SRM UNIVERSITY
(Under section 3 of UGC Act 1956)

18MAB102T- Advanced Calculus and Complex Analysis

UNIT V
COMPLEX INTEGRATION

TOPICS DISCUSED

- Line integral
- Cauchy's integral theorem(without proof)
- Cauchy s integral formula (with proof)
- Application of Cauchy's integral formula
- Taylor s and Laurent's expansion(statements only)
- Singularities
- Poles and Residues
- Cauchy s residue theorem (with proof)
- Evaluation of line integrals

LINE INTEGRAL

Definition :

Let $w = f(z)$ be a continuous function of the complex variable $z = x + iy$ along a curve c with end points A and B

$$\oint_c f(z)\,dz = \int_c (udx - vdy) + i(vdx + udy)$$

EXAMPLE 1

Evaluate $\int_C \bar{z}\,dz$ from $A(0,0)$ to $B(4,2)$ along the curve C and $z = t^2 + it$

Solution:

$$Let\ \bar{z} = x - iy\ , \quad z = x + iy = t^2 + it$$

$$\Rightarrow x = t^2,\ y = t$$

$$dx = 2t\,dt\ ,\ dy = dt \quad and \quad dz = dx + i\,dy$$

$$= 2t\,dt + i\,dt$$

$$= (2t + i)\,dt$$

$$Also \quad x = 0, 4 \ \Longrightarrow t = 0, 2$$

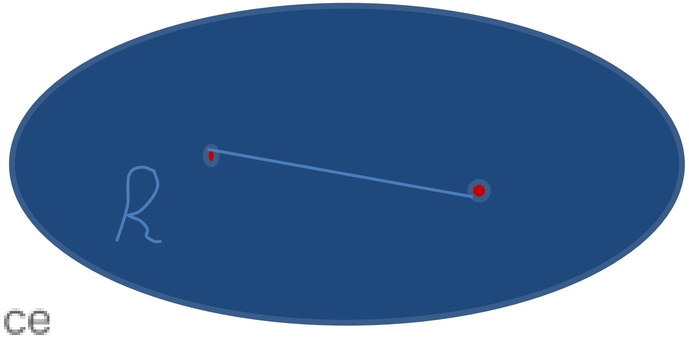
$$y = 0, 2 \ \Longrightarrow t = 0, 2$$

$$I = \oint_c \bar{z}\,dz = \int_0^2 (t^2 - it)(2t + i)\,dt = 10 - \frac{8}{3}i$$

DEFINITIONS

*Connected region :*牋

A region R is said to be connected when two points of it are connected by a curve; the curve should lie inside the region.

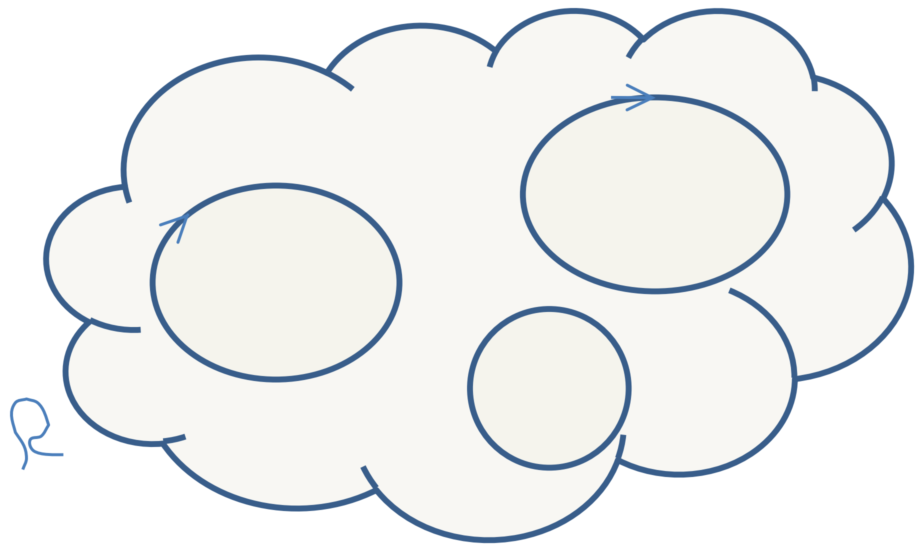


Simply Connected region:

A region R is said to be simply connected if any closed curve which lies in R can be shrunk to a point without leaving R

MultiplyConnected region:

A region which is not simply connected.



NOTE

Multiply connected regions can be converted

into a simply connected region by strip cuts

CAUCHY'S INTEGRAL THEOREM (or)

牋CAUCHY'S FUNDAMENTAL THEOREM

If $f(z)$ is analytic and its derivatives $f'(z)$ is continuous at all points on and inside a simple closed curve C , then

$$\int_{c} f(z)\,dz = 0$$

CAUCHY'S INTEGRAL THEOREM FOR MULTIPLY CONNECTED REGION

If $f(z)$ is analytic and its derivatives $f'(z)$ is continuous at all points in the region bounded by the simple closed curve C_1 & C_2 then

$$\oint_{c_1} f(z)\,dz = \oint_{c_2} f(z)\,dz$$

CAUCHY'S INTEGRAL FORMULA

If $f(z)$ is analytic inside and on a simple closed curve C that encloses a simple connected region R and if "a" is any point in R then

$$f(a) = \frac{1}{2\pi i}\oint_c \frac{f(z)}{z-a}dz$$

Where C is described in the anticlockwise direction

CAUCHY'S INTEGRAL FORMULA FOR THE DERIVATIVES OF AN ANALYTIC FUNCTION

If a function $f(z)$ is analytic within and on a simple closed curve C and 'a' is any point lying in it, then

$$f^{n}(a) = \frac{n!}{2\pi i}\oint_{c}\frac{f(z)}{(z-a)^{n+1}}dz$$

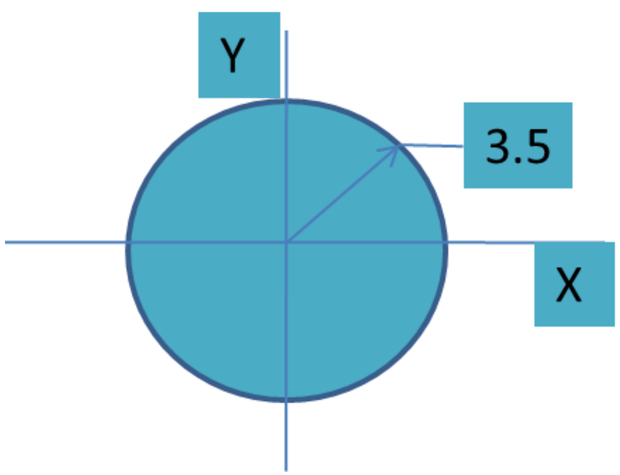
In general, $f^{n}(a) = \frac{n!}{2\pi i}\oint_{c}\frac{f(z)}{(z-a)^{n+1}}dz$

EXAMPLES 1

$$Evaluate \int_c \frac{dz}{z^2 - 7z + 12} \ where\, C \ is \ the$$

$$circle \, |z| = 3.5$$

$$Solution: Singular points: z^2 - 7z + 12 = 0 \Rightarrow z = 4, 3$$



$$z = 4 \ lies \ outside \ the \ circle \ |z| = 3.5$$

$$z = 3 \ lies \ inside \ the \ circle \ |z| = 3.5$$

$$\oint_c \frac{dz}{(z-4)(z-3)} = \oint_c \frac{\left(\frac{1}{z-4}\right)}{z-3} dz$$

$$Here \quad f(z) = \frac{1}{z-4} \quad is\, analytic\ inside\ C$$

$$\oint \frac{f(z)}{z-a} dz = 2\pi i f(a)$$

$$= 2\pi i f(3)$$

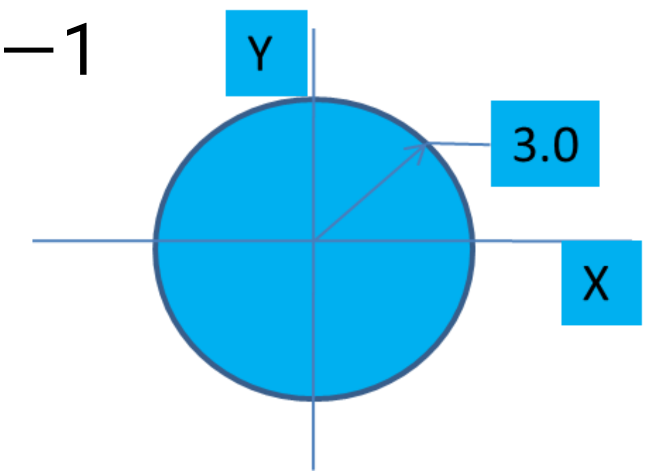
$$= -2\pi i$$

EXAMPLES 2

$$Evaluate \oint_c \frac{z-2}{z(z-1)}\,dz \quad where\ C\ is\ a\ circle\ |z| = 3$$

$Solution:$ Singular points $z = 0,1$ $lies\ inside\ C$

$$Now\ Consider\ \frac{1}{z(z-1)} = \frac{A}{z} + \frac{B}{z-1}$$



$$A = -1 \quad and\ B = 1$$

$$\therefore \quad \frac{1}{z(z-1)} = \frac{-1}{z} + \frac{1}{z-1}$$

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$$\oint \frac{f(z)}{z-a}dz = 2\pi i f(a)$$

$$\therefore \oint_c \frac{(z-2)}{z(z-1)}dz = \oint_c \left(\frac{1}{z-1} - \frac{1}{z}\right)(z-2)\,dz$$

$$= 2\pi i f(0) - 2\pi i f(1)$$

$$= 2\pi i(-2) - 2\pi i(-1)$$

$$= -2\pi i$$

EXAMPLES 3

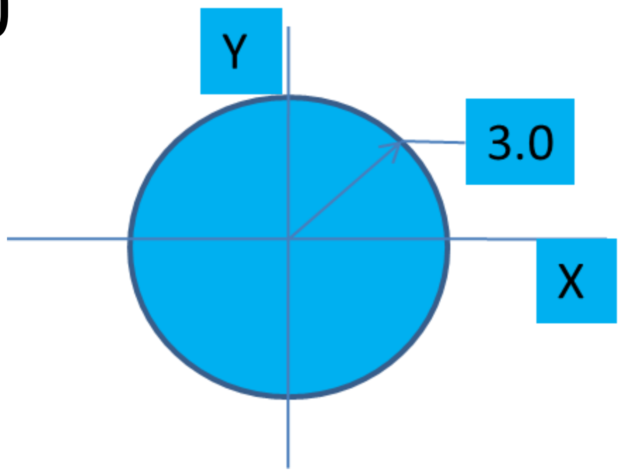
$$Evaluate \oint_c \frac{\cos \pi z^2}{(z-1)(z-2)} dz \quad where \ |z| = 3$$

using cauchy residues theorem

soln :

Singular points: $(z-1)(z-2)=0$

$\Rightarrow z = 1, 2 \quad lies \, inside \ |z| = 3$



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$$Now\ Consider\ \frac{1}{(z-1)(z-2)} = \frac{A}{z-1} + \frac{B}{z-2}$$

$$A = -1 \quad and\ B = 1$$

$$\therefore \quad \frac{1}{(z-1)(z-2)} = \frac{-1}{z-1} + \frac{1}{z-2}$$

$$\oint_c \frac{\cos \pi z^2}{(z-1)(z-2)} dz = -\oint_c \frac{\cos \pi z^2}{(z-1)} dz + \oint_c \frac{\cos \pi z^2}{(z-2)} dz$$

$$= -2\pi i f(1) + 2\pi i f(2)$$

$$= 4\pi i$$

TAYLORS SERIES

A function $f(z)$ be *analytic at all points inside a circle C with its center at a and radius* r , *we can expand as*

$$f(z) = f(a) + \frac{f'(a)}{1!}(z-a) + \frac{f''(a)}{2!}(z-a)^2 + \ldots\ldots\ldots$$

$$\ldots + \frac{f^n(a)}{n!}(z-a)^n + \ldots.\infty$$

$$= \sum_{n=1}^{\infty} \frac{(z-a)^n}{n!} f^n(a)$$

EXAMPLE 1

Expand $\dfrac{1}{z-2}$ *at* $z = 1$ *is a Taylor's series.*

Solution : Let

$$f(z) = \frac{1}{z-2} \quad \Rightarrow \quad f(1) = -1$$

$$f'(z) = \frac{-1}{(z-2)^2} \quad \Rightarrow \quad f'(1) = -1$$

$$f''(z) = \frac{2}{(z-2)^3} \quad \Rightarrow \quad f''(1) = 2$$

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$$f'''(z) = \frac{-6}{(z-2)^4} \Rightarrow f'''(1) = -6$$

Taylor's series of $f(z)$ *about the point* $z = 1$ *is*

$$f(z) = -1 + \frac{(-1)}{1!}(z-1) + \frac{(2)}{2!}(z-1)^2 + \frac{(-6)}{3!}(z-1)^3 + $$

$$f(z) = -1 - (z-1) + (z-1)^2 + (z-1)^3 + \ldots$$

EXAMPLE 2

Expand $\cos z$ at $z = 0$ is a Taylor sseries

Solution : Let

$$f(z) = \cos z \quad \Rightarrow \quad f(0) = 1$$

$$f^{'}(z) = -\sin z \quad \Rightarrow \quad f^{'}(1) = 0$$

$$f^{''}(z) = -\cos z \quad \Rightarrow \quad f^{''}(1) = -1$$

$$f^{'''}(z) = \sin z \quad \Rightarrow \quad f^{'''}(1) = 0$$

Taylor's series of $f(z)$ *about the point* $z = 0$ *is*

$$f(z) = 1 + \frac{(0)}{1!}(z-0) + \frac{(-1)}{2!}(z-0)^2 + \frac{(0)}{3!}(z-0)^3 +$$

$$f(z) = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \ldots\ldots\ldots$$

NOTE

If $a = 0$ *then the taylor's series become Maclaurin's series*

$$f(z) = f(0) + \frac{f'(0)}{1!}(z) + \frac{f''(0)}{2!}(z)^2 + \ldots\ldots\ldots$$

$$\ldots + \frac{f^n(0)}{n!}(z)^n + \ldots\infty$$

LAURENTS SERIES:

If $f(z)$ is analytic on two concentric circle C_1 and C_2 of radii r_1 and r_2 with center at a and also on the annular region R bounded by C_1 and C_2 then for all Z in R

$$f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z-a)^n} \quad \text{where}$$

$$a_n = \frac{1}{2\pi i} \oint_{c_1} \frac{f(z)}{(z-a)^{n+1}} dz \; ; \; b_n = \frac{1}{2\pi i} \oint_{c_2} \frac{f(z)}{(z-a)^{1-n}} dz$$

Both the integral being taken anticlockwise direction

EXAMPLE 1

Find the Laurent's series for $f(z) = \dfrac{z-1}{(z+2)(z+3)}$

in the region $2 < |z| < 3$

Soln : Let $\dfrac{z-1}{(z+2)(z+3)} = \dfrac{A}{z+2} + \dfrac{B}{z+3}$

$$\Rightarrow z-1 = A(z+3) + B(z+2)$$

$$\Rightarrow A = -3 \quad , \quad B = 4$$

$$\therefore \quad f(z) = \frac{-3}{z+2} + \frac{4}{z+3}$$

$$Let \ 2 < |z| < 3 \implies |z| > 2 \quad and \quad |z| < 3$$

$$\implies \frac{2}{|z|} < 1 \quad and \quad \frac{|z|}{3} < 1$$

$$f(z) = \frac{-3}{z\left(1+\frac{2}{z}\right)} + \frac{4}{3\left(1+\frac{3}{z}\right)}$$

$$f(z) = \frac{-3}{z}\left(1+\frac{2}{z}\right)^{-1} + \frac{4}{3}\left(1+\frac{3}{z}\right)^{-1}$$

EXAMPLE 2

Find the Laurent's series for $f(z)=\dfrac{1}{z^2-3z+2}$

in the region $(i)\ 1<|z|<2 \quad (ii)\ |z|>2 \quad (iii)\ |z-1|<1$

Soln : Let $f(z)=\dfrac{1}{z^2-3z+2}$

$$\text{Consider } \frac{1}{z^2-3z+2}=\frac{A}{z-1}+\frac{B}{z-2}$$

$$\Rightarrow A=-1\ ,\ B=1$$

$$\therefore\ f(z)=\frac{-1}{z-1}+\frac{1}{z-2}$$

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$$(i)\ 1 < |z| < 2 \Rightarrow |z| > 1 \quad and \quad |z| < 2$$

$$\Rightarrow \frac{1}{|z|} < 1 \quad and \quad \frac{|z|}{2} < 1$$

$$f(z) = \frac{-1}{z\left(1 - \frac{1}{z}\right)} + \frac{1}{2\left(\frac{z}{2} - 1\right)}$$

$$f(z) = \frac{-1}{z}\left(1 - \frac{1}{z}\right)^{-1} - \frac{1}{2}\left(1 - \frac{z}{2}\right)^{-1}$$

$$(ii)\ |z| > 2 \Rightarrow \frac{2}{|z|} < 1$$

$$f(z) = \frac{-1}{z\left(1-\frac{1}{z}\right)} + \frac{1}{z\left(1-\frac{2}{z}\right)}$$

$$f(z) = \frac{-}{z}\left(1-\frac{1}{z}\right)^{-1} + \frac{1}{z}\left(1-\frac{2}{z}\right)^{-1}$$

$$(iii)\ |z-1| < 1 \Rightarrow Put\ z-1 = u \Rightarrow z = u+1\ \ \&\ \ |u| < 1$$

$$f(z) = \frac{-1}{z-1} + \frac{1}{z-2}$$

$$f(z) = \frac{-1}{u+1-1} + \frac{1}{u+1-2}$$

$$= \frac{-1}{u} + \frac{1}{u-1}$$

$$= \frac{-1}{u} - (1-u)^{-1}$$

$$f(z) = \frac{-1}{z-1} - \left[1 + (z-1) + (z-1)^2 + \ldots\ldots\right]$$

SINGULAR POINTS

A point $z = z_0$ at which a function $f(z)$ fails to be analytic is called a singular point or singularity of $f(z)$

Example

$f(z) = \dfrac{1}{z-3}$, here $z = 3$ is a singular point of f(z)

TYPES OF SINGULAR POINTS

ISOLATED SINGULARITY

A point $z = z_0$ is said to be an isolated singularity of $f(z)$ if (i) $f(z)$ is not analytic at $z = z_0$

(ii) There exist a neighbourhood of $z = z_0$ containing no other singularity

Example:$-$ $f(z) = \dfrac{1}{z}$ is an analytic every where except at $z = 0$

$\therefore$ $z = 0$ is an isolated singularity

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NOTE

If $z = z_0$ *is an isolated singular point of a function* $f(z)$ *then the singularity is called*

(i) *Removable singularity*

(ii) *A pole*

(iii) *An essential singularity.*

REMOVABLE SINGULARITY

A singular point $z = z_0$ is called a removable singularity of $f(z)$ if $\lim_{z \to z_0} f(z)$ exist and is finite

Example: $f(z) = \dfrac{\sin z}{z}$

$$= \frac{1}{z}\left(z - \frac{z^3}{3!} + \frac{z^5}{5!} + \ldots\ldots \right)$$

$$= 1 - \frac{z^2}{3!} + \frac{z^4}{5!} + \ldots$$

There is no negative power of Z.

Therefore $z = 0$ is removable singularity

POLES

An analytic function $f(z)$ with a singularity at $z = a$ if $\lim_{z \to a} f(z) = \infty$ then $z = a$ is a pole of $f(z)$.

SIMPLE POLES

A pole of order one is called a simple pole

ESSENTIAL SINGULARITY

If the principal part contains an infinite no of non − zero terms then $z = z_0$ is known as an essential singularity

$$Example: \ f(z) = e^{\frac{1}{z}}$$

$z = 0$ *is a* singular points

$$But\ e^{\frac{1}{z}} = 1 + \frac{\frac{1}{z}}{1!} + \frac{\frac{1}{z^2}}{2!} + \ldots\ldots$$

$$= 1 + \frac{1}{z} + \frac{1}{2z^2} + \ldots\ldots$$

Here $f(z)$ *has* infinite number of -ve powers of z

$\therefore$ z = 0 is a essential singularity.

EVALUATION OF RESIDUES OF f(z)

(i) *Residue of* $f(z)$ *at its simple pole* $z = z_0$ *is given by*

$$R = \operatorname{Re}(z = z_0) = \lim_{z \to z_0} (z = z_0) f(z)$$

(ii) *Residue of* $f(z)$ *at its pole* $z = z_0$ *of order* n *is given by*

$$R = \operatorname{Re}(z = z_0) = \lim_{z \to z_0} \left[\frac{1}{(n-1)!} \frac{d^{n-1}}{dz^{n-1}} (z = z_0)^n f(z) \right]$$

CAUCHY RESIDUES THEOREM

Let $f(z)$ be analytic at all points inside and on a simple closed curve C except for a finite no of isolated singularity $z_1, z_2, \ldots\ldots z_n$ inside C, then

$$\int_c f(z)\,dz = 2\pi i(\text{sum of the residue of } f(z) \text{ at } z_1, z_2, \ldots\ldots z_n)$$

$$= 2\pi i \sum_{i=1}^{n} R_i \text{, where } R_i \text{ is the residue of } f(z) \text{ at } z = z_i$$

EXAMPLE-1

$$\text{Evaluate} \oint_c \frac{\cos \pi z^2}{(z-1)(z-2)} dz \quad \text{where } |z| = 3$$

using cauchy residues theorem

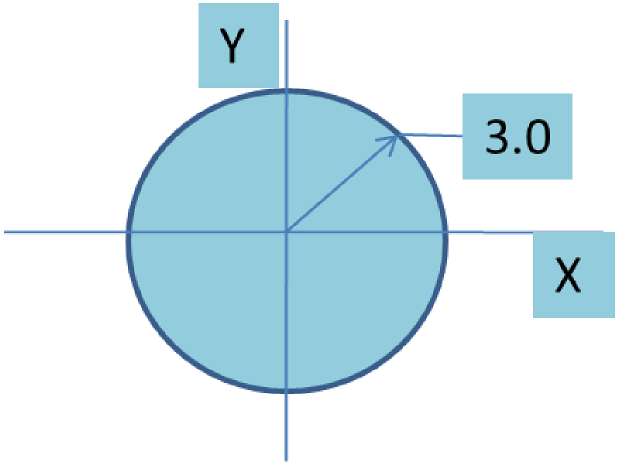
soln :

Singular points: (z-1)(z-2)=0

$\Rightarrow z = 1, 2$ *is a pole of order one.*

$\therefore$ *Its a simple pole*

$$z = 1\ ,\ 2 \quad \textit{both lies inside the circle} \ |z| = 3$$



Now

$$\mathrm{Res}_1\,(z = 1) = \lim_{z \to 1} (z - 1)\ f(z)$$

$$= \lim_{z \to 1} (z - 1) \frac{\cos \pi z^2}{(z - 1)(z - 2)}$$

$$= 1$$

$$\operatorname{Res}_2(z=2) = \lim_{z\to 2}(z-2)\, f(z)$$

$$= \lim_{z\to 2}(z-2)\frac{\cos \pi z^2}{(z-1)(z-2)}$$

$$= 1$$

$$\oint_c \frac{\cos \pi z^2}{(z-1)(z-2)} = \oint_c f(z)\,dz$$

$$= 2\pi i(sum\ of\ \ residues)$$

$$= 2\pi i(R_1 + R_2)$$

$$= 4\pi i$$

EXAMPLE-2

(ii) Evaluate $\oint_c \frac{\sin \pi z + \cos \pi z^2}{z+z^2} dz$ where C is a circle $|z| = 2$

soln : (Hint)

$z = 0, 1$ are simple pole & both lies inside the circle $|z| = 2$

$R_1(z=0) = 1$ & $R_2(z=1) = 1$

$\therefore \oint_c f(z)\,dz = 4\pi i$

EXAMPLE-3

$$\text{Find the residues at their poles of } f(z) = \frac{z}{(z-1)^2}$$

$$\text{soln: The poles are given by } (z-1)^2 = 0$$

$$\text{So } z = 1 \text{ is a pole of order } 2$$

$$R = \mathrm{Re}(z = z_0) = \lim_{z \to z_0} \left[\frac{1}{(n-1)!} \frac{d^{n-1}}{dz^{n-1}} (z = z_0)^n f(z) \right]$$

$$\mathrm{Re}(z = 1) = \lim_{z \to 1} \left[\frac{1}{(2-1)!} \frac{d^{2-1}}{dz^{2-1}} (z = 1)^2 \frac{z}{(z-1)^2} \right]$$

$$= \lim_{z \to 1} \frac{d}{dz}(z) = 1$$

APPLICATION OF RESIDUES TO EVALUATE REAL INTEGRALS CONTOUR INTEGRATION 牋(UNIT CIRCLE)

Type 1: $\int_0^{2\pi} f(\cos\theta, \sin\theta)\,d\theta$

Here $z = e^{i\theta} \Rightarrow dz = ie^{i\theta}d\theta = iz\,d\theta$

$$\Rightarrow d\theta = \frac{1}{iz}dz$$

$$Now\ let\quad z = e^{i\theta} = \cos\theta + i\sin\theta$$

$$\frac{1}{z} = e^{-i\theta} = \cos\theta - i\sin\theta$$

$$\therefore \quad \cos\theta = \frac{1}{2}\left(z + \frac{1}{z}\right) \ \& \ \sin\theta = \frac{1}{2i}\left(z - \frac{1}{z}\right)$$

$$\therefore \int_0^{2\pi} f\left(\frac{1}{2}\left[z + \frac{1}{z}\right], \frac{1}{2i}\left[z - \frac{1}{z}\right]\right)\frac{dz}{zi}$$

EXAMPLE

$$Evaluate \int_0^{2\pi} \frac{d\theta}{5+3\cos\theta}$$

$Soln:$

$$Let \quad z = e^{i\theta} \Rightarrow dz = ie^{i\theta} d\theta = iz\, d\theta$$

$$\Rightarrow d\theta = \frac{1}{iz} dz$$

$$\cos\theta = \frac{1}{2}\left(z + \frac{1}{z}\right) \;\&\; \sin\theta = \frac{1}{2i}\left(z - \frac{1}{z}\right)$$

Now

$$I = \int_0^{2\pi} \frac{d\theta}{5+3\cos\theta} = \oint_c \frac{1}{5+\frac{3}{2}\left(z+\frac{1}{z}\right)} \frac{dz}{iz}$$

$$= \frac{2}{i}\oint_c \frac{dz}{3z^2+10z+3} = \frac{2}{i}\oint_c f(z)\,dz$$

$$= \frac{2}{i}[2\pi i(sum\ of\ the\ residues\ of\ f(z)]$$

$$= 4\pi\,[\,sum\ of\ \ the\ residues\,of\ \ f\,(z)\,]$$

Hence

$$\operatorname{Re}\left(z = -\frac{1}{3}\right) = \lim_{z \to -\frac{1}{3}} \left(z + \frac{1}{3}\right) \frac{1}{(3z+1)(z+3)}$$

$$= \frac{1}{8}$$

$$\int_0^{2\pi} \frac{d\theta}{5 + 3\cos\theta} = 4\pi\left(\frac{1}{8}\right) = \frac{\pi}{2}$$

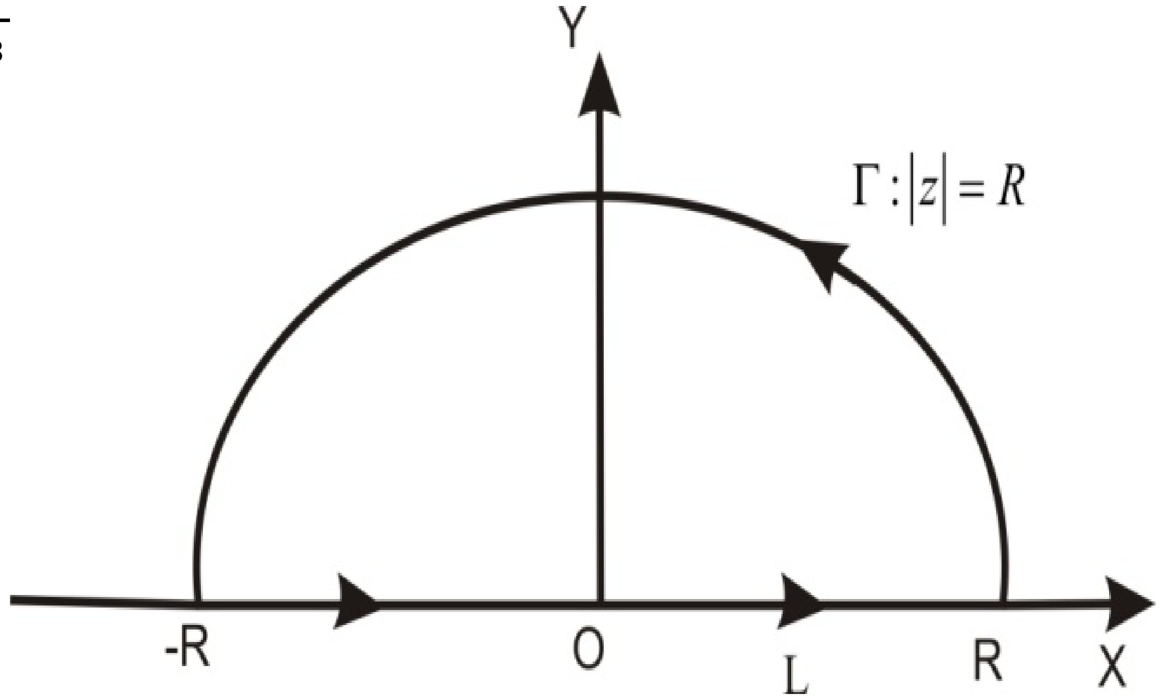
$$\int_0^{\infty} \frac{dx}{(x^2 + a^2)^3}$$

$$I = \int_C \frac{dz}{(z^2 + a^2)^3}$$

$$|z| = R$$

$$(z^2 + a^2)^3 = 0$$

$$z^2 = -a^2$$



$$z = \pm ia$$

$$I = \int_C f(z)\,dz = 2\pi i\, R_1$$

$$\int_\Gamma f(z)\,dz + \int_L f(z)\,dz = 2\pi i\, R_1 \qquad \ldots(1)$$

$$R_1 = \frac{1}{\angle(n-1)} \lim_{z \to ai} \frac{d^{\,n-1}}{dz^{n-1}} (z-ai)^n\, f(z)$$

$$= \frac{1}{\angle 2} \lim_{z \to ai} \frac{d^{\,2}}{dz^{2}} (z-ai)^3 \frac{1}{(z-ai)^3 (z+ai)^3}$$

$$= \frac{1}{2} \lim_{z \to ai} \frac{d^2}{dz^2} (z + ai)^3$$

$$= \frac{1}{2} \lim_{z \to ai} \frac{d}{dz} [-3(z + ai)^{-4}](1)$$

$$= \frac{1}{2} \lim_{z \to ai} [12(z + ai)^{-5}]$$

$$= 6(2ai)^{-5} = \frac{6}{2^5 (ai)^5}$$

$$= \frac{3}{16 a^5 (i^2)^2 i} = \frac{3}{16 a^5 i}$$

$$\int_{\Gamma} f(z)\, dz + \int_{L} f(z)\, dz = 2\pi i \frac{3}{16 a^5 i}$$

$$\int_{\Gamma} f(z)\,dz + \int_{-R}^{R} f(x)\,dx = \frac{3\pi}{8a^5}$$

$R \to \infty$

$$\lim_{R\to\infty} \int_{\Gamma} f(z)\,dz + \int_{-\infty}^{\infty} f(x)\,dx = \frac{3\pi}{8a^5} \qquad \ldots(2)$$

$$\lim_{R\to\infty} \int_{\Gamma} f(z)\,dz \to 0$$

$$\int_{-\infty}^{\infty} \frac{dx}{(x^2 + a^2)^3} = \frac{3\pi}{8a^5}$$

$$2\int_{0}^{\infty} \frac{dx}{(x^2 + a^2)^3} = \frac{3\pi}{8a^5} = \frac{3\pi}{16a^5}$$

$$\int_{0}^{\infty} \frac{x \sin x dx}{x^2 + a^2}$$

$$\int_{c} \frac{ze^{iz} dz}{z^2 + a^2}$$

Γ

$|z| = R$

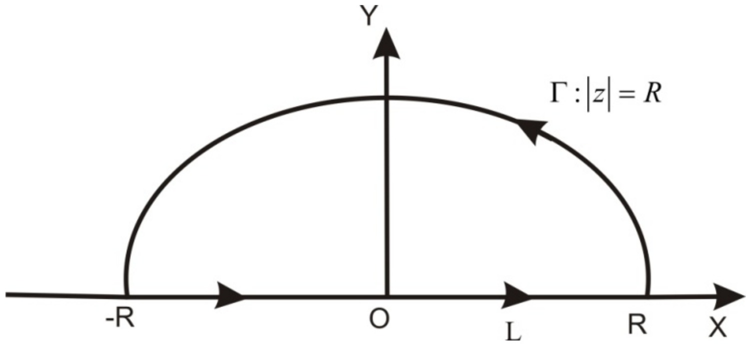
$$z^2 + a^2 = 0$$

$$z^2 = -a^2$$

$$\int_C f(z)\,dz = 2\pi i\, R_1$$

$$R_1 = \lim_{z \to ai} (z - ai) \frac{ze^{iz}}{(z - ai)(z + ai)}$$

$$= ai \frac{d^{i(ai)}}{2ai} = \frac{e^{-a}}{2}$$



$$\int_{\Gamma} f(z)\,dz + \int_{L} f(z)\,dz = 2\pi i \frac{e^{-a}}{2}$$

$$\int_{\Gamma} f(z)\,dz + \int_{-R}^{R} f(x)\,dx = \pi i e^{-a}$$

$$R \to \infty$$

$$\lim_{R\to\infty} \int_{\Gamma} f(z)\,dz + \int_{-\infty}^{\infty} f(x)\,dx = \pi i e^{-a}$$

$$\lim_{R\to\infty} \int_{\Gamma} f(z)\,dz \to 0$$

$$\int_{-\infty}^{\infty} f(x)\,dx = \pi i e^{-a}$$

$$\int_{-\infty}^{\infty} \frac{xe^{ix}}{x^2 + a^2} dx = \pi i e^{-a}$$

$$\int_{-\infty}^{\infty} \frac{x(\cos x + i \sin x)}{x^2 + a^2} dx = \pi i e^{-a}$$

$$\int_{-\infty}^{\infty} \frac{x \sin x}{x^2 + a^2} dx = \pi e^{-a}$$

$$\int_{0}^{\infty} \frac{x \sin x}{x^2 + a^2} dx = \frac{\pi}{2} e^{-a}$$

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(Under section 3 of UGC Act 1956)

Unit V - Completed

*** THANK YOU ***