

# 18MAB102T- ADVANCED CALCULUS AND COMPLEX ANALYSIS UNIT- IV

## ANALYTIC FUNCTIONS



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# Analytical functions :- ( Regular functions or Holomorphic functions)

## DEFINITION:-

A SINGLE VALUED FUNCTION  $f(z)$  IS SAID TO BE ANALYTIC AT A POINT  $z_0$ , IF IT HAS A DERIVATIVE AT  $z_0$  AND AT EVERY POINT IN SOME NEIGHBOURHOOD OF  $z_0$ .

## NOTE:

IF IT IS ANALYTICAL AT EVERY POINT IN A REGION R, THEN IT IS SAID TO BE ANALYTIC IN THE REGION R.



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Necessary condition for a complex function  $f(z)$  to be analytic:-

Derivation of Cauchy-Riemann equations:-

STATEMENT:-

IF  $F(Z) = U(X,Y) + i V(X,Y)$  IS ANALYTIC IN A REGION R OF THE Z-PLANE THEN

- I)  $U_X, U_Y, V_X, V_Y$  EXIST AND
- II)  $U_X = V_Y$  AND  $U_Y = -V_X$  AT EVERY POINT IN THAT REGION.



# Necessary condition for a complex function $f(z)$ to be analytic:-

## Derivation of Cauchy-Riemann equations:-

PROOF:-

$$\text{LET } F(Z) = U(X,Y) + i V(X,Y)$$

WE FIRST ASSUME  $F(Z)$  IS ANALYTIC IN A REGION R. THEN BY THE DEFINITION,  $F(Z)$  HAS A DERIVATIVE  $F'(Z)$  EVERYWHERE IN R.

$$\text{NOW } f'(z) = \lim_{\Delta z \rightarrow 0} \frac{f(z + \Delta z) - f(z)}{\Delta z}$$



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## Derivation of Cauchy-Riemann equations:-

LET  $Z = X + iY$

$$\Delta Z = \Delta X + i \Delta Y$$

$$\therefore (Z + \Delta Z) = (X + \Delta X) + i(Y + \Delta Y)$$

$$\therefore f(z + \Delta z) = u(x + \Delta x, y + \Delta y) + i v(x + \Delta x, y + \Delta y)$$

WE KNOW THAT,  $F(Z) = U(X, Y) + i V(X, Y)$

NOW

$$f'(z) = \lim_{(\Delta x + i \Delta y) \rightarrow 0} \frac{[u(x + \Delta x, y + \Delta y) + i v(x + \Delta x, y + \Delta y)] - [u(x, y) + i v(x, y)]}{\Delta x + i \Delta y}$$



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# Derivation of Cauchy-Riemann equations:-

CASE (I) :- IF  $\Delta Z \rightarrow 0$ , FIRST WE ASSUME THAT  $\Delta Y = 0$  AND  $\Delta X \rightarrow 0$

$$\begin{aligned}
 \therefore f'(z) &= \lim_{\Delta x \rightarrow 0} \frac{[u(x + \Delta x, y) + i v(x + \Delta x, y)] - [u(x, y) + i v(x, y)]}{\Delta x} \\
 &= \lim_{\Delta x \rightarrow 0} \frac{[u(x + \Delta x, y) - u(x, y)] + i [v(x + \Delta x, y) - v(x, y)]}{\Delta x} \\
 &= \lim_{\Delta x \rightarrow 0} \frac{[u(x + \Delta x, y) - u(x, y)]}{\Delta x} + i \lim_{\Delta x \rightarrow 0} \frac{[v(x + \Delta x, y) - v(x, y)]}{\Delta x} \\
 &= \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x}
 \end{aligned}$$

$$\therefore F'(Z) = U_X + i V_X \quad \text{-----} \rightarrow (1)$$

## Derivation of Cauchy-Riemann equations:-

CASE (II) :- IF  $\Delta Z \rightarrow 0$ , NOW WE ASSUME  $\Delta X=0$  AND  $\Delta Y \rightarrow 0$

$\therefore$

$$\begin{aligned}
 f'(z) &= \lim_{\Delta y \rightarrow 0} \frac{[u(x, y + \Delta y) + i v(x, y + \Delta y)] - [u(x, y) + i v(x, y)]}{i \Delta y} \\
 &= \lim_{\Delta y \rightarrow 0} \frac{[u(x, y + \Delta y) - u(x, y)] + i [v(x, y + \Delta y) - v(x, y)]}{i \Delta y} \\
 &= \lim_{\Delta y \rightarrow 0} \frac{[u(x, y + \Delta y) - u(x, y)]}{i \Delta y} + i \lim_{\Delta y \rightarrow 0} \frac{[v(x, y + \Delta y) - v(x, y)]}{i \Delta y} \\
 &= \frac{1}{i} \frac{\partial u}{\partial y} + \frac{\partial v}{\partial y} \\
 \therefore F'(Z) &= -I U_Y + V_Y \quad \text{-----} \rightarrow (2) \quad (\text{SINCE } 1/I = -I)
 \end{aligned}$$

## Derivation of Cauchy-Riemann equations:-

FROM (1) & (2), WE GET

$$U_x + iV_x = -iU_y + V_y$$

EQUATING REAL AND IMAGINARY PARTS WE GET,

$$U_x = V_y \text{ AND } U_y = -V_x$$

THE ABOVE EQUATIONS ARE CALLED CAUCHY-RIEMANN EQUATIONS (OR) C-R EQUATIONS.

THEREFORE THE FUNCTION  $F(z)$  TO BE ANALYTIC AT THE POINT  $z$ , IT IS NECESSARY THAT THE FOUR PARTIAL DERIVATIVES  $U_x, U_y, V_x, V_y$  SHOULD EXIST AND SATISFY THE C-R EQUATIONS.



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## Sufficient condition for $f(z)$ to be analytic

STATEMENT:- THE SINGLED VALUED CONTINUOUS FUNCTION  $F(Z) = U + iV$  IS ANALYTIC IN A REGION R OF THE Z-PLANE, IF THE FOUR PARTIAL DERIVATIVES  $U_x, U_y, V_x, V_y$ , (I) EXIST , (II) CONTINUOUS , (III) THEY SATISFY THE C-R EQUATIONS  $U_x = V_y$  AND  $U_y = -V_x$  AT EVERY POINT OF R.

NOTE:- ALL POLYNOMIALS, TRIGONOMETRIC, EXPONENTIAL FUNCTIONS ARE CONTINUOUS.



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# Cauchy-Riemann Equations in Polar form

STATEMENT:- IF  $F(Z) = U(R,\Theta) + iV(R,\Theta)$  IS DIFFERENTIAL AR  
 $Z = RE^{i\Theta}$ , THEN

$$\frac{\partial u}{\partial r} = \left( \frac{1}{r} \right) \frac{\partial v}{\partial \theta} \Rightarrow u_r = \left( \frac{1}{r} \right) v_\theta$$

$$\frac{\partial v}{\partial r} = - \left( \frac{1}{r} \right) \frac{\partial u}{\partial \theta} \Rightarrow v_r = - \left( \frac{1}{r} \right) u_\theta$$

PROOF:- LET  $Z = RE^{i\Theta}$

AND  $F(Z) = U+iV$

I.E.,  $U+iV = F( RE^{i\Theta} )$

(1)



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# Cauchy-Riemann Equations in Polar form

DIFFERENTIATING PARTIALLY W.R.T. `R' WEGET,

$$\frac{\partial u}{\partial r} + i \frac{\partial v}{\partial r} = f'(r e^{i\theta}) e^{i\theta} \longrightarrow$$

DIFFERENTIATING PARTIALLY W.R.T. `θ' WEGET,

$$\begin{aligned}
 \frac{\partial u}{\partial \theta} + i \frac{\partial v}{\partial \theta} &= f'(r e^{i\theta})(r e^{i\theta})(i) \\
 &= (ri) f'(r e^{i\theta})(e^{i\theta}) \\
 &= (ri) \left[ \frac{\partial u}{\partial r} + i \frac{\partial v}{\partial r} \right] \quad (\text{from eqn. (2)}) \\
 &= ir \left( \frac{\partial u}{\partial r} \right) - r \left( \frac{\partial v}{\partial r} \right) \quad \rightarrow (3)
 \end{aligned}$$

# Cauchy-Riemann Equations in Polar form

EQUATING REAL AND IMAGINARY PARTS IN EQN. (3),  
WE GET,

$$\frac{\partial u}{\partial \theta} = -r \frac{\partial v}{\partial r} \quad \text{and} \quad \frac{\partial v}{\partial \theta} = r \frac{\partial u}{\partial r}$$

i.e.,  $u_\theta = -r v_r$  and  $v_\theta = r u_r$

(or) 
$$v_r = \left( \frac{-1}{r} \right) u_\theta \quad \text{and} \quad u_r = \left( \frac{1}{r} \right) v_\theta$$



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# EXAMPLES

1) SHOW THAT  $F(Z) = Z^3$  IS ANALYTIC.

PROOF:- GIVEN  $F(Z) = Z^3 = (X+IY)^3 = X^3 + 3X^2(IY) + 3X(IY)^2 + (IY)^3$   
 $= (X^3 - 3XY^2) + I (3X^2Y - Y^3)$

WE KNOW THAT  $F(Z) = U+IV$

$$U = X^3 - 3XY^2 , \quad V = 3X^2Y - Y^3$$

$$\text{SO , } \frac{\partial U}{\partial X} = 3X^2 - 3Y^2 , \quad \frac{\partial V}{\partial X} = 6XY$$

$$\frac{\partial U}{\partial Y} = -6XY , \quad \frac{\partial V}{\partial Y} = 3X^2 - 3Y^2$$



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## EXAMPLES

from the above equations we get,

$$u_x = v_y \quad \text{and} \quad u_y = -v_x$$

$\therefore C - R$  equations are satisfied.

HERE  $U_X, U_Y, V_X, V_Y$  EXISTS AND CONTINUOUS.

HENCE THE GIVEN FUNCTION  $F(Z)$  IS ANALYTIC.

2) EXAMINE THE ANALYTICITY OF THE FOLLOWING FUNCTIONS AND FIND ITS DERIVATIVES.

i)  $f(z) = e^z$

ii)  $f(z) = \cos z$

iii)  $f(z) = \sinh z$



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# EXAMPLES

## I) SOLUTION:-

$$f(z) = e^z = e^{x+iy} = e^x e^{iy} = e^x (\cos y + i \sin y)$$

Here  $u = e^x \cos y$  and  $v = e^x \sin y$

$$u_x = e^x \cos y$$

$$v_x = e^x \sin y$$

$$u_y = -e^x \sin y$$

$$v_y = e^x \cos y$$

$$\therefore u_x = v_y \text{ and } u_y = -v_x$$

$\Rightarrow C - R$  equations are satisfied.

$\Rightarrow f(z)$  is analytic everywhere in the complex plane.



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# EXAMPLES

$$\begin{aligned} \text{Now } f'(z) &= u_x + i v_x \\ &= e^x \cos y + i e^x \sin y \\ &= e^x (\cos y + i \sin y) \\ &= e^x e^{iy} \\ &= e^{x+iy} \\ &= e^z \end{aligned}$$



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# EXAMPLES

## II) SOLUTION:-

$$\begin{aligned}
 f(z) &= \cos z \\
 &= \cos(x + iy) \\
 &= \cos x \cos(iy) - \sin x \sin(iy) \\
 &= \cos x \cosh y - i \sin x \sinh y \quad (\because \cos(ix) = \cosh x \\
 &\qquad\qquad\qquad \sin(ix) = i \sinh x)
 \end{aligned}$$

$$\therefore u = \cos x \cosh y \qquad v = -\sin x \sinh y$$

$$u_x = -\sin x \cosh y \qquad v_x = -\cos x \sinh y$$

$$u_y = \cos x \sinh y \qquad v_y = -\sin x \cosh y$$

$$\Rightarrow u_x = v_y \quad \text{and} \quad u_y = -v_x$$



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# EXAMPLES

$\therefore C - R$  equations satisfied

$\Rightarrow$  It is analytic

$$\begin{aligned}
 \text{Also } f'(z) &= u_x + i v_x \\
 &= (-\sin x \cosh y) + i(-\cos x \sinh y) \\
 &= -\sin x \cos iy + i(-\cos x \left(\frac{1}{i}\right) \sin(iy)) \\
 &= -\sin x \cos(iy) - \cos x \sin(iy) \\
 &= -[\sin(x+iy)] \\
 &= -\sin z
 \end{aligned}$$



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# EXAMPLES

iii) SOLUTION:-

$$\begin{aligned}
 f(z) &= \sinh z = \frac{1}{i} \sin(iz) \\
 &= -i(\sin i(x+iy)) \\
 &= -i(\sin(ix)\cos y - \cos(ix)\sin y) \\
 &= -i(i\sinh x \cos y - \cosh x \sin y) \\
 &= \sinh x \cos y + i \cosh x \sin y
 \end{aligned}$$

$$\therefore u = \sinh x \cos y, \quad v = \cosh x \sin y$$

$$u_x = \cosh x \cos y, \quad v_x = \sinh x \sin y$$

$$u_y = -\sinh x \sin y, \quad v_y = \cosh x \cos y$$

$$\Rightarrow u_x = v_y \quad \text{and} \quad u_y = -v_x$$



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# EXAMPLES

$\therefore C - R$  equations are satisfied

$\Rightarrow f(z)$  is analytic.

$$\text{Now } f'(z) = u_x + i v_x$$

$$= (\cosh x \cos y) + i (\sinh x \sin y)$$

$$= (\cos(ix) \cos y) + i \left( \left( \frac{1}{i} \right) \sin(ix) \sin y \right)$$

$$= \cos(ix - y)$$

$$= \cos i(x + iy) \quad (\because (1/i) = -i)$$

$$= \cos iz$$

$$= \cosh z$$



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# TRY IT

EXAMINE THE ANALYTICITY OF THE FOLLOWING FUNCTIONS AND FIND ITS DERIVATIVES.

$$i) \quad f(z) = e^x (\cos y + i \sin y)$$

$$ii) \quad f(z) = e^{-x} (\cos y - i \sin y)$$

$$iii) \quad f(z) = \sin x \cosh y + i \cos x \sinh y$$



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# EXAMPLES

- 3) Show that the function  $f(z) = \sqrt{|xy|}$  is not regular (analytic) at the origin, although C – R equations are satisfied at the origin.

*Solution :-*

$$\text{Given } f(z) = \sqrt{|xy|}$$

$$\text{Hence } u = \sqrt{|xy|} \quad \text{and} \quad v = 0$$

$$\text{Now, } u_x = \frac{\partial u}{\partial x} = \lim_{\Delta x \rightarrow 0} \frac{u(x + \Delta x, y) - u(x, y)}{\Delta x}$$



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# EXAMPLES

$$u_x(0, 0) = \lim_{\Delta x \rightarrow 0} \frac{u(\Delta x, 0) - u(0, 0)}{\Delta x} = 0$$

///  $y \quad u_y(0, 0) = 0$

$$v_x(0, 0) = 0$$

$$v_y(0, 0) = 0$$

$\Rightarrow u_x = v_y$  and  $u_y = -v_x$  at the origin.

$\therefore C - R$  equations are satisfied at the origin.



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# EXAMPLES

$$\text{But } f'(0) = \lim_{\Delta z \rightarrow 0} \frac{f(0 + \Delta z) - f(z)}{\Delta z}$$

$$= \lim_{(\Delta x + i\Delta y) \rightarrow 0} \frac{\sqrt{|\Delta x \Delta y|} - 0}{\Delta z}$$

Along the curve  $y = mx$

$$f'(0) = \lim_{\substack{\Delta y = m\Delta x \\ \Delta x \rightarrow 0}} \frac{\sqrt{|m||\Delta x|^2}}{\Delta x(1 + im)} = \frac{\sqrt{|m|}}{1 + im}$$

$\therefore$  The limit is not unique, since it depends on 'm'.

$\therefore$   $f'(0)$  does not exist.

Hence  $f(z)$  is not regular at the origin.



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# C-R equations in polar form

## EXAMPLES

1) CHECK FOR ANALYTICITY OF  $\log z$

(OR) SHOW THAT  $f(z) = \log z$  IS ANALYTIC EVERYWHERE EXCEPT AT THE ORIGIN AND FIND ITS DERIVATIVES.

SOLUTION:-

$$\begin{aligned}f(z) &= \log z \\&= \log(r e^{i\theta}) \quad (\because z = r e^{i\theta}) \\&= \log r + \log e^{i\theta} \\&= \log r + i\theta\end{aligned}$$

w.k.t.  $f(z) = u + iv$

Here  $u = \log r$  and  $v = \theta$



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# EXAMPLES

$$\therefore \quad u_r = \frac{1}{r} \quad v_r = 0$$

$$u_\theta = 0 \quad v_\theta = 1$$

$\therefore u_r, u_\theta, v_r, v_\theta$  exist, are continuous and satisfy C-R equations

$$u_r = \left( \frac{1}{r} \right) v_\theta \text{ and } v_r = - \left( \frac{1}{r} \right) u_\theta \text{ everywhere except at } r=0 \text{ (i.e.) } z=0.$$

$\therefore f(z)$  is analytic everywhere except at  $z=0$ .



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# EXAMPLES

2) PROVE THAT  $f(z) = z^n$  IS ANALYTIC FUNCTION AND FIND ITS DERIVATIVES.

PROOF:-  $f(z) = z^n = (re^{i\theta})^n$

$$= r^n e^{in\theta}$$

$$= r^n [\cos n\theta + i \sin n\theta]$$

$$\therefore u = r^n \cos n\theta \quad ; \quad v = r^n \sin n\theta$$

$$u_r = nr^{n-1} \cos n\theta \quad ; \quad v_r = nr^{n-1} \sin n\theta$$

$$u_\theta = -nr^n \sin n\theta \quad ; \quad v_\theta = nr^n \cos n\theta$$

$$\Rightarrow u_r = \left( \frac{1}{r} \right) v_\theta \quad \text{and} \quad v_r = -\left( \frac{1}{r} \right) u_\theta$$



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# EXAMPLES

Thus  $u_r, u_\theta, v_r, v_\theta$  exist, are continuous and satisfy C – R equations everywhere.

∴  $f(z)$  is analytic.

$$\begin{aligned}
 \text{Also } f'(z) &= \left( \frac{u_r + i v_r}{e^{i\theta}} \right) \\
 &= \frac{(nr^{n-1} \cos n\theta) + i(nr^{n-1} \sin n\theta)}{e^{i\theta}} \\
 &= \frac{nr^{n-1} [\cos n\theta + i \sin n\theta]}{e^{i\theta}} \\
 &= \frac{nr^{n-1} e^{in\theta}}{e^{i\theta}} = n(r e^{i\theta})^{n-1} = nz^{n-1}
 \end{aligned}$$



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# Laplace Equations

In Cartesian form :

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0$$

i.e.,  $\nabla^2 \phi = 0$

In Polar form :

$$\frac{\partial^2 \phi}{\partial r^2} + \frac{1}{r} \frac{\partial \phi}{\partial r} + \frac{1}{r^2} \frac{\partial^2 \phi}{\partial \theta^2} = 0$$



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# HARMONIC FUNCTIONS

A REAL VALUED FUNCTION OF TWO REAL VARIABLES X AND Y IS SAID TO BE HARMONIC, IF

i) THE SECOND ORDER PARTIAL DERIVATIVES  $U_{xx}$ ,  $U_{xy}$ ,  $U_{yx}$ ,  $U_{yy}$  EXIST AND THEY ARE CONTINUOUS.

AND

ii) THE LAPLACE EQUATION  $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$  SATISFIES.

## CONJUGATE HARMONIC FUNCTIONS:-

IF  $U+iV$  IS AN ANALYTIC FUNCTION OF Z THEN V IS CALLED A CONJUGATE HARMONIC FUNCTION OF U; (OR) U IS CALLED A CONJUGATE HARMONIC FUNCTION OF V; (OR) U AND V ARE CALLED CONJUGATE HARMONIC FUNCTIONS.



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# Properties of Analytic functions

PROPERTY (1) :- THE REAL AND IMAGINARY PARTS OF AN ANALYTIC FUNCTION  $F(Z) = U+IV$  SATISFY THE LAPLACE EQUATION (OR) REAL PART “U” AND IMAGINARY PART “V” OF AN ANALYTIC FUNCTION  $F(Z) = U+IV$  ARE HARMONIC FUNCTIONS.

PROOF:-

GIVEN  $F(Z) = U+IV$  IS AN ANALYTIC FUNCTION.

I.E.,  $U$  AND  $V$  ARE CONTINUOUS,  $U_x, U_y, V_x, V_y$  ARE EXIST AND THEY SATISFY THE C-R EQUATIONS  $U_x = V_y$  AND  $U_y = -V_x$

(1)

(2)



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# Properties of Analytic functions

Diff. eqn.(1) partially w.r.t. x, we get,

$$u_{xx} = v_{yx} \rightarrow (3)$$

Diff. eqn.(2) partially w.r.t. y, we get,

$$u_{yy} = -v_{xy} \rightarrow (4)$$

Adding (3) & (4) we get,

$$u_{xx} + u_{yy} = v_{yx} - v_{xy} = 0 \quad [\because v_{yx} = v_{xy}]$$

$\therefore u$  satisfies Laplace equation.

Hence  $u$  is a Harmonic function.



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# Properties of Analytic functions

NOW ,

Diff. eqn.(1) partially w.r.t. y, we get,

$$u_{xy} = v_{yy} \rightarrow (5)$$

Diff. eqn.(2) partially w.r.t. x, we get,

$$u_{yx} = -v_{xx} \rightarrow (6)$$

subtracting (5) & (6) we get,

$$v_{yy} + v_{xx} = u_{xy} - u_{yx} = 0 \quad [\because u_{xy} = u_{yx}]$$

$\therefore v$  satisfies Laplace equation.

Hence  $v$  is a Harmonic function.

Thus  $u$  and  $v$  are harmonic functions.

NOTE:- THE CONVERSE OF THE ABOVE RESULT NEED NOT BE TRUE.



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# Properties of Analytic functions

## TRY IT

PROVE THAT THE REAL AND IMAGINARY PARTS OF AN ANALYTIC FUNCTION  $f(z) = u(r,\theta) + i v(r, \theta)$  SATISFY THE LAPLACE EQUATION IN POLAR COORDINATES.

i.e., To prove that

$$u_{rr} + \left(\frac{1}{r}\right) u_r + \left(\frac{1}{r^2}\right) u_{\theta\theta}$$

and

$$v_{rr} + \left(\frac{1}{r}\right) v_r + \left(\frac{1}{r^2}\right) v_{\theta\theta}$$



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# Properties of Analytic functions

## ORTHOGONAL CURVES:-

TWO CURVES ARE SAID TO BE ORTHOGONAL TO EACH OTHER THEN THEY INTERSECT AT RIGHT ANGLES. [ PRODUCT OF SLOPES  $M_1 M_2 = -1$ ]

## PROPERTY (2) :-

IF  $F(Z) = U + iV$  IS AN ANALYTIC FUNCTION THEN THE FAMILY OF CURVES

$U(X,Y) = A$  AND  $V(X,Y) = B$  (WHERE A&B ARE CONSTANTS)  
CUT EACH OTHER ORTHOGONALLY.

Proof:- Given:  $u(x, y) = a$  and  $v(x, y) = b$

*Taking differentials on both sides, we get,*

$$du = 0$$



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# Properties of Analytic functions

$$\Rightarrow \frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy = 0$$

$$\therefore \boxed{\frac{dy}{dx} = \frac{-u_x}{u_y} = m_1}$$

$$//y - v(x, y) = b$$

$$\Rightarrow \frac{\partial v}{\partial x} + \frac{\partial v}{\partial y} \frac{dy}{dx} = 0$$

$$\therefore \boxed{\frac{dy}{dx} = \frac{-v_x}{v_y} = m_2}$$



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# Properties of Analytic functions

$$\text{Product of slopes, } m_1 m_2 = \left( \begin{array}{c} -u_x \\ u_y \end{array} \right) \left| \left( \begin{array}{c} -v_x \\ v_y \end{array} \right) \right.$$

$$= \frac{(-u_x)(u_y)}{(u_y)(u_x)} \quad \left[ \begin{array}{l} \because u_x = v_y \\ \text{and } u_y = -v_x \end{array} \right]$$

HENCE THE TWO CURVES  $\text{EQNS. (3) \& (4)}$  ARE ORTHOGONAL CURVES.



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# Properties of Analytic functions

RESULT :- (1) AN ANALYTIC FUNCTION WITH CONSTANT MODULUS IS CONSTANT.

PROOF:-

LET  $F(z) = U + iV$  BE AN ANALYTIC FUNCTION

$$\therefore |f(z)| = \sqrt{u^2 + v^2}$$

Given:       $|f(z)| = c$

$$i.e., \sqrt{u^2 + v^2} = c$$

$$\Rightarrow u^2 + v^2 = c^2 \rightarrow (1)$$



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# Properties of Analytic functions

Diff. eqn.(1) partially w.r.t. x, we get,

$$2uu_x + 2vv_x = 0 \Rightarrow \boxed{uu_x + vv_x = 0} \rightarrow (2)$$

Diff. eqn.(1) partially w.r.t. y, we get,

$$2uu_y + 2vv_y = 0 \Rightarrow \boxed{uu_y + vv_y = 0} \rightarrow (3)$$

Since  $f(z)$  is analytic, it satisfies C – R equations

i.e.,  $u_x = v_y$  and  $u_y = -v_x$

$$\therefore (2) \Rightarrow uu_x + v(-u_y) = 0 \Rightarrow uu_x - vu_y = 0$$

$$(3) \Rightarrow uu_y + v(-u_x) = 0 \Rightarrow uu_y - vu_x = 0$$

# Properties of Analytic functions

SQUARING AND ADDING THE ABOVE EQUATIONS, WE GET,

$$(uu_x - vu_y)^2 + (uu_y + vu_x)^2 = 0$$

$$\Rightarrow u^2 u_x^2 + v^2 u_y^2 - 2uvu_x u_y + u^2 u_y^2 + v^2 u_x^2 + 2uvu_y u_x = 0$$

$$\Rightarrow u^2 [u_x^2 + u_y^2] + v^2 [u_x^2 + u_y^2] = 0$$

$$\Rightarrow (u^2 + v^2) (u_x^2 + u_y^2) = 0$$

But  $u^2 + v^2 = c^2 \neq 0$  (from eqn. (1))

$$\therefore \boxed{u_x^2 + u_y^2 = 0} \rightarrow (4)$$



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# Properties of Analytic functions

SINCE

$$f(z) = u + i v$$

$$\begin{aligned} f'(z) &= u_x + i v_x \\ &= u_x - i u_y \quad (\text{by C-R eqns.}) \end{aligned}$$

$$\therefore |f'(z)| = \sqrt{u_x^2 + u_y^2}$$

$$\begin{aligned} \Rightarrow |f'(z)|^2 &= u_x^2 + u_y^2 \\ &= 0 \quad (\text{from (4)}) \end{aligned}$$

$$\Rightarrow f'(z) = 0$$

$\Rightarrow f(z)$  is a constant

$\therefore$  An analytic function with constant modulus is constant.



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# Properties of Analytic functions

RESULT :- (2) IF  $F(z) = U+iV$  IS A REGULAR FUNCTION OF  $Z = X+iY$  THEN  $\nabla^2 [ |f(z)|^2] = 4 |f'(z)|^2$

Proof :-

$$\text{To prove that} \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) |f(z)|^2 = 4 |f'(z)|^2$$

$$\text{Let } f(z) = u + iv$$

$$\overline{f(z)} = u - iv$$

$$\therefore f(z) \overline{f(z)} = (u + iv)(u - iv) = u^2 + v^2$$

$$\therefore |f(z)|^2 = u^2 + v^2$$



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# Properties of Analytic functions

Now,

$$\begin{aligned}
 \left[ \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right] |f(z)|^2 &= \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) (u^2 + v^2) \\
 &= \frac{\partial^2}{\partial x^2}(u^2) + \frac{\partial^2}{\partial x^2}(v^2) + \frac{\partial^2}{\partial y^2}(u^2) + \frac{\partial^2}{\partial y^2}(v^2) \\
 &\rightarrow (1)
 \end{aligned}$$

Now, consider,  $\frac{\partial}{\partial x}(u^2) = 2uu_x$

$$\therefore \frac{\partial^2}{\partial x^2}(u^2) = \frac{\partial}{\partial x}(2uu_x) = 2uu_{xx} + 2u_x^2$$

///y  $\frac{\partial^2}{\partial y^2}(u^2) = 2uu_{yy} + 2u_y^2$



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# Properties of Analytic functions

$$\begin{aligned}
 \therefore \frac{\partial^2 u^2}{\partial x^2} + \frac{\partial^2 u^2}{\partial y^2} &= 2u(u_{xx} + u_{yy}) + 2(u_x^2 + u_y^2) \\
 &= 2[u(0) + u_x^2 + u_y^2] \quad \left[ \begin{array}{l} \because f(z) \text{ is analytic} \\ u \text{ is harmonic} \end{array} \right] \\
 &= 2[u_x^2 + (-v_x)^2] \quad \left[ \begin{array}{l} \because f(z) \text{ is analytic,} \\ \Rightarrow C - R \text{ eqns. satisfied} \end{array} \right] \\
 &= 2[u_x^2 + v_x^2] \\
 &= 2|f'(z)|^2 \quad \left[ \begin{array}{l} \because f'(z) = u_x + i v_x \\ \Rightarrow |f'(z)| = \sqrt{u_x^2 + v_x^2} \end{array} \right]
 \end{aligned}$$



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# Properties of Analytic functions

$$///y \quad \frac{\partial^2 v^2}{\partial x^2} + \frac{\partial^2 v^2}{\partial y^2} = 2 |f'(z)|^2$$

$$\therefore (1) \Rightarrow \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) |f(z)|^2 = 2 |f'(z)|^2 + 2 |f'(z)|^2 \\ = 4 |f'(z)|^2$$

*Thus proved*



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# EXAMPLES

1) If  $f(z) = e^x (\cos y + i \sin y)$  is analytic function prove that  $u, v$  are harmonic functions.

Solutions:-

To prove that  $u$  and  $v$  are Harmonic functions

i.e., T.P.T.  $u_{xx} + u_{yy} = 0$  and  $v_{xx} + v_{yy} = 0$

$$\text{Here } u = e^x \cos y \quad v = e^x \sin y$$

$$u_x = e^x \cos y \quad v_x = e^x \sin y$$

$$u_{xx} = e^x \cos y \quad v_{xx} = e^x \sin y$$

$$u_y = -e^x \sin y \quad v_y = e^x \cos y$$

$$u_{yy} = -e^x \cos y \quad v_{yy} = -e^x \sin y$$



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# EXAMPLES

$$\therefore u_{xx} + u_{yy} = e^x \cos y - e^x \cos y = 0$$

and  $v_{xx} + v_{yy} = e^x \sin y - e^x \sin y = 0$

$\therefore$  Both  $u$  &  $v$  satisfies Laplace equation

Hence  $u$  &  $v$  are Harmonic functions.



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# EXAMPLES

**Prove that  $u = \frac{1}{2} \log (x^2 + y^2)$  is harmonic and find its conjugate harmonic.**

Solution: Given  $u = \frac{1}{2} \log (x^2 + y^2)$

To prove  $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$

Consider  $u = \frac{1}{2} \log (x^2 + y^2)$

Differentiating this w.r.to x and y partially, we get

$$\frac{\partial u}{\partial x} = \frac{x}{x^2+y^2} \quad \frac{\partial u}{\partial y} = \frac{y}{x^2+y^2}$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{y^2 - x^2}{(x^2 + y^2)^2}$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{x^2 - y^2}{(x^2 + y^2)^2}$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{y^2 - x^2 + x^2 - y^2}{(x^2 + y^2)^2} = 0$$

$u$  is harmonic.



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# EXAMPLES

To find the harmonic conjugate

Let  $v(x,y)$  be the conjugate harmonic. Then  $w = u+iv$  is analytic.

By C-R equations,  $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$  and  $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$

We have

$$dv = \frac{\partial v}{\partial x} dx + \frac{\partial v}{\partial y} dy$$

$$dv = \frac{-\partial u}{\partial y} dx + \frac{\partial u}{\partial x} dy$$

$$dv = \frac{-y}{x^2+y^2} dx$$

Integrating, we get

$$v = \tan^{-1}(y/x) + c \text{ where } c \text{ is a constant.}$$



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# CONSTRUCTION OF ANALYTIC FUNCTION

MILNE-THOMSON METHOD :-

TO FIND THE ANALYTIC FUNCTION  $F(z)$ :

i) WHEN  $U(X, Y)$  IS GIVEN (I.E., REAL PART IS GIVEN)

$$f(z) = \int u_x(z, 0) dz - i \int u_y(z, 0) dz$$

II) WHEN  $V(X, Y)$  IS GIVEN (I.E., IMAGINARY PART IS GIVEN)

$$f(z) = \int v_y(z, 0) dz + i \int v_x(z, 0) dz$$



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# CONSTRUCTION OF ANALYTIC FUNCTION

## METHOD TO FIND OUT THE HARMONIC CONJUGATE:

LET  $F(Z) = U + iV$  BE AN ANALYTIC FUNCTION.

GIVEN:  $U(X, Y)$

$$\therefore V = \int -u_y dx + \int u_x dy$$



$\left( \begin{array}{l} \text{treating } y \\ \text{as constant} \end{array} \right) \left( \begin{array}{l} \text{Integrating the terms} \\ \text{independent of } x \end{array} \right)$

# EXAMPLES

1) IF  $U(X,Y) = X^2 + Y^2$ , FIND  $V(X,Y)$  AND HENCE FIND  $F(Z)$ .

Given:  $u = x^2 - y^2$

$$\Rightarrow u_x = 2x, \quad u_y = -2y$$

we know that,

$$V = \int -u_y dx + \int u_x dy$$

↓                          ↓

$\begin{pmatrix} \text{treating } y \\ \text{as constant} \end{pmatrix}$ 
 $\begin{pmatrix} \text{Integrating the terms} \\ \text{independent of } x \end{pmatrix}$



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# EXAMPLES

$$\therefore v = \int -(-2y) dx + \int 2x dy$$

$$= 2xy + 0 \quad \left( \begin{array}{l} \text{Ind integral is zero since} \\ \text{there is no term indep. of "x"} \end{array} \right)$$

$$\Rightarrow v = 2xy$$

$$\therefore f(z) = u + i v$$

$$\Rightarrow f(z) = (x^2 - y^2) + i(2xy)$$

$$= x^2 + i^2 y^2 + 2x(iy)$$

$$= (x + iy)^2$$

$\therefore f(z) = z^2$



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# EXAMPLES

1) FIND  $F(Z)$ , WHEN  $U(X,Y) = X^2 + Y^2$ .

(SAME EXAMPLE, USING MILNE-THOMSON METHOD,  
FINDING  $F(Z)$ )

Solution: –

$$\text{Given: } u = x^2 - y^2$$

$$\Rightarrow u_x = 2x, \quad u_y = -2y$$

$$\therefore u_x(z,0) = 2z, \quad u_y(z,0) = 0$$

By Milne-Thomson method ,

$$\begin{aligned} f(z) &= \int u_x(z,0) dz - i \int u_y(z,0) dz \\ &= \int 2z dz - i \int 0 dz \end{aligned}$$

∴

$f(z) = z^2$



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# EXAMPLES

2) SHOW THAT THE FUNCTION  $U(X,Y) = \sin x \cosh y$  IS HARMONIC.

FIND ITS HARMONIC CONJUGATE  $V(X,Y)$  AND THE ANALYTIC FUNCTION  $F(Z) = U + iV$ .

Solution :-

Given:  $u = \sin x \cosh y$

$$u_x = \cos x \cosh y$$

$$u_y = \sin x \sinh y$$

$$u_{xx} = -\sin x \cosh y$$

$$u_{yy} = \sin x \cosh y$$

$$\therefore u_{xx} + u_{yy} = 0$$

$\Rightarrow u$  is harmonic.



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# EXAMPLES

To find  $v(x, y)$  :-

$$\text{we know that, } v = \int -u_y \, dx + \int u_x \, dy$$

↓                      ↓

$\begin{pmatrix} \text{treating } y \\ \text{as constant} \end{pmatrix}$ 
 $\begin{pmatrix} \text{Integrating the terms} \\ \text{independent of } x \end{pmatrix}$

$$\therefore V = \int -(\sin x \sinh y) \, dx + \int (\cos x \cosh y) \, dy$$

$$= -\sinh y \int \sin x \, dx + 0 \quad \left[ \begin{array}{l} \text{since no term is} \\ \text{independent of } x \end{array} \right]$$

$$= -\sinh y (-\cos x)$$

$$\therefore V = \cos x \sinh y$$



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# EXAMPLES

Now,

$$f(z) = u + i v = \sin x \cosh y + i \cos x \sinh y$$

$$= \sin x \cos(iy) + i \cos x \left( \frac{\sin(iy)}{i} \right)$$

$$= \sin x \cos(iy) + \cos x \sin(iy)$$

$$= \sin(x + iy)$$

$$= \sin z$$

$$\therefore f(z) = \sin z$$



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# EXAMPLES

3) CONSTRUCT ANALYTIC FUNCTION  $f(z)$  OF WHICH IMAGINARY PART  $v(x,y) = -2 \sin x (e^y - e^{-y})$ .

Solution :-

$$\text{Given: } v(x, y) = -2 \sin x (e^y - e^{-y})$$

$$\text{i.e., } v = -4 \sin x \sinh y \quad [\because e^y - e^{-y} = 2 \sinh y]$$

$$v_x = -4 \cos x \sinh y, \quad v_y = -4 \sin x \cosh y$$

$$\therefore v_x(z, 0) = 0, \quad v_y(z, 0) = -4 \sin z$$

$$\begin{aligned}\therefore f(z) &= \int v_y(z, 0) dz + i \int v_x(z, 0) dz \\ &= \int -4 \sin z dz\end{aligned}$$

$\Rightarrow$

$f(z) = 4 \cos z + c$



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# EXAMPLES

4) FIND THE ANALYTIC FUNCTION  $F(z) = U+iV$  SUCH THAT,

$$U+V = X^3 + 3X^2 Y - 3XY^2 - Y^2 + 4X + 5 \text{ AND } F(0) = 2+3i .$$

Solution :-

$$\text{we know that, } f(z) = u + i v$$

$$i f(z) = iu - v$$

$$\therefore f(z) + i f(z) = u + i v + iu - v$$

$$\Rightarrow f(z)(1+i) = (u-v) + i(u+v)$$

$$F(z) = U + i V$$

$$\text{where } F(z) = f(z)(1+i)$$

$$U = (u-v), \quad V = u+v = x^3 + 3x^2y - 3xy^2 - y^2 + 4x + 5$$



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# EXAMPLES

By Milne-thomson method,

$$F(z) = \int v_y(z, 0) dz + i \int v_x(z, 0) dz$$

$$\text{Now, } v_x = 3x^2 + 6xy - 3y^2 + 4$$

$$v_y = 3x^2 - 6xy - 2y$$

$$v_x(z, 0) = 3z^2 + 4$$

$$v_y(z, 0) = 3z^2$$

$$\therefore F(z) = \int 3z^2 dz + i \int (3z^2 + 4) dz$$

$$= \frac{3z^3}{3} + i \left( \frac{3z^3}{3} + 4z \right)$$



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## EXAMPLES

$$\therefore F(z) = z^3 + i(z^3 + 4) + c$$

$$\therefore (1+i)f(z) = z^3(1+i) + i4z + c$$

$$\begin{aligned}\therefore f(z) &= z^3 + \frac{i4z}{(1+i)} + \frac{c}{(1+i)} \\ &= z^3 + \frac{i4z(1-i)}{(1+i)(1-i)} + c_1 \\ &= z^3 + \frac{4z(i+1)}{2} + c_1\end{aligned}$$

$$\therefore f(z) = z^3 + 2z(1+i) + c_1 \rightarrow (1)$$



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# EXAMPLES

Given:  $f(0) = 2 + 3i$

put  $z=0$  in (1), we get,  $f(0) = c_1$

$$\therefore \boxed{c_1 = 2 + 3i}$$

$$\therefore f(z) = z^3 + 2z(1+i) + (2+3i)$$

$$\therefore \boxed{f(z) = (z^3 + 2z + 2) + i(2z + 3)}$$



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**5. Determine the analytic function  $f(z) = u + iv$  such that  $u - v = e^x(\cos y - \sin y)$**

Solution:  $f(z) = u + iv$  ----- (1)

$\dot{f}(z) = iu - v$  ----- (2)

Adding (1) and (2)

$$\therefore (1 + i)f(z) = (u - v) + i(u + v)$$

$F(z) = U + iV$

Where  $F(z) = (1 + i)f(z)$ ,  $U = u - v$   $V = u + v$

Given  $u - v = e^x(\cos y - \sin y)$

**Step 1:**

$$\frac{\partial u}{\partial x} = e^x(\cos y - \sin y)$$

$$\frac{\partial u}{\partial y} = e^x(-\sin y - \cos y)$$

**Step 2:**

$$U_x(z, 0) = e^z \quad U_y(z, 0) = -e^z$$



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# EXAMPLES

## Step3:

$$\int F'(z) dz = \int U_x(z, 0) dz - i \int U_y(z, 0) dz$$

$$\int F'(z) dz = \int e^z dz - i \int -e^z dz$$

$$\text{Integrating } F(z) = [(i + 1)e^z] + c$$

$$(1+i) f(z) = [(i + 1)e^z] + c$$



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# EXAMPLES

6. Find the analytic function  $f(z) = u+iv$  given that  $2u + 3v = \frac{\sin 2x}{\cosh 2y + \cos 2x}$

$$\text{Solution: } 3f(z) = 3u + 3iv \quad \dots \quad (1)$$

$$2if(2) = 2iu - 2v \quad \dots \quad (2)$$

Adding (1) and (2)

$$(3+2i)f(z) = (3u-2v) + i(2u+3v)$$

$$F(z) = U+iV$$

$$\text{Where } F(z) = (3+2i)f(z), \quad U = 3u - 2v \quad V = 2u + 3v$$

$$\text{Given } 2u + 3v = \frac{\sin 2x}{\cosh 2y + \cos 2x}$$

$$\text{i.e., } V = 2u + 3v = \frac{\sin 2x}{\cosh 2y + \cos 2x}$$

**Step 1:**

$$\frac{\partial V}{\partial x} = \frac{(\cosh 2y - \cos 2x)2 \cos 2x - \sin 2x(2 \sin 2x)}{(\cosh 2y - \cos 2x)^2}$$

$$\frac{\partial V}{\partial y} = \frac{(\cosh 2y - \cos 2x)(0) - \sin 2x(2 \sinh 2y)}{(\cosh 2y - \cos 2x)^2}$$



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# EXAMPLES

## Step 2:

$$V_x(z, 0) = \frac{2(\cos 2z - 1)}{(1 - \cos 2z)^2} = \frac{-2}{2\sin^2 z} = -\operatorname{cosec}^2 z$$

$$V_y(z, 0) = 0$$

## Step 3:

$$\int F'(z) dz = \int V_y(z, 0) dz + i \int V_x(z, 0) dz$$

$$\int F'(z) dz = \int -\operatorname{cosec}^2 z dz + i \int 0 dz$$

$$\text{Integrating } F(z) = i \cot z + c$$

$$(3+2i) f(z) = i \cot z + c$$

$$f(z) = \frac{i}{3+2i} \cot z + \frac{c}{3+2i}$$

$$f(z) = \frac{2+3i}{13} \cot z + \frac{3-2i}{13} c$$



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# CONFORMAL MAPPING

**INTRO.:** SUPPOSE TWO CURVES  $C_1, C_2$  IN THE

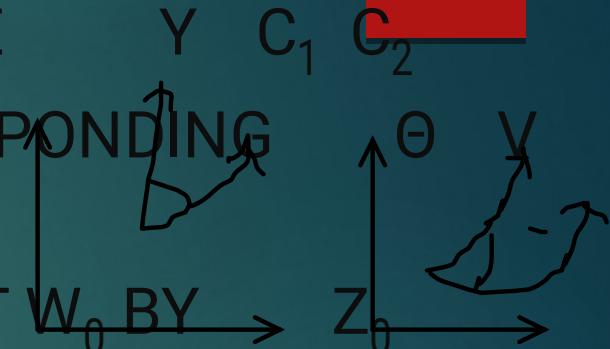
Z-PLANE INTERSECT AT  $Z_0$  AND THE CORRESPONDING

$Y_1 Y_2$

CURVES  $Y_1, Y_2$  IN THE W-PLANE INTERSECT AT

$W_0$  BY

THE TRANSFORMATION  $W = F(Z)$ .



X                  U

IF THE ANGLE BETWEEN THE TWO CURVES IN THE Z-PLANE IS SAME AS THE ANGLE BETWEEN THE CURVES IN THE W-PLANES BOTH IN MAGNITUDE AND IN DIRECTION, THEN THE TRANSFORMATION  $W = F(Z)$  IS SAID TO BE CONFORMAL MAPPING.

## DEFINITION:-

A TRANSFORMATION THAT PRESERVES ANGLES BETWEEN EVERY PAIR OF CURVES THROUGH A POINT BOTH IN MAGNITUDE AND SENSE OF ROTATION IS SAID TO BE CONFORMAL AT THAT POINT.

# CONFORMAL MAPPING

## ISOGONAL TRANSFORMATION:-

THE TRANSFORMATION WHICH PRESERVES ANGLE BETWEEN EVERY PAIR OF CURVES IN MAGNITUDE AND NOT IN DIRECTION(SENSE) IS CALLED AN ISOGONAL TRANSFORMATION.

## THEOREM:-

IF  $F(Z)$  IS ANALYTIC AND  $F'(Z) \neq 0$  IN A REGION R OF THE Z-PLANE THEN THE MAPPING PERFORMED BY  $W=F(Z)$  IS CONFORMAL AT ALL POINTS OF R.



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# CONFORMAL MAPPING

## CRITICAL POINTS:-

THE POINT AT WHICH THE MAPPING  $W=F(Z)$  IS NOT CONFORMAL, I.E.,  $F'(Z) = 0$  IS CALLED A CRITICAL POINT OF THE MAPPING.

Eg.: Consider  $w = f(z) = \sin z$

$$\therefore f'(z) = \cos z$$

$$\Rightarrow f'(0) = 0, \text{ when } z = \pm \frac{\pi}{2}, \pm \frac{3\pi}{2}, \dots$$

$$\text{i.e., } z = \frac{(2n-1)\pi}{2}, \text{ where } n \text{ is an integer,}$$

which are the critical points of the given transformation.



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➤ **TRANSLATION**

MAPS OF THE FORM  $Z \rightarrow Z + K$ , WHERE  $K \in C$

➤ **MAGNIFICATION AND ROTATION**

MAPS OF THE FORM  $Z \rightarrow KZ$ , WHERE  $K \in C$

➤ **INVERSION**

MAPS OF THE FORM  $Z \rightarrow 1/Z$



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# EXAMPLE FOR TRANSLATION

1) FIND THE REGION OF THE W-PLANE INTO WHICH THE RECTANGULAR

REGION IN THE Z-PLANE BOUNDED BY THE LINES  $X=0$ ,  $Y=0$ ,  $X=1$ ,  $Y=2$  IS MAPPED UNDER THE TRANSFORMATION  $W=Z+2-I$ .

SOLUTION:- GIVEN:  $W = Z + 2 - I$

$$\rightarrow (U+IV) = (X+IY) + (2-I)$$

$$= (X+2) + I(Y-1)$$

EQUATING REAL AND IMAGINARY PARTS, WE GET,

$$U = X + 2 \quad \text{AND} \quad V = Y - 1$$

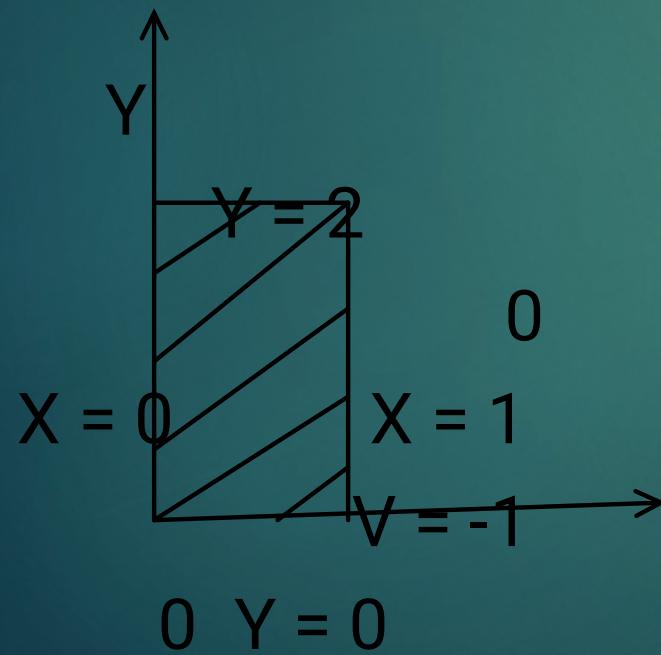


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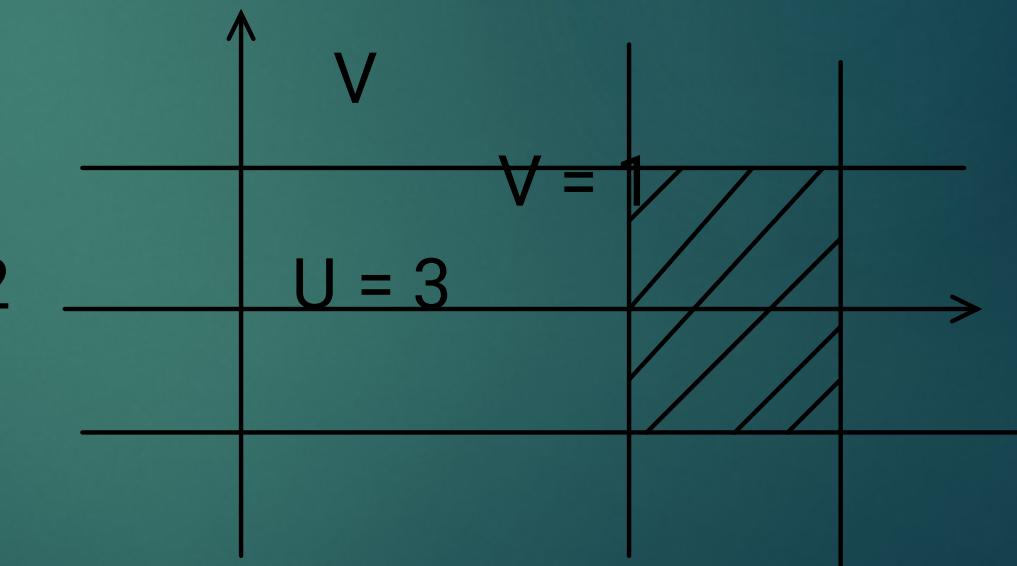
# EXAMPLE FOR TRANSLATION

GIVEN BOUNDARY LINES ARE:  
LINES ARE:

$$\begin{array}{ll} X = 0 & U = 2 \\ Y = 0 & V = -1 \\ X = 1 & U = 3 \\ Y = 2 & V = 1 \end{array}$$



TRANSFORMAL BOUNDARY



# MAGNIFICATION AND ROTATION

LET  $W = A Z$ , WHERE  $A \neq 0$

IF  $A = |A| e^{i\alpha}$  AND,  $Z = |z| e^{i\theta}$ , THEN

$$W = |A| |z| e^{i(\theta + \alpha)}$$

THE IMAGE OF  $Z$  IS OBTAINED BY ROTATING THE VECTOR  $Z$  THROUGH THE ANGLE  $\alpha$  AND MAGNIFYING OR CONTRACTING THE LENGTH OF  $Z$  BY THE FACTOR  $|A|$ .

THUS THE TRANSFORMATION  $W = A Z$  IS REFERRED TO AS A ROTATION OR MAGNIFICATION.



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# EXAMPLE FOR MAGNIFICATION

2) DETERMINE THE REGION R OF THE W PLANE INTO WHICH THE TRIANGULAR REGION D ENCLOSED BY THE LINES  $X = 0$ ,  $Y = 0$ ,  $X + Y = 3$  IS TRANSFORMED UNDER THE TRANSFORMATION  $W = 2Z$ .

SOLUTION:

LET  $W = U + iV$ ;  $Z = X + iY$

GIVEN:  $W = 2Z$

I.E.,  $U + iV = 2(X + iY)$

I.E.,  $U = 2X$ ;  $V = 2Y$  AND  $U + V = 2(X + Y)$



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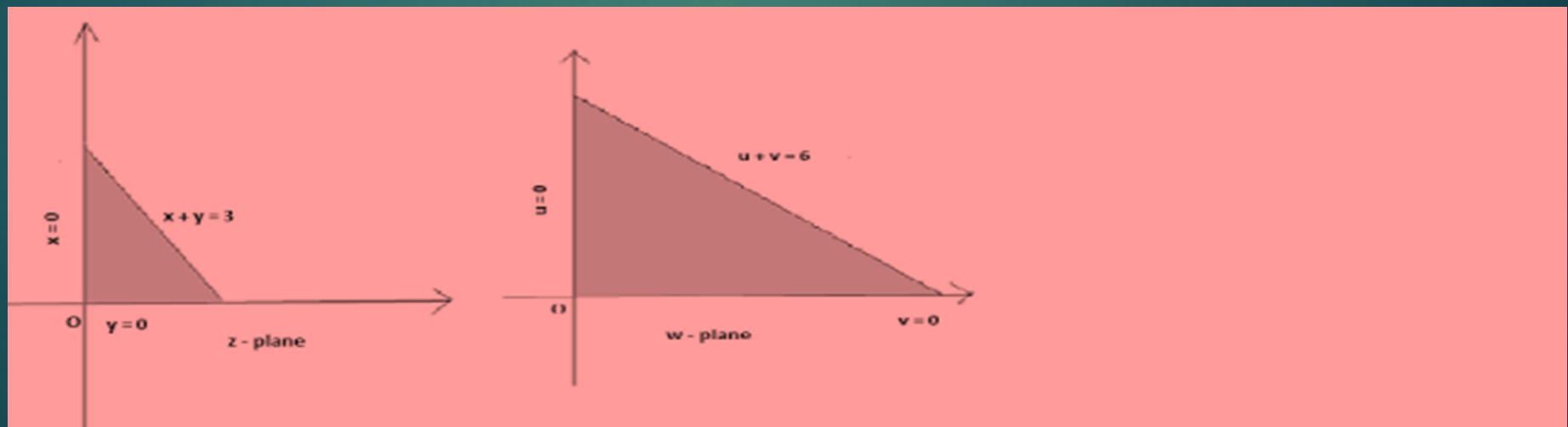
# EXAMPLE FOR MAGNIFICATION

WHEN  $X = 0$  ,  $U = 0$

$Y = 0$  ,  $V = 0$

$X + Y = 3$  ,  $U + V = 6$

THUS THE TRANSFORMATION  $W = 2Z$  MAPS A TRIANGLE IN THE Z-PLANE INTO A 2-TIMES MAGNIFIED TRIANGLE IN THE W-PLANE.



# EXAMPLE FOR ROTATION

- 3) CONSIDER THE TRANSFORMATION  $w = e^{i\pi/4} z$  AND DETERMINE THE REGION IN THE W-PLANE CORRESPONDING TO TRIANGLE REGION BOUNDED BY THE LINES  $X=0$ ,  $Y=0$ ,  $X+Y=1$ .

Solution :-

$$\text{Given: } w = e^{i\pi/4} z$$

$$\therefore (u + iv) = e^{i\pi/4} (x + iy)$$

$$= \left( \cos\left(\frac{\pi}{4}\right) + i \sin\left(\frac{\pi}{4}\right) \right) (x + iy)$$

$$= \left( \frac{1}{\sqrt{2}} + i \frac{1}{\sqrt{2}} \right) (x + iy)$$

$$= \left( \frac{x - y}{\sqrt{2}} \right) + i \left( \frac{x + y}{\sqrt{2}} \right)$$

# EXAMPLE FOR ROTATION

$$\therefore \boxed{U = \frac{x - y}{\sqrt{2}}} \text{ and } \boxed{V = \frac{x + y}{\sqrt{2}}}$$

Y  
X+Y=1  
X=0

when  $x = 0$ ,     $u = \frac{-y}{\sqrt{2}}$  and  $v = \frac{y}{\sqrt{2}}$

0    Y=0    X

$$\Rightarrow y = -\sqrt{2} u \text{ and } y = \sqrt{2} v$$

$$\Rightarrow -\sqrt{2} u = \sqrt{2} v$$

$$\Rightarrow \boxed{u = -v}$$



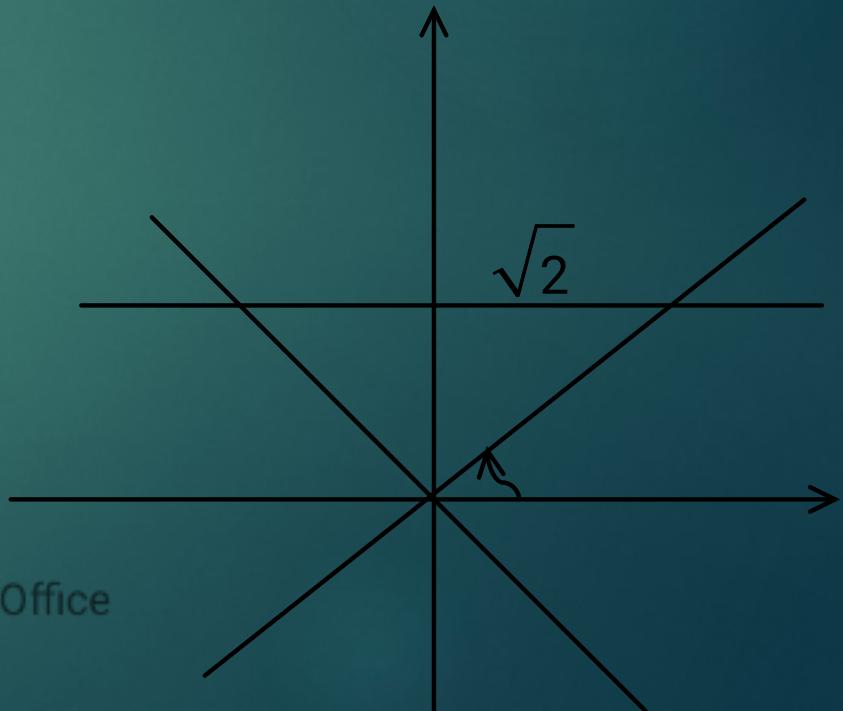
when  $y = 0$ ,  $\boxed{V = \frac{x}{\sqrt{2}}} \text{ and } v = \frac{x}{\sqrt{2}}$

$$\Rightarrow \boxed{U = -V} \quad \boxed{U = V}$$

$\Pi/4$

when  $x + y = 1$   $\Rightarrow \boxed{V = \frac{1}{\sqrt{2}}}$

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# EXAMPLE FOR ROTATION

THE REGION IN THE Z-PLANE IS MAPPED ON TO THE  
REGION BOUNDED BY  $U = -\frac{1}{\sqrt{2}} \leq U \leq V$ ,  
IN THE W-  
PLANE.

$\therefore$  The mapping  $w = ze^{i\pi/4}$  performs a rotation  
of  $R$  through an angle  $\pi / 4$ .



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# INVERSE TRANSFORMATION

## THE RECIPROCAL TRANSFORMATION $W = 1/Z$

THE MAPPING

$$w = \frac{1}{z}$$

CALLED THE RECIPROCAL

TRANSFORMATION AND MAPS THE Z-PLANE ONE-TO-ONE AND ONTO THE W-PLANE EXCEPT FOR THE POINT  $Z=0$ , WHICH HAS NO IMAGE, AND THE POINT  $W=0$ , WHICH HAS NO PREIMAGE OR INVERSE IMAGE. USE THE EXPONENTIAL NOTATION

WE HAVE

$$z = r e^{i\theta} \neq 0$$

IN THE W-PLANE. IF ,

$$w = \rho e^{i\phi} = \frac{1}{z} = \frac{1}{r} e^{-i\theta}$$

$$w = \rho e^{i\phi}$$



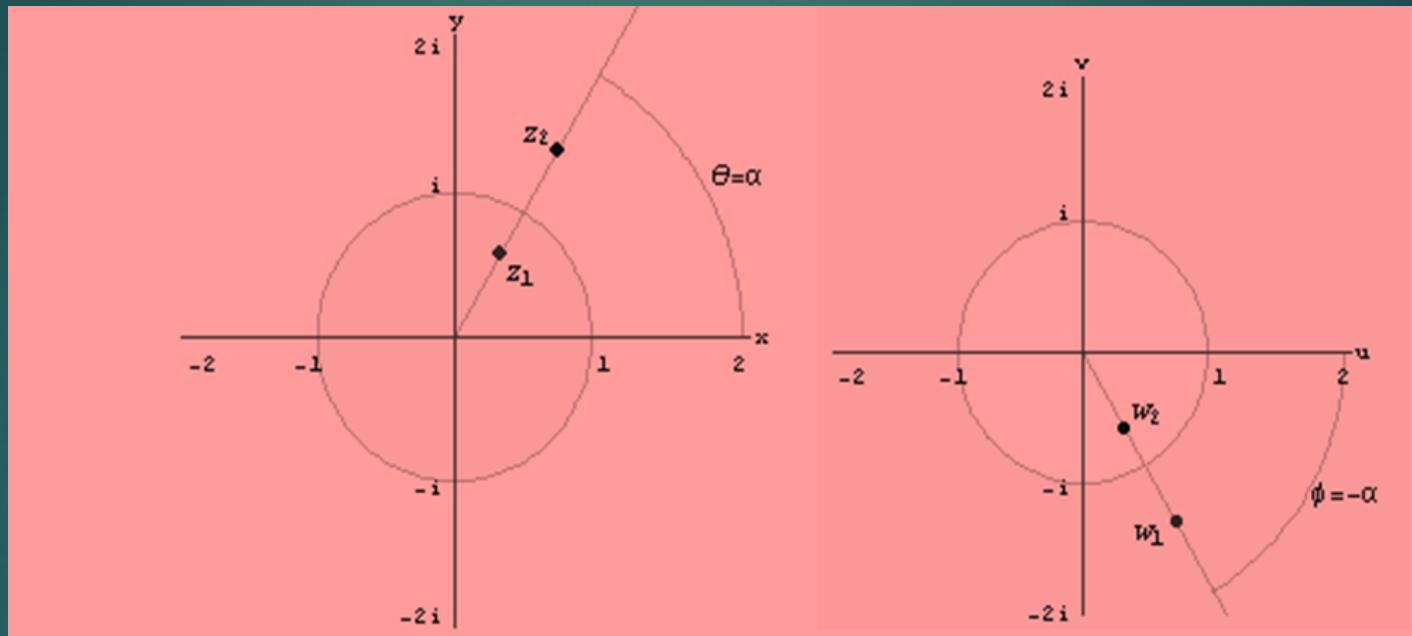
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# INVERSE TRANSFORMATION

THE GEOMETRIC DESCRIPTION OF THE RECIPROCAL TRANSFORMATION IS NOW EVIDENT. IT IS AN INVERSION (THAT IS, THE MODULUS OF  $\frac{1}{z}$  IS THE RECIPROCAL OF THE MODULUS OF Z) FOLLOWED BY A REFLECTION THROUGH THE X AXIS. THE RAY  $r > 0, \theta = \alpha$  IS MAPPED ONE-TO-ONE AND ONTO THE RAY  $r > 0, \theta = -\alpha$ . POINTS THAT LIE INSIDE THE UNIT CIRCLE ARE MAPPED ONTO POINTS THAT LIE OUTSIDE THE UNIT CIRCLE AND VICE VERSA. THE SITUATION IS ILLUSTRATED IN FIGURE.



# INVERSE TRANSFORMATION



# EXAMPLE OF INVERSE TRANSFORMATION

- 1) SHOW THAT THE IMAGE OF THE RIGHT-HANDED HALF-PLANE  
 UNDER THE MAPPING  $w = f(z) = \frac{1}{z}$  IS THE CLOSED DISK  
 $\overline{D_1(1)} = \{w : |w - 1| \leq 1\}$  IN THE W-PLANE.

**SOLUTION:-**

$$u + iv = w = f(z) = \frac{1}{z}$$

THEN,

$$z = f^{-1}(w) = \frac{1}{w}$$

# EXAMPLE OF INVERSE TRANSFORMATION

$$u + \bar{z}v = w = f(z) \in \overline{D_1(1)}$$

$$\iff f^{-1}(w) = x + \bar{z}y \in A$$

$$\iff \frac{1}{u + \bar{z}v} = x + \bar{z}y \in A$$

$$\iff \frac{u}{u^2 + v^2} + \bar{z} \frac{-v}{u^2 + v^2} = x + \bar{z}y \in A$$

$$\iff \frac{u}{u^2 + v^2} = x \geq \frac{1}{2}, \quad \text{and} \quad \frac{-v}{u^2 + v^2} = y$$

$$\iff \frac{u}{u^2 + v^2} \geq \frac{1}{2}$$

$$\iff u^2 - 2u + 1 + v^2 \leq 1$$

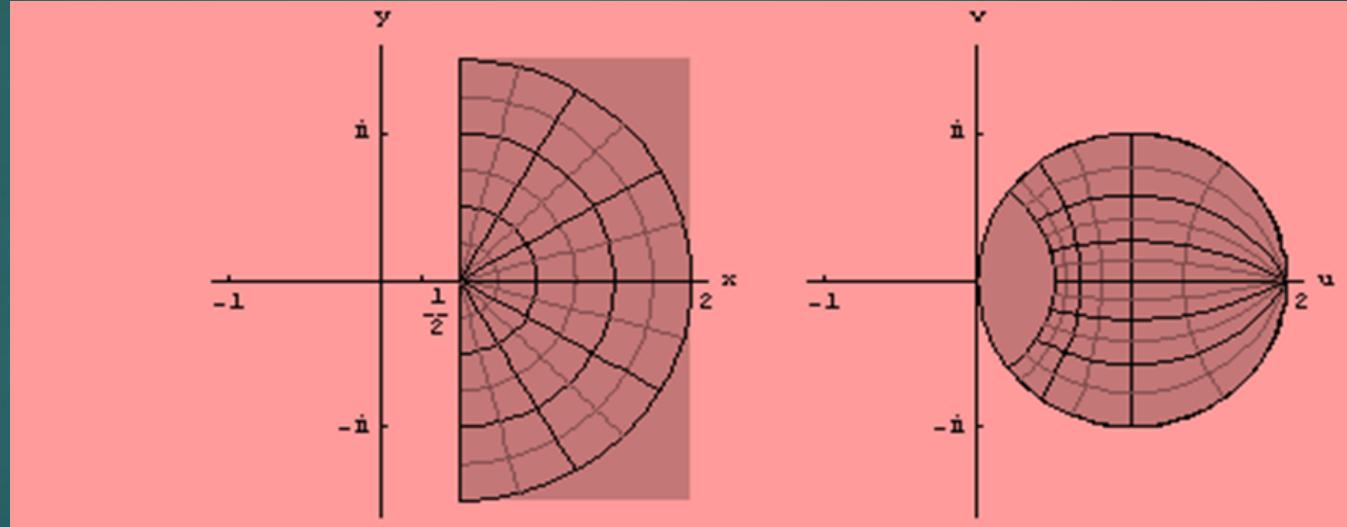
$$\iff (u-1)^2 + v^2 \leq 1$$



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# EXAMPLE OF INVERSE TRANSFORMATION

WHICH DESCRIBES THE DISK. AS THE RECIPROCAL TRANSFORMATION IS ONE-TO-ONE, PREIMAGES OF THE POINTS IN THE DISK  $\overline{D_1(1)}$  WILL LIE IN THE RIGHT HALF-PLANE . FIGURE ILLUSTRATES THIS RESULT.



# EXAMPLE OF INVERSE TRANSFORMATION

2) FIND THE IMAGES OF THE FINITE STRIPS,

$$\frac{1}{4} \leq y \leq \frac{1}{2} \quad \text{under the transformation } w = \frac{1}{z}.$$

Solution: –      Given:  $w = \frac{1}{z}$

$$\Rightarrow z = \frac{1}{w}$$

$$\text{i.e., } x + iy = \frac{1}{u + iv} = \frac{u - iv}{u^2 + v^2}$$

$$\therefore x = \frac{u}{u^2 + v^2} \quad \text{and} \quad y = \frac{-v}{u^2 + v^2}$$

$\downarrow \rightarrow (1)$

$\downarrow \rightarrow (2)$



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# EXAMPLE OF INVERSE TRANSFORMATION

Given:  $\frac{1}{4} < y < \frac{1}{2}$

when  $y = \frac{1}{4}$  eqnuation (2) becomes,

$$\frac{1}{4} = \frac{-v}{u^2 + v^2}$$

$$\Rightarrow u^2 + v^2 = -4v$$

$$\Rightarrow u^2 + v^2 + 4v + 4 - 4 = 0$$

$$\Rightarrow u^2 + (v + 2)^2 = 4$$

which is a circle whose centre at  $(0, -2)$  and radius is 2 in  $w$ - plane.



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# EXAMPLE OF INVERSE TRANSFORMATION

when  $y = \frac{1}{2}$ , equation (2) becomes,

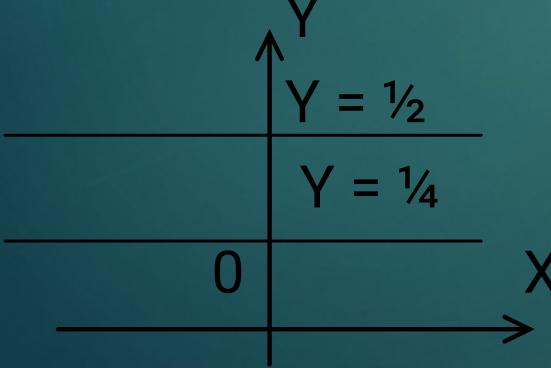
$$\frac{1}{2} = \frac{-v}{u^2 + v^2}$$

$$\Rightarrow u^2 + v^2 = -2v$$

$$\Rightarrow u^2 + v^2 + 2v + 1 - 1 = 0$$

$$\Rightarrow \boxed{u^2 + (v+1)^2 = 1}$$

which is a circle whose centre is at  $(0, -1)$  and radius is 1.



# BILINEAR TRANSFORMATION

Def.:-

The transformation  $w = \frac{az + b}{cz + d}$ , where  $a, b, c, d$

are complex constants and  $ad - bc \neq 0$  is known as bilinear transformation.

Note:-

(i) A bilinear transformation is also called as Möbius transformation or a linear fractional transformation.

(ii) The inverse mapping of  $w = \frac{az + b}{cz + d}$  is  $z = \frac{-wd + b}{cw - a}$

is also called as a bilinear transformation.

# BILINEAR TRANSFORMATION

## FIXED POINTS (OR) INVARIANT POINTS :-

IF THE IMAGE OF A POINT  $Z$  UNDER A TRANSFORMATION  $w=F(z)$  IS ITSELF, THEN THE POINT IS CALLED A FIXED POINT OR AN INVARIANT POINT OF THE TRANSFORMATION.

THUS FIXED POINT OF THE TRANSFORMATION  $w=F(z)$  IS GIVEN BY  $Z = F(Z)$ .

Eg.: Let  $w = \frac{z}{z-2}$ , find the fixed point (or)

invariant point.

$$\text{Solution:--} \quad \text{put } w = z \text{ then } z = \frac{z}{z-2} \Rightarrow z^2 - 2z = z$$

$$\Rightarrow z(z-3) = 0$$

$$\Rightarrow z = 0, z = 3 \text{ are two fixed points.}$$



# BILINEAR TRANSFORMATION

Definition of cross ratio:-

If  $z_1, z_2, z_3, z_4$  are four points in the z-plane then

the ratio  $\frac{(z_1 - z_2)(z_3 - z_4)}{(z_1 - z_4)(z_3 - z_2)}$  is called the cross ratio of these points.

Cross Ratio Property of a bilinear transformation:-

The cross ratio of four points is invariant under a bilinear transformation.

i.e., If  $w_1, w_2, w_3, w_4$  are the images of  $z_1, z_2, z_3, z_4$  respectively under a bilinear transformation then

$$\left( \frac{(w_1 - w_2)(w_3 - w_4)}{(w_1 - w_4)(w_3 - w_2)} \right) = \left( \frac{(z_1 - z_2)(z_3 - z_4)}{(z_1 - z_4)(z_3 - z_2)} \right)$$



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# BILINEAR TRANSFORMATION

NOTE:-

THE BILINEAR TRANSFORMATION WHICH TRANSFORMS THE POINTS

$z_1, z_2, z_3$  OF Z-PLANE RESPECTIVELY INTO THE POINTS  $w_1, w_2, w_3$  OF

W-PLANE IS GIVEN BY 
$$\left( \frac{(w - w_1)(w_2 - w_3)}{(w_1 - w_2)(w_3 - w)} \right) = \left( \frac{(z - z_1)(z_2 - z_3)}{(z_1 - z_2)(z_3 - z)} \right)$$



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# BILINEAR TRANSFORMATION

## EXAMPLES

- 1) Find the bilinear transformation which maps the points  $z = 0, -i, -1$  into  $w = i, 1, 0$ .

Solution:-

$$\text{Given: } z_1 = 0, z_2 = -i, z_3 = -1$$

$$\text{and } w_1 = i, w_2 = 1, w_3 = 0.$$

The bilinear transformation is got by using the relation

$$\frac{(w - w_1)(w - w_3)}{(w_1 - w_2)(w_3 - w)} = \frac{(z - z_1)(z - z_3)}{(z_1 - z_2)(z_3 - z)}$$

$$\Rightarrow \frac{(w - i)(1 - 0)}{(i - 1)(0 - w)} = \frac{(z - 0)(-i + 1)}{(0 - 1)(\bar{z})}$$

# BILINEAR TRANSFORMATION

## EXAMPLES

$$\Rightarrow (-i)(w - i)(1 + z) = z(1 - i)(-w)(i - 1)$$

$$\Rightarrow -i - iwz - 1 - z = -2iwz$$

$$\Rightarrow -iw + iwz = 1 + z$$

$$\Rightarrow w(zi - i) = (1 + z)$$

$$\Rightarrow w = \frac{1 + z}{zi - i}$$

$$\Rightarrow w = \frac{1 + z}{(-i)(1 - z)}$$



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# BILINEAR TRANSFORMATION

## EXAMPLES

2) Find the bilinear transformation which transforms the points  $z = \infty, i, 0$  into the points  $w = 0, i, \infty$  respectively.

Solution :-

Given:  $z_1 = \infty, z_2 = i, z_3 = 0$

and  $w_1 = 0, w_2 = i, w_3 = \infty$ .

The bilinear transformation is got by using the relation

$$\frac{(w - w_1)(w_2 - w_3)}{(w_1 - w_2)(w_3 - w)} = \frac{(z - z_1)(z_2 - z_3)}{(z_1 - z_2)(z_3 - z)}$$



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# BILINEAR TRANSFORMATION

## EXAMPLES

$$\frac{(w - w_1)(w_3) \left( \frac{w^2}{w_3} - 1 \right)}{(w_1 - w_2)(w_3) \left( 1 - \frac{w}{w_3} \right)} = \frac{(z_1) \left( \frac{z}{z_1} - 1 \right) (z_2 - z_3)}{(z_1) \left( 1 - \frac{z^2}{z_1} \right) (z_3 - z)}$$

$$\Rightarrow \frac{(w - 0)(0 - 1)}{(0 - 1)(i - 0)} = \frac{(0 - 1)(i - 0)}{(z - 0)(0 - 1)}$$

$$\Rightarrow \frac{(-w)}{(-i)} = \frac{(-i)}{(-z)}$$

$$\Rightarrow \boxed{w = \frac{-1}{z}}$$



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# BILINEAR TRANSFORMATION

## TRY IT

3) FIND THE BILINEAR TRANSFORMATION WHICH MAPS THE POINTS,

- I)  $1, -1, 2$  onto  $0, 2, i$  respectively.
- II)  $-1, 0, i$  into  $-1, i, 1$  respectively.
- III)  $0, 1, -i$  into  $i, -1, -i$  respectively.



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# THANK YOU



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