2. Reduce the quadratic form $8x_1^2 + 7x_2^2 + 3x_3^2 - 12x_1x_2 - 8x_2x_3 + 4x_3x_1$ to a canonical form and state its nature. [Ans: $\lambda = 15, 0, 3, (1,2,2)^\gamma, (2,1,-2)^\gamma, (2,-2,1)^\gamma$ positive semi definite]

3. Reduce the following to a canonical forms by orthogonal transformation:
(i) $Q = 7x_1^2 + 10x_2^2 + 7x_3^2 - 4x_1x_2 + 4x_2x_3 + 2x_3x_1$ (ii) $Q = x_1^2+4x_2^2+9x_3^2+4x_1x_2+12x_2x_3+6x_3x_1$ (iii) $Q = 7x_1^2 - 8x_2^2 - 8x_3^2 + 8x_1x_2 - 2x_2x_3 - 6x_3x_1$ [Ans: i) $2y_1^2 + 4y_2^2 + 6y_3^2$ ii) $14y_1^2$ iii) $9(y_1^2 - y_2^2 + y_3^2)$]

4. Find the nature of Q.F without reducing to canonical form i) $x_1^2+2x_1x_2+x_2^2$ ii) $2x_1^2+x_2^2-3x_3^2+12x_1x_2-8x_2x_3-4x_3x_1$

2 FUNCTIONS OF SEVERAL VARIABLES

2.1 Introduction

This chapter is devoted to a study of functions depending on more than one independent variable. A real function $z = f(x,y)$ of two independent variables x and y, can be thought to represent a surface in the three-dimensional space referred to a set of co-ordinate axes X, Y, Z.

A simple example of a function of two independent variables x and y is $z = xy$, which represents the area of a rectangle whose sides are x and y.

Continuity of a function of two variables:

Definition: A function $z = f(x, y)$ is said to be continuous at the point (x_0, y_0) provided that a small change in the values of x and y produces a corresponding (small) change in the value of z. More precisely, if the value of $z = f(x, y)$ at (x_0, y_0) is z_0, then the continuity of the function at the point (x_0, y_0) means that

$$\lim_{x \to x_0, y \to y_0} f(x, y) = f(x_0, y_0) = z_0$$

If a function is continuous at all points of some region R in the XY plane, then it is said to be continuous in the region R.

The definition of continuity of a function of more than two independent variables is similar.

2.2 Partial Derivatives

The analytical definition of the derivatives of a function $y = f(x)$ of a single variable x is

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

Let Δz_x denote the increment in the function $z = f(x, y)$ where y is kept fixed and x is changed by an amount Δx

(i.e.) $\Delta z = f(x + \Delta x, y) - f(x, y)$.

Then,

$$\lim_{\Delta x \to 0} \frac{\Delta z}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(x + \Delta x, y) - f(x, y)}{\Delta x}$$

is called the partial derivative of z with request to x and is denoted by $\frac{\partial z}{\partial x}$ or z_x. Similarly, the partial derivatives of z with respect y is defined and denoted as $\frac{\partial z}{\partial y}$ or z_y.

In general, if $z = f(x_1, x_2, \cdots, x_n)$ is a function of n independent variables $x_1, x_2, \cdots, x_n$, then $\frac{\partial z}{\partial x_i}$ denote the partial derivatives of z with respect to $x_i (i = 1, 2, \cdots n)$, when the remaining variables are treated as constants.

Example 1:

If $z = x^2 + y^2 + 3xy$, then $\frac{\partial z}{\partial x} = 2x + 3y$, $\frac{\partial z}{\partial y} = 2y + 3x$

Example 2:

If $u = e^x \sin y \cos z$, then $\frac{\partial u}{\partial x} = e^x \sin y \cos z$ (when y and z are held constants), $\frac{\partial u}{\partial y} = -e^x \cos y \cos z$ (Here x and z are treated as constants) and $\frac{\partial u}{\partial z} = e^x \sin y \sin z$ (both x and y are held constants).

Total Differential: If $z = f(x, y)$ is a function of two independent variables x and y, then the total differential of z is denoted as dz and defined as

$$dz = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy$$

In general, if $u = f(x_1, x_2, \cdots, x_n)$, then the total differential of u is given by $du = \frac{\partial f}{\partial x_1}dx_1 + \frac{\partial f}{\partial x_2}dx_2 + \cdots + \frac{\partial f}{\partial x_n}dx_n$

It may thus be noted that the total differential is equal to the sum of the partial differentials.

Example 3:

A metal box without a top has inside dimensions 6 ft, 4 ft and 2 ft. If the metal is 0.1 ft. thick, find the approximate volume by using the differential.

Let x, y, z be the dimensions of a metal box. Then its volume is $V = xyz$.

More its differential is given by

$$dV = \frac{\partial V}{\partial x}dx + \frac{\partial V}{\partial y}dy + \frac{\partial V}{\partial z}dz$$

$$= yzdx + xzdy + xydz$$

$$= 8 \times 0.2 + 12 \times 0.2 + 24 \times 0.1$$

$$= 1.6 + 2.4 + 2.4$$

$$= 6.4 cu.ft$$

2.3 Total Differentiation:

Let $z = f(x, y)$ and let x and y be both functions of one independent variable t such that z becomes a function of this single independent variable and therefore, z may have a derivative with respect to t, called total derivative of z with respect to t, obtained via partial derivatives of f with respect to x and y.

(i. e) when $z = f(x, y)$, $x = g(t)$ and $y = h(t)$, then $\frac{dz}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$ In general, if $z = f(x_1, x_1, \cdots, x_n)$ with $x_1 = g(t), x_2 = g(t), \cdots, x_n = g(t)$, then $\frac{dz}{dt} = \frac{\partial f}{\partial x_1}\frac{dx_1}{dt} + \frac{\partial f}{\partial x_2}\frac{dx_2}{dt} + \cdots + \frac{\partial f}{\partial x_n}\frac{dx_n}{dt}$.

Example 2.1. *If $f(x, y) = x^2 + y^2$, where $x = r\cos\theta$ and $y = r\sin\theta$, find (i) $\frac{df}{dr}$ and $\frac{df}{d\theta}$ and also (ii) df.*

Solution: Given $f(x, y) = x^2 + y^2$, where $x = r\cos\theta$ and $y = r\sin\theta$ (i) Now, $\frac{df}{dr} = \frac{\partial f}{\partial x}\frac{dx}{dr} + \frac{\partial f}{\partial y}\frac{dy}{dr} = 2x\cos\theta + 2y\sin\theta$

$$= 2r(\cos^2\theta + \sin^2\theta) = 2r$$

(ii) $\frac{df}{d\theta} = \frac{\partial f}{\partial x}\frac{dx}{d\theta} + \frac{\partial f}{\partial y}\frac{dy}{d\theta} = 2x(-r\sin\theta) + 2y(r\cos\theta)$

$$= -2r^2\cos\theta\sin\theta + 2r^2\cos\theta\sin\theta = 0$$

(iii) $df = 2rdr$ ($\because$ from (i) $df/dr = 2r$)

$= 2xdx + 2ydy$ ($\because r^2 = x^2 + y^2$)

2.3.1 Homogeneous Function:

A function $f(x, y)$ is said to be a homogeneous function of degree $'n'$ if it can be expressed of the form $x^n \phi\left(\frac{y}{x}\right)$.

2.4 Euler's theorem:

If u is a homogeneous function of degree $'n'$, then

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = nu$$

and $x^2\dfrac{\partial^2 u}{\partial x^2} + 2xy\dfrac{\partial^2 u}{\partial x \partial y} + y^2\dfrac{\partial^2 u}{\partial y^2} = n(n-1)u$

Example 2.2. *If* $u = \sin^{-1}\left(\dfrac{x^2 + y^2}{x - y}\right)$, *then show that* $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = \tan u$.

Solution: Given $u = \sin^{-1}\left(\dfrac{x^2 + y^2}{x - y}\right)$ (i.e.) $\sin u = \dfrac{x^2 + y^2}{x - y} = x'\phi\left(\frac{y}{x}\right)$, $\sin u$ is a homogeneous function of degree 1.

Using Euler's theorem for the homogeneous function $\sin u$, we have

$$x\frac{\partial}{\partial x}(\sin u) + y\frac{\partial}{\partial y}(\sin u) = 1.\sin u$$

$$\cos u\left[x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}\right] = \sin u$$

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{\sin u}{\cos u} = \tan u$$

2.4.1 Chain Rule:

If $z = f(x, y)$, where x and y are functions of given two variables u and v, then $\dfrac{\partial z}{\partial u} = \dfrac{\partial z}{\partial x}\dfrac{\partial x}{\partial u} + \dfrac{\partial z}{\partial y}\dfrac{\partial y}{\partial u}$ and $\dfrac{\partial z}{\partial v} = \dfrac{\partial z}{\partial x}\dfrac{\partial x}{\partial v} + \dfrac{\partial z}{\partial y}\dfrac{\partial y}{\partial v}$

Example 2.3. *Find* $\dfrac{dz}{dt}$, *when* $z = xy^2 + x^2y$, *where* $x = at^2, y = 2at$

Solution: We know that $\dfrac{dz}{dt} = \dfrac{\partial z}{\partial x}\dfrac{dx}{dt} + \dfrac{\partial z}{\partial y}\dfrac{dy}{dt}$

$= (y^2 + 2xy)(2at) + (2xy + x^2)(2a)$

$= (2at)(2at)^2 + 4at(at^2)(2at) + 4a(at^2)(2at) + 2a(at^2)^2$

$= 8a^3t^3 + 8a^3t^4 + 8a^3t^3 + 2a^3t^4 = 16a^3t^3 + 10a^3t^4$

$= 2a^3t^3(8+5t)$

Example 2.4. *If* $u = \sin\left(\frac{x}{y}\right)$, $x = e^t, y = t^2$, *find* $\frac{du}{dt}$

Solution: Let $u = \sin\left(\frac{x}{y}\right)$, $x = e^t, y = t^2$ $\frac{du}{dt} = \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt}$

$$= \cos\left(\frac{x}{y}\right)\left(\frac{1}{y}\right)(e^t) + \cos\left(\frac{x}{y}\right)\left(\frac{-x}{y^2}\right)(2t)$$
$$= \cos\left(\frac{e^t}{t^2}\right)\left(\frac{e^t}{t^2}\right) - \cos\left(\frac{e^t}{t^2}\right).2.\left(\frac{e^t}{t^3}\right)$$
$$= \frac{e^t}{t^2}\cos\left(\frac{e^t}{t^2}\right)\left[1 - \frac{2}{t}\right]$$

Example 2.5. *If* $u = x^2 - y^2$, $v = 2xy$, $f(x,y) = \varphi(u,v)$ *show that* $\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 4(x^2+y^2)\left(\frac{\partial^2\varphi}{\partial u^2} + \frac{\partial^2\varphi}{\partial v^2}\right)$

Solution:

Let $\frac{\partial f}{\partial x} = \frac{\partial \varphi}{\partial u}.\frac{\partial u}{\partial x} + \frac{\partial \varphi}{\partial v}.\frac{\partial v}{\partial x} = \frac{\partial \varphi}{\partial u}(2x) + \frac{\partial \varphi}{\partial v}(2y)$

$$\therefore \frac{\partial^2 f}{\partial x^2} = \frac{\partial^2\varphi}{\partial u^2}(4x^2) + 2\frac{\partial\varphi}{\partial u}(1) + \frac{\partial^2\varphi}{\partial v^2}(4y^2) \tag{2.1}$$

$$\frac{\partial f}{\partial y} = \frac{\partial \varphi}{\partial u}.\frac{\partial u}{\partial x} + \frac{\partial \varphi}{\partial v}.\frac{\partial v}{\partial x} = \frac{\partial \varphi}{\partial u}(-2y) + \frac{\partial \varphi}{\partial v}(2x)$$

$$\therefore \frac{\partial^2 f}{\partial y^2} = \frac{\partial^2\varphi}{\partial u^2}(4y^2) + \frac{\partial\varphi}{\partial u}(-2) + \frac{\partial^2\varphi}{\partial v^2}(4x^2) \tag{2.2}$$

Equation (2.1) + (2.2) gives,

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = \left(\frac{\partial^2\varphi}{\partial u^2} + \frac{\partial^2\varphi}{\partial v^2}\right)(4x^2+4y^2)$$

$$= 4(x^2+y^2)\left(\frac{\partial^2\varphi}{\partial u^2} + \frac{\partial^2\varphi}{\partial v^2}\right)$$

Example 2.6. *If* $u = f(r)$, $x = r\cos\theta$, $y = r\sin\theta$, *prove that*

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f''(r) + \left(\frac{1}{r}\right)f'(r)$$

Solution: Given $u = f(r)$ where $r = \sqrt{x^2+y^2}$

Now, $\frac{\partial u}{\partial x} = \frac{\partial u}{\partial r}\frac{\partial r}{\partial x} = f'(r).\frac{2x}{2\sqrt{x^2+y^2}} = f'(r)\frac{x}{r}$

$$\therefore \frac{\partial^2 u}{\partial x^2} = \frac{x}{r}f''(r)\frac{\partial r}{\partial x} + f'(r)\frac{1}{r} + xf'(r)\frac{(-1)}{r^2}\frac{\partial r}{\partial x}$$

$$= \frac{x^2}{r^2}f''(r) + f'(r)\frac{1}{r} - \frac{x^2}{r^3}f'(r)\left[\because \frac{\partial r}{\partial x} = \frac{x}{r}\right] \tag{2.3}$$

Similarly,

$$\frac{\partial^2 u}{\partial y^2} = \frac{y^2}{r^2}f''(r) + f'(r)\frac{1}{r} - \frac{y^2}{r^3}f'(r) \tag{2.4}$$

Equations (2.3) + (2.4) gives, $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = f''(r) + \dfrac{1}{r} f'(r)$

2.4.2 Differentiation of Implicit Functions

Consider the implicit function $f(x, y) = 0$. Then, $\dfrac{dy}{dx} = -\dfrac{(\partial f/\partial x)}{(\partial f/\partial y)}$

Example 2.7. *Find* $\dfrac{dy}{dx}$, *if* $xe^{-y} - 2ye^{x} = 1$.

Solution: Given $f(x, y) = xe^{-y} - 2ye^{x} - 1 = 0$

$\dfrac{\partial f}{\partial x} = e^{-y} - 2ye^{x}$ and $\dfrac{\partial f}{\partial y} = -xe^{-y} - 2e^{x}$

$$\frac{dy}{dx} = -\frac{(\partial f/\partial x)}{(\partial f/\partial y)} = -\frac{(e^{-y} - 2ye^{x})}{(-xe^{-y} - 2e^{x})} = \frac{e^{-y} - 2ye^{x}}{xe^{-y} + 2e^{x}}$$

Example 2.8. *If* $u = x\log(xy)$, *where* $x^3 + y^3 + 3xy = 1$, *find* $\dfrac{du}{dx}$

Solution:

$$\frac{du}{dx} = \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}\frac{dy}{dx} \tag{2.5}$$

Now $f(x, y) = 0$ is $x^3 + y^3 + 3xy - 1 = 0$

$$\frac{dy}{dx} = -\frac{(\partial f/\partial x)}{(\partial f/\partial y)} = -\frac{(3x^2 + 3y)}{(3y^2 + 3x)} = -\left(\frac{x^2 + y}{y^2 + x}\right) \tag{2.6}$$

Equation (2.5) becomes,

$$\begin{aligned}\frac{du}{dx} &= x.\frac{1 \times y}{xy} + \log(xy) + x.\frac{1 \times x}{xy}\left(-\frac{x^2 + y}{y^2 + x}\right)\\ &= 1 + \log(xy) - \frac{x}{y}\left(\frac{x^2 + y}{y^2 + x}\right)\end{aligned}$$

Example 2.9. *Find* $\dfrac{dy}{dx}$, *if* $(\cos x)^y = (\sin y)^x$

Solution: Given $(\cos x)^y = (\sin y)^x$

Taking log on both sides, we get, $y\log(\cos x) = x\log(\sin y)$

(i.e.) $f(x, y) = y\log(\cos x) - x\log(\sin y) = 0$

$$\frac{dy}{dx} = -\frac{(\partial f/\partial x)}{(\partial f/\partial y)} = -\frac{\left[y\dfrac{(-\sin x)}{\cos x} - \log(\sin y)\right]}{\left[\log(\cos x) - x\dfrac{\cos y}{\sin y}\right]}$$

(i.e.) $\dfrac{dy}{dx} = \dfrac{y\tan x + \log(\sin y)}{\log(\cos x) - x\cot y}$

2.4.3 Taylor's theorem for functions of two variables

We know that by Taylor's theorem for a function $f(x)$ of a single variable x, we have

$$f(x+h) = f(x) + hf'(x) + \frac{h^2}{2!}f''(x) + \frac{h^3}{3!}f'''(x) + \cdots$$

Now let $f(x, y)$ be a function of two independent variables x and y. If y is kept constant, then by Taylor's theorem for a function of a single variable x, we have

$$f(x+h, y+k) = f(x, y+k) + h\frac{\partial}{\partial x}f(x, y+k)$$

$$+\frac{h^2}{2!}\frac{\partial^2}{\partial x^2}f(x, y+k) + \frac{h^3}{3!}\frac{\partial^3}{\partial x^3}f(x, y+k) + \cdots \tag{2.7}$$

Now keeping x constant and applying Taylor's theorem for a function of single variable y, we have

$$f(x, y+k) = f(x, y) + k\frac{\partial}{\partial y}f(x, y) + \frac{k^2}{2!}\frac{\partial^2}{\partial y^2}f(x, y)$$

$$+\frac{k^3}{3!}\frac{\partial^2}{\partial y^3}f(x, y) + \cdots \tag{2.8}$$

Using (4.7), we can write (4.6) as

$$f(x+h, y+k) = \left[f(x, y) + k\frac{\partial}{\partial y}f(x, y) + \frac{k^2}{2!}\frac{\partial^2}{\partial y^2}f(x, y)\right.$$

$$\left.+\frac{k^3}{3!}\frac{\partial^2}{\partial y^3}f(x, y) + \cdots\right] + h\frac{\partial}{\partial x}\left[f(x, y) + k\frac{\partial}{\partial y}f(x, y)\right.$$

$$\left.+\frac{k^2}{2!}\frac{\partial^2}{\partial y^2}f(x, y) + \frac{k^3}{3!}\frac{\partial^2}{\partial y^3}f(x, y) + \cdots\right]$$

$$+\frac{h^2}{2!}\frac{\partial^2}{\partial x^2}\left[f(x, y) + k\frac{\partial}{\partial y}f(x, y) + \frac{k^2}{2!}\frac{\partial^2}{\partial y^2}f(x, y) + \frac{k^3}{3!}\frac{\partial^2}{\partial y^3}f(x, y) + \cdots\right]$$

$$+\frac{h^3}{3!}\frac{\partial^3}{\partial x^3}\left[f(x, y) + k\frac{\partial}{\partial y}f(x, y) + \frac{k^2}{2!}\frac{\partial^2}{\partial y^2}f(x, y) + \cdots\right] + \cdots$$

$$= \left[f(x, y) + k\frac{\partial}{\partial y}f(x, y) + \frac{k^2}{2!}\frac{\partial^2}{\partial y^2}f(x, y) + \cdots\right]$$

$$+\left[h\frac{\partial f}{\partial x} + hk\frac{\partial^2 f}{\partial x \partial y} + h\frac{k^2}{2!}\frac{\partial^3 f}{\partial x \partial y^2} + \cdots\right] + \left[\frac{k^2}{2!}\frac{\partial^2 f}{\partial x^2} + \cdots\right]$$

$$+\frac{h^3}{3!}\frac{\partial^3 f}{\partial x^3} + \cdots$$

$$= f(x, y) + \left(h\frac{\partial f}{\partial x} + k\frac{\partial f}{\partial y}\right) + \left(\frac{h^2}{2!}\frac{\partial^2 f}{\partial x^2} + hk\frac{\partial^2 f}{\partial x \partial y} + \frac{k^2}{2!}\frac{\partial^2 f}{\partial y^2}\right)$$

$$+\left(\frac{h^3}{3!}\frac{\partial^3 f}{\partial x^3} + \frac{h^2 k}{2!}\frac{\partial^3 f}{\partial x^2 \partial y} + \frac{hk^2}{2!}\frac{\partial^3 f}{\partial x \partial y^2} + \frac{k^3}{3!}\frac{\partial^3 f}{\partial y^3}\right)$$

$$= f(x, y) + \left(h\frac{\partial f}{\partial x} + k\frac{\partial f}{\partial y}\right) + \frac{1}{2!}\left(\frac{h^2\partial^2 f}{\partial x^2} + 2hk\frac{\partial^2 f}{\partial x \partial y} + \frac{k^2\partial^2 f}{\partial y^2}\right)$$

$$+\frac{1}{3!}\left(\frac{h^3\partial^3 f}{\partial x^3} + 3h^2k\frac{\partial^3 f}{\partial x^2 \partial y} + 3hk^2\frac{\partial^3 f}{\partial x \partial y^2} + k^3\frac{\partial^3 f}{\partial y^3}\right)$$

$$= f(x, y) + \left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)f + \frac{1}{2!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^2 f$$

$$+\frac{1}{3!}\left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right)^3 f+\cdots$$

Corollary 1: Putting $x=a$ and $y=b$, we have

$$\begin{aligned}f(a+h,b+k) &= f(a,b)+[hf_x(a,b)+kf_y(a,b)]\\ &+\frac{1}{2!}[h^2f_{xx}(a,b)+2hkf_{xy}(a,b)+k^2f_{yy}(a,b)]\\ &+\frac{1}{3!}[h^3f_{xxx}(a,b)+3h^2kf_{xxy}(a,b)+3hk^2f_{xyy}(a,b)\\ &+k^3f_{yyy}(a,b)]+\cdots\end{aligned}$$

Corollary 2:

In cor 1, putting $a+h=x$ and $b+k=y$ so that $h=x-a$ and $k=y-b$, we have

$$\begin{aligned}f(x,y) &= f(a,b)+[(x-a)f_x(a,b)+(y-b)f_y(a,b)]\\ &+\frac{1}{2!}[(x-a)^2f_{xx}(a,b)+2(x-a)(y-b)f_{xy}(a,b)\\ &+(y-b)^2f_{yy}(a,b)]+\cdots\end{aligned}$$

Corollary 3:

Putting $a=0, b=0$ in cor. 2, we have

$$\begin{aligned}f(x,y) &= f(0,0)+[xf_x(0,0)+yf_y(0,0)]\\ &+\frac{1}{2!}[x^2f_{xx}(0,0)+2xyf_{xy}(0,0)+y^2f_{yy}(0,0)]+\cdots\end{aligned}$$

This is called **Maclaurin's theorem** for two variables.

Note: Cor 3 is used to expand $f(x,y)$ in powers of x and y (i. e) near origin), whereas cor 2 is used to expand $f(x,y)$ in the neighbourhood of (a,b).

Example 2.10. *Expand $x^2y+3y-2$ in powers of $(x-1)$ and $(y+2)$ using Taylor's theorem upto terms of third degree.*

Solution: : The Taylor series expansion of $f(x,y)$ in powers of $(x-a)$ and $(y-b)$ is given by

$$f(x,y)=f(a,b)+[(x-a)f_x(a,b)+(y-b)f_y(a,b)]$$

$$+\frac{1}{2!}[(x-a)^2f_{xx}(a,b)+2(x-a)(y-b)f_{xy}(a,b)$$

$$+(y-b)^2f_{yy}(a,b)]+\cdots \tag{2.9}$$

Here $f(x,y)=x^2y+3y-2$, $a=1, b=-2 \Rightarrow f(1,-2)=-10$

$f_x(x,y)=2xy$	$f_x(1,-2)=-4$
$f_y(x,y)=x^2+3$	$f_{xy}(x,y)=2x$
$f_y(1,-2)=4$	$f_{xy}(1,-2)=2$
$f_{xx}(x,y)=2y$	$f_{yy}(x,y)=0$
$f_{xx}(1,-2)=-4$	$f_{yy}(1,-2)=0$
$f_{xxx}(x,y)=0$	$f_{yyy}(x,y)=0$
$f_{xxy}(x,y)=2$	$f_{xyy}(x,y)=0$

From (4.8)

$$\begin{aligned}x^2y+3y-2 &= -10+\frac{1}{1!}[(x-1)(-4)+(y+2)(4)]\\ &+\frac{1}{2!}[(x-1)^2(-4)+2(x-1)(y+2)(2)]\\ &+\frac{1}{3!}[3(x-1)^2(y+2)(2)]+\cdots\end{aligned}$$

(or) $x^2y+3y-2=-10-4(x-1)+4(y+2)-2(x-1)^2+2(x-1)(y+2)$

$+(x-1)^2(y+2)+\cdots$

Example 2.11. *Expand $e^x\cos y$ in powers of x and y as far as the terms of the third degree.*

Solution: Let $f(x,y)=e^x\cos y$

$f_x(x,y)=e^x\cos y$	$f_x(0,0)=1$
$f_y(x,y)=-e^x\sin y$	$f_{xxx}(x,y)=e^x\cos y$
$f_y(0,0)=0$	$f_{xxx}(0,0)=1$
$f_{xx}(x,y)=e^x\cos y$	$f_{xxy}(x,y)=-e^x\sin y$
$f_{xx}(0,0)=1$	$f_{xxy}(0,0)=0$
$f_{xy}(x,y)=-e^x\sin y$	$f_{xyy}(x,y)=-e^x\cos y$
$f_{xy}(0,0)=0$	$f_{xyy}(0,0)=-1$
$f_{yy}(x,y)=-e^x\cos y$	$f_{yyy}(x,y)=e^x\sin y$
$f_{yy}(0,0)=-1$	$f_{yyy}(0,0)=0$

Taylor's series of $f(x,y)$ in powers of x and y is

$$\begin{aligned}f(x,y) &= f(0,0)+[xf_x(0,0)+yf_y(0,0)]\\ &+\frac{1}{2!}[x^2f_{xx}(0,0)+2xyf_{xy}(0,0)+y^2f_{yy}(0,0)]+\cdots\end{aligned}$$

$$\begin{aligned}\therefore e^x\cos y &= 1+\frac{1}{1!}[x.1+y.0]+\frac{1}{2!}[x^2.1+2xy.0+y^2(-1)]\\ &+\frac{1}{3!}[x^3.1+3x^2y.0+3xy^2(-1)+y^3.0]+\cdots\\ &= 1+\frac{x}{1!}+\frac{x^2-y^2}{2!}+\frac{x^3-3xy^2}{3!}+\cdots\end{aligned}$$

Example 2.12. *Expand $\tan^{-1}\left(\frac{y}{x}\right)$ using Taylor's series near $(1,1)$ upto quadratic terms.*

Solution: The Taylor series expansion for $f(x,y)$ in powers of $(x-a)$ and $(y-b)$ is given by

$$f(x,y)=f(a,b)+[(x-a)f_x(a,b)+(y-b)f_y(a,b)]$$

$$+\frac{1}{2!}[(x-a)^2f_{xx}(a,b)+2(x-a)(y-b)f_{xy}(a,b)$$

$$+(y-b)^2f_{yy}(a,b)]+\cdots \tag{2.10}$$

Here $f(x,y)=\tan^{-1}\left(\frac{y}{x}\right), a=1=b \Rightarrow f(a,b)=f(1,1)=\tan^{-1}\left(\frac{1}{1}\right)=\frac{\pi}{4}$

$$f_x(x,y) = \frac{1}{1+\left(\frac{y^2}{x^2}\right)}\left(\frac{-y}{x^2}\right) = \frac{x^2}{x^2+y^2}\left(\frac{-y}{x^2}\right) = \frac{-y}{x^2+y^2}$$

$$\Rightarrow f_x(1,1) = -\frac{1}{2},\ f_y(x,y) = \frac{1}{1+\left(\frac{y^2}{x^2}\right)} \times \frac{1}{x} = \frac{x}{x^2+y^2}$$

$$\Rightarrow f_y(1,1) = \frac{1}{2}$$

$$f_{xx}(x,y) = (-y)\frac{1}{(x^2+y^2)^2}.2x = -\frac{2xy}{(x^2+y^2)^2}$$

$$\Rightarrow f_{xx}(1,1) = -\frac{2}{4} = -\frac{1}{2}$$

$$f_{xy}(x,y) = \frac{(x^2+y^2)(1) - x(2x)}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2}$$

$$\Rightarrow f_{xy}(1,1) = 0$$

$$f_{yy}(x,y) = (x)\frac{1}{(x^2+y^2)^2}.2y = \frac{2xy}{(x^2+y^2)^2}$$

$$\Rightarrow f_{yy}(1,1) = \frac{1}{2}$$

Using these values in (4.10) we get,

$$\tan^{-1}\left(\frac{y}{x}\right) = \frac{\pi}{4} + \frac{1}{1!}\left[(x-1)\left(-\frac{1}{2}\right) + (y-1)\frac{1}{2}\right]$$

$$+\frac{1}{2!}[(x-1)^2\left(-\frac{1}{2}\right)$$

$$+(x-1)(y-1)\times 0 + (y-1)^2\left(\frac{1}{2}\right) + \cdots]$$

$$\text{(or) } \tan^{-1}\left(\frac{y}{x}\right) = \frac{\pi}{4} - \left(\frac{x-1}{2}\right) + \left(\frac{y-1}{2}\right)$$

$$-\frac{(x-1)^2}{4} + \frac{(y-1)^2}{4} + \cdots$$

Example 2.13. *Using Taylor's series expand $e^x \log(1+y)$ up to terms of the third degree in the neighborhood of origin.*

Solution: The Taylor series expansion for $f(x,y)$ near $(0,0)$ is

$$f(x,y) = f(0,0) + \frac{1}{1!}[(x-0)f_x(0,0) + (y-0)f_y(0,0)]$$

$$+\frac{1}{2!}[(x-0)^2 f_{xx}(0,0) + 2(x-0)(y-0)f_{xy}(0,0)$$

$$+(y-0)^2 f_{yy}(0,0)] + \cdots \tag{2.11}$$

Now $f(x,y) = e^x \log(1+y) \Rightarrow f(0,0) = 0$ as $\log 1 = 0$

$f_x(x,y) = e^x \log(1+y)$	$f_{xxx}(x,y) = e^x \log(1+y)$
$f_y(x,y) = \dfrac{e^x}{1+y}$	$f_{xxy}(x,y) = \dfrac{e^x}{1+y}$
$f_{xx}(x,y) = e^x \log(1+y)$	$f_{xyy}(x,y) = -\dfrac{e^x}{(1+y)^2}$
$f_{xy}(x,y) = \dfrac{e^x}{(1+y)}$	$f_{yyy}(x,y) = \dfrac{2e^x}{(1+y)^3}$
$f_{yy}(x,y) = -\dfrac{e^x}{(1+y)^2}$	

$f_x(0,0) = 0$	$f_{xx}(0,0) = 0$	$f_{xxx}(0,0) = 0$
$f_y(0,0) = 0$	$f_{xy}(0,0) = 1$	$f_{xxy}(0,0) = 1$
	$f_{yy}(0,0) = -1$	$f_{xyy}(0,0) = -1$
		$f_{yyy}(0,0) = 2$

Using in (2.11),

$$\begin{aligned} e^x \log(1+y) &= 0 + \frac{1}{1!}[(x-0)\times 0 + (y-0)(1)] \\ &+ \frac{1}{2!}[(x-0)^2 \times 0 + 2(x-0)(y-0)(1) + (y-0)^2(-1)] \\ &+ \frac{1}{3!}[(x-0)^3 \times 0 + 3x^2y(1) + 3xy^2(-1) + (y-0)^3(2)] + \cdots \end{aligned}$$

$$\text{(or) } e^x \log(1+y) = y + xy - \frac{y^2}{2} + \frac{x^2y}{2} - \frac{xy^2}{2} + \frac{y^3}{3} + \cdots$$

Example 2.14. *Expand $xy^2 + 2x - 3y$ in powers of $(x+2)$ and $(y-1)$ up to third degree terms.*

Solution: The Taylor series expansion of $f(x,y)$ in powers of $(x-a)$ and $(y-b)$ is given by

$$\begin{aligned} f(x,y) &= f(a,b) + [(x-a)f_x(a,b) + (y-b)f_y(a,b)] \\ &+ \frac{1}{2!}[(x-a)^2 f_{xx}(a,b) + 2(x-a)(y-b)f_{xy}(a,b) \\ &+ (y-b)^2 f_{yy}(a,b)] + \cdots \end{aligned} \tag{2.12}$$

Here $f(x,y) = xy^2 + 2x - 3y; a = -2, b = 1 \therefore f(-2,1) = (-2)(1) + 2(-2) - 3(1) = -9$

$f_x(x,y) = y^2 + 2$	$f_{xx}(x,y) = 0$
$f_x(-2,1) = 3$	$f_{xy}(x,y) = 2y$
$f_y(x,y) = 2xy - 3$	$f_{xy}(-2,1) = 2$
$f_y(-2,1) = -7$	$f_{yy}(x,y) = 2x$
$f_{xxx}(x,y) = 0$	$f_{yy}(-2,1) = -4$
$f_{xyy}(x,y) = 2$	$f_{yyy}(x,y) = 0$
$f_{xxy}(x,y) = 0$	

Substituting in (4.11)

$$\begin{aligned} xy^2 + 2x - 3y = &-9 + 3(x+2) - 7(y-1) + 2(x+2)(y-1) \\ &- 2(y-1)^2 + (x+2)(y-1)^2 + \cdots \end{aligned}$$

Example 2.15. *Find the Taylor series expansion of e^{xy} at $(1,1)$ up to third degree terms.*

Solution: The Taylor series expansion of $f(x,y)$ at $(1,1)$ is given by

$$f(x,y) = f(1,1) + [(x-1)f_x(1,1) + (y-1)f_y(1,1)]$$

$$+\frac{1}{2!}[(x-1)^2 f_{xx}(1,1) + 2(x-1)(y-1)f_{xy}(1,1)$$

$$+(y-1)^2 f_{yy}(1,1)] + \cdots \tag{2.13}$$

Here $f(x,y) = e^{xy}; a = 1 = b$

$f(1,1) = e$

$f_x(x,y) = e^{xy}.y \Rightarrow f_x(1,1) = e \times 1 = e$

$f_y(x,y) = e^{xy}.x \Rightarrow f_x(1,1) = e \times 1 = e$

$f_{xx}(x,y) = ye^{xy}.y \Rightarrow f_{xx}(1,1) = 1^2 \times e = e$

$f_{xy}(x,y) = e^{xy}.1 + xe^{xy} \Rightarrow f_{xy}(1,1) = 2e$

$f_{yy}(x,y) = x^2 e^{xy} \Rightarrow f_{yy}(1,1) = e$

$f_{xxx}(x,y) = y^3 e^{xy} \Rightarrow f_{xxx}(1,1) = e$

$f_{xxy}(x,y) = e^{xy}.y + y(e^{xy}.1 + xye^{xy}) \Rightarrow f_{xxy}(1,1) = 3e$

$f_{xyy}(x,y) = x^2 e^{xy}.y + e^{xy}(2x) \Rightarrow f_{xyy}(1,1) = 3e$

$f_{yyy}(x,y) = x^3 e^{xy} \Rightarrow f_{yyy}(1,1) = e$

Using these values in (4.12)

$$\begin{aligned} e^{xy} &= e + \frac{1}{1!}[(x-1)e + (y-1)e] \\ &+ \frac{1}{2!}[(x-1)^2 e + 2(x-1)(y-1)2e + (y-1)^2 e] \\ &+ \frac{1}{3!}[(x-1)^3 e + 3(x-1)^2(y-1)3e \\ &+ 3(x-1)(y-1)^2 3e + (y-1)^3 e] + \cdots \end{aligned}$$

$\Rightarrow e^{xy} = e[1 + (x-1) + (y-1) + \frac{(x-1)^2}{2} + 2(x-1)(y-1)$
$+(y-1)^2 + \frac{(x-1)^3}{6} + \frac{3}{2}(x-1)^2(y-1) + \frac{3}{2}(x-1)(y-1)^2 +$
$(y-1)^3 + \cdots]$

Example 2.16. *Using Taylor's series, verify that* $\cos(x+y) = 1 - \frac{(x+y)^2}{2!} + \frac{(x+y)^4}{4!} - \cdots$

Solution: The required expansion is possible if we obtain the expansion in powers of x and y.

$f(x,y) = \cos(x+y) \Rightarrow f(0,0) = \cos 0 = 1$

$f_x(x,y) = -\sin(x+y)(1) \Rightarrow f_x(0,0) = \sin 0 = 0$

$f_y(x,y) = -\sin(x+y)(1) \Rightarrow f_y(0,0) = 0$

$f_{xx}(x,y) = -\cos(x+y) \Rightarrow f_{xx}(0,0) = -\cos 0 = -1$

$f_{xy}(x,y) = -\cos(x+y) \Rightarrow f_{xy}(0,0) = -\cos 0 = -1$

$f_{yy}(x,y) = -\cos(x+y) \Rightarrow f_{yy}(0,0) = -\cos 0 = -1$

$f_{xxx}(x,y) = \sin(x+y) \Rightarrow f_{xxx}(0,0) = \sin 0 = 0$

$f_{xxy}(x,y) = \sin(x+y) \Rightarrow f_{xxy}(0,0) = \sin 0 = 0$

$f_{xyy}(x,y) = \sin(x+y) \Rightarrow f_{xyy}(0,0) = \sin 0 = 0$

$f_{yyy}(x,y) = \sin(x+y) \Rightarrow f_{yyy}(0,0) = \sin 0 = 0$

$f_{xxxx}(x,y) = \cos(x+y) \Rightarrow f_{xxxx}(0,0) = \cos 0 = 1$

$f_{xxxy}(x,y) = \cos(x+y) \Rightarrow f_{xxxx}(0,0) = \cos 0 = 1$

Similarly other terms can be found.

$$\therefore \cos(x+y) = 1 + \frac{1}{2!}[(x-0)^2(-1) + 2(x-0)(y-0)(-1)$$

$$+(y-0)^2(-1)] + \frac{1}{4!}[(x-0)^4(1) + \cdots]$$

$$\therefore \cos(x+y) = 1 - \frac{(x+y)^2}{2} + \frac{(x+y)^4}{4} - \cdots$$

Example 2.17. *Expand* $\frac{(x+h)(y+k)}{x+h+y+k}$ *in a series of powers of h and k up to the second degree terms.*

Solution:

Let $f(x+h, y+k) = \frac{(x+h)(y+k)}{x+h+y+k}$ and $f(x,y) = \frac{xy}{x+h}$.

Taylor's series of $f(x+h, y+k)$ in powers of h and k is

$$f(x+h, y+k) = f(x,y) + \frac{1}{1!}\left(h\frac{\partial f}{\partial x} + k\frac{\partial f}{\partial y}\right)$$

$$+\frac{1}{2!}\left(\frac{h^2\partial^2 f}{\partial x^2} + 2hk\frac{\partial^2 f}{\partial x \partial y} + \frac{k^2\partial^2 f}{\partial y^2}\right)$$

$$+\frac{1}{3!}\left(\frac{h^3\partial^3 f}{\partial x^3} + 3h^2k\frac{\partial^3 f}{\partial x^2 \partial y} + 3hk^2\frac{\partial^3 f}{\partial x \partial y^2} + \frac{k^3\partial^3 f}{\partial y^3}\right) + \cdots \tag{2.14}$$

$$f_x(x,y) = y\left[\frac{(x+y)(1) - x(1)}{(x+y)^2}\right] = \frac{y^2}{(x+y)^2}$$

By symmetry

$$f_y(x,y) = \frac{x^2}{(x+y)^2}, f_{xx}(x,y) = \frac{-2y^2}{(x+y)^3}, f_{yy}(x,y) = \frac{-2x^2}{(x+y)^3},$$

$$f_{xy}(x,y) = \frac{(x+y)^2(2x) - x^2(x+y)(2)}{(x+y)^4} = \frac{2xy}{(x+y)^3}$$

Using in (4.13)

$$\frac{(x+h)(y+k)}{x+h+y+k} = \frac{xy}{x+y} + \frac{1}{1!}\left(\frac{hy^2}{(x+y)^2} + \frac{kx^2}{(x+y)^2}\right)$$
$$+\frac{1}{2!}\left(h^2\left(\frac{-2y^2}{(x+y)^3}\right) + 2hk\frac{2xy}{(x+y)^3}\right.$$
$$\left. + k^2\left(\frac{-2x^2}{(x+y)^3}\right)\right) + \cdots$$

Example 2.18. *Expand* $(x^2y + \sin y + e^x)$ *in powers of* $(x-1)$ *and* $(y-\pi)$.

Solution: The Taylor series expansion of $f(x,y)$ in powers of $(x-a)$ and $(y-b)$ is given by

$$f(x,y) = f(a,b) + [(x-a)f_x(a,b) + (y-b)f_y(a,b)]$$

$$+\frac{1}{2!}[(x-a)^2 f_{xx}(a,b) + 2(x-a)(y-b)f_{xy}(a,b) + (y-b)^2 f_{yy}(a,b)] + \cdots \tag{2.15}$$

Here $f(x,y) = x^2y + \sin y + e^x$; $a = 1, b = \pi$ and

$f(1,\pi) = 1^2\pi + \sin\pi + e = \pi + e$, as $\sin\pi = 0$

$f_x(x,y) = 2xy + e^x \Rightarrow f_x(1,\pi) = 2\pi + e$

$f_y(x,y) = x^2 + \cos y \Rightarrow f_y(1,\pi) = 1 + \cos\pi = 0$ $[\because \cos\pi = -1]$

$f_{xx}(x,y) = 2y + e^x \Rightarrow f_{xx}(1,\pi) = 2\pi + e$

$f_{xy}(x,y) = 2x \Rightarrow f_{xy}(1,\pi) = 2$

$f_{yy}(x,y) = -\sin y \Rightarrow f_{yy}(1,\pi) = 0$

Using these values in (4.14)

$$x^2y + \sin y + e^x = \pi + e + (x-1)(2\pi + e) + \frac{1}{2}(x-1)^2(2\pi + e) + (x-1)(y-\pi) + \cdots$$

Example 2.19. *Find the expansion for* $\cos x \cos y$ *in powers of* x *and* y *up to terms of 3rd degree.*

Solution: Let $f(x,y) = \cos x \cos y \Rightarrow f(0,0) = \cos 0 \cos 0 = 1$

$f_x(x,y) = -\sin x \cos y \Rightarrow f_x(0,0) = 0$;

$f_y(x,y) = -\cos x \sin y \Rightarrow f_y(0,0) = 0$

$f_{xx}(x,y) = -\cos x \cos y \Rightarrow f_{xx}(0,0) = -1$;

$f_{xy}(x,y) = \sin x \sin y \Rightarrow f_{xy}(0,0) = 0$;

$f_{yy}(x,y) = -\cos x \cos y \Rightarrow f_{yy}(0,0) = -1$;

$f_{xxx}(x,y) = \sin x \cos y \Rightarrow f_{xxx}(0,0) = 0$;

$f_{xxy}(x,y) = \cos x \sin y \Rightarrow f_{xxy}(0,0) = 0$;

$f_{xyy}(x,y) = \sin x \cos y \Rightarrow f_{xyy}(0,0) = 0$;

and $f_{yyy}(x,y) = \cos x \sin y \Rightarrow f_{yyy}(0,0) = 0$ The required expansion is $\cos x \cos y = 1 - \dfrac{x^2}{2} - \dfrac{y^2}{2} - \cdots$

Example 2.20. *Find the Taylor's series expansion of* $e^x \sin y$ *near the point* $\left(-1, \frac{\pi}{4}\right)$ *up to the 3rd degree terms.*

Solution: Taylor's series of $f(x,y)$ near the point $\left(-1,\frac{\pi}{4}\right)$ is

$$f(x,y)=f\left(-1,\frac{\pi}{4}\right)+\frac{1}{1!}\left\{(x+1)f_x\left(-1,\frac{\pi}{4}\right)\right.$$

$$\left.+\left(y-\frac{\pi}{4}\right)f_y\left(-1,\frac{\pi}{4}\right)\right\}+\frac{1}{2!}\left[(x+1)^2f_{xx}\left(-1,\frac{\pi}{4}\right)\right.$$

$$+2(x+1)\left(y-\frac{\pi}{4}\right)f_{xy}\left(-1,\frac{\pi}{4}\right)$$

$$\left.+\left(y-\frac{\pi}{4}\right)^2f_{yy}\left(-1,\frac{\pi}{4}\right)\right]+\cdots \tag{2.16}$$

$$f(x,y)=e^x\sin y;\ f_x=e^x\sin y;\ f_y=e^x\cos y$$

$$f_{xx}=e^x\sin y;\ f_{xy}=e^x\cos y;\ f_{yy}=-e^x\sin y$$

$$f_{xxx}(x,y)=e^x\sin y;\ f_{xxy}=e^x\cos y;$$

$$f_{xyy}(x,y)=-e^x\sin y;\ f_{yyy}(x,y)=-e^x\cos y$$

$$f\left(-1,\frac{\pi}{4}\right)=\frac{1}{e\sqrt{2}};\ f_x\left(-1,\frac{\pi}{4}\right)=\frac{1}{e\sqrt{2}};$$

$$f_{xy}\left(-1,\frac{\pi}{4}\right)=\frac{1}{e\sqrt{2}};$$

$$f_{xx}\left(-1,\frac{\pi}{4}\right)=\frac{1}{e\sqrt{2}};\ f_{xy}\left(-1,\frac{\pi}{4}\right)=\frac{1}{e\sqrt{2}}$$

$$f_{yy}\left(-1,\frac{\pi}{4}\right)=-\frac{1}{e\sqrt{2}}$$

$$f_{xxx}\left(-1,\frac{\pi}{4}\right)=\frac{1}{e\sqrt{2}};\ f_{xxy}\left(-1,\frac{\pi}{4}\right)=\frac{1}{e\sqrt{2}};$$

$$f_{xyy}\left(-1,\frac{\pi}{4}\right)=-\frac{1}{e\sqrt{2}};\ f_{yyy}\left(-1,\frac{\pi}{4}\right)=-\frac{1}{e\sqrt{2}}$$

Using these values in (4.15), we get

$$e^x\sin y=\frac{1}{e\sqrt{2}}\left\{1+\frac{1}{1!}(x+1)+\left(y-\frac{\pi}{4}\right)\right\}$$

$$+\frac{1}{2!}\left\{(x+1)^2+2(x+1)\left(y-\frac{\pi}{4}\right)-\left(y-\frac{\pi}{4}\right)^2\right\}$$

$$+\frac{1}{3!}\left\{(x+1)^3+3(x+1)^2\left(y-\frac{\pi}{4}\right)\right.$$

$$\left.-3(x+1)\left(y-\frac{\pi}{4}\right)^2-\left(y-\frac{\pi}{4}\right)^3\right\}+\cdots$$

EXERCISE

1. Find the Taylor's series expansion of x^y near the point $(1,1)$ up to the second degree terms. [Ans: $x^y=1+(x-1)+(x-1)(y-1)$].

2. Using Taylor's series, verify that $\log(1+x+y)=(x+y)-\frac{1}{2}(x+y)^2+\frac{1}{3}(x+y)^3-\cdots$

3. Using Taylor's series, verify that $\tan^{-1}(x+y)=(x+y)+\frac{1}{3}(x+y)^3+\cdots+\infty$

4. Expand $e^x \sin y$ in powers of x and y as far as terms of 3rd degree. [Ans: $e^x \sin y = y + xy + \frac{1}{2}x^2y - \frac{1}{6}y^3 + \cdots$]

5. Expand $f(x,y) = \sin(xy)$ in powers of $(x-1)$ and $\left(y - \frac{\pi}{2}\right)$ [Ans: $1 - \frac{1}{8}\pi^2(x-1)^2 - \frac{1}{2}\pi(x-1)\left(y-\frac{\pi}{2}\right) - \frac{1}{2}\left(y-\frac{\pi}{2}\right)^2 + \cdots$]

6. Expand $xy^2 + \sin(xy)$ at the point $\left(1, \frac{\pi}{2}\right)$ up to terms of second degree. [Ans: $1+\pi^2+\frac{\pi^2}{4}(x-1)+\frac{1}{2!}[-\frac{\pi^2}{4}(x-1)^2 + \pi\left(y-\frac{\pi}{2}\right) + \pi(x-1)\left(y-\frac{\pi}{2}\right) + \left(y-\frac{\pi}{2}\right)^2]$]

7. Find the Taylor's for the function $f(x,y) = \frac{1}{xy}$ upto terms of degree two for $(2,-1)$. [Ans: $\frac{1}{2} + \frac{1}{4}(x-2) - \frac{1}{2}(y+1) - \frac{1}{8}(x-2)^2 + \frac{1}{4}(x-2)(y+1) - \frac{1}{2}(y+1)^2 + \cdots$]

8. Expand $\frac{y^2}{x^3}$ at $(1,-1)$ as a Taylor's series. [Ans: $1 - 3(x-1) - 2(y+1) + 6(x+1)^2 + 6(x-1)(y+1) + (y+1)^2 + \cdots$]

9. If $f(x,y) = \tan^{-1}(xy)$, compute an approximate value of $f(0.9,-1.2)$. [Ans: -0.823] [**Hint:** expand near $(1,-1)$; take $h=-0.1, k=-0.2$]

10. Expand $f(x,y) = 21 + x - 20y + 4x^2 + xy + 6y^2$ in Taylor series of maximum order about the point $(-1,2)$. [Ans: $f(x,y) = 6 - 5(x+1) + 3(y-2) + 4(x+1)^2 + (x+1)(y-2) + 6(y-2)^2 + \cdots$]

11. Expand $e^{ax}\sin(by)$ at $(0,0)$ using Taylor's series. [Ans: $by + abxy + \frac{1}{3!}(3a^2bx^2y - b^3y^3) + \cdots$]

12. Expand e^{xy} at $(1,1)$ upto 3 terms. [Ans: $e\Big\{1 + (x-1) + (y-1) + \frac{1}{2!}[(x-1)^2 + 4(x-1)(y-1) + (y-1)^2\Big\}$]

13. Expand $e^x \log(1+y)$ as the Taylor's series in the neighborhood of $(0,0)$ [Ans: $y + xy - \frac{y^2}{2} + \frac{x^2y}{2} - \frac{xy^2}{2} + \cdots$]

2.5 Maxima and Minima of Functions of Two Variables

A function $f(x,y)$ is said to have a maximum value at $x=a, y=b$ if $f(a,b) > f(a+h, b+k)$, for small and independent values of h and k, positive or negative. A function $f(x,y)$ is said to have a minimum value at $x=a, y=b$ if $f(a,b) < f(a+h,b+k)$, for small and independent values of h and k positive or negative. Thus, $f(x,y)$ has a maximum or minimum value at a point (a,b) according as $\Delta f = f(a+h,b+k) - f(a,b) <$ or > 0. A minimum or a maximum value of a function is called its extreme value.

2.5.1 Conditions for f(x, y) to be maximum or minimum

By Taylor's theorem, we have

$$\Delta f = f(a+h, b+k) - f(a,b)$$

$$= [hf_x(a,b) + kf_y(a,b)]$$

$$+\frac{1}{2!}[h^2 f_{xx}(a,b) + 2hk f_{xy}(a,b) + k^2 f_{yy}(a,b)] + \cdots \quad (2.17)$$

For small values of h and k, the second and higher order terms are still smaller and may be neglected.

Thus sign of Δf = sign of $[hf_x(a,b) + kf_y(a,b)]$.

Taking $h = 0$, the sign of Δf changes with sign of k. Similarly taking $k = 0$, the sign of Δf changes with the sign of h. Since Δf changes sign with h and k, $f(x,y)$ can not have a maximum (or) minimum value at (a,b) unless $f_x(a,b) = 0 = f_y(a,b)$.

Hence the necessary conditions for $f(x,y)$ to have a maximum (or) a minimum value at (a,b) are $f_x(a,b) = 0, f_y(a,b) = 0$.

If these conditions are satisfied, then for small values of h and k we have from (4.16),

$$\Delta f = \tfrac{1}{2}[h^2 f_{xx}(a,b) + 2hk f_{xy}(a,b) + k^2 f_{yy}(a,b)]$$

$= \frac{1}{2}[h^2 r + 2hks + k^2 t$ where $r = f_{xx}(a,b), s = f_{xy}(a,b), t =$

$f_{yy}(a,b)$

$$\Delta f = \frac{1}{2r}[h^2r^2 + 2hkrs + k^2rt]$$

$$= \frac{1}{2r}[(h^2r^2 + 2hkrs + k^2s^2) + k^2rt - k^2s^2] + \cdots$$

$$= \frac{1}{2r}[(hr+ks)^2 + k^2(rt - s^2)] + \cdots \quad (2.18)$$

Now $(hr+ks)^2$ is always positive and $k^2(rt - s^2)$ will be positive if $rt - s^2 > 0$.

In this case, Δf will have the same sign as that of r for all values of h and k.

Hence if $rt - s^2 > 0$, then $f(x,y)$ has a maximum or a minimum at (a,b) according as $r < 0$ or $r > 0$.

If $rt - s^2 < 0$, then Δf changes sign with h and k. Hence there is neither a maximum nor a minimum value at (a,b). The point (a,b) is a saddle point in this case.

If $rt - s^2 = 0$, no conclusion can be drawn about a maximum or minimum value at (a,b) and hence further investigation is required.

Note: The point (a,b) is called a stationary point if $f_x(a,b) = 0, f_y(a,b)$. The value $f(a,b)$ is called a stationary value. Thus every value is a stationary value but the converse may not be true.

2.5.2 Rule to find the extreme value of a function z = f(x, y)

i) Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$

ii) Solve $\frac{\partial z}{\partial x} = 0$ and $\frac{\partial z}{\partial y} = 0$, simultaneously. Let $(a,b); (c,d), \cdots$ be the solutions of these equations.

iii) For each solution in step (ii) find $r = \frac{\partial^2 f}{\partial x^2}, s = \frac{\partial^2 f}{\partial x \partial y}, t = \frac{\partial^2 f}{\partial y^2}$

iv) a) If $rt - s^2 > 0$ and $r < 0$ for a particular solution (a,b) of step (ii) then z has a maximum value at (a,b).

b) If $rt - s^2 > 0$ and $r > 0$ for a particular solution (a,b) of step (ii), then z has a minimum value at (a,b).

c) If $rt - s^2 < 0$ for a particular solution (a,b) of step (ii) then z has no extreme value at (a,b). In this case, the points are called saddle points.

d) If $rt - s^2 = 0$, this case is doubtful and requires further investigation.

Example 2.21. *Examine for extreme values of* $x^2+y^2+6x+12$

Solution: Let $f(x,y) = x^2+y^2+6x+12$ and $f_x = 2x+6, f_y = 2y$

For maximum or minimum, $f_x = 0, f_y = 0 \Rightarrow 2x+6 = 0, 2y = 0$

Solving we get $x = -3, y = 0$. Stationary point is $(-3,0)$

$r = f_{xx} = 2$ at $(-3,0)$, $rt - s^2$ gives 4 (i.e) $rt - s^2 > 0, t = f_{xy} = 0$

$s = f_{yy} = 2$. As r is > 0, and $rt - s^2 > 0 \Rightarrow (-3,0)$ is a minimum point.

And $f(-3,0) = 9 - 18 + 12 = 3$

$\therefore$ Minimum value $= 3$.

Example 2.22. *Examine for extreme values of* $xy + \frac{a^3}{x} + \frac{a^3}{y}$

Solution: Let $z = f(x,y) = xy + \frac{a^3}{x} + \frac{a^3}{y}$

$f_x = y - \frac{a^3}{x^2}; f_y = x - \frac{a^3}{y^2}$

Equation f_x and f_y to zero

$$y - \frac{a^3}{x^2} = 0 \tag{2.19}$$

$$x - \frac{a^3}{y^2} = 0 \tag{2.20}$$

From (??), $y = \frac{a^3}{x^2}$.

Put this value in (4.19), we get $x - \frac{a^3}{\left(\frac{a^6}{x^4}\right)} = 0$

(i.e.) $x - \frac{a^3 x^4}{a^6} = 0$

(i.e.) $x\left(1 - \frac{a^3 x^3}{a^6}\right) = 0$

(i.e.) $x\left(1 - \frac{x^3}{a^3}\right) = 0$

(i.e.) $x = 0, a$.

When $x = 0 \Rightarrow y = \infty$, when $x = a \Rightarrow y = a$.

Omitting $(0, \infty) \Rightarrow$ stationary point is (a, a)

$r = 2\frac{a^3}{x^3}; t = 2\frac{a^3}{y^3}; s = 1$ at (a, a)

$r = \frac{2a^3}{a^3} = 2; t = 2; s = 1.$

$\therefore rt - s^2$ gives $4 - 1 = 3 > 0$

As $rt - s^2 > 0, r > 0 \Rightarrow$ the point (a, a) is a minimum point.

$f(a, a) = a^2 + \frac{a^3}{a} + \frac{a^3}{a}$

(i.e.) $3a^2$ is the minimum value.

Example 2.23. *Examine the function $x^3 + y^3 - 12x - 3y + 20$ for extreme values.*

Solution: Let $f(x, y) = x^3 + y^3 - 12x - 3y + 20$

$f_x = 3x^2 - 12; f_y = 3y^2 - 3; f_{xx} = 6x; f_{yy} = 6y; f_{xy} = 0$

The stationary points are given by $f_x = 0, f_y = 0$

$\Rightarrow 3x^2 - 12 = 0 \Rightarrow x^2 - 4 = 0 \Rightarrow x = \pm 2$

Also $3y^2 - 3 = 0 \Rightarrow y^2 = 1 \Rightarrow y = \pm 1$

The critical points are $(2, 1), (2, -1), (-2, 1), (-2, -1)$.

Now at $(2, 1); r = f_{xx} = 12, t = f_{yy} = 6, s = 0$

$\therefore \Delta = rt - s^2 \Rightarrow (12 \times 6) - 0^2 = 72 > 0$

(i. e) $(2, 1)$ is a minimum point as $r > 0, \Delta > 0$

$\therefore f(2, 1) = 8 + 1 - 24 - 3 + 20 = 2$

$\therefore$ Minimum value $= 2$.

At $(2, -1) : r = 12, t = -6, s = 0$ and

$\Delta < 0 \Rightarrow (2, -1)$ is a saddle point.

Similarly $(-2,1)$ is also a saddle point.

At $(-2,-1): r=-12, t=-6, s=0$

$\therefore \Delta = rt - s^2 = 72 - 0 = 72$

$\Rightarrow \Delta > 0, r < 0$ at $(-2,1)$

$\therefore (-2,-1)$ is a maximum point.

$\therefore f(-2,-1) = -8-1+24+3+20 = 38$

$\therefore$ At $(-2,-1)$, the maximum value $= 38$.

Example 2.24. *Identify the saddle point and extreme points of* $f(x,y) = x^4 - y^4 - 2x^2 + 2y^2$

Solution: Let $f(x,y) = x^4 - y^4 - 2x^2 + 2y^2$

$f_x = 4x^3 - 4x, f_y = -4y^3 + 4y$

$f_{xx} = 12x^2 - 4, f_{xy} = 0, f_{yy} = -12y^2 + 4$

The stationary points are given by $f_x = 0, f_y = 0$

$\therefore 4x^3 - 4x = 0 \Rightarrow x^3 - x = 0 \Rightarrow x(x^2-1) = 0$

$\therefore x = 0, \pm 1$

and $4y - 4y^3 = 0 \Rightarrow y - y^3 = 0 \Rightarrow y(1-y^2) = 0$

$\therefore y = 0, \pm 1$

The stationary points are $(0,0), (0,\pm1), (\pm1,0), (\pm1,\pm1)$

At $(0,0): r=-4, t=4, s=0$

$\therefore \Delta = rt - s^2 = -16 < 0$

$(0,0)$ is a saddle point.

At $(0,1): r=-4, t=-8, s=0$

$\therefore \Delta = rt - s^2 > 0$

As $rt - s^2 > 0, r < 0, (0,1)$ is a maximum point.

Similarly $(0,-1)$ is a maximum point.

At $(\pm1,0): r = 12-4 = 8, t=4, s=0$

$\therefore \Delta = rt - s^2 = 32 > 0$. $\Delta > 0, r > 0 \Rightarrow (\pm1,0)$ is a minimum point.

The points $(\pm1,\pm1)$ are saddle points.

Example 2.25. *Examine* $f(x,y) = x^3 + y^3 - 3axy$ *for maxima and minima.*

Solution: Given $f(x,y) = x^3 + y^3 - 3axy$.

$f_x(x,y) = 3x^2 - 3ay$	$f_y(x,y) = 3y^2 - 3ax$
$f_{xx}(x,y) = 6x$	$f_{yy}(x,y) = 6y$
$f_{xy}(x,y) = -3a$	

The stationary points are obtained from $f_x = \dfrac{\partial f}{\partial x} = 0, f_y = \dfrac{\partial f}{\partial y} = 0$ where $3x^2 - 3ay = 0$ and $3y^2 - 3ax = 0$

(i.e.) $x^2 - ay = 0$ and $y^2 - ax = 0$.

Using these two equations, we get the stationary points as $(0,0)$ and (a,a).

Now, at $(0,0) : r = f_{xx} = 0, t = f_{yy} = 0$ and $s = f_{xy} = -3a$

$\therefore rt - s^2 = 0 - 9a^2 < 0$

The point $(0,0)$ is neither a maximum nor a minimum point.

At $(a,a) : r = f_{xx} = 6a, t = f_{yy} = 6a$ and $s = f_{xy} = -3a$ such that $rt - s^2 = 36a^2 - 9a^2 > 0$.

Also, $r = f_{xx}$ at $(a,a) = 6a$

$\Rightarrow$ r is positive when a is positive and r is negative when a is negative.

(i. e) The point (a,a) is a minimum if $a > 0$ and (a,a) is a maximum if $a < 0$.

Example 2.26. *Find the maximum or minimum value of* $\sin x + \sin y + \sin(x+y)$.

Solution: Given $f(x,y) = \sin x + \sin y + \sin(x+y)$

$f_x = \cos x + \cos(x+y); f_y = \cos y + \cos(x+y)$

$f_{xx} = -\sin x - \sin(x+y); f_{yy} = -\sin y - \sin(x+y)$ and

$f_{xy} = -\sin(x+y)$

The stationary points are obtained by equating f_x to 0 and f_y to 0.

(i.e.)

$$f_x = 0 \Rightarrow \cos x + \cos(x+y) = 0 \tag{2.21}$$

$$f_y = 0 \Rightarrow \cos y + \cos(x+y) = 0 \tag{2.22}$$

From (??),

$$\cos x = -\cos(x+y) = \cos[\pi - (x+y)] \Rightarrow x = \pi - (x+y) \tag{2.23}$$

(i.e.)

$$2x + y = \pi \tag{2.24}$$

Solving (4.22) and (4.23), we get, $x = \dfrac{\pi}{3}, y = \dfrac{\pi}{3}$

Then the stationary point is $\left(\dfrac{\pi}{3}, \dfrac{\pi}{3}\right)$

At $\left(\dfrac{\pi}{3}, \dfrac{\pi}{3}\right) : r = f_{xx} = -\dfrac{\sqrt{3}}{2} - \dfrac{\sqrt{3}}{2} = -\sqrt{3}$,

$s = f_{xy} = -\dfrac{\sqrt{3}}{2}, t = f_{yy} = -\sqrt{3}$

$\therefore rt - s^2 = 3 - \dfrac{3}{4} = \dfrac{9}{4} > 0$

Also, $r < 0$ at $\left(\dfrac{\pi}{3}, \dfrac{\pi}{3}\right)$

$\therefore$ The point $\left(\dfrac{\pi}{3}, \dfrac{\pi}{3}\right)$ is a maximum point.

Hence the maximum value of the given function is

$$\begin{aligned} f\left(\frac{\pi}{3}, \frac{\pi}{3}\right) &= \sin\frac{\pi}{3} + \sin\frac{\pi}{3} + \sin\frac{2\pi}{3} \\ &= \sin\frac{\pi}{3} + \sin\frac{\pi}{3} + \sin\left(\pi - \frac{\pi}{3}\right) \\ &= \sin\frac{\pi}{3} + \sin\frac{\pi}{3} + \sin\frac{\pi}{3} \\ &= 3\sin\frac{\pi}{3} = 3\frac{\sqrt{3}}{2} \end{aligned}$$

Example 2.27. *Determine the maxima or minima of the function $x^3y^2(12 - x - y)$ where $x > 0, y > 0$. Also find the extreme value.*

Solution: Given $f(x, y) = x^3y^2(12 - x - y) = 12x^3y^2 - x^4y^2 - x^3y^3$

Now, $f_x = 36x^2y^2 - 4x^3y^2 - 3x^2y^3$ and $f_y = 24x^3y - 2x^4y - 3x^3y^2$

$\therefore f_x = 0$ and $f_y = 0 \Rightarrow 36x^2y^2 - 4x^3y^2 - 3x^2y^3 = 0$

$\Rightarrow x^2y^2(36 - 4x - 3y) = 0 \Rightarrow 36 - 4x - 3y = 0$ (since $x > 0, y > 0$)

(i.e.)

$$4x + 3y = 36 \qquad (2.25)$$

and $24x^3y - 2x^4y - 3x^3y^2 = 0$

$\Rightarrow x^3y(24 - 2x - 3y) = 0$

$\Rightarrow 24 - 2x - 3y = 0$ (since $x > 0, y > 0$)

(i.e.)

$$2x + 3y = 24 \qquad (2.26)$$

Solving (4.24) and (4.25), we get, $y = 4$ and $x = 6$.

Now, $f_{xx} = 72xy^2 - 12x^2y^2 - 6xy^3, f_{xy} = 72x^2y - 8x^3y - 9x^2y^2$

$f_{yy} = 24x^3 - 2x^4 - 6x^3y$

Then at $(6, 4) : r = -2204, s = -1728, t = -2592$

$\therefore rt - s^2 > 0$. Also $r < 0$.

Then point $(6, 4)$ is a maximum point.

The maximum value of the given function is 6912.

2.5.3 Lagrange's method of undetermined multipliers

This method is to find the maximum or minimum value of a function of three or more variables, given the constraints.

Let $f(x,y,z)$ be a function of three variables which is to be tested for maximum or minimum value. Let the variables x, y, z be connected by a relation

$$\varphi(x,y,z) = 0 \tag{2.27}$$

The conditions for $f(x,y,z)$ to have a maximum point, or a minimum point are

$$\frac{\partial f}{\partial x} = 0, \frac{\partial f}{\partial y} = 0, \frac{\partial f}{\partial z} = 0$$

By total differentials, we have

$$\frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz = 0 \tag{2.28}$$

Similarly from (4.26), we have that

$$\frac{\partial \varphi}{\partial x}dx + \frac{\partial \varphi}{\partial y}dy + \frac{\partial \varphi}{\partial z}dz = 0 \tag{2.29}$$

Equation (4.27) + λ equation (4.28), ultimately gives the following:

$$\frac{\partial f}{\partial x} + \lambda\frac{\partial \varphi}{\partial x} = 0, \frac{\partial f}{\partial y} + \lambda\frac{\partial \varphi}{\partial y} = 0 \text{ and } \frac{\partial f}{\partial z} + \lambda\frac{\partial \varphi}{\partial z} = 0$$

(Here 'λ' is called the Lagrange multiplier).

Solving the above equations along with the given relation, we will get the values of x, y, z and the Lagrange's multiplier λ.

These values give finally the required maximum or minimum value of the function $f(x,y,z)$.

Example 2.28. *A rectangular box open at the top is to have volume of 32 cubic ft. Find the dimensions in order that the total surface area is minimum.*

Solution: Given volume,

$$\varphi(x,y,z) = xyz - 32 = 0 \tag{2.30}$$

The required function is the total surface area

$$S = f(x,y,z) = xy + 2xz + 2yz \tag{2.31}$$

At the critical points, we have

$$\frac{\partial f}{\partial x} + \lambda\frac{\partial \varphi}{\partial x} = 0 \Rightarrow y + 2z + \lambda yz = 0 \tag{2.32}$$

$$\frac{\partial f}{\partial y} + \lambda\frac{\partial \varphi}{\partial y} = 0 \Rightarrow x + 2z + \lambda xy = 0 \tag{2.33}$$

$$\frac{\partial f}{\partial z} + \lambda\frac{\partial \varphi}{\partial z} = 0 \Rightarrow 2x + 2y + \lambda xy = 0 \tag{2.34}$$

(??) $\times x -$ (??) $\times y$ gives, $2(zx - zy) = 0, z \neq 0$

$$\Rightarrow x - y = 0 \Rightarrow x = y \tag{2.35}$$

(4.32) $\times y -$ (4.33) $\times z$ gives, $xy - 2xz = 0$

$y^2 - 2yz = 0$ (using (4.33)) $\Rightarrow y(y-2z) = 0 \Rightarrow y - 2z = 0 (y \neq 0)$

$$\Rightarrow z = \frac{y}{2} \tag{2.36}$$

Using (4.34) and (4.35) in (4.29), we get,

$x.x.\frac{x}{2} = 32 \Rightarrow x^3 = 64 \Rightarrow x = 4$

$\therefore y = 4$ and $z = 2$.

The dimensions are 4 cm, 4 cm and 2 cm.

Example 2.29. *Find the volume of the largest rectangular parallelopiped that can be inscribed in the ellipsoid* $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2} = 1$.

Solution: The given ellipsoid is

$$\varphi(x, y, z) = \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1 = 0 \tag{2.37}$$

The required function is the volume of the parallelopiped, given by

$$V = 8xyz = f(x, y, z) \tag{2.38}$$

where the dimensions are 2x, 2y and 2z.

At the critical points we have

$$\frac{\partial f}{\partial x} + \lambda\frac{\partial \varphi}{\partial x} = 0 \Rightarrow 8yz + \lambda\left(\frac{2x}{a^2}\right) = 0 \tag{2.39}$$

$$\frac{\partial f}{\partial y} + \lambda\frac{\partial \varphi}{\partial y} = 0 \Rightarrow 8xz + \lambda\left(\frac{2y}{b^2}\right) = 0 \tag{2.40}$$

$$\frac{\partial f}{\partial z} + \lambda\frac{\partial \varphi}{\partial z} = 0 \Rightarrow 8xy + \lambda\left(\frac{2z}{c^2}\right) = 0 \tag{2.41}$$

Equation (4.38) $\times x +$ (4.40) $\times y +$ (??) $\times z$ gives

$24xyz + 2\lambda\left(\frac{x}{a^2} + \frac{y}{b^2} + \frac{z}{c^2}\right) = 0$

(i.e.) $2\lambda = -24xyz$ (using (4.36))

$$\therefore \lambda = -12xyz \tag{2.42}$$

Using (4.44) in (4.41), we get

$$8xy + (-12xyz)\left(\frac{2z}{c^2}\right) = 0 \Rightarrow 8xy\left(1 - \frac{3z^2}{c^2}\right) = 0$$

$$\Rightarrow \frac{3z^2}{c^2} = 1 \text{ (or) } z = \frac{c}{\sqrt{3}} \; (\because x \neq 0, y \neq 0)$$

Similarly by using (4.44) in (4.40) and (4.38), we get

$$y = \frac{b}{\sqrt{3}} \text{ and } x = \frac{a}{\sqrt{3}}$$

$\therefore$ The maximum volume of rectangular parallelepiped is

$$V = 8xyz = \frac{abc}{3\sqrt{3}} \text{ cu.units.}$$

Example 2.30. *Find the dimensions of the rectangular box, open at the top of maximum capacity whose surface is 432 sq.cm.*

Solution: Let x, y, z be the dimensions of the rectangular box, open at the top.

Given its surface area

$$\varphi(x, y, z) = xy + 2yz + 2zx - 432 = 0 \tag{2.43}$$

The required function is its volume

$$V = xyz = f(x, y, z) \tag{2.44}$$

At the critical point we get

$$yz + \lambda(y + 2z) = 0 \tag{2.45}$$
$$xz + \lambda(x + 2z) = 0 \tag{2.46}$$
$$xy + \lambda(2y + 2x) = 0 \tag{2.47}$$

Equation (4.47) $\times\, x$ − (4.48) $\times\, y$ gives,

$$2\lambda z(x - y) = 0 \Rightarrow x = y (\because z \neq 0, \lambda \neq 0) \tag{2.48}$$

Equation (4.47) $\times\, x$ − (4.49) $\times\, z$ gives,

$$\lambda y(x - 2z) = 0 \Rightarrow z = \frac{x}{2} (\because y \neq 0, \lambda \neq 0) \tag{2.49}$$

Using (4.50) and (4.51) in (4.45), we get,

$$x^2 + x^2 + x^2 = 432 \Rightarrow 3x^2 = 432 \Rightarrow x^2 = 144$$

$\therefore x = 12$

Hence $y = 12$ and $z = 6$.

Thus, the dimensions of the rectangular box open at the top of maximum capacity are 12 cm, 12 cm and 6 cm.

Example 2.31. *Find the maximum and minimum distance of the point (3, 4, 12) from the sphere $x^2 + y^2 + z^2 = 1$.*

Solution: Given

$$\varphi(x,y,z) = x^2 + y^2 + z^2 - 1 = 0 \tag{2.50}$$

The required equation is $f(x,y,z)$ = square of the distance from the point $(3,4,12)$ to the sphere

$$= (x-3)^2 + (y-4)^2 + (z-12)^2 \tag{2.51}$$

At the critical points we have,

$$2(x-3) + 2\lambda x = 0 \tag{2.52}$$

$$2(y-4) + 2\lambda y = 0 \tag{2.53}$$

$$2(z-12) + 2\lambda z = 0 \tag{2.54}$$

From (4.54), (4.55) and (4.56), we get,

$$x = \frac{3}{1+\lambda}, y = \frac{4}{1+\lambda} \quad \text{and} \quad z = \frac{12}{1+\lambda} \tag{2.55}$$

Using these values in (4.52), we get,

$$\frac{9}{(1+\lambda)^2} + \frac{16}{(1+\lambda)^2} + \frac{144}{(1+\lambda)^2} = 1$$

(i.e.)

$$(1+\lambda)^2 = 169 \Rightarrow 1+\lambda = \pm 13 \tag{2.56}$$

Using (4.58) in (4.57) we get the points as:

$$\left(\frac{3}{13}, \frac{4}{13}, \frac{12}{13}\right) \text{ and } \left(\frac{-3}{13}, \frac{-4}{13}, \frac{-12}{13}\right)$$

$\therefore$ The distances are:

$$\sqrt{\left(3-\frac{3}{13}\right)^2 + \left(4-\frac{4}{13}\right)^2 + \left(12-\frac{12}{13}\right)^2} = 12$$

$$\text{and } \sqrt{\left(3+\frac{3}{13}\right)^2 + \left(4+\frac{4}{13}\right)^2 + \left(12+\frac{12}{13}\right)^2} = 14$$

Thus, the maximum distance is 14 and the minimum distance is 12.

Example 2.32. *If* $\frac{3}{x} + \frac{4}{y} + \frac{5}{z} = 6$ *find the values of* x, y, z *which make* $x + y + z$ *is minimum.*

Solution: Given

$$\varphi(x,y,z) = \frac{3}{x} + \frac{4}{y} + \frac{5}{z} - 6 = 0 \tag{2.57}$$

The required function is

$$f(x,y,z) = x + y + z \tag{2.58}$$

$$\frac{\partial f}{\partial x} + \lambda \frac{\partial \varphi}{\partial x} = 0 \Rightarrow \left(1 - \frac{3\lambda}{x^2}\right) = 0 \tag{2.59}$$

$$\frac{\partial f}{\partial y} + \lambda \frac{\partial \varphi}{\partial y} = 0 \Rightarrow \left(1 - \frac{4\lambda}{y^2}\right) = 0 \tag{2.60}$$

$$\frac{\partial f}{\partial z} + \lambda \frac{\partial \varphi}{\partial z} = 0 \Rightarrow \left(1 - \frac{5\lambda}{z^2}\right) = 0 \tag{2.61}$$

From (4.61), (4.62) and (2.61), we get

$$3y^2 = 4x^2 \tag{2.62}$$

$$4z^2 = 5y^2 \tag{2.63}$$

(i.e.)

$$y = \pm \frac{2}{\sqrt{3}} x \tag{2.64}$$

and

$$y = \pm \frac{2}{\sqrt{5}} z \tag{2.65}$$

Using (2.64) and (2.65) in (??), we get,

$$\frac{3}{x} + \frac{2\sqrt{3}}{x} + \frac{5}{\sqrt{5}\left(\frac{x}{\sqrt{3}}\right)} = 6$$

$$\Rightarrow \frac{3}{x} + \frac{2\sqrt{3}}{x} + \frac{\sqrt{15}}{x} = 6$$

$$\sqrt{3}(\sqrt{3} + \sqrt{5} + 2) = 6x \Rightarrow 3 + 2\sqrt{3} + \sqrt{15} = 6x$$

(i.e.) $x = \dfrac{\sqrt{3}}{6}(\sqrt{3} + \sqrt{5} + 2)$

$y = \dfrac{1}{3}(\sqrt{3} + \sqrt{5} + 2)$

and $z = \dfrac{\sqrt{5}}{6}(\sqrt{3} + \sqrt{5} + 2)$

EXERCISE

1. If $u \sin\left(\frac{x}{y}\right)$, $x = e^t, y = t^2$, find $\frac{du}{dt}$ $\left[Ans : e^t \cos\left(\frac{e^t}{t^2}\right)\left(\frac{t-2}{t^3}\right)\right]$

2. If $x = u^2 - v^2, y = 2uv$, find $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}$ and $\frac{\partial v}{\partial y}$
$\left[Ans : \frac{\partial u}{\partial x} = \frac{u}{2(u^2+v^2)}, \frac{\partial u}{\partial y} = \frac{v}{2(u^2+v^2)},\right.$
$\left.\frac{\partial v}{\partial x} = \frac{-v}{2(u^2+v^2)}, \frac{\partial v}{\partial y} = \frac{u}{2(u^2+v^2)}\right]$

3. Find $\frac{du}{dx}$, if $u = x^2y$, where $x^2+xy+y^2 = 1$ $\left[Ans : \frac{x(xy+4y^2-2x^2)}{x+2y}\right.$

4. Find $\frac{du}{dx}$, if $u = \sin(x^2+y^2)$, where $a^2x^2 + b^2y^2 = c^2$
$\left[Ans : 2\left(1 - \frac{a^2}{b^2}\right) x \cos(x^2+y^2)\right]$

5. If $u = f(x, y)$ and $x = r\cos\theta, y = r\sin\theta$, show that

$$\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial u}{\partial y}\right)^2=\left(\frac{\partial u}{\partial r}\right)^2+\frac{1}{r^2}\left(\frac{\partial u}{\partial \theta}\right)^2$$

6. If $z=f(x,y)$ where $x=e^u\cos v, y=e^u\sin v$, show that
$$\left(\frac{\partial z}{\partial u}\right)^2+\left(\frac{\partial z}{\partial v}\right)^2=e^{2u}\left[\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2\right]$$

7. If $u=u\left(\frac{y-x}{xy},\frac{z-x}{zx}\right)$, show that $x^2\frac{\partial u}{\partial x}+y^2\frac{\partial u}{\partial y}+z^2\frac{\partial u}{\partial z}=$ 0

8. Examine $x^4+y^4-2x^2+4xy-2y^2$ for maxima and minima. Also find the extreme values. [Ans: The function is minimum at the points $(\sqrt{2},-\sqrt{2}),(-\sqrt{2},\sqrt{2})$ and the minimum value is -8]

9. Find the maximum and minimum values of $x^3y^2(1-x-y)$ [Ans: $(1/2,1/3)$ is a maximum point and the maximum value is $1/432$]

10. Determine the maximum and minimum values of $x^2y+xy^2-axy, a>0$. [Ans: The function is minimum at (a/3, a/3) and the minimum value is $-\frac{a^3}{27}$]

11. Find the minimum value of $x^2+y^2+z^2$ subject to the condition $\frac{1}{x}+\frac{1}{y}+\frac{1}{z}=1$. [Ans: The function is maximum at (3, 3, 3) and the maximum value is 27]

12. The temperature at any point (x,y,z) in space is $T=400xyz^2$. Find the highest temperature on the surface of the sphere $x^2+y^2+z^2=1$. [Ans: 50]

13. Find the maximum value of $x^my^nz^p$ given that $x+y+z=a$ [Ans: The maximum value is $\left[a^{m+n+p}\frac{m^mn^np^p}{(m+n+p)^{m+n+p}}\right]$]

14. Find the dimensions of the rectangular box without a top of maximum capacity, whose surface is 108 sq.cm. [Ans: 6cm, 6cm and 3cm]

2.6 JACOBIANS

Definition: If u and v are functions of two independent variables x and y, then the determinant $\begin{vmatrix}\frac{\partial u}{\partial x} & \frac{\partial u}{\partial y}\\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y}\end{vmatrix}$ is called Jacobian of u,v with aspect to x,y and is denoted by the symbol $J\left(\frac{u,v}{x,y}\right)$ or $\frac{\partial(u,v)}{\partial(x,y)}$.

Similarly if u,v,w be functions of x,y,z then the Jacobian of u,v,w with respect to x,y,z is $J\left(\frac{u,v,w}{x,y,z}\right)$ or

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix}$$

2.6.1 Properties of Jacobians

Property 2.1. *If J_1 is the Jacobian of u, v with respect to x, y and J_2 is the Jacobian of x, y with respect to u, v then $J_1 J_2 = 1$.*

(i.e.) $\dfrac{\partial(u,v)}{\partial(x,y)} \times \dfrac{\partial(x,y)}{\partial(u,v)} = 1.$

Proof: Let $u = u(x,y)$ and $v = v(x,y)$, so that u and v are functions of x, y.

Differentiating partially w. r. to u and v, we get

$$1 = \frac{\partial u}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial u} = u_x x_u + u_y y_u$$

$$0 = \frac{\partial u}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial v} = u_x x_v + u_y y_v$$

$$0 = \frac{\partial v}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial v}{\partial y}\frac{\partial y}{\partial u} = v_x x_u + v_y y_u$$

$$1 = \frac{\partial v}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial v}{\partial y}\frac{\partial y}{\partial v} = v_x x_v + v_y y_v$$

Now $\dfrac{\partial(u,v)}{\partial(x,y)} \times \dfrac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix} \begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix}$ (Interchanging rows and columns in the second determinant)

$$\frac{\partial(u,v)}{\partial(x,y)} \times \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix} \begin{vmatrix} x_u & y_u \\ x_v & y_v \end{vmatrix}$$

$$= \begin{vmatrix} u_x x_u + u_y y_u & u_x x_v + u_y y_v \\ v_x x_u + v_y y_u & v_x x_v + v_y y_v \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1$$

Property 2.2. *If u, v are functions of r, s where r, s are functions of x, y then* $\dfrac{\partial(u,v)}{\partial(x,y)} = \dfrac{\partial(u,v)}{\partial(r,s)} \times \dfrac{\partial(r,s)}{\partial(x,y)}$.

Proof: Since u and v are composite functions of x and y.

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial u}{\partial s}\frac{\partial s}{\partial x} = u_r r_x + u_s s_x$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial u}{\partial s}\frac{\partial s}{\partial y} = u_r r_y + u_s s_y$$

$$\frac{\partial v}{\partial x} = \frac{\partial v}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial v}{\partial s}\frac{\partial s}{\partial x} = v_r r_x + v_s s_x$$

$$\frac{\partial v}{\partial y} = \frac{\partial v}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial v}{\partial s}\frac{\partial s}{\partial y} = v_r r_y + v_s s_y$$

Now $\dfrac{\partial(u,v)}{\partial(r,s)} \times \dfrac{\partial(r,s)}{\partial(x,y)} = \begin{vmatrix} u_r & u_s \\ v_r & v_s \end{vmatrix} \begin{vmatrix} r_x & r_y \\ s_x & s_y \end{vmatrix}$ (Interchanging rows and columns in the second determinant.

$$= \begin{vmatrix} u_r & u_s \\ v_r & v_s \end{vmatrix} \begin{vmatrix} r_x & r_y \\ s_x & s_y \end{vmatrix}$$

$$= \begin{vmatrix} u_r r_x + u_s s_y & u_r r_y + u_s s_y \\ v_r r_x + v_s s_x & v_r r_y + v_s s_y \end{vmatrix}$$

$$= \begin{vmatrix} \dfrac{\partial u}{\partial x} & \dfrac{\partial u}{\partial y} \\ \dfrac{\partial v}{\partial x} & \dfrac{\partial v}{\partial y} \end{vmatrix}$$

$= \dfrac{\partial(u,v)}{\partial(x,y)}$, using (1)

Note: If the Jacobian value is zero then u and v are functionally dependent.

Example 2.33. *If $x = u^2 - v^2$ and $y = 2uv$, find the Jacobian of x and y with respect to u and v.*

Solution: Let $J\left(\dfrac{\partial(x,y)}{\partial(u,v)}\right) = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix}$

$$= \begin{vmatrix} 2u & -2v \\ 2v & 2u \end{vmatrix}$$

$= 4u^2 + 4v^2 = 4(u^2 + v^2)$

Example 2.34. *If $x = r\cos\theta$, $y = r\sin\theta$, verify that $\dfrac{\partial(x,y)}{\partial(r,\theta)} \times \dfrac{\partial(r,\theta)}{\partial(x,y)} = 1$*

Solution: Let $x = r\cos\theta$, $y = r\sin\theta$. Then

$\dfrac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \dfrac{\partial x}{\partial r} & \dfrac{\partial x}{\partial \theta} \\ \dfrac{\partial y}{\partial r} & \dfrac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r(\cos^2\theta + \sin^2\theta) = r$

Now $r^2 = x^2 + y^2$ and $\theta = \tan^{-1}\dfrac{y}{x}$.

Then $2r\dfrac{\partial r}{\partial x} = 2x \Rightarrow \dfrac{\partial r}{\partial x} = \dfrac{x}{r}$

and $\dfrac{\partial \theta}{\partial x} = \dfrac{1}{1 + \dfrac{y^2}{x^2}} \times \dfrac{-y}{x^2} = \dfrac{-y}{x^2 + y^2} = \dfrac{-y}{r^2}$

Similarly $\dfrac{\partial r}{\partial y} = \dfrac{y}{r}$. Also $\dfrac{\partial \theta}{\partial y} = \dfrac{x}{r^2}$

$$\frac{\partial(r,\theta)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial r}{\partial x} & \frac{\partial r}{\partial y} \\ \frac{\partial \theta}{\partial x} & \frac{\partial \theta}{\partial y} \end{vmatrix} = \begin{vmatrix} \frac{x}{r} & \frac{y}{r} \\ \frac{-y}{r^2} & \frac{x}{r^2} \end{vmatrix}$$

$$= \frac{x^2+y^2}{r^2} = \frac{1}{r}$$

$$\therefore \frac{\partial(x,y)}{\partial(r,\theta)} \times \partial(x,y) = r.\frac{1}{r} = 1$$

Example 2.35. *If* $u = \frac{yz}{x}, v = \frac{zx}{y}, w = \frac{xy}{z}$, *show that* $\frac{\partial(u,v,w)}{\partial(x,y,z)}$ 4.

Solution:

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix}$$

$$= \begin{vmatrix} \frac{-yz}{x^2} & \frac{z}{x} & \frac{y}{x} \\ \frac{z}{y} & \frac{-xz}{y^2} & \frac{x}{y} \\ \frac{y}{z} & \frac{x}{z} & \frac{-xy}{z^2} \end{vmatrix}$$

$$= \frac{1}{x^2y^2z^2} \begin{vmatrix} -yz & xz & xy \\ yz & -xz & xy \\ yz & xz & -xy \end{vmatrix}$$

$$= \frac{(yz)(xz)(xy)}{x^2y^2z^2} \begin{vmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{vmatrix}$$ (Taking yz, xz, xy as common factors from 1st, 2nd and 3rd columns respectively)

$= -1(1-1) - 1(-1-1) + 1(1+1)$, Expanding through 1st row.

$= 4$

Example 2.36. *If* $x = r\sin\theta\cos\phi, y = r\sin\theta\sin\phi, z = r\cos\theta$, *show that* $\frac{\partial(x,y,z)}{\partial(r,\theta,\phi)} = r^2\sin\theta$.

Solution:

Let $$\frac{\partial(x,y,z)}{\partial(r,\theta,\phi)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \phi} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \phi} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial \phi} \end{vmatrix}$$

$$= \begin{vmatrix} \sin\theta\cos\phi & r\cos\theta\cos\phi & -r\sin\theta\sin\phi \\ \sin\theta\sin\phi & r\cos\theta\sin\phi & r\sin\theta\cos\phi \\ \cos\theta & -r\sin\theta & 0 \end{vmatrix}$$

Taking out common factors (r from second column and $r\sin\theta$ from third column)

$$= r^2\sin\theta \begin{vmatrix} \sin\theta\cos\phi & \cos\theta\cos\phi & -\sin\phi \\ \sin\theta\sin\phi & \cos\theta\sin\phi & \cos\phi \\ \cos\theta & -\sin\theta & 0 \end{vmatrix}$$

Expanding by third row

$$= r^2\sin\theta\left\{\cos\theta \begin{vmatrix} \cos\theta\cos\phi & -\sin\phi \\ \cos\theta\sin\phi & \cos\phi \end{vmatrix} + \sin\theta \begin{vmatrix} \sin\theta\cos\phi & -\sin\phi \\ \sin\theta\sin\phi & \cos\phi \end{vmatrix}\right\}$$

$$= r^2\sin\theta[\cos\theta(\cos\theta\cos^2\phi + \cos\theta\sin^2\phi)$$

$$+\sin\theta(\sin\theta\cos^2\phi + \sin\theta\sin^2\phi)]$$

$$= r^2\sin\theta(\cos^2\theta + \sin^2\theta) = r^2\sin\theta$$

Example 2.37. *If* $x = r\cos\theta, y = r\sin\theta, z = z$, *evaluate* $\dfrac{\partial(x,y,z)}{\partial(r,\theta,z)}$

Solution: Let

$$\frac{\partial(x,y,z)}{\partial(r,\theta,z)} = \begin{vmatrix} \dfrac{\partial x}{\partial r} & \dfrac{\partial x}{\partial \theta} & \dfrac{\partial x}{\partial z} \\ \dfrac{\partial y}{\partial r} & \dfrac{\partial y}{\partial \theta} & \dfrac{\partial y}{\partial z} \\ \dfrac{\partial z}{\partial r} & \dfrac{\partial z}{\partial \theta} & \dfrac{\partial z}{\partial z} \end{vmatrix}$$

$$= \begin{vmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix}$$
$$= r(\cos^2\theta + \sin^2\theta) = r$$

Example 2.38. *If* $u = xyz, v = xy + yz + zx, w = x + y + z$, *find* $\dfrac{\partial(u,v,w)}{\partial(x,y,z)}$

Solution:

$$\text{Let } \frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} \dfrac{\partial u}{\partial x} & \dfrac{\partial u}{\partial y} & \dfrac{\partial u}{\partial z} \\ \dfrac{\partial v}{\partial x} & \dfrac{\partial v}{\partial y} & \dfrac{\partial v}{\partial z} \\ \dfrac{\partial w}{\partial x} & \dfrac{\partial w}{\partial y} & \dfrac{\partial w}{\partial z} \end{vmatrix}$$

$$= \begin{vmatrix} yz & xz & xy \\ y+z & x+z & y+x \\ 1 & 1 & 1 \end{vmatrix}$$

Using $C_1 \to C_2 - C_1, C_2 \to C_3 - C_2$, we get

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} z(x-y) & x(y-z) & xy \\ x-y & y-z & y+x \\ 0 & 0 & 1 \end{vmatrix}$$

Taking $(x-y),(y-z)$ as common factors from column 1 and column 2, we get

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = (x-y)(y-z)\begin{vmatrix} z & x & xy \\ 1 & 1 & y+x \\ 0 & 0 & 1 \end{vmatrix}$$

$$= (x-y)(y-z)(z-x)$$

Example 2.39. *Are the functions* $u = \dfrac{x+y}{1-xy}$ *and* $v = \tan^{-1} x + \tan^{-1} y$ *functionally dependent? If so, find the relation between them.*

Solution: Let $u = \dfrac{x+y}{1-xy}$

Differentiate w. r. to x partially we get,

$$\frac{\partial u}{\partial x} = \frac{(1-xy)(1)-(x+y)(-y)}{(1-xy)^2} = \frac{1+y^2}{(1-xy)^2}$$

Similarly $\dfrac{\partial u}{\partial y} = \dfrac{1+x^2}{(1-xy)^2}$ (since u is symmetric)

$v = \tan^{-1} x + \tan^{-1} y$

Differentiate w. r. to x partially we get,

$$\frac{\partial v}{\partial x} = \frac{1}{1+x^2} \text{ and } \frac{\partial v}{\partial y} = \frac{1}{1+y^2}$$

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} \frac{1+y^2}{(1-xy)^2} & \frac{1+x^2}{(1-xy)^2} \\ \frac{1}{1+x^2} & \frac{1}{1+y^2} \end{vmatrix} = 0$$

$\Rightarrow$ u and v are functionally dependent.

Now $v = \tan^{-1} x + \tan^{-1} y = \tan^{-1}\left\{\dfrac{x+y}{1-xy}\right\}$

$\Rightarrow v = \tan^{-1} u \Rightarrow u = \tan v.$

Example 2.40. *If* $u = y+z; v = x+2z^2; w = x-4yz-2y^2$, *find the Jacobian of* u,v,w *with respect to* x,y,z. *Comment on the result.*

Solution: Let $u = y+z; v = x+2z^2; w = x-4yz-2y^2$.

Differentiate them partially w. r. to x,y,z, we get

$$\frac{\partial u}{\partial x} = 0; \frac{\partial u}{\partial y} = 1; \frac{\partial u}{\partial z} = 1$$

$$\frac{\partial v}{\partial x}=1;\ \frac{\partial v}{\partial y}=0;\ \frac{\partial v}{\partial z}=4z$$

$$\frac{\partial w}{\partial x}=1;\ \frac{\partial w}{\partial y}=-4z-4y;\ \frac{\partial w}{\partial z}=-4y$$

$$\frac{\partial(u,v,w)}{\partial(x,y,z)}=\begin{vmatrix}\frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z}\\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z}\\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z}\end{vmatrix}$$

$$=\begin{vmatrix}0 & 1 & 1\\ 1 & 0 & 4z\\ 1 & -4y-4z & -4y\end{vmatrix}=-1(-4y-4z)+(-4y-4z)=0$$

$\therefore u,v$ and w are functionally dependent.

Now $v-w=2z^2+4yz+2y^2=2(y+z)^2=2u^2$

Hence the functional relationship is $2u^2=v-w$.

Example 2.41. *If $u=x+y+z, uv=y+z, uvw=z$, show that* $\frac{\partial(x,y,z)}{\partial(u,v,w)}=u^2v$.

Solution: Given $z=uvw\Rightarrow \frac{\partial z}{\partial u}=vw;\ \frac{\partial z}{\partial v}=uw;\ \frac{\partial z}{\partial w}=uv$

and $y+z=uv\Rightarrow y+uvw=uv\Rightarrow y=uv-uvw$

$\Rightarrow \frac{\partial y}{\partial u}=v-vw;\ \frac{\partial y}{\partial v}=u-uw;\ \frac{\partial y}{\partial w}=-uv$

Also $u=x+y+z\Rightarrow x=u-y-z=u-uv+uvw-uvw$

$\Rightarrow x=u-uv\Rightarrow \frac{\partial x}{\partial u}=1-v;\ \frac{\partial x}{\partial v}=-u;\ \frac{\partial x}{\partial w}=0$

Now $$\frac{\partial(x,y,z)}{\partial(u,v,w)}=\begin{vmatrix}\frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w}\\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w}\\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w}\end{vmatrix}$$

$$=\begin{vmatrix}1-v & -u & 0\\ v-vw & u-uw & -uv\\ vw & uw & uv\end{vmatrix}$$

$$=\begin{vmatrix}1-vw & -uw & -uv\\ v & u & 0\\ vw & uw & uv\end{vmatrix}\ \text{(Using } R_1\to R_1+R_2, R_2\to R_2+R_3\text{)}$$

From $R_1\to R_1+R_3$

$$=\begin{vmatrix}1 & 0 & -uv\\ v & u & 0\\ vw & uw & uv\end{vmatrix}=u^2v\ \text{(by expanding through 1st row)}$$

Example 2.42. *If $u=x^2-2y, v=x+y+z, w=x-2y+3z$, find* $\frac{\partial(u,v,w)}{\partial(x,y,z)}$

Solution: Let $\frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix}$

$= \begin{vmatrix} 2x & -2 & 0 \\ 1 & 1 & 1 \\ 1 & -2 & 3 \end{vmatrix} = 2x(3+2) + 2(3-1)$ (by expanding through 1st row) $= 10x + 4$

EXERCISE

1. If $u = x^2 - 2y, v = x + y$, prove that $\frac{\partial(u,v)}{\partial(x,y)} = 2x + 2$

2. If $u = x(1-y), v = xy$, prove that $JJ' = 1$, where $J = \frac{\partial(u,v)}{\partial(x,y)}$ and $J' = \frac{\partial(x,y)}{\partial(u,v)}$

3. If $u = \frac{x}{y-z}, v = \frac{y}{z-x}, w = \frac{z}{x-y}$, show that $\frac{\partial(u,v,w)}{\partial(x,y,z)} = 0$

4. Determine whether $u = \sin^{-1} x + \sin^{-1} y$ and $v = x\sqrt{1-y^2} + y\sqrt{1-x^2}$ are functionally dependent. If so, find the relation between them: [Ans: $v = \sin u$]

5. If $u = xy + yz + zx, v = x^2 + y^2 + z^2$ and $w = x + y + z$, Determine whether there is a functional relationship between u,v,w and if so, find it. [Ans: $w^2 - v - 2u = 0$]

6. For the transformation $x = e^u \cos v, v = e^v \sin v$, prove that $\frac{\partial(x,y)}{\partial(u,v)} \times \frac{\partial(u,v)}{\partial(x,y)} = 1$

7. Find the value of $\frac{\partial(y_1,y_2,y_3)}{\partial(x_1,x_2,x_3)}$, if $y_1 = 1 - x_1, y_2 = x_1(1 - x_2), y_3 = x_1 x_2 (1 - x_3)$ [Ans: $-x_1^2 x_2$]

8. If $u = x+y+z, u^2 v = y+z, u^3 w = z$, show that $\frac{\partial(u,v,w)}{\partial(x,y,z)} = u^{-5}$