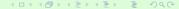
Unit-5 Sequences and Series

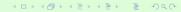
Dr. V. Visalakshi
Assistant Professor
Department of Mathematics
SRM Institute of Science and Technology, KTR
E-mail:visalakv@srmist.edu.in

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Definition:

A set of numbers $a_1, a_2, ... a_n, ...$ such that to each positive integer n, there corresponds a number a_n of the set , is called a sequence and it is denoted by $\{a_n\}$. In otherwords, a sequence of real numbers is a function s from the set of natural numbers N into the set of real numbers R.

Examples:

- **1** If $a_n = \frac{1}{n}$, then the sequence is $1, \frac{1}{2}, \frac{1}{3}, \dots, \frac{1}{n}$...
- ② If $a_n = (-1)^n$, then the sequence is -1, 1, -1, ...
- **3** If $a_n = k$, then the sequence is k, k, ...

Note:

- 1. A finite sequence has a finite number of terms.
- 2. A sequence which is not finite, is an infinite sequence.
- 3. Example 3 is a constant sequence.



Operations on Sequences:

If $\{s_n\}$ and $\{t_n\}$ are sequences then

- **1** Sum sequence is $\{s_n + t_n\} = \{s_n\} + \{t_n\}$.
- 2 Product sequence is $\{s_n t_n\} = \{s_n\}.\{t_n\}.$
- **3** If $c \in R$, then $c\{s_n\} = \{cs_n\}$.
- $\{\frac{s_n}{t_n}\}$ is defined as the quotient of sequence $\{s_n\}$ and $\{t_n\}$, $t_n \neq 0$.

Bounded Sequence

A sequence $\{s_n\}$ is said to be bounded if there exist numbers m, M such that $m < a_n < M$ for all $n \in N$. Otherwise it is said to be unbounded.

Example:

- 1. $\{\frac{1}{n}\}$ which is bounded by 1.
- 2. $\{2^n\}$ which is unbounded.



Monotonic Sequence:

A sequence $\{s_n\}$ is said to be

- (i) Monotonically increasing if $s_{n+1} \ge s_n$ for every n, $s_1 \le s_2 \le s_3.... \le s_n \le s_{n+1} \le$
- (ii) Monotonically decreasing if $s_{n+1} \le s_n$ for every n, $s_1 \ge s_2 \ge s_3.... \ge s_n \ge s_{n+1} \ge$
- (iii) Monotonic if it is either monotonically increasing or monotonically decreasing.

Example:

- 1,4,7,10,.... is a monotonic sequence.
- 2 $1, \frac{1}{2}, \frac{1}{3}, \dots$ is monotonic sequence.
- $\mathbf{3}$ 1,-1,1,-1,.... is not a monotonic sequence.

Limit of a sequence:

Let $\{s_n\}$ be a sequence. I is said to be limit of the sequence $\{s_n\}$, if to each $\varepsilon > 0$ there exists $m \in z^+$ such that $|s_n - I| < \varepsilon$, $\forall n \ge m$. That is $\lim_{n \to \infty} s_n = I$.

Convergent Sequence:

A sequence $\{s_n\}$ is said to be convergent if it has a finite limit.

That is $\lim_{n\to\infty} s_n = I$.

Divergent Sequence:

If $\lim_{n\to\infty} s_n = \infty$, $\{s_n\}$ is divergent.

Oscillatory Sequence:

If $\lim_{n\to\infty}$ is not unique (oscillates finitely) or $\pm\infty$ (oscillates infinitely) then $\{s_n\}$ is oscillatory sequence.

Examples:

- $\{\frac{1}{n^2}\}$ is a convergent sequence.
- 2 {n} is a divergent sequence.
- $(-1)^n$ oscillates finitely.
- $\{(-1)^n n^2\}$ oscillates infinitely.



Problems:

Which of the following sequence are convergent?

(i)
$$\{\frac{n}{n^2+1}\}$$
 (ii) $\{(-1)^{n+1}\}$ (iii) $\{\frac{n}{n+1}\}$ (iv) $\{1+\frac{(-1)^n}{n}\}$ (v) $\{(\frac{1}{2})^n\}$ Solution: (i)

$$\lim_{n\to\infty} s_n = \lim_{n\to\infty} \frac{n}{n^2 + 1}$$

$$= \lim_{n\to\infty} \frac{n}{n(n + \frac{1}{n})}$$

$$= \lim_{n\to\infty} \frac{1}{(n + \frac{1}{n})}$$

$$= 0$$

Hence the sequence is convergent.

(ii)
$$\{(-1)^{n+1}\}=1,-1,1,-1,...$$
 is an oscillating sequence.



(iii)

$$\lim_{n\to\infty} s_n = \lim_{n\to\infty} \frac{n}{n+1}$$

$$= \lim_{n\to\infty} \frac{n}{n\left(1+\frac{1}{n}\right)}$$

$$= \lim_{n\to\infty} \frac{1}{\left(1+\frac{1}{n}\right)}$$

$$= 1$$

Hence the sequence is convergent.

(iv)

$$\lim_{n\to\infty} s_n = \lim_{n\to\infty} 1 + \frac{(-1)^n}{n}$$
$$= 1$$

Hence the sequence is convergent.

(v)The sequence $\{(\frac{1}{2})^n\}$ is convergent.

Definition:

 $u_1, u_2,, u_n...$ is an infinite sequence. The expression $u_1 + u_2 + + u_n + ...$ is called the series. It is denoted by $\sum_{n=1}^{\infty} u_n$.

Note:

- 1. If the number of terms are finite in a series then the series is called a finite series.
- 2. If the number of terms are infinite in a series then the series is called an infinite series.

Definition:

The sum of a finite number of terms (the first n-terms) of a series is called the n^{th} partial sum of the series. $S_n = u_1 + u_2 + + u_n = \sum_{n=1}^{\infty} u_n$.

- 1. If $\lim_{n\to\infty} S_n = S(\text{finite})$, then the series $\sum_{n=1}^{\infty} u_n$ converges.
- 2. If $\lim_{n\to\infty} S_n = \pm \infty$, then the series $\sum_{n=1}^{\infty} u_n$ diverges.
- 3. If $\lim_{n\to\infty} S_n$ is more than one limit (or) $\pm \infty$, then $\sum_{n=1}^{\infty} u_n$ is oscillatory (or) non converges.



Problems:

1. Examine the nature of the series $1+3+5+7+....\infty$

Solution:

The n^{th} partial sum is $S_n = 1 + 3 + 5 + 7 + ... + n$. It is an arithmetic series with a = 1, d = 2, $S_n = \frac{n}{2}[2a + (n-1)d] = n^2$. It follows that $\lim_{n\to\infty} S_n = \infty$. Hence the series is divergent.

- 2. Show that the series $1 + r + r^2 + ... \infty$ (i) Converges if |r| < 1
- (ii) Diverges if $r \ge 1$ and (iii) Oscillatory if r < -1

Solution:

- (i) If |r| < 1, the n^{th} partial sum is $S_n = 1 + r + r^2 + \cdots + r^{n-1} = 1$ $\frac{1(1-r^n)}{1-r}. \ lim_{n\to\infty}r^n=0, \ \text{if} \ |r|<1. \ \text{Thus the series is convergent}.$ (ii) If r>1, $lim_{n\to\infty}r^n=\infty$. If r=1, then $S_n=1+1+\cdots+1=n$.
- Hence $\lim_{n\to\infty} r^n = \infty$. Hence the series is divergent if $r \ge 1$.
- (iii) If r < -1, then $\lim_{n \to \infty} S_n = \begin{cases} \infty, & \text{if n is odd} \\ -\infty, & \text{if n is even} \end{cases}$

If
$$r = -1$$
, then $S_n = 1 - 1 + 1 - 1 + \dots + 1$. $S_n = \begin{cases} 1, & \text{if n is odd} \\ -1, & \text{if n is even} \end{cases}$

Hence the series $1 + r + r^2 + ... \infty$ is oscillatory if $r \le -1$

3. Examine the converges of the following series.

(i)
$$5-4-1+5-4-1+\cdots \infty$$
 (ii) $1+\frac{5}{4}+\frac{6}{4}+\cdots \infty$ (iii) $\sum_{n=1}^{\infty} \frac{1}{n(n+2)}$

(iv)
$$1 + \frac{1}{2} + \frac{1}{2^2} \cdots \infty$$
 (v) $1 + \frac{4}{3} + \left(\frac{4}{3}\right)^2 + \cdots \infty$

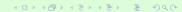
Solution:

- (i) n^{th} partial sum is S_n =5 or 1 or 0. Hence the series is oscillatory.
- (ii) The series is arithmetic series a=1, $d=\frac{1}{4}$, $S_n=\frac{n}{2}[2a+(n-1)d]=\frac{n}{8}[7+n]$. Therefore $\lim_{n\to\infty}S_n=\infty$. Hence the series is divergent.
- (iii) $u_n = \frac{1}{n(n+2)}$. By using partial fractions $\frac{1}{n(n+2)} = \frac{A}{n} + \frac{B}{n+2}$. Which implies 1 = A(n+2) + Bn. When n = 0, $A = \frac{1}{2}$ and when

$$n = -2$$
, $B = \frac{-1}{2}$.

Therefore $u_n=\frac{1}{n(n+2)}=\frac{1}{2n}-\frac{1}{2(n+2)}$. The n^{th} partial sum is $S_n=u_1+u_2+\cdots+u_n=\frac{1}{2}-\frac{1}{2(n+2)}$. $\lim_{n\to\infty}S_n=\frac{1}{2}$. Hence the given series is convergent.

- (iv) The n^{th} partial sum is $S_n = 1 + \frac{1}{2} + \left(\frac{1}{2}\right)^2 + \dots + \left(\frac{1}{2}\right)^{n-1}$.
- $S_n = \frac{1-\frac{1}{2^n}}{\frac{1}{2}} = 2\left(1-\frac{1}{2^n}\right)$. $\lim_{n\to\infty} S_n = 2$. Hence the given series is convergent.
- (v) The n^{th} partial sum is $S_n = 1 + \frac{4}{3} + \left(\frac{4}{3}\right)^2 + \dots + \left(\frac{4}{3}\right)^{n-1}$.
- $S_n = \frac{\left(\frac{4}{3}\right)^n 1}{\frac{1}{3}} = 3\left(\left(\frac{4}{3}\right)^n 1\right)$. $\lim_{n \to \infty} S_n = \infty$. Hence the given series is divergent.



Series of Positive terms

Properties of Series:

- 1. Convergence of a series remains unchanged by the replacement, inclusion or omission of a finite number of terms.
- 2. A series remains convergent, divergent or oscillatory when each term of it is multiplied by a fixed number other than zero.
- 3. A series of positive terms either converges or diverges to $+\infty$. That is omitting the negative terms the sum of first n terms tends to either a finite limit or $+\infty$
- 4. Every finite series is a convergent series.

Definition:

If all terms after few positive terms in an infinite series are positive, such a series is a positive term series.

Example: -10-6-1+5+12+20+···.

Series of positive terms

Necessary Condition for Convergence:

If a positive term series $\sum_{n=1}^{\infty} u_n$ is convergent, then $\lim_{n\to\infty} u_n = 0$.

Note:

Converse of the above theorem is not true.

Example:

The series $1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \cdots + \frac{1}{\sqrt{n}} + \cdots$ is divergent eventhough $\lim_{n \to \infty} u_n = \lim_{n \to \infty} \frac{1}{\sqrt{n}} = 0$.

Test for Divergence:

If $\lim_{n\to\infty} u_n \neq 0$, the series $\sum_{n=1}^{\infty} u_n$ must be divergent.

Comparison Test for Convergence:

If two positive term series $\sum_{n=1}^{\infty} u_n$ and $\sum_{n=1}^{\infty} v_n$ be such that $\sum_{n=1}^{\infty} v_n$ converges and $u_n \leq v_n$ for all values of n, then $\sum_{n=1}^{\infty} u_n$ also converges.



Series of positive terms

Comparison Test for divergence:

If two positive term series $\sum_{n=1}^{\infty} u_n$ and $\sum_{n=1}^{\infty} v_n$ be such that $\sum_{n=1}^{\infty} v_n$ diverges and $u_n \geq v_n$ for all values of n, then $\sum_{n=1}^{\infty} u_n$ also diverges.

Limit Comparision test:

If two positive term series $\sum_{n=1}^{\infty} u_n$ and $\sum_{n=1}^{\infty} v_n$ be such that $\lim_{n\to\infty} \frac{u_n}{v_n} =$ non zero finite value, then $\sum_{n=1}^{\infty} u_n$ and $\sum_{n=1}^{\infty} v_n$ converges or diverges together.

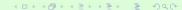
Auxiliary Series:

(a) p-series:

The harmonic series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ is convergent for p > 1 and divergent for $p \le 1$.

(b) Geometric series:

The geometric series $\sum_{n=1}^{\infty} r^{n-1}$ is convergent if r > 1 and divergent if $r \ge 1$.



1. Examine the series $\frac{1}{1.3.5} + \frac{2}{3.5.7} + \frac{3}{5.7.9} + \cdots$ for convergence.

Solution:

To find u_n :

1,2,3,... is an arithmetic series where
$$a = 1, d = 1$$
,

$$t_n = a + (n-1)d = n$$

1,3,5,.... is an arithmetic series where
$$a = 1, d = 2$$
,

$$t_n = a + (n-1)d = 2n-1$$

3,5,7,.... is an arithmetic series where
$$a = 3, d = 2$$
,

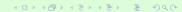
$$t_n = a + (n-1)d = 2n+1$$

5,7,9,... is an arithmetic series where
$$a = 5$$
, $d = 2$,

$$t_n = a + (n-1)d = 2n+3$$

Therefore
$$u_n = \frac{n}{(2n-1)(2n+1)(2n+3)} = \frac{1}{n^2(2-\frac{1}{n})(2+\frac{1}{n})(2+\frac{3}{n})}$$
.

Choose
$$v_n = \frac{1}{n^2}$$
.



$$\lim_{n\to\infty} \frac{u_n}{v_n} = \lim_{n\to\infty} \frac{1}{(2-\frac{1}{n})(2+\frac{1}{n})(2+\frac{3}{n})} = \frac{1}{8} \neq 0$$
. By comparison test, $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is convergent, implies that $\sum_{n=1}^{\infty} \frac{n}{(2n-1)(2n+1)(2n+3)}$ is convergent.

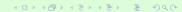
2. Discuss the convergence or divergence of the series

$$\frac{2}{1^p} + \frac{3}{2^p} + \frac{4}{3^p} + \cdots \infty$$
.

Solution:

$$u_n = \frac{(n+1)}{n^p}$$
. Choose $v_n = \frac{1}{n^{p-1}}$. We have

 $\lim_{n\to\infty}\frac{u_n}{v_n}=\lim_{n\to\infty}(1+\frac{1}{n})=1\neq 0.$ The series $\sum_{n=1}^{\infty}\frac{1}{n^{p-1}}$ is convergent if p>2 and it is divergent if $p\leq 2$. Hence the given series $\sum_{n=1}^{\infty}\frac{(n+1)}{n^p}$ is convergent if p>2 and it is divergent if $p\leq 2$.



3. Determine whether the following series is convergent or di-

vergent
$$\frac{\sqrt{2}-1}{3^3-1} + \frac{\sqrt{3}-1}{4^3-1} + \frac{\sqrt{4}-1}{5^3-1} + \cdots$$

Solution:

$$u_n = \frac{\sqrt{n+1}-1}{(n+1)^3-1} = \frac{\sqrt{1+\frac{1}{n}-\frac{1}{\sqrt{n}}}}{n^{\frac{5}{2}}\left((1+\frac{2}{n})^3-\frac{1}{n^3}\right)}$$
. Choose $v_n = \frac{1}{n^{\frac{5}{2}}}$. $\lim_{n\to\infty} \frac{u_n}{v_n} = \frac{1}{n^{\frac{5}{2}}}$.

$$\lim_{n\to\infty} = \frac{\sqrt{1+\frac{1}{n}-\frac{1}{\sqrt{n}}}}{\left((1+\frac{2}{n})^3-\frac{1}{n^3}\right)} = 1 \neq 0.$$
 Since $\sum_{n=1}^{\infty} v_n$ is convergent.

Hence by comparison test $\sum_{n=1}^{\infty} u_n$ is convergent.

4. Examine the nature of series $\sum_{n=1}^{\infty} \frac{1}{(a+n)^p(b+n)^q}$ where a,b,p,q all positive.

Solution:

$$u_n = \frac{1}{(a+n)^p (b+n)^q}$$

$$= \frac{1}{n^{p+q} \left(1 + \frac{a}{n}\right)^p \left(1 + \frac{b}{n}\right)^q}$$

$$Choose v_n = \frac{1}{n^{p+q}}$$

$$\lim_{n \to \infty} \frac{u_n}{v_n} = \lim_{n \to \infty} \frac{1}{\left(1 + \frac{a}{n}\right)^p \left(1 + \frac{b}{n}\right)^q} = 1$$

Since $\sum_{n=1}^{\infty} \frac{1}{n^{p+q}}$ is convergent if p+q>1 and is divergent if $p+q\leq 1$. Which implies $\sum_{n=1}^{\infty} \frac{1}{(a+n)^p(b+n)^q}$ is convergent if p+q>1 and is divergent if $p+q\leq 1$.

5. Determine whether the following series is convergent or divergent $\sum_{n=1}^{\infty} \left(\sqrt{n^4 + 1} - \sqrt{n^4 - 1} \right)$.

Solution:

$$\begin{split} u_n &= \left(\sqrt{n^4+1} - \sqrt{n^4-1}\right) \left(\frac{\sqrt{n^4+1} + \sqrt{n^4-1}}{\sqrt{n^4+1} + \sqrt{n^4-1}}\right) \\ &= \frac{(n^4+1) - (n^4-1)}{\sqrt{n^4+1} + \sqrt{n^4-1}} = \frac{2}{\sqrt{n^4+1} + \sqrt{n^4-1}} \\ &= \frac{2}{n^2} \frac{1}{\sqrt{1+\frac{1}{n^4}} + \sqrt{1-\frac{1}{n^4}}} \\ \text{Choose } v_n &= \frac{1}{n^2} \\ \lim_{n \to \infty} \frac{u_n}{v_n} &= \lim_{n \to \infty} \frac{2}{2} = 1. \end{split}$$

Since the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is convergent, $\sum_{n=1}^{\infty} \left(\sqrt{n^4 + 1} - \sqrt{n^4 - 1} \right)$ is convergent.

6. Determine whether the following series is convergent or divergent

$$\sum_{n=1}^{\infty} \sqrt{\frac{3^n-1}{2^n+1}}.$$
 Solution:

$$\begin{split} u_n &= \sqrt{\frac{3^n-1}{2^n+1}} \\ &= \left(\sqrt{\frac{3}{2}}\right)^n \sqrt{\frac{1-\frac{1}{3^n}}{1+\frac{1}{2^n}}} \\ \text{Let } v_n &= \left(\sqrt{\frac{3}{2}}\right)^n \end{split}$$

$$\lim_{n\to\infty} \frac{u_n}{v_n} = \lim_{n\to\infty} \sqrt{\frac{1-\frac{1}{3^n}}{1+\frac{1}{2^n}}} = 1$$

Since $\sum_{n=1}^{\infty} v_n$ is divergent, $\sum_{n=1}^{\infty} \sqrt{\frac{3^n-1}{2^n+1}}$ is also divergent.

7. Examine the nature of the series $\sum_{n=1}^{\infty} \sin(\frac{1}{n})$.

Solution:

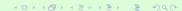
$$u_n = sin(\frac{1}{n})$$
, choose $v_n = \frac{1}{n}$. $lim_{n \to \infty} \frac{u_n}{v_n} = lim_{n \to \infty} \frac{sin(\frac{1}{n})}{\frac{1}{n}} = 1$.

Since $\sum_{n=1}^{\infty} \frac{1}{n}$ is divergent. Hence $\sum_{n=1}^{\infty} sin(\frac{1}{n})$ is also divergent.

Cauchy's Integral Test

If $\sum_{n=1}^{\infty} u_n$ is a series of positive terms and if u(x) = f(x) be such that

- f(x) is continuous in $1 < x < \infty$.
- f(x) decreases as x increases, then the series $\sum_{n=1}^{\infty} u_n$ is convergent or divergent according as the integral $\int_{1}^{\infty} f(x) dx$ is finite or infinite.



1. Apply Cauchy's integral test to discuss the nature of the harmonic series (p-series) $\sum_{n=1}^{\infty} \frac{1}{n^p}$.

Solution:

Let $u_x = f(x) = \frac{1}{n^p}$. As x increases f(x) decreases.

$$\int_{1}^{\infty} f(x)dx = \int_{1}^{\infty} \frac{1}{x^{p}} dx$$

$$= \int_{1}^{\infty} x^{-p} dx$$

$$= \left[\frac{x^{-p+1}}{-p+1} \right]$$

$$= \frac{-1}{p-1} [x^{-p+1}]_{1}^{\infty}$$

$$= \begin{cases} \frac{1}{p-1} & \text{if } p > 1 \\ \infty & \text{if } p < 1 \end{cases}$$

Therefore $\sum u_n$ is convergent if p > 1 and divergent if $p \le 1$.

2. Find the nature of the series $\sum_{n=2}^{\infty} \frac{1}{n(\log n)^p}$.

Solution:

Let $u_x = f(x) = \frac{1}{x(log_x)^p}$. As x increases f(x) decreases.

$$\int_{2}^{\infty} \frac{1}{x(\log x)^{p}} = \int_{\log 2}^{\infty} \frac{dt}{t^{p}}$$

$$= \left[\frac{t^{-p+1}}{-p+1}\right]$$

$$= \begin{cases} \text{finite,} & \text{if } p = 1\\ \text{finite,} & \text{if } p > 1\\ \text{infinite,} & \text{if } p \leq 1 \end{cases}$$

Therefore $\sum u_n$ is convergent if p > 1 and divergent if $p \le 1$.



3. Discuss the convergence of the series $\sum_{n=2}^{\infty} \frac{1}{n log n}$.

Solution:

Let $u_x = f(x) = \frac{1}{x \log x}$. As x increases f(x) decreases.

$$\int_{2}^{\infty} f(x)dx = \int_{2}^{\infty} \frac{1}{x(logx)} dx$$
$$= \int_{log2}^{\infty} \frac{dt}{t}$$
$$= infinite$$

Therefore $\sum u_n$ is divergent.



4. Examine the convergence of the series $1 + \frac{1}{4^{\frac{2}{3}}} + \frac{1}{9^{\frac{2}{3}}} + \frac{1}{16^{\frac{2}{3}}} + \dots$

Solution:

Let $u_x = f(x) = \frac{1}{x^{\frac{4}{3}}}$. As x increases f(x) decreases.

$$\int_{1}^{\infty} f(x)dx = \int_{1}^{\infty} \frac{1}{x^{\frac{4}{3}}} dx$$

$$= \int_{1}^{\infty} x^{\frac{-4}{3}} dx$$

$$= -3[x^{\frac{-1}{3}}]_{1}^{\infty}$$

$$= 3(finite)$$

Therefore $\sum u_n$ is convergent.



D'Alembert's Ratio Test:

D'Alembert's Ratio Test

The series $\sum_{n=1}^{\infty} u_n$ of positive terms is convergent if $\lim_{n\to\infty} \frac{u_{n+1}}{u_n} < u_n$

- 1 is divergent if $\lim_{n\to\infty}\frac{u_{n+1}}{u_n}>1$. If $\lim_{n\to\infty}\frac{u_{n+1}}{u_n}=1$, the test fails.
- 1. Test the convergence of the series $1 + \frac{1+\alpha}{1+\beta} + \frac{(1+\alpha)(1+2\alpha)}{(1+\beta)(1+2\beta)} + \dots$

Solution:

Let
$$u_n = \frac{(1+\alpha)(1+2\alpha)....(1+(n)\alpha)}{(1+\beta)(1+2\beta).....(1+(n)\beta)}$$
 then
$$u_{n+1} = \frac{(1+\alpha)(1+2\alpha)....(1+(n)\alpha)(1+(n+1)\alpha)}{(1+\beta)(1+2\beta).....(1+(n)\beta)(1+(n+1)\beta)}.$$
 It follows that
$$\frac{u_{n+1}}{u_n} = \frac{(1+(n+1)\alpha)}{(1+(n+1)\beta)} \Rightarrow \lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \frac{\alpha}{\beta}.$$
 Thus the series converges if $\frac{\alpha}{\beta} < 1$ and diverges if $\frac{\alpha}{\beta} > 1$. If $\frac{\alpha}{\beta} = 1$, then $\alpha = \beta$. Therefore the series $\sum_{n=1}^{\infty} u_n = 1+1+1+...$ is a divergent series.

D'Alembert's Ratio Test:

2. Test the convergence of the series $x + \frac{2^2x^2}{2!} + \frac{3^3x^3}{3!} + \dots$

Solution:

Let
$$u_n = \frac{x^n n^n}{n!}$$
 then $u_{n+1} = \frac{x^{n+1}(n+1)^{(n+1)}}{(n+1)!}$. It follows that $\frac{u_{n+1}}{u_n} = x\left(\frac{n+1}{n}\right)^n \Rightarrow \lim_{n\to\infty} \frac{u_{n+1}}{u_n} = x\lim_{n\to\infty} \left(1+\frac{1}{n}\right)^n = xe$. Thus the series converges if $ex < 1$ and diverges if $ex > 1$. If $ex = 1$, then the ratio test fails.

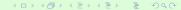
3. Find the nature of the series $\frac{2}{1} + \frac{2.5.8}{1.5.9} + \frac{2.5.8.11}{1.5.9.13} + \dots$

Solution:

$$u_n = \frac{2.5.8...(3n-1)}{1.5.9...(4n-3)}$$
 and $u_{n+1} = \frac{2.5.8...(3n-1)(3n+2)}{1.5.9...(4n-3)(4n+1)}$. Thus

$$\frac{u_{n+1}}{u_n} = \frac{3n+2}{4n+1}. \text{ Hence } \lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \lim_{n\to\infty} \frac{3+\frac{2}{n}}{4+\frac{1}{n}} = \frac{3}{4} < 1. \text{ Hence}$$

the given series is convergent.



Raabe's Test:

Raabe's Test

The positive termed series $\sum_{n=1}^{\infty} u_n$ is convergent or deivergent according as $\lim_{n\to\infty} n\left(\frac{u_{n+1}}{u_n}\right) - 1 > 1$ or < 1. If D'Alemberts test fails then use Raabe's test.

1. Test the convergence of the series $\frac{2}{3.4} + \frac{2.4}{3.5.6} + \frac{2.4.6}{3.5.7.8} + \dots$

Solution:
Let
$$u_n = \frac{2.4.6.8....2n}{3.5.7....(2n+1)} \frac{1}{2n+2}$$
 then $u_{n+1} = \frac{2.4.6.8....2n.2(n+1)}{3.5.7....(2n+1).(2n+3)} \frac{1}{2n+4}$. It follows that $\frac{u_{n+1}}{u_n} = \frac{\left(1 + \frac{1}{n}\right)^2}{\left(1 + \frac{3}{2n}\right)\left(1 + \frac{4}{2n}\right)} \Rightarrow \lim_{n \to \infty} \frac{u_{n+1}}{u_n} = 1$. Hence the ratio test fails. $n\left(\frac{u_{n+1}}{u_n}\right) - 1 = n\left[\frac{(2n+3)(2n+4)}{(2n+2)^2} - 1\right] \to \frac{3}{2}$ as $n \to \infty$.

Therefore by Raabe's test the given series is convergent.

Raabe's Test:

2. Test the convergence of the series $\sum_{n=1}^{\infty} \frac{1.3.5....(2n-1)}{2.4.6....(2n)} x^n$.

Solution:

$$\frac{u_{n+1}}{u_n} = x\left(\frac{2n+1}{2n+2}\right) \Rightarrow \lim_{n\to\infty} \frac{u_{n+1}}{u_n} = x$$
. Thus the series converges if $0 < x < 1$ and diverges if $x > 1$. If $x = 1$, then the D'Alemberts ratio test fails. Apply Raabe's test.

$$n\left(\frac{u_n}{u_{n+1}}-1\right)=\frac{n}{2n+1}\to\frac{1}{2}$$
 as $n\to\infty$. Therefore when $x=1$ the given series is divergent.

Cauchy's root Test:

Cauchy's root Test

If $\sum_{n=1}^{\infty} u_n$ is a series of positive terms, then the series is convergent or deivergent according as $\lim_{n\to\infty} u_n^{\frac{1}{n}} < 1$ or > 1.

1. Test the convergence of the series $\sum_{n=1}^{\infty} \frac{1}{(logn)^n}$.

Solution:

Let
$$u_n = \frac{1}{(logn)^n}$$
. By Cauchy's root test $u_n^{\frac{1}{n}} = \frac{1}{(logn)^n}^{\frac{1}{n}} = \frac{1}{logn}$. It

follows that $u_n^{\frac{1}{n}} \to 0$ as $n \to \infty$. By Cauchy's root test the given series is convergent.

2. Test the convergence of the series $\sum_{n=1}^{\infty} \frac{1}{\left(1+\frac{1}{n}\right)^{n^2}}$.

Solution:

Let
$$u_n = \frac{1}{\left(1 + \frac{1}{n}\right)^{n^2}}$$
. By Cauchy's root test $u_n^{\frac{1}{n}} = \frac{1}{\left(1 + \frac{1}{n}\right)^n}$. It fol-

lows that $u_n^{\frac{1}{n}} \to \frac{1}{e} < 1$ as $n \to \infty$. By Cauchy's root test the given series is convergent.

Alternating Series-Lebnitz's Test:

Alternating Series

A series in which the terms are alternatively positive or negative is called an alternating series.

Lebnitz's Rule

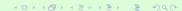
An alternating series $u_1 - u_2 + u_3 - u_4 + ...$ converges if $u_n - u_{n-1} < 0$ and $\lim_{n \to \infty} u_n = 0$. The alternating series is not convergent if one of the condition is satisfied. If $\lim_{n \to \infty} u_n \neq 0$, then the given series is oscillatory.

1. Discuss the convergence of the series $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots$

Solution:

$$u_n = \frac{1}{n}, u_{n-1} = \frac{1}{n-1}$$
. Then $u_n - u_{n-1} = \frac{-1}{n(n-1)} < 0$. $\lim_{n \to \infty} u_n = 0$.

Therefore by Lebnitz's Rule the given series is convergent.



Alternating Series-Lebnitz's Test:

2. Examine the nature of the series $\sum_{n=2}^{\infty} \frac{(-1)^{n-1} x^n}{n(n-1)}$, 0 < x < 1.

Solution:

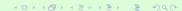
$$u_n = \frac{x^n}{n(n-1)}, \ u_{n-1} = \frac{x^{n-1}}{(n-2)(n-1)}. \ u_n - u_{n-1}$$
 is less than zero.

Also $\lim_{n\to\infty} u_n = \lim_{n\to\infty} \frac{x^n}{n(n-1)} = 0$. Thus the Lebnitz's Conditions are satisfied. Hence the given series is convergent.

3. Examine the nature of the series $\sum_{n=1}^{\infty} (-1)^{n-1} \left[\sqrt{n^2 + 1} - n \right]$. **Solution:**

$$\begin{aligned} u_n &= \sqrt{n^2 + 1} - n, \ u_{n-1} &= \sqrt{n^2 - 2n + 2} - n + 1. \ \text{Also} \ u_n - u_{n-1} < 0. \\ lim_{n \to \infty} u_n &= lim_{n \to \infty} \left[\sqrt{n^2 + 1} - n \right] = lim_{n \to \infty} \left[\frac{(\sqrt{n^2 + 1})^2 - n^2}{\sqrt{n^2 + 1} + n} \right] = 0. \end{aligned}$$

Thus the Lebnitz's Conditions are satisfied. Hence the given series is convergent.



Absolute Convergence and Conditional Convergence:

Absolute Convergence: If the series of arbitrary terms $u_1 + u_2 + + u_n + ...$ be such that the series $|u_1| + |u_2| + + |u_n| + ...$ is convergent, then then series $\sum_{n=1}^{\infty} u_n$ is absolutely convergent. **Conditional Convergence:** If the series $\sum_{n=1}^{\infty} |u_n|$ is divergent but $\sum_{n=1}^{\infty} u_n$ is convergent, then then series $\sum_{n=1}^{\infty} u_n$ is conditionally convergent.

1. Test the series $1 - \frac{1}{2\sqrt{2}} + \frac{1}{3\sqrt{3}} - \frac{1}{4\sqrt{4}} + \dots$ for (i) Absolute Convergence (ii) Conditional Convergence.

Solution:

(i)
$$\sum_{n=1}^{\infty} u_n = 1 - \frac{1}{2\sqrt{2}} + \frac{1}{3\sqrt{3}} - \frac{1}{4\sqrt{4}} + \dots$$
 Thus $\sum_{n=1}^{\infty} |u_n| = \sum_{n=1}^{\infty} \frac{1}{n^{\frac{3}{2}}}$.

Which harmonic p-series with $p=\frac{3}{2}>1$. Hence the series $\sum_{n=1}^{\infty}|u_n|$ is convergent which implies $\sum_{n=1}^{\infty}u_n$ is absolutely convergent. (ii) $\sum_{n=1}^{\infty}u_n=1-\frac{1}{2\sqrt{2}}+\frac{1}{3\sqrt{3}}-\frac{1}{4\sqrt{4}}+...$ is an alternating series.

(ii)
$$\sum_{n=1} u_n = 1 - \frac{1}{2\sqrt{2}} + \frac{1}{3\sqrt{3}} - \frac{1}{4\sqrt{4}} + \dots$$
 is an alternating series. $u_n = \frac{1}{n\sqrt{n}}$ and $u_{n-1} = \frac{1}{n-1\sqrt{n-1}}$. Also $u_n - u_{n-1} < 0$ and $\lim_{n \to \infty} u_n = \frac{1}{n-1\sqrt{n-1}}$.

0. Hence $\sum_{n=1}^{\infty} u_n$ is also convergent. Hence the given series is not conditionally convergent.

Absolute Convergence and Conditional Convergence:

Absolute Convergence:

2. Prove that the exponential series $1 + \frac{x}{1!} + \frac{x}{2!} + \dots + \frac{x}{n!} + \dots$ is absolutely convergent and hence convergent for all values of x.

Solution:

Let
$$u_n = \frac{x^{n-1}}{(n-1)!}$$
, $u_{n+1} = \frac{x^n}{n!}$. Thus $\frac{u_{n+1}}{u_n} = \frac{x}{n}$. $\lim_{n \to \infty} \left| \frac{u_{n+1}}{u_n} \right| =$

 $\lim_{n\to\infty}\frac{|x|}{n}=0<1, \forall x.$ Hence the series is absolutely convergent and hence convergence for all real x.