1}{3\sqrt{3}} - \frac{1}{4\sqrt{4}} + \dots$. Thus $\sum_{n=1}^{\infty} |u_n| = \sum_{n=1}^{\infty} \frac{1}{n^2}$.

Which harmonic p-series with $p = \frac{3}{2} > 1$. Hence the series $\sum_{n=1}^{\infty} |u_n|$ is convergent which implies $\sum_{n=1}^{\infty} u_n$ is absolutely convergent.

(ii) $\sum_{n=1}^{\infty} u_n = 1 - \frac{1}{2\sqrt{2}} + \frac{1}{3\sqrt{3}} - \frac{1}{4\sqrt{4}} + \dots$ is an alternating series.

$u_n = \frac{1}{n\sqrt{n}}$ and $u_{n-1} = \frac{1}{(n-1)\sqrt{n-1}}$. Also $u_n - u_{n-1} < 0$ and $\lim_{n \rightarrow \infty} u_n = 0$. Hence $\sum_{n=1}^{\infty} u_n$ is also convergent. Hence the given series is not conditionally convergent.

Absolute Convergence and Conditional Convergence:

Absolute Convergence:

2. Prove that the exponential series $1 + \frac{x}{1!} + \frac{x}{2!} + \dots + \frac{x}{n!} + \dots$ is absolutely convergent and hence convergent for all values of x .

Solution:

Let $u_n = \frac{x^{n-1}}{(n-1)!}$, $u_{n+1} = \frac{x^n}{n!}$. Thus $\frac{u_{n+1}}{u_n} = \frac{x}{n}$. $\lim_{n \rightarrow \infty} \left| \frac{u_{n+1}}{u_n} \right| = \lim_{n \rightarrow \infty} \frac{|x|}{n} = 0 < 1, \forall x$. Hence the series is absolutely convergent and hence convergence for all real x .