2.4 Integration of Vectors

Line Integral

Let $\vec{F}(x, y, z)$ be a vector function and a curve AB.

Line integral of a vector function $\vec{F}$ along the curve AB is defined as the integral component of $\vec{F}$ along the tangent to the curve AB.

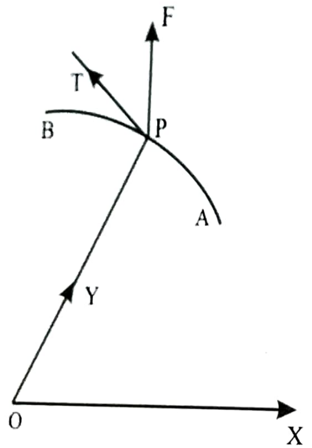

Fig.1

Component of $\vec{F}$ along a tagnent PT at P = Dot product of $\vec{F}$ and unit vector along $PT = \vec{F}.\frac{d\vec{r}}{ds}$ [$\frac{d\vec{r}}{ds}$ is a unit vector along tangent PT]

Line integral $= \sum \vec{F}.\frac{d\vec{r}}{ds}$ from A to B along the curve.

$\therefore$ Line integral $= \int_C \left(\vec{F}.\frac{d\vec{r}}{ds}\right) ds = \int_C \vec{F}.d\vec{r}$

Since $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$, therefore $d\vec{r} = dx\vec{i} + dy\vec{j} + dz\vec{k}$

$\therefore \vec{F}.d\vec{r} = (F_1\vec{i} + F_2\vec{j} + F_3\vec{k}).(dx\vec{i} + dy\vec{j} + dz\vec{k})$

$= F_1dx + F_2dy + F_3dz$

Therefore in components from the above line integral is written as $\int_C \vec{F}.d\vec{r} = \int_C (F_1dx + F_2dy + F_3dz)$

Note 1: $\int_C \phi dr$, where ϕ is a scalar point functions and $\int_C \vec{F} \times d\vec{r}$ are also line integral.

Note 2: If $\vec{F}$ represents the variable force acting on a particle along arc AB, then the total work done $= \int_A^B \vec{F}.d\vec{r}$

Note 3: If $\vec{F}$ represents the velocity of a liquid then $\oint_C \vec{F}.d\vec{r}$ is called the circulation of $\vec{F}$ round the curve C.

Note 4: If $\oint_C \vec{F}.d\vec{r} = 0$, then the field F is called consecutive. (i.e.) $\nabla \times \vec{F} = 0$

Note 5: If $\int_A^B \vec{F}.d\vec{r}$ is said to be independent of path then $\vec{F} = \nabla\phi$.

2.4.1 Surface Integral

Let vector $\overrightarrow{F}$ be a vector point function and S be the given surface. Surface integral of a vector point function $\overrightarrow{F}$ over the surface S is defined as the integral of the component $\overrightarrow{F}$ along the normal to the surface.

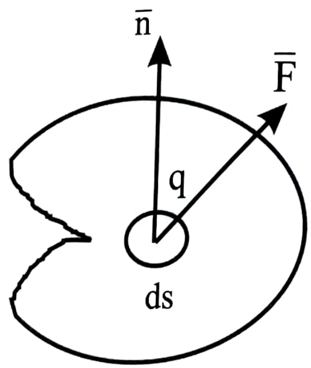

Fig.2

Component of $\overrightarrow{F}$ along the normal $= \overrightarrow{F}.n$ where n is the unit normal vector to an element dS and $\widehat{n} = \dfrac{grad\, f}{|grad\, f|}$, $dS = \dfrac{dxdy}{(\widehat{n}.k)}$

Surface integral of $\overrightarrow{F}$ over $S = \sum \overrightarrow{F}.\widehat{n} = \int\int_S (\overrightarrow{F}.\widehat{n})ds$

2.4.2 Volume Integral

Let vector $\overrightarrow{F}$ be a vector point function and volume V enclosed by a closed surface.

The volume integral $= \int\int\int_V \overrightarrow{F} dv$

Example 2.23. *Evaluate* $\int_C \overrightarrow{F}.d\overrightarrow{r}$, *where* $\overrightarrow{F} = x^2\overrightarrow{i} + y^3\overrightarrow{j}$ *and curve C is the arc of the parabola* $y = x^2$ *in the x - y plane from* $(0,0)$ *to* $(1,1)$.

Solution: In the x - y plane, we have $\overrightarrow{r} = x\overrightarrow{i} + y\overrightarrow{j}$

$\therefore d\overrightarrow{r} = dx\overrightarrow{i} + dy\overrightarrow{j}$

$\therefore \overrightarrow{F}.d\overrightarrow{r} = (x^2\overrightarrow{i} + y^3\overrightarrow{j}).(dx\overrightarrow{i} + dy\overrightarrow{j}) = x^2dx + y^3dy$

$\Rightarrow \int \overrightarrow{F}.d\overrightarrow{r} = \int_C (x^2dx + y^3dy)$

Now along the C, $y = x^2 \Rightarrow dy = 2xdx$

$\therefore \int_C \overrightarrow{F}.d\overrightarrow{r} = \int_0^1 [x^2dx + x^6(2x)dx] = \int_0^1 (x^2 + 2x^7)dx$

$$= \left[\frac{x^3}{3} + \frac{2x^8}{8}\right]_0^1 = \frac{7}{12}.$$

Example 2.24. *If* $\overrightarrow{F} = (3x^2 + 6y)\overrightarrow{i} - 14yz\overrightarrow{j} + 20xz^2\overrightarrow{k}$, *evaluate* $\int_C \overrightarrow{F}.d\overrightarrow{r}$, *where C is the straight line joining* $(0,0,0)$ *to* $(1,1,1)$.

Solution: The equation of the straight line joining (0, 0, 0) and (1, 1, 1) are $\dfrac{x-0}{1-0} = \dfrac{y-0}{1-0} = \dfrac{z-0}{1-0} = t$ (say)

Then along C, $x = t, y = t, z = t$

Also $\overrightarrow{r} = x\overrightarrow{i} + y\overrightarrow{j} + z\overrightarrow{k}$ $\therefore d\overrightarrow{r} = \overrightarrow{i}dx + \overrightarrow{j}dy + \overrightarrow{k}dz$

$x = t \Rightarrow dx = dt; y = t \Rightarrow dy = dt; z = t \Rightarrow dz = dt \; d\vec{r} = (\vec{i} + \vec{j} + \vec{k})dt$

Also along C, $\vec{F} = (3t^2 + 6t)\vec{i} - 14t^2\vec{j} + 20t^3\vec{k}$

At $(0,0,0), t = 0$ and at $(1,1,1), t = 1$

$\therefore \int_C \vec{F}.d\vec{r} = \int_{t=0}^{t=1}[(3t^2 + 6t) - 14t^2 + 20t^3]dt$

$$= \left[3\frac{t^3}{3} + 6\frac{t^2}{2} - 14\frac{t^3}{3} + 20\frac{t^4}{4}\right]_0^1 = 1 + 3 - \frac{14}{3} + 5 = \frac{13}{3}$$

Example 2.25. *Find the total work done in moving a particle in a force field given by $\vec{F} = 3xy\vec{i} - 5z\vec{j} + 10x\vec{k}$ along the curve $x = t^2 + 1, y = 2t^2, z = t^3$ from $t = 1$ to $t = 2$.*

Solution: Let C denote the arc of the given curve from $t = 1$ to $t = 2$.

Then the total work done $\int_C \vec{F}.d\vec{r}$

$= \int_C (3xy\vec{i} - 5z\vec{j} + 10x\vec{k}).(dx\vec{i} + dy\vec{j} + dz\vec{k})$

$= \int_C (3xydx - 5zdy + 10xzdz)$

$$= \int_1^2 \left[3xy\frac{dx}{dt} - 5z\frac{dy}{dt} + 10x\frac{dz}{dt}\right] dt$$

$= \int_1^2 [3(t^2+1)(2t)^2(2t) - (5t^3)(4t) + 10(t^2+1)(3t^2)]dt$

$= \int_1^2 (12t^5 + 12t^3 - 20t^4 + 30t^4 + 30t^2)dt$

$= \int_1^2 (12t^5 + 10t^4 + 12t^3 + 30t^2)dt$

$$= 12\left[\frac{t^6}{6}\right]_1^2 + 10\left[\frac{t^5}{5}\right]_1^2 + 12\left[\frac{t^4}{4}\right]_1^2 + 30\left[\frac{t^3}{3}\right]_1^2 = 303$$

Example 2.26. *Show that $\vec{F} = (2xy + z^3)\vec{i} - x^2\vec{j} + 3xz^2\vec{k}$ is a conservative force field. Find the scalar potential. Find also the work done in moving an object in this field from $(1,-2,1)$ to $(3,1,4)$.*

Solution: The field F will be conservative if $\nabla \times \vec{F} = 0$.

$$\text{We have } \nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2xy + z^3 & x^2 & 3xz^2 \end{vmatrix}$$

$$= \vec{i}\left[\frac{\partial}{\partial y}(3xz^2) - \frac{\partial}{\partial z}(x^2)\right] - \vec{j}\left[\frac{\partial}{\partial x}(3xz^2) - \frac{\partial}{\partial z}(2xy + z^3)\right]$$

$$+\vec{k}\left[\frac{\partial}{\partial x}(x^2) - \frac{\partial}{\partial y}(2xy + z^3)\right]$$

$= \vec{i}(0 - 0) - \vec{j}(3z^2 - 3z^2) + \vec{k}(2x - 2x) = 0$

Therefore $\vec{F}$ is conservative force field.

Let $\vec{F} = \nabla\phi \Rightarrow (2xy + z^3)\vec{i} - x^2\vec{j} + 3xz^2\vec{k}$

$$= \frac{\partial \phi}{\partial x}\vec{i} + \frac{\partial \phi}{\partial y}\vec{j} + \frac{\partial \phi}{\partial z}\vec{k}$$

Then $\dfrac{\partial\phi}{\partial x} = 2xy + z^3$ whence $\phi = x^2y + z^3x + f_1(y,z)$ (2.12)

$\dfrac{\partial\phi}{\partial y} = x^2$ whence $\phi = x^2y + f_2(x,z)$ (2.13)

$\dfrac{\partial\phi}{\partial z} = 3xz^2$ whence $\phi = xz^3 + f_3(x,y)$ (2.14)

Then (2.12), (2.13) and (2.14) each represents ϕ. These agree if we choose $f_1(y,z) = 0, f_2(x,z) = xz^3, f_3(x,y) = x^2y$

$\therefore \phi = x^2y + xz^3 + c$

Work done $= \int_{(1,-2,1)}^{(3,1,4)} \vec{F}.d\vec{r} = \int_{(1,-2,1)}^{(3,1,4)} d\phi = [\phi]_{(1,-2,1)}^{(3,1,4)}$

(i.e.) Work done $= [x^2y+xz^3]_{(1,-2,1)}^{(3,1,4)} = (9+192)-(-2+1) = 202.$

Example 2.27. *A vector field is given by* $\vec{F} = \sin y\,\vec{i} + x(1+\cos y)\,\vec{j}$. *Evaluate the line integral over the circular path given by* $x^2 + y^2 = a^2, z = 0$.

Solution: The parametric equation of the circular path are $x = a\cos t, y = a\sin t$ and $z = 0$ where t varies from 0 to 2π. Since the particle moves in the xy plane ($z = 0$), we can take $\vec{r} = x\,\vec{i} + y\,\vec{j}$, so that $d\vec{r} = dx\,\vec{i} + dy\,\vec{j}$.

$\therefore \oint_C \vec{F}.d\vec{r} = \oint_C [\sin y\,\vec{i} + x(1+\cos y)\,\vec{j}].[dx\,\vec{i} + dy\,\vec{j}]$

$= \oint_C [\sin y dx + x(1+\cos y)dy]$

$= \oint_C [(\sin y dx + x\cos y dy) + xdy] = \oint_C d(x\sin y) + \oint_C xdy$

$= \int_0^{2\pi} d[a\cos t\sin(a\sin t)] + \int_0^{2\pi} a\cos t a\cos t dt$

$= [a\cos t\sin(a\sin t)]_0^{2\pi} + a^2\int_0^{2\pi}\cos^2 t dt$

[$\because$ using $x = a\cos t, y = a\sin t$ so that $dx = -a\sin t dt$,

$dy = a\cos t$]

$= \dfrac{a^2}{2}\int_0^{2\pi}(1+\cos 2t)dt = \dfrac{a^2}{2}\left[t + \dfrac{\sin 2t}{2}\right]_0^{2\pi} = \pi a^2$

Example 2.28. *If* $\vec{F} = 2y\,\vec{i} - z\,\vec{j} + x\,\vec{k}$, *evaluate* $\oint_C \vec{F}\times d\vec{r}$ *along the curve* $x = \cos t, y = \sin t, z = 2\cos t$ *from* $t = 0$ *to* $t = \dfrac{\pi}{2}$

Solution: Let $\vec{F}\times d\vec{r} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2y & -z & x \\ dx & dy & dz \end{vmatrix}$

$= (-zdx - xdy)\,\vec{i} + (xdx - 2ydz)\,\vec{j} + (2ydz + zdx)\,\vec{k}$

In terms of t,

$\vec{F}\times d\vec{r} = [-2\cos t(-2\sin t)dt - \cos t(\cos t)dt]\,\vec{i}$

$+[\cos t(-\sin t) - 2\sin t(-2\sin t)dt]\,\vec{j}$

$+[2\sin t(\cos t)dt + 2\cos t(-\sin t)dt]\,\vec{k}$

$= [(4\cos t\sin t - \cos^2 t)\,\vec{i} + (4\sin^2 t - \cos t\sin t)\,\vec{j}\,dt]$

$$\oint_C \vec{F} \times d\vec{r} = \int_0^{\pi/2} [(4\cos t \sin t - \cos^2 t)\,\vec{i} + (4\sin^2 t - \cos t \sin t)\,\vec{j}\,]dt$$

$$= \left[4.\frac{1}{2} - \frac{1}{2}\frac{\pi}{2}\right]\vec{i} + \left[4.\frac{1}{2}.\frac{\pi}{2} - \frac{1}{2}\right]\vec{j} = \left[2 - \frac{\pi}{4}\right]\vec{i} + \left[\pi - \frac{1}{2}\right]\vec{j}$$

Example 2.29. *Find the work done when a force $\vec{F} = (x^2 - y^2 + x)\,\vec{i} - (2xy + y)\,\vec{j}$ moves particle in the xy plane from $(0,0)$ along the parabola $y^2 = x$. Is the work done different when the path is the straight line $y = x$?*

Solution: As $\vec{F} = (x^2 - y^2 + x)\,\vec{i} - (2xy + y)\,\vec{j}$, in the xy plane $z = 0$, $\vec{r} = x\,\vec{i} + y\,\vec{j}$.

$\therefore d\vec{r} = dx\,\vec{i} + dy\,\vec{j}$ and $\vec{F}.d\vec{r} = (x^2 - y^2 + x)dx - (2xy + y)dy$.

(i) Along the parabola $x = y^2, dx = 2ydy$.

$\therefore$ Work done $= W = \int_C \vec{F}.d\vec{r}$

$$= \int_0^1 (y^4 - y^2 + y^2)(2ydy) - (2y^3 + y)dy$$

$$= \left[2\frac{y^6}{6} - 2\frac{y^4}{4} - \frac{y^2}{2}\right]_0^1 = \frac{1}{3} - \frac{1}{2} - \frac{1}{2} = -\frac{2}{3}$$

(ii) Along the straight line $y = x, dy = dx$

$\therefore$ Work done $= \int_C \vec{F}.d\vec{r}$

$$= \int_0^1 (y^2 - y^2 + y)dy - (2y^2 + y)dy$$

$$= \left[\frac{y^2}{2} - 2\frac{y^3}{3} - \frac{y^2}{2}\right]_0^1 = -\frac{2}{3}$$

We observe that the work done is the same.

Example 2.30. *Evaluate $\int_C \vec{F}.d\vec{r}$, given that $\vec{F} = z\,\vec{i} + x\,\vec{j} + y\,\vec{k}$ and C being the arc of the curve $\vec{r} = \cos t\,\vec{i} + \sin t\,\vec{j} + t\,\vec{k}$ from $t = 0$ to $t = 2\pi$.*

Solution: Let $\vec{r} = \cos t\,\vec{i} + \sin t\,\vec{j} + t\,\vec{k}$, $\frac{d\vec{r}}{dt} = -\sin t\,\vec{i} + \cos t\,\vec{j} + \vec{k}$.

Now $\int_C \vec{F}.d\vec{r} = \oint_C \left(\vec{F}.\frac{d\vec{r}}{dt}\right) dt$

$$= \int_0^{2\pi} [(t\,\vec{i} + \cos t\,\vec{j} + \sin t\,\vec{k}).(-\sin t\,\vec{i} + \cos t\,\vec{j} + \vec{k}]dt$$

$$= \int_0^{2\pi} (-t\sin t + \cos^2 t + \sin t)dt$$

$$= \int_0^{2\pi} (-t\sin t)dt + \int_0^{2\pi} \cos^2 t dt + \int_0^{2\pi} \sin t dt$$

$$= -\int_0^{2\pi} t d(-\cos t) + \int_0^{2\pi} \left(\frac{1 + \cos 2t}{2}\right) dt + \int_0^{2\pi} \sin t dt$$

$$= -\left[(-t\cos t)_0^{2\pi} + \int_0^{2\pi} \cos t dt\right] + \left[\frac{1}{2}t + \frac{\sin 2t}{4}\right]_0^{2\pi} - (\cos t)_0^{2\pi}$$

$= 2\pi + \pi = 3\pi$ as $\sin 2\pi = 0, \cos 2\pi = 1, \sin 0 = 0, \cos 0 = 1$.

Example 2.31. *Find the work done by the force $\vec{F} = (x^2 + y^2)\,\vec{i} + (x^2 + z^2)\,\vec{j} + y\,\vec{k}$ when it moves a particle along the* *upper half of the circle $x^2 + y^2 = 1$ from the point $(-1,0)$ to $(1,0)$.*

Solution: As the circle is in the xy plane $z = 0$.

We use $x = \cos t, y = \sin t, 0 < t < \pi$.

$\frac{dx}{dt} = -\sin t, \frac{dy}{dt} = \cos t, d\vec{r} = dx\,\vec{i} + dy\,\vec{j}$

$\vec{F} = (x^2+y^2)\vec{i} + (x^2+z^2)\vec{j} + y\vec{k}$ and $\vec{F}.d\vec{r} = (x^2+y^2)dx + x^2dy$

Work done $\int_C \vec{F}.d\vec{r} = \int_0^\pi (-\sin t + \cos^2 t \cos t)dt$

$$= \int_0^\pi (-\sin t + \cos^3 t)dt \quad \left[\begin{array}{l} \because \cos 3A = 4\cos^3 A - 3\cos A \\ \therefore \cos^3 A = \frac{3}{4}\cos A + \frac{1}{4}\cos 3A \end{array}\right]$$

$$= \int_0^\pi \left(-\sin t + \frac{3}{4}\cos t + \frac{1}{4}\cos 3t\right) dt$$

$$= \left[\cos t + \frac{3}{4}\sin t + \frac{1}{12}\sin 3t\right]_0^\pi = -1 - 1 = -2$$

$\therefore$ Work done $= -2$.

Example 2.32. *If $\phi = 2xyz^2$ and C is the curve $x = t^2, y = 2t, z = t^3$ from $t = 0$ to $t = 1$, evaluate the line integral $\int_C \phi d\vec{r}$*

Solution: Let $\phi = 2xyz^2$, $x = t^2, y = 2t, z = t^3$ so that $\phi = 2xyz^2 = 2t^2.2t.t^6 = 4t^9$

$\therefore \int_C \phi d\vec{r} = \int_C [(4t^9)(2tdt)\vec{i} + (2dt)\vec{j} + (3t^2dt)\vec{k}]$

$= \int_0^1 (8t^{10}\vec{i} + 8t^9\vec{j} + 12t^{11}\vec{k})dt$

$$= \left[8\frac{t^{11}}{11}\vec{i} + 8\frac{t^{10}}{10}\vec{j} + 12\frac{t^{12}}{12}\vec{k}\right]_0^1 = \frac{8}{11}\vec{i} + \frac{4}{5}\vec{j} + \vec{k}$$

Example 2.33. *If $\vec{F} = (2x+y)\vec{i} + (3y-x)\vec{j}$, evaluate $\int_C \vec{F}.d\vec{r}$, where C is the curve in the xy plane consisting of the straight lines from $(0,0)$ to $(2,0)$ and then to $(3,2)$.*

Solution: The path of integration C has been shown in the figure.

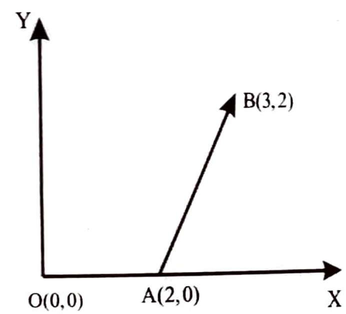

Fig.3

It consists of the straight lines OA and AB, we have

$\int_C \vec{F}.d\vec{r} = \int_C [(2x+y)\vec{i} + (3y-x)\vec{j}].[dx\,\vec{i} + dy\,\vec{j}]$

$= \int_C [(2x+y)dx + (3y-x)dy]$

Now along the straight line OA, $y = 0, dy = 0$ and x varies from

0 to 2.

The equation of the straight line AB is

$$y - 0 = \frac{2-0}{3-2}(x-2) \text{ (i.e.) } y = 2x - 4$$

Along AB, $y = 2x - 4, dy = 2dx$ and x varies from 2 to 3.

$$\int_C \vec{F}.d\vec{r} = \int_0^2 [(2x+0)dx + 0] + \int_2^3 [(2x + 2x - 4)dx + (6x - 12 - x)2dx]$$

$$= [x^2]_0^2 + \int_2^3 (14x - 28)dx = 4 + 14\left[\frac{(x-2)^2}{2}\right]_2^3 = 4 + 7 = 11$$

Example 2.34. *Find the work done by the force $\vec{F} = (e^x z - 2xy)\vec{i} + (1 - x^2)\vec{j} + (e^x + z)\vec{k}$ when it moves a particle from $(0, 1, -1)$ to $(2, 3, 0)$ along any path.*

Solution: Since the path of integration is not given we may assume that the work done depends on end points only, for which $curl\,\vec{F} = 0, \vec{F} = \nabla\phi$.

$\therefore$ We first check if $\vec{F}$ is conservative.

$$\therefore \nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ e^x z - 2xy & 1 - x^2 & e^x + z \end{vmatrix}$$

$$= \vec{i}\left[\frac{\partial}{\partial y}(e^x + z) - \frac{\partial}{\partial z}(1 - x^2)\right] - \vec{j}\left[\frac{\partial}{\partial x}(e^x + z) - \frac{\partial}{\partial z}(e^x z - 2xy)\right]$$

$$+\vec{k}\left[\frac{\partial}{\partial x}(1 - x^2) - \frac{\partial}{\partial y}(e^x z - 2xy)\right]$$

$$= (0-0)\vec{i} - (e^x - e^x)\vec{j} + (-2x + 2x)\vec{k} = 0$$

$\Rightarrow \vec{F}$ is conservative force.

Let $\vec{F} = \nabla\phi$

$$\text{(i.e.) } (e^x z - 2xy)\vec{i} + (1 - x^2)\vec{j} + (e^x + z)\vec{k} = \sum \vec{i}\frac{\partial\phi}{\partial x}$$

$$\Rightarrow \frac{\partial\phi}{\partial x} = e^x z - 2xy; \frac{\partial\phi}{\partial y} = 1 - x^2; \frac{\partial\phi}{\partial z} = e^x + z$$

Integrating, $\phi = e^x z - x^2 y + f_1(y, z); \phi = y - x^2 y + f_2(x, z);$

$$\phi = e^x z + \frac{z^2}{2} + f_3(x, y)$$

We choose $\phi = e^x z - x^2 y + y + \frac{z^2}{2} + c$

Now work done $W = \int_C \vec{F}.d\vec{r} = [\phi]_{(0,1,-1)}^{(2,3,0)}$

$$= \left[e^x z - x^2 y + y + \frac{z^2}{2}\right]_{(0,1,-1)}^{(2,3,0)} = (0 - 12 + 3) - \left(-1 + 1 + \frac{1}{2}\right)$$

$$= -9 - \frac{1}{2} = -\frac{19}{2}$$

Example 2.35. *Find the work done by the force $\vec{F} = x\vec{i} - z\vec{j} + 2y\vec{k}$ in the displacement along the closed path C, consisting of*

segments C_1, C_2 and C_3, where
on $C_1, 0 \le x \le 1, y = x, z = 0$
on $C_2, 0 \le z \le 1, x = 1, y = 1$
on $C_3, 1 \ge x \ge 0, y = z = x$.

Solution: Total work done $= \oint_C \vec{F}.d\vec{r}$

$$= \oint_C (x\vec{i} - z\vec{j} + 2y\vec{k})(dx\vec{i} + dy\vec{j} + dz\vec{k})$$

$$= \oint_C (xdx - zdy + 2ydz)$$

on $C_1, 0 \le x \le 1, y = x, z = 0, dz = 0$

$$\therefore W_1 = \int_0^1 xdx = \frac{1}{2}$$

on $C_2, 0 \le z \le 1, x = 1, y = 1, dx = 0, dy = 0$

$$\therefore W_2 = 2\int_0^1 dz = 2$$

on $C_3, 1 \ge x \ge 0, y = z = x, dy = dz = dx$

$$\therefore W_3 = \int_0^1 (xdx - xdx + 2xdx) = 2\int_0^1 xdx = -1$$

Total work done $= W_1 + W_2 + W_3 = \frac{1}{2} + 2 - 1 = \frac{3}{2}$

Example 2.36. *Evaluate $\int\int_S \vec{A}.\hat{n}ds$, where $\vec{A} = (x + y^2)\vec{i} - 2x\vec{j} + 2yz\vec{k}$ and S is the surface of the plane $2x + y + 2z = 6$ in the first octant.*

Solution: A unit vector normal to the surface S is given by $\hat{n} = \frac{\nabla\phi}{|\nabla\phi|}$ where $\phi = 2x + y + 2z - 6$

$$\Rightarrow \hat{n} = \frac{2\vec{i} + \vec{j} + 2\vec{k}}{\sqrt{4+1+4}} = \frac{2}{3}\vec{i} + \frac{1}{3}\vec{j} + \frac{2}{3}\vec{k}$$

$$\vec{k}.\hat{n} = \vec{k}.\left(\frac{2}{3}\vec{i} + \frac{1}{3}\vec{j} + \frac{2}{3}\vec{k}\right) = \frac{2}{3}$$

$\therefore \int\int_S \vec{A}.\hat{n}dS = \int\int_R \vec{A}.\hat{n}.\frac{dxdy}{|\vec{k}.\hat{n}|}$ where R is the projection of S on the xy plane. The region R is bounded by x axis, y axis, line $2x + y = 6, z = 0$.

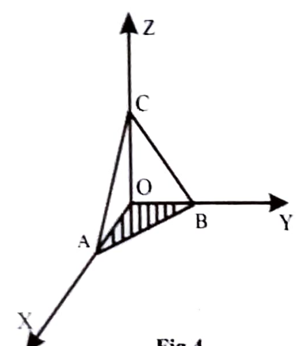

Fig.4

Now $\vec{A}.\hat{n} = [(x + y^2)\vec{i} - 2x\vec{j} + 2yz\vec{k}].\left[\frac{2}{3}\vec{i} + \frac{1}{3}\vec{j} + \frac{2}{3}\vec{k}\right]$

$$= \frac{2}{3}(x + y^2) - \frac{2}{3}x + \frac{4}{3}yz = \frac{2}{3}y^2 + \frac{4}{3}y\left(\frac{6 - 2x - y}{2}\right)$$

$$= \frac{4}{3}y(3 - x)$$

Hence $\int\int_S \vec{A}.\hat{n}dS = \int\int_R \vec{A}.\hat{n}.\frac{dxdy}{|\vec{k}.\hat{n}|} = \int\int_R \frac{4}{3}y(3 - x).\frac{3}{2}dxdy$

$$= \int_0^3 \int_0^{6-2x} 2y(3-x)dydx = \int_0^3 2(3-x)\left[\frac{y^2}{2}\right]_0^{6-2x}$$

$$= \int_0^3 (3-x)(6-2x)^2 dx = 4\int_0^3 (3-x)^3 dx$$

$$= 4\left[\frac{(3-x)^4}{4(-1)}\right]_0^3 = -(0-81) = 81$$

Example 2.37. *Evaluate $\int\int_S \vec{A}.\hat{n}ds$, where $\vec{A} = z\vec{i} + x\vec{j} - 3y^2z\vec{k}$ and S is the surface of the cylinder $x^2+y^2 = 16$ included in the first octant between $z = 0$ and $z = 5$.*

Solution: A vector normal to the surface S is given by $\nabla(x^2 + y^2) = 2x\vec{i} + 2y\vec{j}$.

$\therefore \hat{n} =$ a unit vector normal to surface.

$$= \frac{2x\vec{i} + 2y\vec{j}}{\sqrt{(2x)^2+(2y)^2}} = \frac{x\vec{i} + y\vec{j}}{\sqrt{x^2+y^2}} = \frac{x\vec{i} + y\vec{j}}{4} \text{ as } x^2+y^2 = 16$$

Let R be the projection of S on yz plane, then

$$\int\int_S \vec{A}.\hat{n}ds = \int\int_R \vec{A}.\hat{n}.\frac{dydz}{|\vec{i}.\hat{n}|}$$

The region R is $OBB'O'$ enclosed by $y = 0$ to $y = 4$ and $z = 0$ to $z = 5$.

$$\text{Now } \vec{i}.\hat{n} = \vec{i}\left(\frac{1}{4}x\vec{i} + \frac{1}{4}\vec{j}\right) = \frac{1}{4}x$$

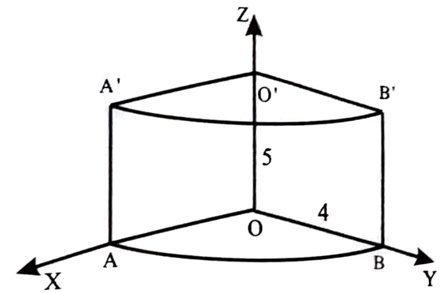

Fig.5

$$\vec{A}.\hat{n} = (z\vec{i} + x\vec{j} - 3y^2z\vec{k}).\left(\frac{1}{4}x\vec{i} + \frac{1}{4}\vec{j}\right)$$

$$= \frac{1}{4}xz + \frac{1}{4}xy = \frac{1}{4}x(y+z)$$

$$\text{Hence } \int\int_S \vec{A}.\hat{n}ds = \int\int_R \vec{A}.\hat{n}.\frac{dydz}{|\vec{i}.\hat{n}|} = \int\int_R \frac{1}{4}x(y+z)\frac{dydz}{\frac{1}{4}x}$$

$$= \int_0^5\int_0^4 (y+z)dydz = \int_0^5 \left[\frac{y^2}{2} + zy\right]_0^4$$

$$= \int_0^5 (8+4z)dz = \left[8z + 4\frac{z^2}{2}\right]_0^5 = 40 + 50 = 90$$

Example 2.38. *Evaluate $\int\int_S \vec{F}.d\vec{s}$, where $\vec{F} = yz\vec{i} + zx\vec{j} + xy\vec{k}$ and S is the part of the sphere $x^2+y^2+z^2 = 1$ that lies in the first octant.*

Solution: Let $I = \int\int_S \vec{F}.\hat{n}.d\vec{s}$

$\hat{n} = \frac{\nabla\phi}{|\nabla\phi|}$ where $\phi = x^2 + y^2 + z^2 = 1$

$\Rightarrow \hat{n} = \frac{2x\vec{i} + 2y\vec{j} + 2z\vec{k}}{\sqrt{4(x^2+y^2+z^2)}} = x\vec{i} + y\vec{j} + z\vec{k}$ as (x, y, z) lies on S.

$\therefore I = \int\int_S (yz\vec{i} + zx\vec{j} + xy\vec{k}).(x\vec{i} + y\vec{j} + z\vec{k}) = \int\int_S 3xyzds$

$= \int\int_R 3xyz\frac{dxdy}{|\vec{k}.\hat{n}|}$ where R is the projection in the xy - plane bounded by the circle $x^2+y^2 = 1$ and lying in the first quadrant.

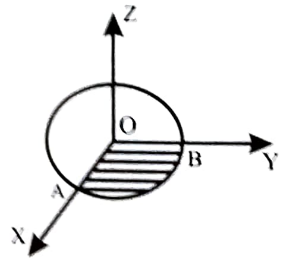

Fig.6

$$I = \int\int_R 3xyz\frac{dxdy}{z} = \int_0^1\int_0^{\sqrt{1-y^2}} 3xydxdy = \int_0^1 3y\left[\frac{x^2}{2}\right]_0^{\sqrt{1-y^2}} dy$$

$$= \frac{3}{2}\int_0^1 y(1-y^2)dy = \frac{3}{2}\left[\frac{y^2}{2} - \frac{y^4}{4}\right]_0^1 = \frac{3}{8}.$$

Example 2.39. *Evaluate $\int\int\int_V \phi dv$, where $\phi = 45x^2y$ and V is the closed origin bounded by the planes $4x + 2y + z = 8$, $x = 0, y = 0, z = 0$.*

Solution: Let $\int\int\int_V \phi dv = \int_{x=0}^{2}\int_{y=0}^{4-2x}\int_{z=0}^{8-4x-2y} 45x^2ydzdydx$

$$= 45\int_{x=0}^{2}\int_{y=0}^{4-2x} x^2y[z]_0^{8-4x-2y}dydx$$

$$= 45\int_{x=0}^{2}\int_{y=0}^{4-2x} x^2y(8-4x-2y)dydx$$

$$= 45\int_0^2\left[x^2(8-4x)\frac{y^2}{2} - 2x^2\frac{y^3}{3}\right]_0^{4-2x} dx$$

$$= 45\int_0^2 \frac{x^2}{3}(4-2x)^3dx = 128$$

Example 2.40. *If $\vec{F} = (2x^2 - 3z)\vec{i} - 2xy\vec{j} - 4x\vec{k}$, then evaluate $\int\int\int_V \nabla.\vec{F}dv$, where V is bounded by the planes $x = 0, y = 0, z = 0$ and $2x + 2y + z = 4$.*

Solution: Let $\nabla.\vec{F} = \frac{\partial}{\partial x}(2x^2-3z) + \frac{\partial}{\partial y}(-2xy) + \frac{\partial}{\partial z}(-4x) = 2x$

Hence $\int\int\int_V \nabla.\vec{F}dv = \int\int\int_V 2xdxdydz$

$$= \int_0^2\int_0^{2-x} 2x[z]_0^{4-2x-2y}dydx = \int_0^2\int_0^{2-x} 2x(4-2x-2y)dydx$$

$$= \int_0^2\int_0^{2-x}[4x(2-x) - 4xy]dydx$$

$$= \int_0^2[4x(2-x)y - 2xy^2]_0^{2-x}dx$$

$$= \int_0^2[4x(2-x)^2 - 2x(2-x)^2]dx$$

$$= \int_0^2 2x(2-x)^2dx = 2\int_0^2(4x - 4x^2 + x^3)dx$$

$$= 2\left[2x^2 - 4\frac{x^3}{3} + \frac{x^4}{4}\right]_0^2 = 2\left[8 - \frac{32}{4} + 4\right] = \frac{8}{3}$$

EXERCISE

1. If $f(\vec{t}) = t\vec{i} + (t^2-2t)\vec{j} + (3t^2+3t^3)\vec{k}$, find the $\int_0^1 f(\vec{t})dt$ $\left[\text{Ans:} \frac{1}{2}\vec{i} - \frac{2}{3}\vec{j} + \frac{7}{4}\vec{k}\right]$

2. If $\vec{r} = t\vec{i} + t^2\vec{j} + (t-1)\vec{k}$ and $\vec{s} = 2t^2\vec{i} + 6t\vec{k}$, evaluate $\int_0^2 \vec{r}.\vec{s}\,dt$. [Ans: 12]

3. If $\vec{F} = 3xy\vec{i} - y^2\vec{j}$, evaluate $\int_C \vec{F}.d\vec{r}$, where C is the arc of the parabola $y = 2x^2$ from $(0,0)$ to $(1,2)$. $\left[\text{Ans:} -\frac{7}{6}\right]$

4. If $\vec{F} = (4xy - 3x^2z^2)\vec{i} + 2x^2\vec{j} - 2x^3z\vec{k}$, show that $\int_C \vec{F}.d\vec{r}$ is independent of the path C.

5. Define the line integral of a vector point function.

6. State the necessary and sufficient condition for the line integral $\int_A^B \vec{F}.d\vec{r}$ to be independent of the path of integration.

7. What is meant by conservative vector field?

8. Evaluate $\int_C xdy - ydx$ around the circle $x^2 + y^2 = 1$.

[Ans: 2π]

9. Find the circulation of $\vec{F}$ around the curve C, where $\vec{F} = y\vec{i} + z\vec{j} + x\vec{k}$ and C is the circle $x^2 + y^2 = 1, z = 0$

[Ans: $-\pi$]

10. If $\vec{A} = (5xy - 6x^2)\vec{i} + (2y - 4x)\vec{j}$, evaluate $\int \vec{A}.d\vec{r}$ where C is the curve $y = x^3$ in the xy- plane from the point $(1,1)$ to $(2,8)$. [Ans: 35]

11. $\vec{F} = (3x^2 + 6y)\vec{i} - 14yz\vec{j} + 20xz^2\vec{k}$, then evaluate $\int_C \vec{F}.d\vec{r}$ from $(0,0,0)$ to $(1,1,1)$ along the curve $x = t, y = t^2, z = t^3$. [Ans: 5]

12. Evaluate $\int_C \vec{F}.d\vec{r}$ where $\vec{F} = (x^2 + y^2)\vec{i} - 2xy\vec{j}$ where C is the rectangle in the xy - plane bounded by $y = 0, x = a, y = b$ and $x = 0$. [Ans: $-2ab^2$]

13. Find the work done in moving a particle once around a circle C in the xy - plane, if the circle has centre at the origin and radius 2 and if the force field $\vec{F}$ is given by $\vec{F} = (2x - y + 2z)\vec{i} + (x + y - z)\vec{j} + (3x - 2y - 5z)\vec{k}$. [Ans: 8π]

14. Evaluate the line integrals $\int_C[(x^2 + xy)dx + (x^2 + y^2)dy]$ where C is the square formed by the lines $y = \pm1$ and $x = \pm1$. [Ans: 0]

15. Evaluate $\int_C[(y + 3z)dx + (2z + x)dy + (3x + 2y)dz]$, where C is the arc of helix $x = a\cos\theta, y = a\sin\theta, z = \frac{2a\theta}{\pi}$ between the points $(a,0,0)$ and $(0,a,a)$ [Ans: 2a]

16. If $\vec{F} = xy\vec{i} - z\vec{j} + x^2\vec{k}$, evaluate $\int_C \vec{F} \times d\vec{r}$, where C is the curve $x = t^2, y = 2t, z = t$ from $(0,0,0)$ to $(1,2,1)$. $\left[\text{Ans: } -\frac{9}{10}\vec{i} - \frac{2}{3}\vec{j} + \frac{7}{5}\vec{k}\right]$

17. Find the work done by the force $\vec{F} = (y^2\cos x + z^3)\vec{i} + (2y\sin x - 4)\vec{j} + (3xz^2 + 2)\vec{k}$, when it moves a particle from $(0, 1, -1)$ to $\left(\frac{\pi}{2}, -1, 2\right)$ along any path. [Ans: $4\pi + 5$]

18. Show that the line integral $\int_C [(2xy + 3)dx + (x^2 - 4z)dy - 4ydz]$, where C is any path joining $(0,0,0)$ to $(1,-1,3)$ does not depend on the path C and evaluate the line integral. [Ans: 14]

19. Find the work done in moving a particle in the force field $\vec{F} = 3x^2\vec{i} + (2xz - y)\vec{j} + z\vec{k}$ along the curve defined by $x^2 = 4y, 3x^3 = 8z$ from $x = 0$ to $x = 2$. [Ans: 16]

20. Show that $\int\int_S \vec{F}.\hat{n}ds = \frac{3}{2}$, where $\vec{F} = 4xz\vec{i} - y^2\vec{j} + yz\vec{k}$ and S is the surface of the cube bounded by planes $x = 0, x = 1, y = 0, y = 1, z = 0$ and $z = 1$.

21. Evaluate $\int\int_S \vec{F}.\hat{n}ds$, where $\vec{F} = 4x\vec{i} - 2y^2\vec{j} + z^2\vec{k}$ and S is the surface bounding the region $x^2 + y^2 = 4, z = 0$ and $z = 3$. [Ans: 84π]

22. Evaluate $\int\int_S \vec{F}.\hat{n}ds$, where $\vec{F} = xy\vec{i} - x^2\vec{j} + (x+z)\vec{k}$ and S is part of the plane $2x + 2y + z = 6$ included in the first octant. $\left[\text{Ans: } \frac{27}{4}\right]$

23. If $\vec{F} = 2z\vec{i} - x\vec{j} + y\vec{k}$, evaluate $\int\int\int_V \vec{F}.dv$, where V is the region bounded by the surfaces $x = 0, y = 0, x = 2, y = 4, z = x^2, z = 2$ $\left[\text{Ans: } \frac{32}{15}(3\vec{i} + 5\vec{j})\right]$

24. If $\vec{F} = (2x^2 - 3z)\vec{i} - 2xy\vec{j} - 4x\vec{k}$, then evaluate $\int\int\int_V \nabla \times \vec{F}$, where V is the closed region bounded by the planes $x = 0, y = 0, z = 0$ and $2x + 2y + z = 4$. $\left[\text{Ans: } \frac{8}{3}(\vec{j} - \vec{k})\right]$

2.5 Integral Theorems

2.5.1 Green's Theorem in the Plane

Statement: If $M(x, y)$ and $N(x, y)$ be continuous functions of x and y having continuous partial derivatives $\frac{\partial M}{\partial y}$ and $\frac{\partial N}{\partial x}$ in a region R of the xy - plane bounded by a closed curve C, then $\oint_C (Mdx + Ndy) = \int\int_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dxdy$, where C is traversed in the counter clockwise direction.

2.5.2 Stoke's Theorem (Relation between Line and Surface Integrals)

Statement: If S be an open surface bounded by a closed curve C and $\vec{F} = F_1\vec{i} + F_2\vec{j} + F_3\vec{k}$ be any vector point function having continuous first order partial derivatives, then $\oint_C \vec{F}.d\vec{r} =$

$\int\int_C curl\,\vec{F}.\hat{n}ds$, where $\hat{n}$ is a unit normal vector at any point of S drawn in the sense in which a right bounded screw would advance when rotated in the sense of description of C.

2.5.3 Gauss Divergence Theorem (Relation between Surface and Volume Integrals)

Statement: If $\vec{F}$ is a vector point function having continuous first order derivatives in the region V bounded by a closed surface S, then $\int\int\int_V \vec{F}.d\vec{v} = \int\int_S \vec{F}.\hat{n}ds$, where $\hat{n}$ is the outwards drawn unit normal to the surface S.

Example 2.41. *Verify Green's theorem in the plane for* $\oint_C (3x^2 - 8y^2)dx + (4y - 6xy)dy$, *where* C *is the boundary of the region defined by* $x = 0, y = 0, x + y = 1$.

Solution: Here $M = 3x^2 - 8y^2, N = 4y - 6xy$

$$\frac{\partial M}{\partial y} = -16y; \frac{\partial N}{\partial x} = -6y$$

$$\therefore \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = 10y$$

$$\therefore \int\int_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dxdy = \int_0^1 \int_0^{1-x} 10ydydx = \int_0^1 5[y^2]_0^{1-x} dx$$

$$= 5\int_0^1 (1-x)^2 dx = 5\left[\frac{(1-x)^3}{-3}\right]_0^1 = -\frac{5}{3}(0-1) = \frac{5}{3}$$

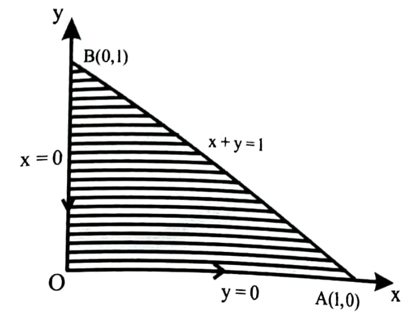

Fig. 7

$$\therefore \int\int_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dxdy = \frac{5}{3} \qquad (2.15)$$

Along OA, $y = 0 \quad \therefore dy = 0$ and the limits of x are from 0 to 1.

$\therefore$ Line integral along OA $= \int_0^1 3x^2 dx = [x^3]_0^1 = 1$

Along AB, $y = 1 - x$. $\therefore dy = -dx$ and the limits of x are from 1 to 0.

$\therefore$ Line integral along AB $= \int_1^0 [3x^2 - 8(1-x)^2]dx + [4(1-x) - 6x(1-x)](-dx)$

$$= \int_1^0 (3x^2 - 8 + 16x - 8x^2 - 4 + 4x + 6x - 6x^2)dx$$

$$= \int_1^0 (-12 + 26x - 11x^2)dx = \left[-12x + 13x^2 - \frac{11}{3}x^3\right]_1^0$$

$$= -\left[-12 + 13 - \frac{11}{3}\right] = \frac{8}{3}$$

$\therefore$ Line integral along BO $= \int_1^0 4y dy$ as along BO.

$x = 0, dx = 0$ and the limits of y are from 1 to 0.

Along BO: $\int_1^0 4y dy = [2y^2]_1^0 = -2$

$\therefore$ Line integral along C ((i.e.) along OABO) $= 1 + \frac{8}{3} - 2 = \frac{5}{3}$

$$\text{(i.e.)} \oint_C (M dx + N dy) = \frac{5}{3} \tag{2.16}$$

The equality of (2.15) and (2.16), verify Green's theorem in the plane.

Example 2.42. *Verify Green's theorem in the plane for $\oint_C [(xy + y^2)dx + x^2 dy]$, where C is the closed curve of the region bounded by $y = x$ and $y = x^2$.*

Solution: By Green's theorem in plane, we have

$$\int\int_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dxdy = \oint_C (M dx + N dy)$$

Here $M = xy + y^2; N = x^2$

The curves $y = x$ and $y = x^2$ intersect at $(0,0)$ and $(1,1)$. The positive direction in traversing C is as shown in the figure.

$$\therefore \int\int_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dxdy = \int\int_R \left[\frac{\partial}{\partial x}(x^2) - \frac{\partial M}{\partial y}(xy + y^2)\right] dxdy$$

$$= \int\int_R (2x - x - 2y) dxdy = \int\int_R (x - 2y) dxdy$$

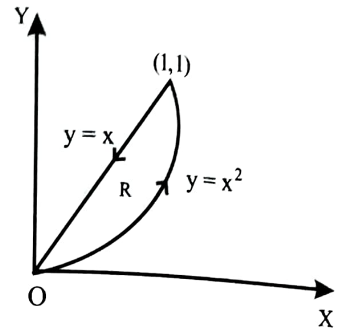

Fig.8

$$= \int_0^1 \int_{y=x^2}^{x} (x - 2y) dy dx = \int_0^1 [xy - y^2]_{x^2}^{x} dx$$

$$= \int_0^1 (x^2 - x^2 - x^3 + x^4) dx = \int_0^1 (x^4 - x^3) dx$$

$$= \left[\frac{x^5}{5} - \frac{x^4}{4}\right]_0^1 = \frac{1}{5} - \frac{1}{4} = \frac{1}{20}$$

Now let us evaluate the line integral along C.

Along $y = x^2, dy = 2x dx$.

$\therefore$ Along $y = x^2$, the line integral equals

$$\int_0^1 [[x(x^2) + x^4] dx + x^2(2x) dx] = \int_0^1 [3x^3 + x^4] dx = \frac{19}{20}$$

Along $y = x, dy = dx$. Therefore along $y = x$, the line integral equals

$$= \int_1^0 [(x(x) + x^2)dx + x^2 dx] = \int_1^0 3x^2 dx = \left[3\frac{x^3}{3}\right]_1^0 = -1$$

Therefore the required line integral $= \dfrac{19}{20} - 1 = -\dfrac{1}{20}$

Hence the theorem is verified.

Example 2.43. *Verify Green's theorem in the plane for* $\int_C (x^2 + 2xy)dx + (y^2 + x^3y)dy$ *where* C *is a square with vertices* $P(0,0)$, $Q(1,0)$, $R(1,1)$ *and* $S(0,1)$.

Solution: The Green's theorem in plane is

$$\int_C (Mdx + Ndy) = \int\int_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dxdy \qquad (2.17)$$

Here $M = x^2 + 2xy;\ N = y^2 + x^3y,\ \dfrac{\partial M}{\partial y} = 2x;\ \dfrac{\partial N}{\partial x} = 3x^2y$

$$\therefore \int\int_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dxdy = \int_0^1 \int_0^1 (3x^2y - 2x)dxdy$$

$$= \int_0^1 \left[3y\left(\frac{x^3}{3}\right)_0^1 - 2\left(\frac{x^2}{2}\right)_0^1\right] dy = \int_0^1 (y-1)dy$$

$$\int\int_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dxdy = \left[\frac{y^2}{2} - y\right]_0^1 = \frac{1}{2} - 1 = \frac{1}{2} \qquad (2.18)$$

Now along PQ, $y = 0, dy = 0$ and x varies from 0 to 1.

$$\therefore \int_{PQ} x^2 dx = \int_0^1 x^2 dx = \left[\frac{y^3}{3}\right]_0^1 = \frac{1}{3}$$

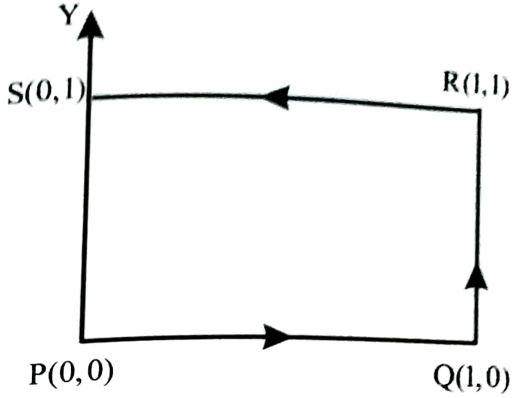

Fig.9

Along QR, $x = 1 \Rightarrow dx = 0, y = 0, 1.$

$$\therefore \int_{QR}(y^2 + y)dy = \int_0^1 (y^2 + y)dy = \left[\frac{y^3}{3} + \frac{y^2}{2}\right]_0^1 = \frac{1}{3} + \frac{1}{2} = \frac{5}{6}$$

Along RS, $y = 1 \Rightarrow dy = 0, x = 1, 0.$

$$\therefore \int_{RS}(x^2 + x)dx = \int_1^0 (x^2 + 2x)dx = \left[\frac{x^3}{3} + \frac{2x^2}{2}\right]_1^0 = -\frac{4}{3}$$

Along SP, $x = 0, dx = 0, y = 1, 0.$

$$\int_{SP} y^2 dy = \int_1^0 y^2 dy = \left[\frac{y^3}{3}\right]_1^0 = -\frac{1}{3}$$

Adding the four line integrals, we have

$$\int_C (Mdx + Ndy) = \frac{1}{3} + \frac{1}{3} + \frac{1}{2} - \frac{1}{3} - 1 - \frac{1}{3} = -\frac{1}{2} \quad (2.19)$$

From (2.18) and (2.19), Green's theorem stands verified.

Example 2.44. *Apply Green's theorem in the plane to* $\int_C[(2x^2 - y^2)dx + (x^2 + y^2)dy]$ *where* C *is the boundary of the surface enclosed by the* x *- axis and the semicircle* $y = (1 - x^2)^{1/2}$.

Solution: Here $M = 2x^2 - y^2; N = x^2 + y^2, \frac{\partial M}{\partial y} = -2y; \frac{\partial N}{\partial x} = 2x$

$\therefore$ By Green's theorem,

$$\int_C[(2x^2 - y^2)dx + (x^2 + y^2)dy] = \int \int_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dxdy$$

$$= \int_{x=-1}^{1} \int_{y=0}^{\sqrt{1-x^2}} (2x + 2y)dydx = 2\int_{-1}^{1} \left[xy + \frac{y^2}{2}\right]_0^{\sqrt{1-x^2}} dx$$

$$\int_C [(2x^2 - y^2)dx + (x^2 + y^2)dy] = 2\int_{-1}^{1} \left[x\sqrt{1-x^2} + \frac{1}{2}(1-x^2)\right] dx \quad (2.20)$$

To evaluate the integral (I) we note that integrand is an odd function

$$\Rightarrow \int_{-1}^{1} x\sqrt{1-x^2} = 0$$

Given integral $I = 2\int_0^1 (1 - x^2)dx$ as the II integrand is even.

$$= 2\left[x - \frac{x^3}{3}\right]_0^1 = 2\left(1 - \frac{1}{3}\right) = \frac{4}{3}.$$

Hence equation (2.20) becomes

$$\int_C[(2x^2 - y^2)dx + (x^2 + y^2)dy] = 0 + \frac{4}{3} = \frac{4}{3}.$$

Example 2.45. *Using Green's theorem evaluate* $\int_C[x^2ydx + x^2dy]$ *where* C *is the boundary described counter clockwise of the triangle with vertices* $(0,0), (1,0), (1,1)$.

Solution:

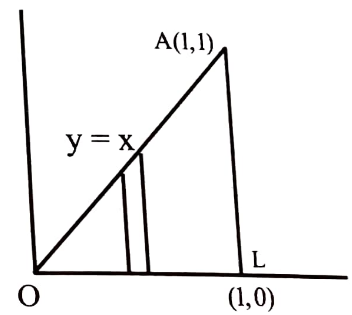

Fig.10

By Green's theorem,

$$\oint_C (Mdx + Ndy) = \int \int_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dxdy$$

$$\int_C[x^2ydx + x^2dy] = \int \int_R (2x - x^2)dxdy$$

$$= \int_0^1 (2x - x^2)dx \int_0^x dy = \int_0^1 (2x - x^2)[y]dx$$

$= \int_0^1 (2x^2 - x^3)dx = \left[2\frac{x^3}{3} - \frac{x^4}{4}\right]_0^1 = \frac{2}{3} - \frac{1}{4} = \frac{5}{12}.$

Example 2.46. *Show that the area bounded by the simple closed curve C is given by* $\frac{1}{2}\oint_C (xdy - ydx)$. *Hence find the area of the ellipse* $x = a\cos\theta, y = b\sin\theta$.

Solution: By Green's theorem in plane, if R is a plane region bounded by a simple closed curve C, then $\oint_C (Mdx + Ndy) = \int\int_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dxdy$ putting $M = -y, N = x$ we get

$\oint_C (xdy - ydx) = \int\int_R \left(\frac{\partial}{\partial x}(x) - \frac{\partial}{\partial y}(-y)\right) dxdy$

$= 2\int\int_R dxdy = 2A$ where A is the area bounded by C.

Hence $A = \frac{1}{2}\oint_C (xdy - ydx)$

The area of the ellipse $= \frac{1}{2}\oint_C (xdy - ydx)$

$= \frac{1}{2}\int_{\theta=0}^{2\pi} \left[a\cos\theta\frac{dy}{d\theta} - b\sin\theta\frac{dx}{d\theta}\right] d\theta$

$= \frac{1}{2}\int_{\theta=0}^{2\pi} (ab\cos^2\theta + ab\sin^2\theta)d\theta$

$= \frac{1}{2}ab\int_0^{2\pi} d\theta = \pi ab$ Sq.units.

Example 2.47. *Verify Stoke's theorem for* $\vec{F} = (x^2 + y^2)\vec{i} - 2xy\vec{j}$ *taken round the rectangle bounded by* $x = \pm a, y = 0, y = b$

Solution: We have $curl\,\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 + y^2 & -2xy & 0 \end{vmatrix}$

$= (-2y - 2y)\,\vec{k} = -4y\,\vec{k}$

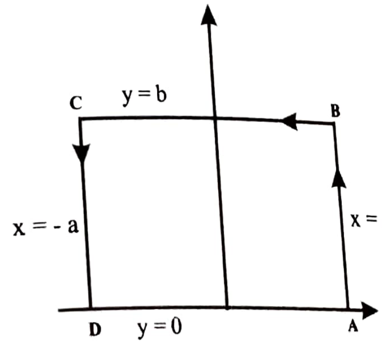

Fig.11

Also $\hat{n} = \vec{k}$

$\therefore \int\int_S (curl\,\vec{F}).\hat{n}ds = \int_{y=0}^{b}\int_{x=-a}^{a} (-4y\,\vec{k})\,\vec{k}\,dxdy$

$= -4\int_{y=0}^{b}\int_{x=-a}^{a} ydxdy = -4\int_0^b [xy]_{-a}^{a}$

$= -4\int_0^b 2aydy = -4[ay^2]_0^b = -4ab^2$

Also $\oint_C \vec{F}.d\vec{r} = \oint_C [(x^2 + y^2)\vec{i} - 2xy\vec{j}].[dx\vec{i} + dy\vec{j}]$

$= \oint_C [(x^2 + y^2)dx - 2xydy]$

$= \int_{DM}[(x^2+y^2)dx - 2xydy] + \int_{AB}[(x^2+y^2)dx - 2xydy]$

$+ \int_{BE}[(x^2+y^2)dx - 2xydy] + \int_{ED}[(x^2+y^2)dx - 2xydy]$

Along DA, $y=0$ and $dy=0$, Along AB, $x=a$ and $dx=0$

Along BE, $y=b$ and $dy=0$, Along ED, $x=-a$ and $dx=0$

$\therefore \oint_C \vec{F}.d\vec{r} = \int_{x=-a}^{x=a} x^2dx + \int_{y=0}^{y=b}(-2ay)dy$

$+ \int_{x=a}^{x=-a}(x^2+b^2)dx + \int_{y=b}^{y=0} 2aydy$

$= \int_{x=-a}^{x=a} x^2dx - \int_{x=-a}^{x=a}(x^2+b^2)dx - 4a\int_{y=0}^{b} ydy$

$= -\int_{-a}^{a} b^2dx - 4a\int_0^b ydy = -2ab^2 - 4a\left[\frac{y^2}{2}\right]_0^b = -4ab^2$

Thus $\oint_C \vec{F}.d\vec{r} = \int\int_S (curl\vec{F}).\hat{n}ds$

Example 2.48. *Verify Stoke's theorem for $\vec{F} = (2x-y)\vec{i} - yz^2\vec{j} - y^2z\vec{k}$ where S is the upper half of the sphere $x^2+y^2+z^2 = 1$ and C is its boundary in the xy - plane.*

Solution: The boundary C of S is a circle in the xy - plane of radius unity and centre at origin. Suppose $x = \cos t, y = \sin t, z = 0, 0 \le t \le 2\pi$ are parametric equations of C.

$\therefore \oint_C \vec{F}.d\vec{r} = \oint_C[(2x-y)\vec{i} - yz^2\vec{j} - y^2z\vec{k}].[dx\vec{i} + dy\vec{j} + dz\vec{k}]$

$= \oint_C[(2x-y)dx - yz^2dy - y^2zdz]$

$= \oint_C(2x-y)dx$, since $z=0$ and $dz=0$

$= \int_0^{2\pi}(2\cos t - \sin t)\frac{dx}{dt}dt = -\int_0^{2\pi}(2\cos t - \sin t)\sin t dt$

$= -\int_0^{2\pi}\left(\sin 2t - \frac{1}{2}(1-\cos 2t)\right)dt$

$= -\left[-\frac{\cos 2t}{2} - \frac{1}{2}t + \frac{1}{2}\frac{\sin 2t}{2}\right]_0^{2\pi}$

$$\oint_C \vec{F}.d\vec{r} = -\left[\left(-\frac{1}{2}+\frac{1}{2}\right) - \frac{1}{2}(2\pi - 0) + \frac{1}{4}(0-0)\right] = \pi \quad (2.21)$$

$$\text{Also } \nabla\times\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2x-y & -yz^2 & -y^2z \end{vmatrix}$$

$= (-2yz+2yz)\vec{i} - (0-0)\vec{j} + (0+1)\vec{k} = \vec{k}$

Also $\hat{n} = \vec{k}$, $ds = dxdy$

$\therefore \int\int_S(\nabla\times\vec{F}).\hat{n}ds = \int\int \vec{k}\,\vec{k}\,dxdy =$ Area of circle

$$\int\int_S(\nabla\times\vec{F}).\hat{n}ds = \pi(1)^2 = \pi \quad (2.22)$$

Hence from (2.21) and (2.22), the theorem is verified.

Example 2.49. *Verify Stoke's theorem for $\vec{F} = (y-z+2)\vec{i} - (yz+4)\vec{j} - (xz)\vec{k}$ over the surface of a cube $x=0, y=0, z=0, x=2, y=2, z=2$ above the XOY plane.*

Solution: Consider the surface of the cube as shown in the figure. Boundary path OABCO shown by arrows.

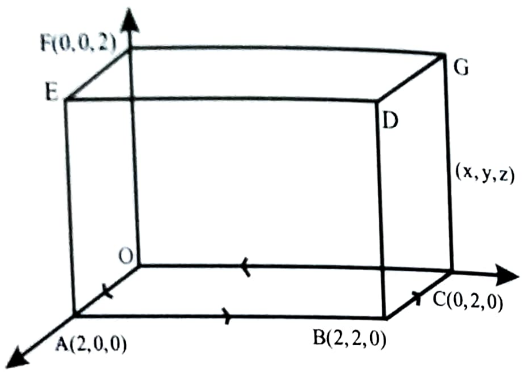

Fig.12

$\int_C \vec{F}.d\vec{r} = \int_C[(y-z+2)\vec{i} - (yz+4)\vec{j} - (xz)\vec{k}].[dx\,\vec{i} + dy\,\vec{j} + dz\,\vec{k}]$

$= \int_C (y-z+2)dx + (yz+4)dy - xzdz$

$$\int_C \vec{F}.d\vec{r} = \int_{OA} \vec{F}.d\vec{r} + \int_{AB} \vec{F}.d\vec{r} + \int_{BC} \vec{F}.d\vec{r} + \int_{CO} \vec{F}.d\vec{r} \qquad (2.23)$$

Along OA, $y = 0, dy = 0, z = 0, dz = 0$

$\int_{OA} \vec{F}.d\vec{r} = \int_0^2 2dx = [2x]_0^2 = 4$

Along AB, $x = 2, dx = 0, z = 0, dz = 0$

$\int_{AB} \vec{F}.d\vec{r} = \int_0^2 4dy = 4[y]_0^2 = 8$

Along BC, $y = 2, dy = 0, z = 0, dz = 0$

$\int_{BC} \vec{F}.d\vec{r} = \int_2^0 (2-0+2)dx = [4x]_2^0 = -8$

Along CO, $x = 0, dx = 0, z = 0, dz = 0$

$\int_{CO} \vec{F}.d\vec{r} = 4\int_2^0 dy = 4[y]_2^0 = -8$

On putting the values of these integrals in (2.23), we get

$\int_C \vec{F}.d\vec{r} = 4 + 8 - 8 - 8 = -4$

To obtain the surface integral

$$\nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2y - z + 2 & yz + 4 & -xz \end{vmatrix}$$

$= (0 - y)\vec{i} - (-z + 1)\vec{j} + (0 - 1)\vec{k} = -y\vec{i} + (z - 1)\vec{j} - \vec{k}$

Here we have to integrate over five surfaces, ABDE, OCGF, BCGD, OAEF, DEFG.

Over the surface ADBE (x=2), $\hat{n} = \vec{i}, ds = dydz$

$\therefore \int\int_S (\nabla \times \vec{F}).\hat{n}ds = \int\int[-y\vec{i} + (z-1)\vec{j} - \vec{k}].\vec{i}\,dydz$

$= \int\int(-y)dydz = -\int_0^2 ydy \int_0^2 dz = -\left[\frac{y^2}{2}\right]_0^2 [z]_0^2 = -4$

Over the surface OCGF (x=0), $\hat{n} = -\vec{i}, ds = dydz$

$$\therefore \int\int_S (\nabla \times \vec{F}).\hat{n}ds = \int\int [-y\vec{i} + (z-1)\vec{j} - \vec{k}].(-\vec{i})dydz$$

$$= \int\int ydydz = \int_0^2 ydy \int_0^2 dz = 2\left[\frac{y^2}{2}\right]_0^2 = 4$$

Over the surface BCGD (y=2), $\hat{n} = \vec{j}$, $ds = dxdz$

$$\therefore \int\int_S (\nabla \times \vec{F}).\hat{n}ds = \int\int [-y\vec{i} + (z-1)\vec{j} - \vec{k}].\vec{j}\,dxdz$$

$$= \int\int (z-1)dxdz = \int_0^2 dx \int_0^2 (z-1)dz = [x]_0^2\left[\frac{z^2}{2} - z\right]_0^2 = 0$$

Over the surface OAEF (y=0), $\hat{n} = -\vec{j}$, $ds = dxdz$

$$\therefore \int\int_S (\nabla \times \vec{F}).\hat{n}ds = \int\int [-y\vec{i} + (z-1)\vec{j} - \vec{k}].(-\vec{j})dxdz$$

$$= -\int\int (z-1)dxdz = -\int_0^2 dx \int_0^2 (z-1)dz$$

$$= -[x]_0^2\left[\frac{z^2}{2} - z\right]_0^2 = 0$$

Over the surface DEFG (z=2), $\hat{n} = \vec{k}$, $ds = dxdy$

$$\therefore \int\int_S (\nabla \times \vec{F}).\hat{n}ds = \int\int [-y\vec{i} + (z-1)\vec{j} - \vec{k}].\vec{k}\,dxdz$$

$$= -\int\int dxdy = -\int_0^2 dx \int_0^2 dy = -[x]_0^2[y]_0^2 = -4$$

Total surface integral $= -4 + 4 + 0 + 0 - 4 = -4$

Thus $\int\int_S curl\,\vec{F}.\hat{n}ds = \int_C \vec{F}.d\vec{r} = -4$ which verifies Stoke's

theorem.

Example 2.50. *Evaluate $\oint_C (xydx + xy^2dy)$ by Stoke's theorem where C is the square in the xy - plane with vertices $(1,0)$, $(-1,0)$, $(0,1)$, $(0,-1)$.*

Solution: Here $\vec{F} = xy\vec{i} + xy^2\vec{j}$

$$curl\,\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xy & xy^2 & 0 \end{vmatrix} = (y^2 - x)\vec{k}$$

Also $\hat{n} = \vec{k}$

$$curl\,\vec{F}.\hat{n} = (y^2 - x)\vec{k}.\vec{k} = y^2 - x$$

The given integral $= \int\int_S (y^2 - x)ds = \int_{y=-1}^{1}\int_{x=-1}^{1}(y^2 - x)dxdy$

$$= \int_{y=-1}^{1}\left[y^2x - \frac{x^2}{2}\right]_{x=-1}^{1} dy = \int_{-1}^{1} 2y^2dy = 2\left[\frac{y^3}{3}\right]_{-1}^{1} = \frac{4}{3}$$

Example 2.51. *Use Stoke's theorem to evaluate $\int\int(\nabla\times\vec{F}).\hat{n}ds$, where $\vec{F} = y\vec{i} + (x - 2xz)\vec{j} - xy\vec{k}$ and S is the surface of the sphere $x^2 + y^2 + z^2 = a^2$ above the xy plane.*

Solution: The boundary C of the surface S is circle $x^2 + y^2 = a^2$, $z = 0$. Suppose $x = a\cos t$, $y = a\sin t$ and $z = 0$, $0 \le t \le 2\pi$ are parametric equations of C. By Stoke's theorem, we have

$$\int\int(\nabla \times \vec{F}).\hat{n}ds = \int_C \vec{F}.d\vec{r}$$

$= \int_C [y\vec{i} + (x-2xz)\vec{j} - xy\vec{k}].[dx\vec{i} + dy\vec{j} + dz\vec{k}]$

$= \int_C [ydx + (x-2xz)dy - xydz]$

$= \int_C (ydx + xdy) \quad (\because$ on C, $z = 0$ and $dz = 0)$

$= \int_0^{2\pi} \left(y\frac{dx}{dt} + x\frac{dy}{dt} \right) dt$

$= \int_0^{2\pi} [a\sin t(-a\sin t) + a\cos t(a\cos t)]dt$

$= a^2 \int_0^{2\pi} (\cos^2 t - \sin^2 t)dt = a^2 \int_0^{2\pi} \cos 2t dt$

$= a^2 \left[\frac{\sin 2t}{2} \right]_0^{2\pi} = 0$

Example 2.52. *Evaluate* $\int_C \vec{F}.d\vec{r}$ *by Stoke's theorem where* $\vec{F} = y^2\vec{i} + x^2\vec{j} - (x+z)\vec{k}$ *and* C *is the boundary of the triangle with vertices at* $(0,0,0)$, $(1,0,0)$, $(1,1,0)$.

Solution: We have $curl\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2 & x^2 & -(x+z) \end{vmatrix} = 0\vec{i} + \vec{j} + 2(x-y)\vec{k}$

Also we note that z - coordinate of each vertex of the triangle is zero. Therefore the triangle lies in the xy - plane. So $\hat{n} = \vec{k}$.

$\therefore curl\vec{F}.\hat{n} = [\vec{j} + 2(x-y)\vec{k}].\vec{k} = 2(x-y)$

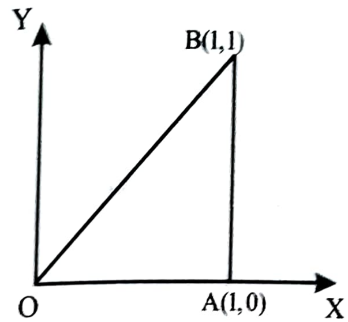

Fig.13

The equation of the line OB is $y = x$. By Stoke's theorem

$\oint_C \vec{F}.d\vec{r} = \int\int_S (curl\vec{F}.\hat{n})ds = \int_{x=0}^{1}\int_{y=0}^{x} 2(x-y)dxdy$

$= 2\int_{x=0}^{1} \left[xy - \frac{y^2}{2} \right]_{y=0}^{x} dx = 2\int_0^1 \left[x^2 - \frac{x^2}{2} \right] dx$

$= 2\int_0^1 \frac{x^2}{2} dx = \frac{1}{3}.$

Note: Cartesian Equivalent of Divergence Theorem

$\int\int\int_V \left(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} \right) dxdydz = \int\int_S (F_1 dydz + F_2 dzdx + F_3 dxdy)$

Example 2.53. *By transforming a triple integral evaluate* $I = \int\int_S (x^3 dydz + x^2 ydzdx + x^2 zdxdy)$ *where* S *is the closed surface bounded by the planes* $z = 0, z = b$ *and the cylinder* $x^2 + y^2 = a^2$.

Solution: By Divergence theorem, the required surface integral I is equal to the volume integral

$$\int\int\int_V \left(\frac{\partial}{\partial x}(x^3) + \frac{\partial}{\partial y}(x^2 y) + \frac{\partial}{\partial z}(x^2 z)\right) dv$$

$$= \int_{z=0}^{b} \int_{y=-a}^{a} \int_{x=-\sqrt{a^2-y^2}}^{\sqrt{a^2-y^2}} (3x^2 + x^2 + x^2) dxdydz$$

$$= 4 \times 5 \int_0^b \int_0^a \int_0^{\sqrt{a^2-y^2}} x^2 dxdydz = 20 \int_0^b \int_0^a \left[\frac{x^3}{3}\right]_0^{\sqrt{a^2-y^2}} dydz$$

$$= \frac{20}{3} \int_0^b \int_0^a (a^2 - y^2)^{3/2} dydz = \frac{20}{3} \int_0^a \left[(a^2 - y^2)^{3/2} z\right]_0^b dy$$

$$= \frac{20}{3} \int_0^a \left[(a^2 - y^2)^{3/2} b\right] dy$$

Put $y = a \sin t$ so that $dy = a \cos t dt$

$$= \frac{20}{3} b \int_0^{\pi/2} a^3 \cos^3 t (a \cos t) dt$$

$$= \frac{20}{3} a^4 b \int_0^{\pi/2} \cos^4 t dt = \frac{20}{3} a^4 b \frac{3}{4.2} \frac{\pi}{2} = \frac{5}{4} \pi a^4 b$$

Example 2.54. *Apply Gauss divergence theorem to evaluate* $\int\int[(x^3 - yz)dydz - 2x^2 ydzdx + zdxdy]$ *over the surface of a cube bounded by the coordinate planes and the plane* $x = y = z = a$.

Solution: By divergence theorem, we have

$$\int\int\int_V \left(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}\right) dxdydz = \int\int_S (F_1 dydz + F_2 dzdx + F_3 dxdy)$$

Here $F_1 = x^3 - yz, F_2 = -2x^2 y, F_3 = z$

$$\therefore \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} = 3x^2 - 2x^2 + 1 = x^2 + 1$$

The given surface integral is equal to the volume integral

$$\int_{x=0}^{a} \int_{y=0}^{a} \int_{z=0}^{a} (x^2 + 1) dxdydz = \int_{z=0}^{a} \int_{y=0}^{a} \left[\frac{x^3}{3} + x\right]_{x=0}^{a} dydz$$

$$= \int_{z=0}^{a} \int_{y=0}^{a} \left[\frac{a^3}{3} + a\right] dydz = a^2 \left[\frac{a^3}{3} + a\right]$$

Example 2.55. *Apply Gauss divergence theorem to evaluate* $\int\int[(x + z)dydz + (y + z)dzdx + (x + y)dxdy]$ *where S is the surface of the sphere* $x^2 + y^2 + z^2 = 4$.

Solution: By divergence theorem, the given surface integral is equal to the volume integral.

$$\int\int\int_V \left(\frac{\partial}{\partial x}(x + z) + \frac{\partial}{\partial y}(y + z) + \frac{\partial}{\partial z}(x + y)\right) dV = \int\int\int_v 2dV$$

$= 2\int\int\int_v dV = 2V$, where V is the volume of the sphere $x^2 + y^2 + z^2 = 4$ $\left[\because \frac{4}{3}\pi r^3 \text{for sphere of radius} = r\right]$

$$= 2\left[\frac{4}{3}\pi (2)^3\right] = \frac{64}{3}\pi$$

Example 2.56. *If* $\vec{F} = x\vec{i} - y\vec{j} + (z^2 - 1)\vec{k}$, *find the value of*

$\int\int_S \vec{F}.\hat{n}ds$, *where S is the closed surface bounded by the planes* $z = 0, z = 1$ *and the cylinder* $x^2 + y^2 = 4$.

Solution: By divergence theorem, we have

$\int\int_S \vec{F}.\hat{n}ds = \int\int\int_V div\vec{F}dv$

Here $div\vec{F} = \frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(-y) + \frac{\partial}{\partial z}(z^2 - 1) = 1 - 1 + 2z = 2z$

$\therefore \int\int\int_V div\vec{F}dv = \int_{z=0}^{1}\int_{y=-2}^{2}\int_{x=-\sqrt{4-y^2}}^{x=\sqrt{4-y^2}} 2zdxdydz$

$= \int_{z=0}^{1}\int_{y=-2}^{2}[2zx]_{-\sqrt{4-y^2}}^{x=\sqrt{4-y^2}} dydz = \int_{z=0}^{1}\int_{y=-2}^{2} 4z\sqrt{4-y^2}dydz$

$= \int_{y=-2}^{2}\left[4\frac{z^2}{2}\sqrt{4-y^2}\right]_0^1 dy = 4\int_0^2[\sqrt{4-y^2}]dy$

$= 4\left[\frac{y}{2}(\sqrt{4-y^2}) + 2\sin^{-1}\frac{y}{2}\right]_0^2 = 4[2\sin^{-1}(1)] = 4\pi$

Example 2.57. *Evaluate* $\int\int_S(ax^2+by^2+cz^2)dS$ *over the sphere* $x^2 + y^2 + z^2 = 1$ *using the divergence theorem.*

Solution: Let $I = \int\int_S(ax^2+by^2+cz^2)dS$. We put I in the form $\int\int_S \vec{F}.\hat{n}dS$

The normal vector to $\phi(x,y,z) = x^2 + y^2 + z^2 - 1 = 0$ is $\nabla\phi = 2x\vec{i} + 2y\vec{j} + 2z\vec{k}$.

$\hat{n} = \frac{\nabla\phi}{|\nabla\phi|} = \frac{2x\vec{i} + 2y\vec{j} + 2z\vec{k}}{\sqrt{4(x^2+y^2+z^2)}} = x\vec{i} + y\vec{j} + z\vec{k}$ as $x^2 + y^2 + z^2 = 1$ on S.

We have to choose $\vec{F}$ such that $\vec{F}.\hat{n} = ax^2+by^2+cz^2$. Obviously $\vec{F} = ax\vec{i} + by\vec{j} + cz\vec{k}$

$\int\int_S \vec{F}.\hat{n}ds = \int\int\int_V div\vec{F}dv$

$= \int\int\int_V(a+b+c)dV$ as $div\vec{F} = a + b + c$

$= (a+b+c)\int\int\int_V dV = (a+b+c)V$

$= (a+b+c)\frac{4}{3}\pi$, (the volume V enclosed by the sphere S of unit radius is $\frac{4}{3}\pi$)

Example 2.58. *Verify divergence theorem for* $\vec{F} = (x^2-yz)\vec{i} + (y^2-zx)\vec{j} + (z^2-xy)\vec{k}$ *taken over the rectangular parallelopiped* $0 \le x \le a, 0 \le y \le b, 0 \le z \le c$.

Solution: By divergence theorem $\int\int_S \vec{F}.\hat{n}ds = \int\int\int_V div\vec{F}dv$

Here $div\vec{F} = \frac{\partial}{\partial x}(x^2 - yz) + \frac{\partial}{\partial y}(y^2 - zx) + \frac{\partial}{\partial z}(z^2 - xy)$

$= 2(x + y + z)$

$\therefore \int\int\int_V div\vec{F}dv = \int_0^c\int_0^b\int_0^a 2(x+y+z)dxdydz$

$$= 2\int_0^c\int_0^b\left[\frac{x^2}{2}+yx+zx\right]_0^a dydz = 2\int_0^c\int_0^b\left[\frac{a^2}{2}+ya+za\right]dydz$$

$$= 2\int_0^c\left[\frac{a^2}{2}y+a\frac{y^2}{2}+azy\right]_0^b dz = 2\int_0^c\left[\frac{a^2}{2}b+a\frac{b^2}{2}+abz\right]dz$$

$$= 2\left[\frac{a^2}{2}bz+a\frac{b^2}{2}z+ab\frac{z^2}{2}\right]_0^a = a^2bc+ab^2c+abc^2$$

$$\therefore \int\int\int_V div\vec{F}dv = abc(a+b+c) \qquad (2.24)$$

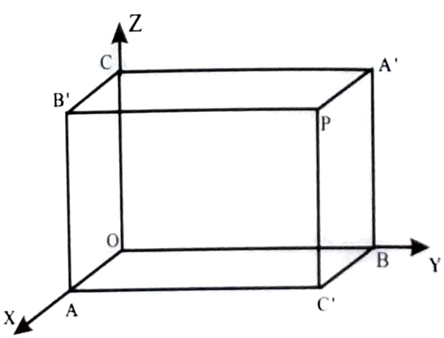

Fig.14

To evaluate the surface integral on S_1 : Face OACB, $z=0, \hat{n} = -\vec{k}$, $\vec{F}.\hat{n} = (x^2\vec{i}+y^2\vec{j}-xy\vec{k}).(-\vec{k}) = xy$

$$\int\int_{S_1}\vec{F}.\hat{n}ds = \int_0^b\int_0^a xydxdy = \frac{a^2b^2}{4}$$

S_2 : Face CB'PA, $z=c, \hat{n}=\vec{k}, \vec{F}.\hat{n} = c^2 - xy$

$$\int\int_{S_2}\vec{F}.\hat{n}ds = \int_0^b\int_0^a(c^2-xy)dxdy = abc^2 - \frac{a^2b^2}{4}$$

S_3 : Face OBA'C, $x=0, \hat{n}=\vec{i}, \vec{F}.\hat{n} = yz$

$$\int\int_{S_3}\vec{F}.\hat{n}ds = \int_0^c\int_0^b yzdydz = \frac{b^2c^2}{4}$$

S_4 : Face AC'PB', $x=a, \hat{n}=\vec{i}, \vec{F}.\hat{n} = a^2 - yz$

$$\int\int_{S_4}\vec{F}.\hat{n}ds = \int_0^c\int_0^b(a^2-yz)dydz = a^2bc - \frac{b^2c^2}{4}$$

S_5 : Face OCB'A, $y=0, \hat{n}=-\vec{j}, \vec{F}.\hat{n} = zx$

$$\int\int_{S_5}\vec{F}.\hat{n}ds = \int_0^a\int_0^c zxdxdz = \frac{a^2c^2}{4}$$

S_6 : Face BA'PC', $y=b, \hat{n}=\vec{j}, \vec{F}.\hat{n} = b^2 - zx$

$$\int\int_{S_6}\vec{F}.\hat{n}ds = \int_0^a\int_0^c(b^2-zx)dzdx = ab^2c - \frac{a^2c^2}{4}$$

Adding the six surface integrals, we get

$$\int\int_S\vec{F}.\hat{n}ds = \frac{a^2b^2}{4} + abc^2 - \frac{a^2b^2}{4} + \frac{b^2c^2}{4} + a^2bc - \frac{b^2c^2}{4} + \frac{a^2c^2}{4} + ab^2c - \frac{a^2c^2}{4}$$

$$\int\int_S\vec{F}.\hat{n}ds = abc(a+b+c) \qquad (2.25)$$

The equality of (2.24) and (2.25), verify divergence theorem.

Example 2.59. *Verify divergence theorem for the function* $\vec{F} = 4xz\,\vec{i} - y^2\vec{j} + yz\,\vec{k}$ *taken over the cube bounded by the planes* $x = 0, x = 1,\ y = 0, y = 1,\ z = 0, z = 1$.

Solution: By divergence theorem, $\int\int_S \vec{F}.\hat{n}ds = \int\int\int_V div\vec{F}dv$

$$div\vec{F} = \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right).(4xz\,\vec{i} - y^2\vec{j} + yz\,\vec{k})$$

$$= \frac{\partial}{\partial x}(4xz) + \frac{\partial}{\partial y}(-y^2) + \frac{\partial}{\partial z}(yz) = 4z - 2y + y = 4z - y$$

$$\therefore \int\int\int_V div\vec{F}dv = \int_0^1\int_0^1\int_0^1(4z - y)dzdydx$$

$$= \int_0^1\int_0^1[2z^2 - yz]_0^1 dydx = \int_0^1\int_0^1(2 - y)dydx$$

$$\therefore \int\int\int_V div\vec{F}dv = \int_0^1\left[2y - \frac{y^2}{2}\right]_0^1 dx = \int_0^1\frac{3}{2}dx = \frac{3}{2} \qquad (2.26)$$

To evaluate $\int\int_S \vec{F}.\hat{n}ds$

Here S is the surface of the cube bounded by the plane surfaces.

Over the face OABC, $z = 0, dz = 0, \hat{n} = -\vec{k}, ds = dxdy$

$$\int\int_S \vec{F}.\hat{n}ds = \int_0^1\int_0^1(-y^2\vec{j}).(-\vec{k})dxdy = 0$$

Over the face BCDE, $y = 1, dy = 0, \hat{n} = \vec{j}, ds = dxdz$

$$\int\int_S \vec{F}.\hat{n}ds = \int_0^1\int_0^1(4xz\,\vec{i} - \vec{j} + z\,\vec{k}).\vec{j}\,dxdz$$

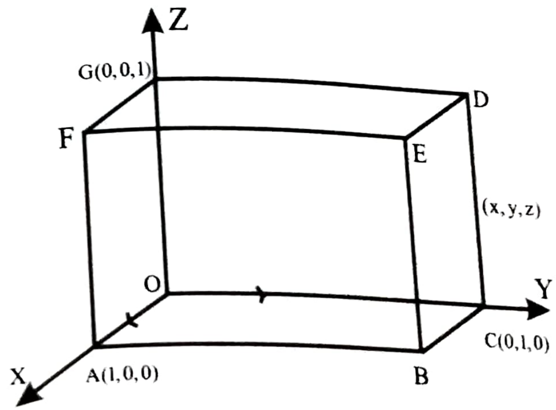

Fig.15

$$= (-1)\int_0^1\int_0^1 dxdz = -(x)_0^1(z)_0^1 = -1$$

Over the face DEFG, $z = 1, dz = 0, \hat{n} = \vec{k}, ds = dxdy$

$$\int\int_S \vec{F}.\hat{n}ds = \int_0^1\int_0^1(4x\,\vec{i} - y^2\vec{j} + y\,\vec{k}).\vec{k}\,dxdz$$

$$= \int_0^1 dx\int_0^1 ydy = (x)_0^1\left(\frac{y^2}{2}\right)_0^1 = \frac{1}{2}$$

Over the face AOGF, $y = 0, dy = 0, \hat{n} = -\vec{j}, ds = dxdz$

$$\int\int_S \vec{F}.\hat{n}ds = \int_0^1\int_0^1(4xz\,\vec{i}).(-\vec{j})dxdz = 0$$

Over the face OCDG, $x = 0, dx = 0, \hat{n} = -\vec{i}, ds = dydz$

$$\int\int_S \vec{F}.\hat{n}ds = \int_0^1\int_0^1(-y^2\vec{j} + yz\,\vec{k}).(-\vec{i})dydz = 0$$

Over the surface ABEF, $x = 1, dx = 0, \hat{n} = \vec{i}, ds = dydz$

$\int\int_S \vec{F}.\hat{n}ds = \int_0^1\int_0^1(4z\vec{i} - y^2\vec{j} + yz\vec{k}).\vec{i}\,dydz$

$= \int_0^1\int_0^1 4zdydz = (y)_0^1(2z^2)_0^1 = 2$

Adding for the whole surface S, we get

$$\int\int_S \vec{F}.\hat{n}ds = 0 - 1 + \frac{1}{2} + 0 + 0 + 2 = \frac{3}{2} \tag{2.27}$$

From the equations (2.26) and (2.27), the divergence theorem stand verified.

EXERCISE

1. Evaluate $\int(xdy - ydx)$, where C is the circle $x^2 + y^2 = a^2$ [Ans: $2\pi a^2$]
2. Evaluate $\int\int_S(xdydz + ydzdx + zdxdy)$ over the surface of the sphere $x^2 + y^2 + z^2 = a^2$. [Ans: $4\pi a^3$]
3. State Gauss divergence theorem.
4. For any closed surface S, prove that $\int\int_S curl\vec{F}.\hat{n}.dS = 0$
5. Evaluate $\int\int_S \vec{r}.\hat{n}.dS$, where S is any closed surface.

   [Ans: 3V]
6. State Green's theorem in plane.
7. If $\vec{F} = \nabla\phi, \nabla^2\phi = -4\pi\rho$, show that

$\int\int_S \vec{F}.\vec{n}.dS = -4\pi\int\int\int_V \rho dV$

8. Use Green's theorem to evaluate $\int_C[(x^2 + xy)dx + (x^2 + y^2)dy]$, where C is the square formed by the lines $y = \pm1, x = \pm1$. [Ans: 0]
9. Find $\int\int_S \vec{F}.\vec{n}.dS$, where $\vec{F} = (2x + 3z)\vec{i} - (xz + y)\vec{j} + (y^2 + 2z)\vec{k}$ and S is the surface of the sphere having centre at $(3, -1, 2)$ and radius 3. [Ans: 108π]
10. The vector field $\vec{F} = 2x^2\vec{i} + z\vec{j} + yz\vec{k}$ is defined over the volume of the cuboid given by $0 \le x \le a, 0 \le y \le b, 0 \le z \le c$ enclosing the surface S, evaluate the surface integral $\int\int_S \vec{F}.\vec{n}.dS$ $\left[\text{Ans: } abc\left(a + \frac{b}{2}\right)\right]$
11. Using Green's theorem, evaluate $\int_C[x^2ydx + x^2dy]$ where C is the boundary described counter clockwise of the triangle with vertices $(0,0), (1,0), (1,1)$. $\left[\text{Ans: } \frac{5}{12}\right]$
12. Prove that $\int_C \vec{r}.d\vec{r} = 0$
13. Evaluate by Stoke's theorem $\oint_C(e^x dx + 2ydy - dz)$, where C is the curve $x^2 + y^2 = 4, z = 2$ [Ans: 0]
14. Prove that $\int\int\int_V \frac{dV}{r^2} = \int\int_S \frac{\vec{r}.\hat{n}}{r^2}dS$.
15. Find $\int\int_S \vec{r}.d\vec{S}$, where S is the surface of the tetrahedron whose vertices are $(0,0,0), (1,0,0), (0,1,0), (0,0,1)$.

[Ans: 3]

16. Verify Green's theorem for $\oint_C (e^{-x} \sin y dx + e^{-x} \cos y dy)$, where C is the rectangle with vertices $(0,0)$, $(\pi,0)$, $\left(\pi, \frac{\pi}{2}\right)$, $\left(0, \frac{\pi}{2}\right)$. [Ans: Common Value $= 2(e^{-\pi} - 1)$]

17. Verify Green's theorem for the integral $\int [(x^2 + y)dx - xy^2 dy]$ taken around the boundary of the square whose vertices $(0,0), (1,0), (1,1)$ and $(0,1)$

$\left[\text{Ans: Common Value} = -\frac{4}{3}\right]$

18. Evaluate $\int_C [(2xy - x^2)dx + (x + y^2)dy]$, where C is the closed curve of the region bounded $y = x^2$ and $y^2 = x$ $\left[\text{Ans:} \frac{1}{30}\right]$

19. Find the area of the four cusped hyper cycloid $x^{2/3} + y^{2/3} = a^{2/3}$ using Green's theorem.

$\left[\text{Ans:} \frac{3\pi a^2}{8} \text{ sq.units}\right]$

20. Verify Green's theorem in the plane for $\int_C [(3x^2 - 8y^2)dx + (4y - 6xy)dy]$, where C is the boundary of the region defined on $y = \sqrt{x}, y = x^2$ $\left[\text{Ans: Common Value} = \frac{3}{2}\right]$

21. Verify Stoke's theorem for a vector field defined by $\vec{F} = (x^2 - y^2)\vec{i} + 2xy\vec{j}$ in the rectangular region in the XOY plane bounded by the lines $x = 0, x = a, y = 0$ and $y = b$. [Ans: Common Value $= 2ab^2$]

22. Verify Stoke's theorem for $\vec{F} = -y^3\vec{i} + x^3\vec{j}$, where S is the circular disc $x^2 + y^2 \leq 1, z = 0$

$\left[\text{Ans: Common Value} = \frac{3\pi}{2}\right]$

23. Apply Stoke's theorem to evaluate $\int_C [(x + y)dx + (2x - z)dy + (y + z)dz]$, where C is the boundary of the triangle with vertices $(2,0,0)$, $(0,3,0)$ and $(0,0,6)$. [Ans: 21]

24. Evaluate $\int_C \vec{F}.d\vec{r}$, $\vec{F} = y\vec{i} + xz^3\vec{j} - zy^3\vec{k}$, C is the circle $x^2 + y^2 = 4, z = 1.5$ $\left[Ans: \frac{19\pi}{2}\right]$

25. Evaluate $\int\int_S (a^2x^2 + b^2y^2 + c^2z^2)^{-1/2} dS$, where S is the surface of the ellipsoid $ax^2 + by^2 + cz^2 = 1$. $\left[Ans: \frac{4\pi}{\sqrt{abc}}\right]$

26. Verify divergence theorem for $\vec{F} = 4x\vec{i} - 2y^2\vec{j} + z^2\vec{k}$ taken over the region bounded by $x^2 + y^2 = 4$, $z = 0$ and $z = 3$ [Ans: Common Value $= 84\pi$]

27. By converting the surface integral into a volume integral evaluate $\int\int_S (x^3 dydz + y^3 dzdx + z^3 dxdy)$, where S is the surface of the sphere $x^2 + y^2 + z^2 = 1$. $\left[\frac{12\pi}{5}\right]$

28. Evaluate $\int\int_S \vec{F}.\hat{n}dS$ over the region entire surface of the

region above xy - plane bounded by the curve $z^2 = x^2 + y^2$ and the plane $z = 4$, if $\vec{F} = 4xz\,\vec{i} + xyz^2\,\vec{j} + 3z\,\vec{k}$.

[Ans: 320π]