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1 MULTIPLE INTEGRALS

INTRODUCTION AND NOTATION

1.1 Double and Triple Integrals

The symbol $\int_{y_1}^{y_2} \int_{x_1}^{x_2} f(x, y) dx dy$ denotes a double integral. It means that $f(x, y)$ is to be integrated first with respect to x , considering y temporarily, as a constant, between the limits x_1 and x_2 and the resulting expression is then to be integrated with respect to y between the limits y_1 and y_2 . Here x_1 and x_2 may be constants or functions of y alone.

(i.e.) $x_1 = f_1(y), x_2 = f_2(y), y_1$ and y_2 are constants.

Similarly the symbol $\int_{z_1}^{z_2} \int_{y_1}^{y_2} \int_{x_1}^{x_2} f(x, y, z) dx dy dz$ denotes a triple integral. This is concluded as follows. First $f(x, y, z)$ is integrated with respect to x between the limits x_1 and x_2 , keeping y and z as constants temporarily. The resulting function is integrated with respect to y between the limits y_1 and y_2 , keeping z constant. The result thus obtained is finally integrated with respect to z between the limits z_1 and z_2 . In this integral, z_1 and z_2 are constants, y_1 and y_2 are either constants or functions of z alone and x_1, x_2 are either constants or functions of y and z .

Note: In $\int \int f(x, y) dx dy$, if the right hand integral has limits involving x , then the first integral is with respect to y .

Differentiation Formulas:

$$(i) \frac{d}{dx}(k) = 0$$

$$(ii) \frac{d}{dx}[f(x) \pm g(x)] = f'(x) \pm g'(x)$$

$$(iii) \frac{d}{dx}[k \cdot f(x)] = k \cdot f'(x)$$

$$(iv) \frac{d}{dx}[f(x) \cdot g(x)] = f(x)g'(x) + g(x)f'(x)$$

$$(v) \frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{g(x)f'(x) - f(x)g'(x)}{[g(x)]^2}$$

$$(vi) \frac{d}{dx}f(g(x)) = f'(g(x)) \cdot g'(x)$$

$$(vii) \frac{d}{dx}(x^n) = nx^{n-1}$$

$$(viii) \frac{d}{dx}(e^x) = e^x$$

$$(ix) \frac{d}{dx}(a^x) = a^x \log a, a > 0$$

$$(x) \frac{d}{dx}(\log_e x) = \frac{1}{x}, x > 0$$

$$(xi) \frac{d}{dx}(\sin x) = \cos x$$

$$(xii) \frac{d}{dx}(\cos x) = -\sin x$$

$$(xiii) \frac{d}{dx}(\tan x) = \sec^2 x$$

$$(xiv) \frac{d}{dx}(\cot x) = -\csc^2 x$$

$$(xv) \frac{d}{dx}(\sec x) = \sec x \tan x$$

$$(xvi) \frac{d}{dx}(\csc x) = -\csc x \cot x$$

$$(xvii) \frac{d}{dx}(\sinh x) = \cosh x$$

$$(xviii) \frac{d}{dx}(\cosh x) = \sinh x$$

$$(xix) \frac{d}{dx}(\tanh x) = \sec h^2 x$$

$$(xx) \frac{d}{dx}(\coth x) = -\csc h^2 x$$

$$(xxi) \frac{d}{dx}(\sec hx) = -\sec hx \tanh x$$

$$(xxii) \frac{d}{dx}(\csc hx) = -\csc hx \coth x$$

$$(xxiii) \frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}} = -\frac{d}{dx}(\cos^{-1} x)$$

$$(xxiv) \frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2} = -\frac{d}{dx}(\cot^{-1} x)$$

$$(xxv) \frac{d}{dx}(\sec^{-1} x) = \frac{1}{x\sqrt{x^2-1}} = -\frac{d}{dx}(\csc^{-1} x)$$

$$(xxvi) \frac{d}{dx}(\sinh^{-1} x) = \frac{1}{\sqrt{1+x^2}}$$

$$(xxvii) \frac{d}{dx}(\cosh^{-1} x) = \frac{1}{\sqrt{x^2-1}}$$

$$(xxviii) \frac{d}{dx}(\tan^{-1} x) = \frac{1}{1-x^2}, |x| < 1$$

$$(xxix) \frac{d}{dx}(\coth^{-1} x) = -\frac{1}{x^2-1}, |x| > 1$$

$$(xxx) \frac{d}{dx}(\sec h^{-1} x) = \frac{-1}{x\sqrt{1-x^2}}$$

$$(xxxi) \frac{d}{dx}(\csc h^{-1} x) = \frac{1}{x\sqrt{x^2+1}}$$

Integration Formulas:

$$1. \int dx = x + C$$

$$2. \int x^n dx = \frac{x^{n+1}}{n+1} + C, n \neq -1$$

$$3. \int \frac{1}{x} dx = \log x + C$$

$$4. \int e^{ax} dx = \frac{e^{ax}}{a} + C$$

$$5. \int \sin x dx = -\cos x + C$$

$$6. \int \cos x dx = \sin x + C$$

$$7. \int \sec^2 x dx = \tan x + C$$

$$8. \int \csc^2 x dx = -\cot x + C$$

$$9. \int \sec x \tan x dx = \sec x + C$$

$$10. \int \csc x \cot x dx = -\csc x + C$$

$$11. \int \frac{dx}{\sqrt{1-x^2}} = \sin^{-1} x + C$$

$$12. \int \frac{dx}{1+x^2} = \tan^{-1} x + C$$

$$13. \int \frac{dx}{x\sqrt{x^2-1}} = \sec^{-1} x + C$$

$$14. \int \sinh x dx = \cosh x + C$$

$$15. \int \cosh x dx = \sinh x + C$$

$$16. \int \sec h^2 x dx = \tanh x + C$$

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17. $\int \csc h^2 x dx = -\coth x dx + C$

18. $\int \sec hx \tanh x dx = -\sec hx + C$

19. $\int \csc hx \coth x dx = -\csc hx + C$

20. $\int \frac{dx}{x^2 - a^2} = \frac{1}{2a} \log \frac{x-a}{x+a} + C$

21. $\int \frac{dx}{a^2 - x^2} = \frac{1}{2a} \log \frac{a+x}{a-x} + C$

22. $\int \frac{dx}{x^2 + a^2} = \frac{1}{a} \tan^{-1} \frac{x}{a} + C$

23. $\int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1} \frac{x}{a} + C$

24. $\int \frac{dx}{\sqrt{x^2 + a^2}} = \sinh^{-1} \frac{x}{a} + C \text{ (or) } \log(x + \sqrt{x^2 + a^2})$

25. $\int \frac{dx}{\sqrt{x^2 - a^2}} = \cosh^{-1} \frac{x}{a} + C \text{ (or) } \log(x + \sqrt{x^2 - a^2})$

26. $\int \tan x dx = \log \sec x = -\log \cos x + C$

27. $\int \cot x dx = \log \sin x + C$

28. $\int \sec x dx = \log(\sec x + \tan x) + C$

29. $\int \csc x dx = \log(\csc x - \cot x) + C$

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30. $\int e^{ax} \cos bx dx = \frac{e^{ax}}{a^2 + b^2} (a \cos bx + b \sin bx) + C$

31. $\int e^{ax} \sin bx dx = \frac{e^{ax}}{a^2 + b^2} (a \sin bx - b \cos bx) + C$

32. $\int \sqrt{a^2 - x^2} dx = \frac{a^2}{2} \sin^{-1} \frac{x}{a} + \frac{x}{2} \sqrt{a^2 - x^2} + C$

33. $\int \sqrt{a^2 + x^2} dx = \frac{a^2}{2} \sinh^{-1} \frac{x}{a} + \frac{x}{2} \sqrt{a^2 + x^2} + C$

34. $\int \sqrt{x^2 - a^2} dx = -\frac{a^2}{2} \cosh^{-1} \frac{x}{a} + \frac{x}{2} \sqrt{x^2 - a^2} + C$

35. $\int a^x dx = \frac{a^x}{\log a} + C$

Example 1.1. Evaluate $\int_0^2 \int_0^1 4xy dx dy$

Solution: Let $I = \int_0^2 \int_0^1 4xy dx dy$

$$= 4 \int_0^2 \left(y \frac{x^2}{2} \right)_0^1 dy = 2 \left(\frac{y^2}{2} \right)_0^2 = 4$$

Example 1.2. Evaluate $\int_0^a \int_0^b \int_0^c (x + y + z) dx dy dz$

Solution: Let $I = \int_0^a \int_0^b \left(\frac{x^2}{2} + yx + zx \right)_{x=0}^{x=c} dy dz$

$$= \int_0^a \int_0^b \left(\frac{c^2}{2} + yc + zc \right) dy dz$$

$$\begin{aligned}
&= \int_0^a \left(\frac{c^2 y}{2} + \frac{y^2 c}{2} + zcy \right)_{y=0}^{y=b} dz \\
&= \int_0^a \left(\frac{c^2 b}{2} + \frac{b^2 c}{2} + zcb \right) dz \\
&= \left(\frac{c^2 bz}{2} + \frac{b^2 cz}{2} + \frac{z^2 cb}{2} \right)_{z=0}^{z=a} = \frac{abc}{2}(a+b+c)
\end{aligned}$$

Example 1.3. Evaluate $\int_0^{2\pi} \int_0^\pi \int_0^a r^4 \sin \phi dr d\phi d\theta$

Solution: The given integral

$$\begin{aligned}
&\int_0^{2\pi} d\theta \int_0^\pi d\phi \int_0^a r^4 \sin \phi dr = \int_0^{2\pi} d\theta \int_0^\pi \left(\frac{r^5}{5} \right)_0^a \sin \phi d\phi \\
&= \frac{a^5}{5} \int_0^{2\pi} d\theta \int_0^\pi \sin \phi d\phi = \frac{a^5}{5} \int_0^{2\pi} (-\cos \phi)_0^\pi d\theta \\
&= \frac{2}{5} a^5 \int_0^{2\pi} d\theta \text{ as } \cos \pi = -1, \cos 0 = 1 \\
&= \frac{4}{5} \pi a^5
\end{aligned}$$

Example 1.4. Evaluate $\int_0^1 \int_0^x dx dy$

Solution: Let I be the given integral.

$\therefore I = \int_0^1 (y)_0^x dx$ as right hand integral has a limit which is function of x.

$$I = \int_0^1 x dx = \left(\frac{x^2}{2} \right)_0^1 = \frac{1}{2}$$

Example 1.5. Evaluate $\int_0^\pi \int_0^{a \sin \theta} r d\theta dr$

$$\begin{aligned}
\text{Solution: Let } I &= \int_0^\pi \left(\frac{r^2}{2} \right)_0^{a \sin \theta} = \frac{1}{2} \int_0^\pi a^2 \sin^2 \theta d\theta \\
&= \frac{a^2}{2} \int_0^\pi \left(\frac{1 - \cos 2\theta}{2} \right) d\theta \\
&= \frac{a^2}{2} \cdot \frac{1}{2} \left(\theta - \frac{\sin 2\theta}{2} \right)_0^\pi \\
&= \frac{\pi a^2}{4} \text{ as } \sin \pi = 0, \sin 0 = 0
\end{aligned}$$

Example 1.6. Evaluate $\int_0^1 \int_0^{\sqrt{1+y^2}} \frac{dxdy}{1+x^2+y^2}$

$$\begin{aligned}
\text{Solution: The given integral} &= \int_0^1 \left[\int_0^{\sqrt{1+y^2}} \frac{1}{(1+y^2)+x^2} dx \right] dy \\
&= \int_0^1 \left[\frac{1}{\sqrt{1+y^2}} \tan^{-1} \left(\frac{x}{\sqrt{1+y^2}} \right) \right]_{x=0}^{x=\sqrt{1+y^2}} dy \\
&= \int_0^1 \left[\frac{1}{\sqrt{1+y^2}} (\tan^{-1}(1) - \tan^{-1} 0) \right] dy \\
&= \frac{\pi}{4} \int_0^1 \frac{dy}{\sqrt{1+y^2}}
\end{aligned}$$

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$$= \frac{\pi}{4} \left[\log(y + \sqrt{1+y^2}) \right]_0^1 = \frac{\pi}{4} \log(1 + \sqrt{2})$$

Example 1.7. Evaluate $\int_0^a \int_0^{\sqrt{a^2-y^2}} \sqrt{a^2-x^2-y^2} dx dy$

$$\begin{aligned} \text{Solution: Let } I &= \int_0^a \int_0^{\sqrt{a^2-y^2}} \sqrt{a^2-x^2-y^2} dx dy \\ &= \int_0^a \left[\frac{x}{2} \sqrt{(a^2-y^2)-x^2} + \left(\frac{a^2-y^2}{2} \right) \sin^{-1} \left(\frac{x}{\sqrt{a^2-y^2}} \right) \right]_0^{\sqrt{a^2-y^2}} dy \\ &= \int_0^a \left(\frac{a^2-y^2}{2} \right) \frac{\pi}{2} y = \frac{\pi}{4} \left[a^2 y - \frac{y^3}{3} \right]_0^a = \frac{\pi a^3}{6} \end{aligned}$$

Example 1.8. Evaluate $\int_0^1 \int_{y^2}^1 \int_0^{1-x} x dz dx dy$

$$\begin{aligned} \text{Solution: Let } I &= \int_0^1 \int_{y^2}^1 \int_0^{1-x} x dz dx dy \\ &= \int_0^1 \int_{y^2}^1 x(z)_0^{1-x} dx dy = \int_0^1 \int_{y^2}^1 (x-x^2) dx dy \\ &= \int_0^1 \left[\left(\frac{1}{2} - \frac{1}{3} \right) - \left(\frac{y^4}{2} - \frac{y^6}{3} \right) \right] dy \\ &= \int_0^1 \left(\frac{1}{6} - \frac{y^4}{2} + \frac{y^6}{3} \right) dy \\ &= \left(\frac{1}{6}y - \frac{1}{2}\frac{y^5}{5} + \frac{1}{3}\frac{y^7}{7} \right)_0^1 \\ &= \frac{1}{6} - \frac{1}{10} + \frac{1}{21} = \frac{4}{35} \end{aligned}$$

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Example 1.9. Evaluate $\int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{\sqrt{1-x^2-y^2}} \frac{dxdydz}{\sqrt{1-x^2-y^2-z^2}}$

Solution:

$$\begin{aligned} \text{Let } I &= \int_0^1 \int_0^{\sqrt{1-x^2}} \left[\sin^{-1} \left(\frac{z}{\sqrt{(1-x^2-y^2)-z^2}} \right) \right]_0^{\sqrt{1-x^2-y^2}} dx dy \\ &= \int_0^1 \int_0^{\sqrt{1-x^2}} \frac{\pi}{2} dx dy = \frac{\pi}{2} \int_0^1 (y)_0^{\sqrt{1-x^2}} dx = \frac{\pi}{2} \int_0^1 \sqrt{1-x^2} dx \\ &= \frac{\pi}{2} \left[\frac{x}{2} \sqrt{1-x^2} + \frac{1}{2} \sin^{-1} x \right]_0^1 = \frac{\pi^2}{8} \end{aligned}$$

Example 1.10. Evaluate $\int_0^{\frac{\pi}{2}} \int_0^{a \cos \theta} r \sqrt{a^2 - r^2} d\theta dr$

$$\begin{aligned} \text{Solution: Let } I &= \int_0^{\frac{\pi}{2}} \left(-\frac{1}{3} \right) [(a^2 - r^2)^{3/2}]_0^{a \cos \theta} d\theta \\ &= -\frac{1}{3} \int_0^{\frac{\pi}{2}} [(a^2 - a^2 \cos^2 \theta)^{3/2} - (a^2)^{3/2}]_0^{a \cos \theta} d\theta \\ &= -\frac{1}{3} \int_0^{\frac{\pi}{2}} (a^3 \sin^3 \theta - a^3) d\theta \\ &= \frac{a^3}{3} \int_0^{\frac{\pi}{2}} (1 - \sin^3 \theta) d\theta \\ &= \frac{a^3}{3} \left[\theta + \frac{1}{4} \left(-\frac{1}{3} \cos 3\theta + 3 \cos \theta \right) \right]_0^{\frac{\pi}{2}} \end{aligned}$$

$$= \frac{a^3}{3} \left(\frac{\pi}{2} - \frac{2}{3} \right)$$

Example 1.11. Evaluate $\int_0^1 \int_0^y xye^{-x^2} dx dy$

Solution: Let $I = \int_0^1 y \left(\frac{e^{-x^2}}{2} \right)_0^y dy$

$$= -\frac{1}{2} \int_0^1 (ye^{-y^2} - y) dy$$

$$= -\frac{1}{2} \left[\frac{e^{-y^2}}{-2} - \frac{y^2}{2} \right]_0^1$$

$$= \frac{1}{4} [(e^{-1} + 1) - (1)] = \frac{1}{4e}$$

Example 1.12. Evaluate $\int_0^a \int_0^{a-x} \int_0^{a-x-y} x^2 dx dy dz$

Solution: Let $I = \int_0^a \int_0^{a-x} [x^2 z]_0^{a-x-y} dy dx$

$$= \int_0^a \int_0^{a-x} x^2 (a - x - y) dy dx$$

$$= \int_0^a \left[x^2 (a - x)y - x^2 \cdot \frac{y^2}{2} \right]_0^{a-x} dx$$

$$= \frac{1}{2} \int_0^a x^2 (a - x)^2 dx$$

$$= \frac{1}{2} [a^2 \frac{x^3}{3} - 2a \frac{x^4}{4} + \frac{x^5}{5}]_0^a = \frac{a^5}{60}$$

Example 1.13. Evaluate $\int_0^a \int_0^x \int_0^y xyz dz dy dx$

Solution: Let $I = \int_0^a \int_0^x \left[\frac{xyz^2}{2} \right]_0^y dy dx$

$$= \frac{1}{2} \int_0^a \int_0^x xy^3 dy dx = \frac{1}{2} \int_0^a \left[\frac{xy^4}{4} \right]_0^x dx$$

$$= \frac{1}{8} \left[\frac{x^6}{6} \right]_0^a = \frac{a^6}{48}$$

Example 1.14. Evaluate $\int_0^{1/2} \int_0^1 \frac{x}{\sqrt{1 - x^2 y^2}} dy dx$

Solution: Let $I = \int_0^{1/2} x \left[\frac{\sin^{-1} xy}{x} \right]_0^1$

$$= \int_0^{1/2} \sin^{-1} x dx$$

$$= \left[x \sin^{-1} x + \sqrt{1 - x^2} \right]_0^{1/2} = \frac{\pi}{12} + \frac{\sqrt{3}}{2} - 1$$

Example 1.15. Evaluate $\int_0^1 \int_0^{1-z} \int_0^{1-y-z} xyz dx dy dz$

Solution: Let $I = \int_0^1 \int_0^{1-z} yz \left(\frac{x^2}{2} \right)_0^{1-y-z} dy dz$

$$= \frac{1}{2} \int_0^1 \int_0^{1-z} yz (1 - y - z)^2 dy dz$$

$$= \frac{1}{2} \int_0^1 yz [(1 - z)^2 - 2(1 - z)y + y^2] dy dz$$

$$\begin{aligned}
&= \frac{1}{2} \int_0^1 \left[z(1-z)^2 \frac{y^2}{2} - 2z(1-z) \frac{y^3}{3} + z \frac{y^4}{4} \right]_0^{1-z} dz \\
&= \frac{1}{2} \int_0^1 \left(\frac{1}{2} z(1-z)^4 - \frac{2}{3} z(1-z)^4 + \frac{1}{4} z(1-z)^4 \right) dz \\
&= \frac{1}{2} \left(\frac{1}{2} - \frac{2}{3} + \frac{1}{4} \right) \int_0^1 z(1-z)^4 dz \\
&= \frac{1}{24} \int_0^1 [1 - (1-z)](1-z)^4 dz \\
&= \frac{1}{24} \left[\frac{(1-z)^5}{-5} + \frac{(1-z)^6}{6} \right] \\
&= \frac{1}{24} \left(\frac{1}{5} - \frac{1}{6} \right) = \frac{1}{720}
\end{aligned}$$

Example 1.16. Evaluate $\int_0^{\log 2} \int_0^x \int_0^{x-y} e^{x+y+z} dx dy dz$

Solution: Let $I = \int_0^{\log 2} \int_0^x \int_0^{x-y} e^{x+y+z} dx dy dz$

$$\begin{aligned}
&= \int_0^{\log 2} dx \int_0^x dy e^{x+y} (e^z)_{z=0}^{z=x+y} \\
&= \int_0^{\log 2} dx \int_0^x (e^{2x+2y} - e^{x+y}) dy \\
&= \int_0^{\log 2} \left[e^{2x} \frac{e^{2y}}{2} - e^x e^y \right]_{y=0}^{y=x} dx \\
&= \int_0^{\log 2} \left(\frac{e^{4x}}{2} - \frac{3e^{2x}}{2} + e^x \right) dx
\end{aligned}$$

$$= \left[\frac{1}{8} e^{4x} - \frac{3e^{2x}}{4} + e^x \right]_0^{\log 2} = \frac{5}{8}$$

Example 1.17. Evaluate $\iint xy dx dy$ over the region bounded by the x -axis, ordinate at $x = 2a$ and the parabola $x^2 = 4ay$.

Solution:

In the region y varies from 0 to $\frac{x^2}{4a}$ and x varies from 0 to $2a$.

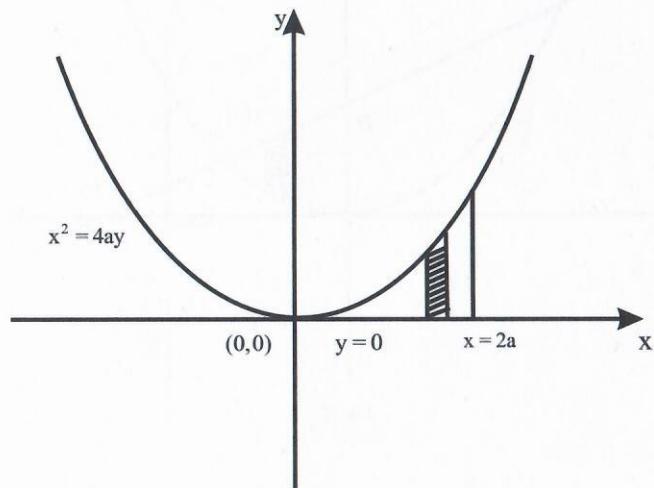


Fig. 1

$$\begin{aligned}
\therefore I &= \int_0^{2a} \int_0^{x^2/4a} xy dy dx = \int_0^{2a} \left[x \frac{y^2}{2} \right]_0^{x^2/4a} dx \\
&= \int_0^{2a} \frac{x}{2} \cdot \frac{x^4}{16a^2} dx = \frac{1}{32a^2} \int_0^{2a} x^5 dx
\end{aligned}$$

$$= \frac{1}{32a^2} \left[\frac{x^6}{6} \right]_0^{2a} = \frac{a^4}{3}.$$

Example 1.18. Evaluate $\int \int y dx dy$ over the area bounded by $x = 0, y = x^2, x + y = 2$ in the first quadrant.

Solution: We have to find the integral over the region OAB.

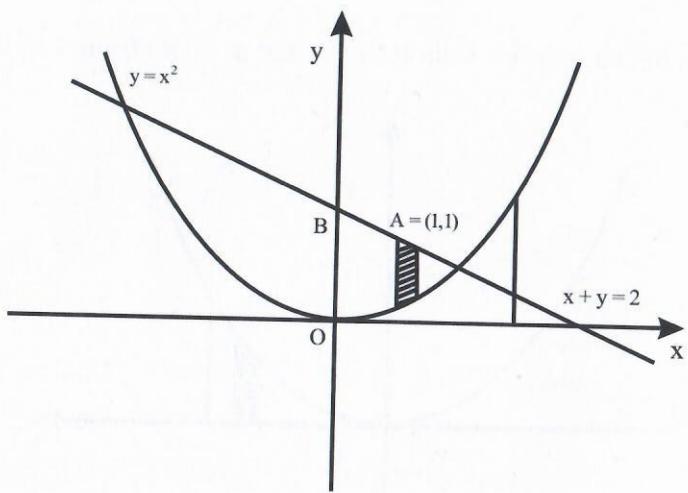


Fig. 2

$$I = \int_0^1 \int_{x^2}^{2-x} y dy dx = \int_0^1 \left[\frac{y^2}{2} \right]_{x^2}^{2-x} dx$$

$$= \frac{1}{2} \int_0^1 [(2-x)^2 - x^4] dx$$

$$= \frac{1}{2} \int_0^1 [4 - 4x + x^2 - x^4] dx$$

$$\begin{aligned} &= \frac{1}{2} \left[4x - 2x^2 + \frac{x^3}{3} - \frac{x^5}{5} \right]_0^1 \\ &= \frac{1}{2} \left[4 - 2 + \frac{1}{3} - \frac{1}{5} \right] = \frac{16}{15} \end{aligned}$$

Example 1.19. Evaluate $\int \int (x^2 + y^2) dx dy$ over the area of the triangle whose vertices are $(0, 1), (1, 1), (1, 2)$.

Solution:

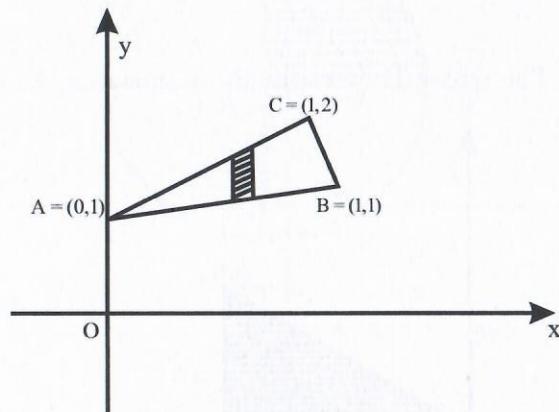


Fig. 3

The equation of the line AC is $\frac{y-2}{1} = \frac{x-1}{1}$

(i.e.) $y = x + 1$.

$$\therefore I = \int_0^1 dx \int_1^{x+1} (x^2 + y^2) dy = \int_0^1 \left[x^2 y + \frac{y^3}{3} \right]_1^{x+1} dx$$

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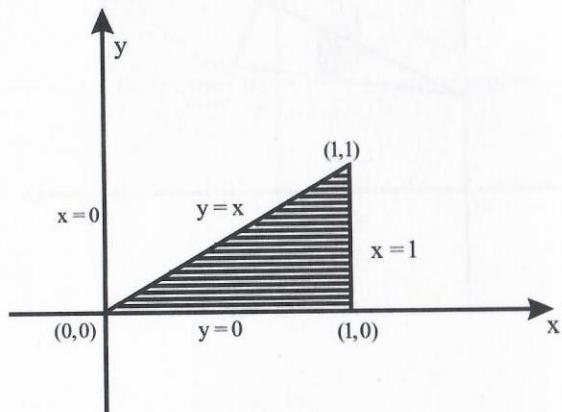
$$= \int_0^1 \left\{ \left(x^2(x+1) + \frac{(x+1)^3}{3} \right) - \left(x^2 + \frac{1}{3} \right) \right\} dx$$

$$= \frac{1}{3} \int_0^1 (4x^3 + 3x^2 + 3x) dx = \frac{1}{3} \left[x^4 + x^3 + \frac{3x^2}{2} \right]_0^1$$

$$= \frac{1}{3} \left[1 + 1 + \frac{3}{2} \right] = \frac{7}{6}$$

Example 1.20. Evaluate $\iint e^{y/x} dxdy$ where D is the region bounded by the straight lines $y = x$, $y = 0$ and $x = 1$.

Solution: The region D is a triangle as shown in the figure.



In this region x varies from 0 to 1. For each fixed x , y varies from 0 to x .

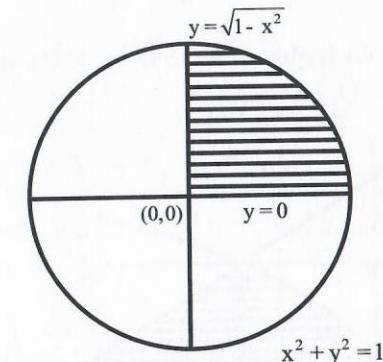
$$\therefore I = \int_0^1 \int_0^x e^{y/x} dy dx = \int_0^1 [xe^{y/x}]_0^x$$

1 MULTIPLE INTEGRALS

$$= \int_0^1 x(e-1) dx = \frac{1}{2}(e-1)$$

Example 1.21. Evaluate $\iint_D x^2 y^2 dxdy$ where D is the circular disc $x^2 + y^2 \leq 1$.

Solution: In D , x varies from -1 to 1. For a fixed x , y varies from $-\sqrt{1-x^2}$ to $\sqrt{1-x^2}$.



$$\therefore \iint_D x^2 y^2 dxdy = \int_{-1}^1 \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} x^2 y^2 dy dx$$

$$= 4 \int_0^1 \int_0^{\sqrt{1-x^2}} x^2 y^2 dy dx \quad [\text{As integrand is even } \int_{-a}^a f(x) dx = 2 \int_0^a f(x) dx \text{ if } f(x) \text{ is even}]$$

$$= 4 \int_0^1 \left[\frac{x^2 y^3}{3} \right]_0^{\sqrt{1-x^2}}$$

$$= \frac{4}{3} \int_0^1 x^2 (1-x^2)^{3/2} dx$$

$$= \frac{4}{3} \int_0^{\pi/2} \sin^2 \theta \cos^4 \theta d\theta \text{ put } x = \sin \theta, dx = \cos \theta d\theta$$

$$= \frac{4}{3} \left(\frac{1.3.1}{2.4.6} \right) \left(\frac{\pi}{2} \right) = \frac{\pi}{24}$$

Example 1.22. Evaluate $I = \iint_D xy dxdy$ where D is the region bounded by the curve $x = y^2$, $x = 2 - y$, $y = 0$ and $y = 1$.

Solution: The given region bounded by the curves is shown in the figure.

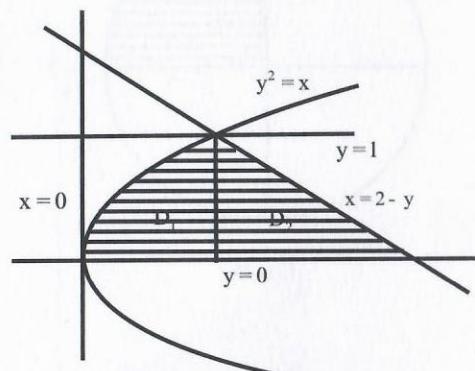


Fig. 5

In this region x varies from 0 to 2 when $0 \leq x \leq 1$ for fixed x, y varies from 0 to \sqrt{x} . When $1 \leq x \leq 2$, y varies from 0 to $2 - x$.

\therefore The region D can be subdivided into two regions D_1 and D_2 as shown in the figure.

$$I = \iint_D xy dxdy = \iint_{D_1} xy dxdy + \iint_{D_2} xy dxdy$$

$$= \int_0^1 \int_0^{\sqrt{x}} xy dy dx + \int_1^2 \int_0^{2-x} xy dy dx$$

$$= \int_0^1 \left[\frac{xy^2}{2} \right]_0^{\sqrt{x}} + \int_1^2 \left[\frac{xy^2}{2} \right]_0^{2-x} dx$$

$$= \frac{1}{2} \int_0^1 x^2 dx + \frac{1}{2} \int_1^2 x(2-x)^2 dx$$

$$= \frac{1}{2} \int_0^1 x^2 dx + \frac{1}{2} \int_1^2 (x^3 - 4x^2 + 4x) dx$$

$$= \left[\frac{x^3}{6} \right]_0^1 + \frac{1}{2} \left[\frac{x^4}{4} - \frac{4x^3}{3} + \frac{4x^2}{2} \right]_1^2 = \frac{3}{8}$$

Example 1.23. Evaluate $I = \iiint_D xyz dxdydz$ where D is the region bounded by the positive octant of the sphere $x^2 + y^2 + z^2 = a^2$.

Solution:

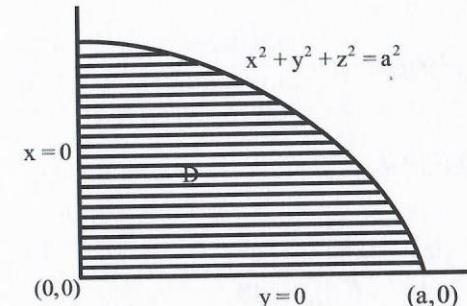


Fig. 6

The projection of the given region in the xy plane ($z = 0$) is the

region bounded by the circle $x^2 + y^2 = a^2$ and lying in the 1st quadrant as shown in the figure.

In the given region x varies 0 to a . For a fixed x , y varies from 0 to $\sqrt{a^2 - x^2}$. For a fixed (x, y) , z varies from 0 to $\sqrt{a^2 - x^2 - y^2}$

$$\therefore I = \int_0^a \int_0^{\sqrt{a^2-x^2}} \int_0^{\sqrt{a^2-x^2-y^2}} xyz dz dy dx$$

$$= \int_0^a \int_0^{\sqrt{a^2-x^2}} xy \left(\frac{z^2}{2} \right)_0^{\sqrt{a^2-x^2-y^2}} dy dx$$

$$= \frac{1}{2} \int_0^a \int_0^{\sqrt{a^2-x^2}} xy(a^2 - x^2 - y^2) dy dx$$

$$= \frac{1}{2} \int_0^a \int_0^{\sqrt{a^2-x^2}} x(a^2y - x^2y - y^3) dy dx$$

$$= \frac{1}{2} \int_0^a x \left[a^2 \frac{y^2}{2} - x^2 \frac{y^2}{2} - \frac{y^4}{4} \right]_0^{\sqrt{a^2-x^2}} dx$$

$$= \frac{1}{8} \int_0^a x(a^2 - x^2)^2 dx$$

$$= \frac{1}{8} \int_0^a (a^4x - 2a^2x^3 + x^5) dx$$

$$= \frac{1}{8} \left[a^4 \frac{x^2}{2} - 2a^2 \frac{x^4}{4} + \frac{x^6}{6} \right]_0^a = \frac{a^6}{48}$$

Example 1.24. Evaluate $I = \iiint_D \frac{dxdydz}{(1+x+y+z)^3}$ where D is the region bounded by the planes $x = 0, y = 0, z = 0$ and $x + y + z = 1$.

Solution: The given region is a tetrahedron. The projection of the given region in the xy plane is the triangle bounded by the lines $x = 0, y = 0$ and $x + y = 1$ as shown in the figure. In the

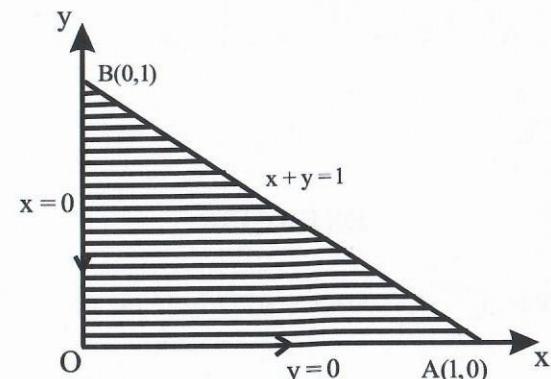


Fig. 7

given region x varies 0 to 1. For each fixed x, y varies from 0 to $1 - x$. For each fixed (x, y) , z varies from 0 to $1 - x - y$.

$$\therefore I = \int_0^1 \int_0^{1-x} \int_0^{1-x-y} \frac{dxdydz}{(1+x+y+z)^3} dy dx$$

$$= -\frac{1}{2} \int_0^1 \int_0^{1-x} [(1+x+y+z)^{-2}]_0^{1-x-y} dy dx$$

$$= -\frac{1}{2} \int_0^1 \int_0^{1-x} \left[\frac{1}{4} - (x+y+1)^{-2} \right] dy dx$$

$$= -\frac{1}{2} \int_0^1 \left[\frac{1}{4}y + (x+y+1)^{-1} \right]_0^{1-x} dx$$

1 MULTIPLE INTEGRALS

$$= -\frac{1}{2} \int_0^1 \left[\frac{1}{4}(1-x) + \frac{1}{2} - (x+1)^{-1} \right] dx$$

$$= -\frac{1}{2} \left[\frac{1}{4}x - \frac{1}{8}x^2 + \frac{1}{2}x - \log(x+1) \right]$$

$$= \frac{1}{2} \log 2 - \frac{5}{16}$$

EXERCISE

1. Evaluate $\int_0^{\pi/2} \int_{\pi/2}^{\pi} \cos(x+y) dx dy$ [Ans: -2]

2. Evaluate $\int_0^1 \int_0^x e^{x+y} dy dx$ [Ans: $\frac{1}{2}(e-1)^2$]

3. Evaluate $\int_0^{\pi} \int_0^{a(1+\cos\theta)} r d\theta dr$ [Ans: $\frac{3}{4}\pi a^2$]

4. Evaluate $\int_0^1 dx \int_0^2 dy \int_1^2 x^2 y z dz$ [Ans: 1]

5. Find the value of $\int_2^3 \int_1^2 \frac{dxdy}{xy}$ [Ans: $\log 2 \log \frac{3}{2}$]

6. Evaluate $\int_0^1 \int_0^2 xy^2 dy dx$. [Ans: 4/3]

7. Find $I = \int_0^{\pi} \int_0^{\pi/2} \int_0^1 r^2 \sin \theta dr d\theta d\phi$ [Ans: $\frac{\pi}{3}$]

8. Find the value of $\int_0^2 \int_1^3 \int_1^2 xy^2 z dz dy dx$ [Ans: 26]

9. Evaluate $\int_0^1 \int_0^z \int_0^{y+z} dz dy dx$ [Ans: 1/2]

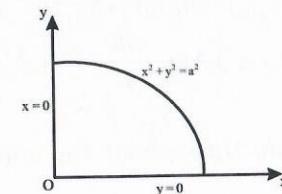
1 MULTIPLE INTEGRALS

10. Find the limits of $\iint_R f(x,y) dx dy$, where R is the 1st quadrant bounded by (a) $x = 0, y = 0, x + y = 1$ (b) $x = 0, x = y, y = 1$.

$$\left[\text{Ans: a) } \int_0^1 \int_0^{1-x} f(x,y) dx dy \text{ b) } \int_0^1 \int_0^{1-y} f(x,y) dy dx \right]$$

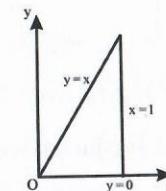
11. Sketch the region by integration to $\int_0^a \int_0^{\sqrt{a^2-x^2}} f(x,y) dx dy$.

Ans:



12. Sketch the region by integration to $\int_0^1 \int_0^x f(x,y) dx dy$.

Ans:



13. Evaluate $\int_0^1 \int_0^x dx dy$

$$14. \text{ Evaluate } \int_1^2 \int_0^x \frac{1}{x^2 + y^2} dx dy \quad \left[\text{Ans: } \frac{\pi}{4} \log 2 \right]$$

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15. Evaluate $\int_0^{\pi/2} \int_0^{a \sin \theta} \int_0^{(a^2 - r^2)/a} r d\theta dr dz$ [Ans: $\frac{5}{64}\pi a^3$]

16. Evaluate $\int_0^2 \int_0^x \int_0^{x+y} e^{x+y+z} dx dy dz$ [Ans: $\frac{e^8 - 3}{8} - \frac{3}{4}e^4 + e^2$]

17. Evaluate $\iint (x^2 - y^2) dx dy$ over the positive quadrant of the circle $x^2 + y^2 = a^2$ [Ans: $\frac{a^5}{15}$]

18. Evaluate $\iint (x + y)^2 dx dy$ over the area bounded by the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ [Ans: $\frac{\pi ab}{4}(a^2 + b^2)$]

19. Evaluate $\iint dx dy$ throughout the area bounded by

$$y = x^2, x + y = 2 \quad \left[\text{Ans: } -\frac{10}{3} \right]$$

20. Evaluate $\iint x^2 y^2 dx dy$ over the area of the circle

$$x^2 + y^2 = 1 \quad \left[\text{Ans: } \frac{\pi}{24} \right]$$

21. Evaluate $\iint (x^2 + y^2) dx dy$ over the area bounded by the parabola $y^2 = 4x$ and its latusrectum. [Ans: $\frac{344}{105}$]

22. Evaluate $\int_0^1 \int_x^1 \frac{y dx dy}{x^2 + y^2}$ and also sketch the region of integration. [Ans: $\frac{\pi}{4}$]

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23. Evaluate $\iint xy dx dy$ over the region in the positive quadrant bounded by the line $2x + 3y = 6$ [Ans: $\frac{3}{2}$]

24. Evaluate $\iint \iint_V (xy + yz + zx) dx dy dz$, where V is the region of sphere bounded by $x = 0, x = 1, y = 0, y = 2, z = 0, z = 3$ [Ans: $\frac{33}{2}$]

25. Evaluate $\iint \iint_D x^2 y z dx dy dz$, where D is the tetrahedron bounded by the planes $x = 0, y = 0, z = 0$ and

$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1. \quad \left[\text{Ans: } \frac{a^2 b^2 c^2}{2520} \right]$$

26. Evaluate $\int_0^{\pi/4} \int_0^{a \sin \theta} \frac{r dr d\theta}{\sqrt{a^2 - r^2}}$ [Ans: $\frac{a(\pi - 3)}{6}$]

27. Evaluate $\int_0^4 \int_{y^2/4}^y \frac{y dx dy}{x^2 + y^2}$ and also sketch the region of integration. [Ans: $2 \log 2$]

28. Evaluate $\iint_R x^2 dx dy$, where R is the region bounded by the hyperbola $xy = 4, y = 0, x = 1$ and $x = 2$. [Ans: 6]

29. Evaluate $\iint \iint_V dx dy dz$, where V is the finite region of space formed by the planes $x = 0, y = 0, z = 0$ and $2x + 3y + 4z = 12$. [Ans: 12]

30. Evaluate $\iint_R xy dx dy$, where R is the region bounded by the parabola $y^2 = x$ and the lines $y = 0$ and $x + y = 2$, lying in the 1st quadrant. [Ans: 3/8]

1.2 Change of Order of Integration

Sometimes a double integral may be conveniently evaluated by changing the order of integration. When the limits of the integration are variables, the change of order of integration will involve a change in the limit also. In such cases, it will be convenient to draw a figure indicating the original region of integration.

Example 1.25. Evaluate $\int_0^a \int_y^a \frac{x dx dy}{x^2 + y^2}$ by changing the order of integration.

Solution: The region D of integration is bounded by the lines $x = y$, $x = a$, $y = 0$ and $y = a$ as shown in the figure.

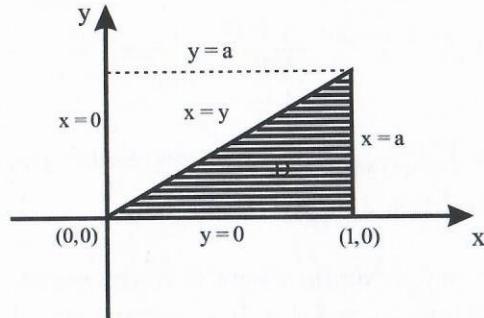


Fig. 8

We have to change the order of integration as $dydx$.

Here in the region D, x varies from 0 to a and for fixed x , y varies from 0 to x .

$$\therefore \int_0^a \int_y^a \frac{x dx dy}{x^2 + y^2} = \int_0^a \int_0^x \frac{x dx dy}{x^2 + y^2} = \int_0^a x \left[\frac{1}{x} \tan^{-1} \left(\frac{y}{x} \right) \right]_0^x$$

$$\begin{aligned} &= \int_0^a x \left[\tan^{-1} \left(\frac{y}{x} \right) \right]_0^x dx = \int_0^a [\tan^{-1} 1 - \tan^{-1} 0] dx \\ &= \int_0^a [\tan^{-1} 1 - \tan^{-1} 0] dx = \frac{\pi}{4} \int_0^a dx \\ &= \frac{\pi}{4} [x]_0^a = \frac{\pi a}{4} \end{aligned}$$

Example 1.26. Evaluate $I = \int_0^\infty \int_x^\infty \frac{e^{-y}}{y} dy dx$ by changing the order of integration.

Solution: The region of integration is as shown. When we change the order x varies from 0 to y and y varies from 0 to ∞ .

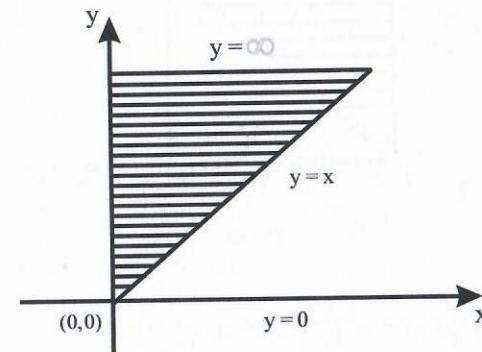


Fig. 9

$$\begin{aligned} I &= \int_0^\infty \int_x^\infty \frac{e^{-y}}{y} dy dx = \int_0^\infty \int_0^y \frac{e^{-y}}{y} dx dy \\ &= \int_0^\infty \left[\frac{e^{-y}}{y} x \right]_0^y dy = \int_0^\infty e^{-y} dy \end{aligned}$$

$$= \left[\frac{e^{-y}}{-1} \right]_0^\infty = 1 \left[\frac{e^{-\infty} - e^0}{-1} \right] = \frac{0 - 1}{-1} = 1$$

Example 1.27. Change the order of integration in $\int_0^a \int_x^a (x^2 + y^2) dx dy$ and hence evaluate the same.

Solution: The region of integration is OAB bounded by $x = 0, x = y, y = a$ and $x = a$.

On changing the order of integration $x = 0, x = y$. Then the strip slides from $y = 0$ to $y = a$.

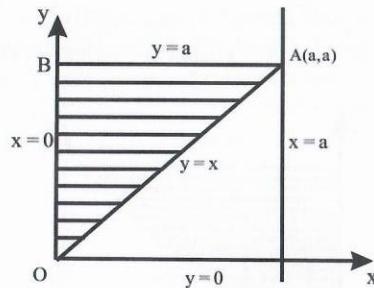


Fig. 10

$$\therefore \int_0^a \int_x^a (x^2 + y^2) dy dx = \int_0^a \int_0^y (x^2 + y^2) dy dx$$

$$= \int_0^a \left[\frac{x^3}{3} + y^2 x \right]_0^y = \int_0^a \left[\frac{y^3}{3} + y^3 \right] dy$$

$$= \frac{4}{3} \int_0^a y^3 dy = \frac{1}{3} [y^4]_0^a = \frac{a^4}{3}$$

Example 1.28. By changing the order of integration, prove that $\int_0^\infty \int_0^y ye^{-y^2/x} dx dy = \frac{1}{2}$

Solution: The region of integration is as shown:

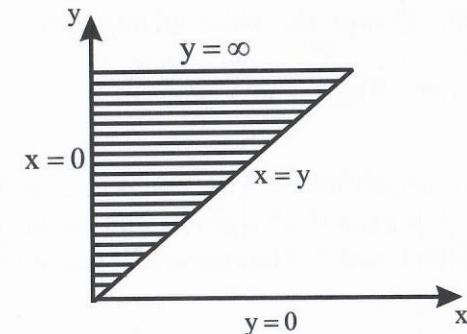


Fig. 11

It is bounded by $x = 0, x = y, y = 0, y = \infty$. On changing the order of integration y ranges from $y = x, y = \infty$ and x ranges from $x = 0$ to $x = \infty$.

$$\int_0^\infty \int_0^y ye^{-y^2/x} dx dy = \int_{x=0}^\infty \int_{y=x}^\infty ye^{-y^2/x} dy dx$$

$$= \int_{x=0}^\infty \left[\int_{y=x}^\infty e^{-y^2/x} d\left(\frac{y^2}{2}\right) \right] dx$$

$$= \int_0^\infty \frac{1}{2} \left[\frac{e^{-y^2/x}}{-1/x} \right]_x^\infty = \int_0^\infty \frac{-x}{2} (0 - e^x) dx$$

$$= \frac{1}{2} \int_0^\infty x e^{-x} dx$$

$$= \frac{1}{2} \left[x \left(\frac{e^{-x}}{-1} \right) - 1 \cdot \frac{e^{-x}}{1} \right]_0^\infty, \text{ on integrating by parts}$$

$$= \frac{1}{2}(0+1) = \frac{1}{2}$$

Example 1.29. Change the order of integration and evaluate
 $I = \int_0^1 \int_0^{\sqrt{1-x^2}} \frac{e^y}{(e^y + 1)\sqrt{1-x^2-y^2}} dx dy$

Solution: The integration is with respect to y between $y = 0, y = \sqrt{1-x^2}$. (i.e.) $y = 0, x^2 + y^2 = 1$ and the next integration is w.r.to x between 0 and 1. The region of integration is as shown here as OAB .

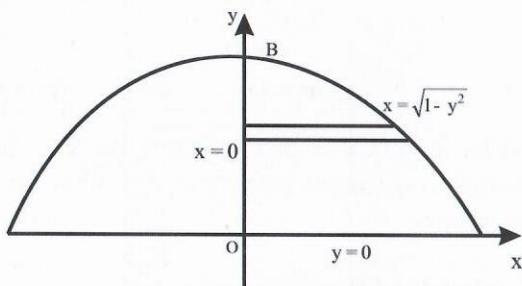


Fig. 12

We now change the order of integration. Integration I w.r.to x between $x = 0, x = \sqrt{1-y^2}$. To cover OAB , y slides from 0 to 1.

$$\begin{aligned} I &= \int_0^1 \int_0^{\sqrt{1-x^2}} \frac{e^y}{(e^y + 1)\sqrt{1-x^2-y^2}} dx dy \\ &= \int_0^1 \int_0^{\sqrt{1-x^2}} \frac{e^y}{(e^y + 1)\sqrt{1-x^2-y^2}} dy dx \end{aligned}$$

$$\begin{aligned} &= \int_0^1 \int_0^{\sqrt{1-y^2}} \frac{e^y}{(e^y + 1)\sqrt{1-x^2-y^2}} dx dy \\ &= \int_0^1 \int_0^{\sqrt{1-y^2}} \left[\frac{e^y}{(e^y + 1)} \frac{dx}{\sqrt{(1-y^2)-x^2}} \right] dy \\ &= \int_0^1 \frac{e^y}{(e^y + 1)} \left[\sin^{-1} \frac{x}{\sqrt{1-y^2}} \right]_0^{\sqrt{1-y^2}} dy = \int_0^1 \frac{e^y}{(e^y + 1)} \frac{\pi}{2} dy \\ &= \frac{\pi}{2} [\log(e^y + 1)]_0^1 = \frac{\pi}{2} \log \left(\frac{e+1}{2} \right) \end{aligned}$$

Example 1.30. By changing the order of integration, evaluate $\int_0^3 \int_1^{4-y} (x+y) dx dy$.

Solution: The region D of integration is bounded by the line $y = 0, y = 3, x = 1$ and by the parabola $x^2 = 4 - y$ as shown in the figure.

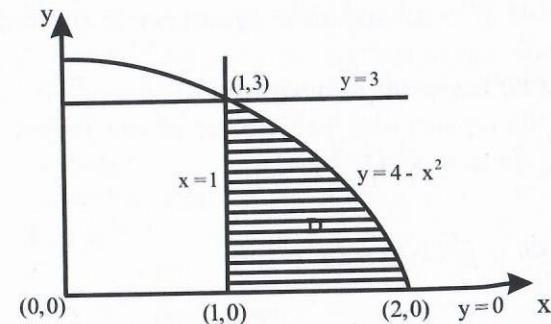


Fig. 13

In this region, x varies from 1 to 2 and y varies from 0 to $4 - x^2$.

$$\begin{aligned} \int_0^3 \int_1^{\sqrt{4-y}} (x+y) dx dy &= \int_1^2 \int_0^{4-x^2} (x+y) dy dx \\ &= \int_1^2 \left[xy + \frac{y^2}{2} \right]_0^{4-x^2} = \int_1^2 \left[x(4-x^2) + \frac{(4-x^2)^2}{2} \right] dx \\ &= \int_1^2 \left[\frac{x^4}{2} - x^3 - 4x^2 + 4x + 8 \right] dx \\ &= \left[\frac{x^5}{10} - \frac{x^4}{4} - 4\frac{x^3}{3} + 2x^2 + 8x \right]_1^2 = \frac{241}{60} \end{aligned}$$

Example 1.31. Change the order of integration and evaluate
 $I = \int_0^4 \int_{x^2/4}^{2\sqrt{x}} dy dx$

Solution: The region of integration is bounded by $y = \frac{x^2}{4}$, $y = 2\sqrt{x}$. (i.e) $x^2 = 4y$ and $y^2 = 4x$ which are parabolas and $x = 0$ and 4. Solving $y^2 = 4x$ and $x^2 = 4y$, we get $(0,0), (4,4)$.

After changing the order of integration

$$\begin{aligned} I &= \int_0^4 \int_{x^2/4}^{2\sqrt{x}} dy dx = \int_0^4 \int_{y^2/4}^{2\sqrt{y}} dx dy \\ &= \int_0^4 [x]_{y^2/4}^{2\sqrt{y}} dy = \int_0^4 (2\sqrt{y} - (y^2/4)) dy \\ &= \left[\frac{4}{3}y^{3/2} - \frac{y^3}{12} \right]_0^4 = \frac{32}{3} - \frac{16}{3} = \frac{16}{3} \end{aligned}$$

Example 1.32. Change the order of integration and evaluate

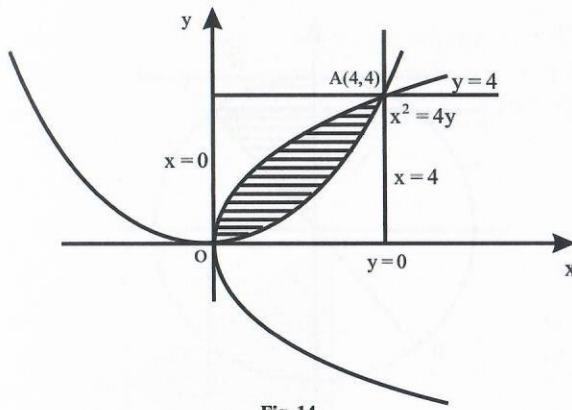


Fig. 14

$$I = \int_0^1 dx \int_x^{\sqrt{2-x^2}} \frac{x}{\sqrt{x^2+y^2}} dy$$

Solution: The limits for y are x and $\sqrt{2-x^2}$ and those for x are 0 and 1. We, therefore, draw the curve $y = x$ which is a straight line and $y = \sqrt{2-x^2}$ which is the upper half of the circle $x^2+y^2 = 2$. The region of integration is OACD. Solving the equation $y = x$ and $x^2+y^2 = 2$, we get the points of intersection $A(1,1)$ and $B(-1,-1)$. If we consider the strip parallel to the x-axis the region has to be divided into two points OAD and ADC. In the region ODA, x varies from 0 to y and y varies from 0 to 1. In the region ADC, x varies from 0 to $\sqrt{2-y^2}$ and y varies from 1 to $\sqrt{2}$.

$$\therefore I = \int_0^1 dx \int_x^{\sqrt{2-x^2}} \frac{x}{\sqrt{x^2+y^2}} dy$$

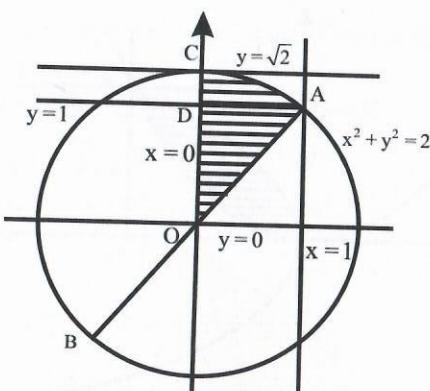


Fig. 15

$$= \int_0^1 dy \int_0^y \frac{x}{\sqrt{x^2 + y^2}} dx + \int_1^{\sqrt{2}} dy \int_0^{\sqrt{2-y^2}} \frac{x}{\sqrt{x^2 + y^2}} dx = I_1 + I_2$$

$$\text{Now } I_1 = \int_0^1 dy [\sqrt{x^2 + y^2}]_0^y = \int_0^1 (\sqrt{2}y - y) dy$$

$$(\text{i.e.}) \quad I_1 = (\sqrt{2} - 1) \left[\frac{y^2}{2} \right]_0^1 = \frac{1}{2}(\sqrt{2} - 1)$$

$$I_2 = \int_1^{\sqrt{2}} dy [\sqrt{x^2 + y^2}]_0^{\sqrt{2-y^2}} = \int_1^{\sqrt{2}} (\sqrt{2}y - y) dy$$

$$= \left[\sqrt{2}y - \frac{y^2}{2} \right]_1^{\sqrt{2}} = \frac{3}{2} - \sqrt{2}$$

$$I = I_1 + I_2 = 1 - \frac{1}{\sqrt{2}}$$

Example 1.33. Change the order of integration and evaluate
 $I = \int_0^5 dx \int_{2-x}^{2+x} dxdy$

Solution: The I integration is w.r.to y, where $y = 2 - x$ and $y = 2 + x$. The II integration is w.r.to x where $x = 0$ to 5. The region integration is as shown in the figure as ABC. Split ABC as ADC, ADB.

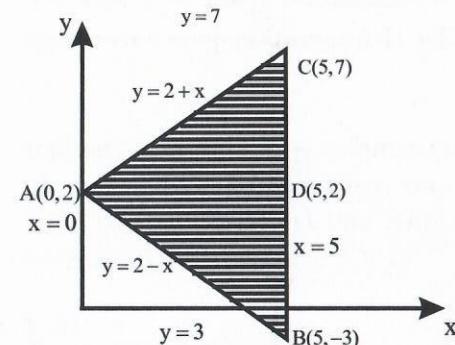


Fig. 16

$$I = I_1 + I_2, \text{ where } I_1 = \int_{-3}^2 \int_{2-y}^5 dx dy \text{ and } I_2 = \int_2^7 \int_{y-2}^5 dx dy$$

$$\text{Now } I_1 = \int_{-3}^2 [x]_{2-y}^5 dy = \int_{-3}^2 (3+y) dy$$

$$= \left[3y + \frac{y^2}{2} \right]_{-3}^2 = 17 - \frac{9}{2} = \frac{25}{2}$$

$$\text{and } I_2 = \int_2^7 [x]_{y-2}^5 dy = \int_2^7 (7-y) dy$$

$$= \left[7y - \frac{y^2}{2} \right]_2^7 = 37 - \frac{49}{2} = \frac{25}{2}$$

$$\therefore I = I_1 + I_2 = \frac{25}{2} + \frac{25}{2} = 25$$

Example 1.34. Change the order of integration and evaluate $\int_0^a \int_{x^2/a}^{2a-x} xy dxdy$

Solution: The I integration is w.r.to y with limits $y = \frac{x^2}{2}$ and $y = 2a - x$. The II integration is w.r.to x between the limits $x = 0, x = a$.

The region of integration is as shown in the figure as OAC. We split this into two regions OAB, ABC. $I = I_1 + I_2$ where I_1 corresponds to OAB and I_2 corresponds to ABC.

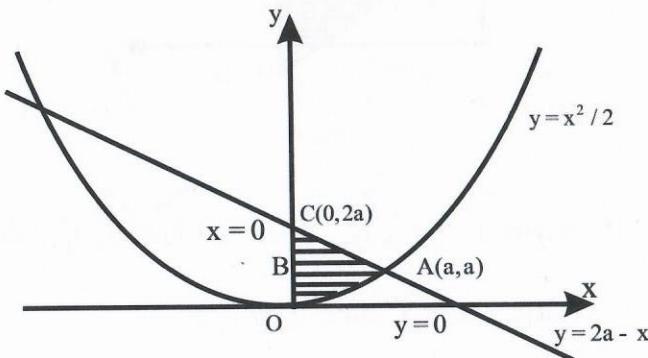


Fig. 17

$$I = \int_0^a \int_0^{\sqrt{ay}} xy dxdy + \int_a^{2a} \int_0^{2a-y} xy dxdy$$

$$= \int_0^a y \left(\frac{x^2}{2} \right)_0^{\sqrt{ay}} dy + \int_a^{2a} y \left(\frac{x^2}{2} \right)_0^{2a-y} dy$$

$$\begin{aligned} &= \int_0^a \frac{a}{2} y^2 dy + \frac{1}{2} \int_a^{2a} (4a^2 y - 4ay^2 + y^3) dy \\ &= \frac{a}{2} \left(\frac{y^3}{3} \right)_0^a + \frac{1}{2} \left[2a^2 y^2 - 4a \frac{y^3}{3} + \frac{y^4}{4} \right]_0^a \\ &= \frac{a^4}{6} + \frac{5a^4}{24} = \frac{3a^4}{8} \end{aligned}$$

EXERCISE

1. Change the order of integration in $\int_{-a}^a \int_0^{\sqrt{a^2-y^2}} f(x, y) dxdy$

$$\left[\text{Ans: } \int_0^a \int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} f(x, y) dydx \right]$$

2. Change the order of integration in $\int_0^a \int_y^a \frac{x}{x^2+y^2} dxdy$

$$\left[\text{Ans: } \int_0^a \int_0^x \frac{x}{x^2+y^2} dydx \right]$$

3. Change the order of integration in $\int_0^\infty \int_0^y f(x, y) dxdy$

$$\left[\text{Ans: } \int_0^\infty \int_x^\infty f(x, y) dxdy \right]$$

4. Change the order of integration in $\int_0^1 \int_0^{2\sqrt{x}} \phi(x, y) dydx$

$$\left[\text{Ans: } \int_0^2 \int_{y^2/4}^1 \phi(x, y) dxdy \right]$$

5. Change the order of integration in $\int_0^1 \int_0^x dydx$

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$\left[\text{Ans: } \int_0^1 \int_y^1 dx dy \right]$

6. Change the order of integration in $\int_0^a \int_{x^2/a}^x f(x, y) dy dx$

$\left[\text{Ans: } \int_0^a \int_y^{\sqrt{ay}} f(x, y) dy dx \right]$

7. Change the order of integration in $\int_0^\infty \int_0^{1/y} f(x, y) dx dy$

$\left[\text{Ans: } \int_0^\infty \int_0^{1/y} f(x, y) dx dy \right]$

8. Change the order of integration and show that $\int_0^1 \int_0^x dy dx = \frac{1}{2}$.

9. Change the order of integration and hence evaluate $\int_1^4 \int_{\sqrt{y}}^2 (x^2 + y^2) dx dy$ $\left[\text{Ans: } \frac{1006}{105} \right]$

10. Change the order of integration and hence evaluate

$\int_0^a \int_0^{\sqrt{ax}} x^2 dy dx$ $\left[\text{Ans: } \frac{2a^4}{7} \right]$

11. Evaluate $\int_0^a \int_0^{\sqrt{1-x^2}} y^2 dy dx$ by interchanging the order of integration. $\left[\text{Ans: } \frac{\pi}{16} \right]$

12. Change the order of integration in $\int_0^a \int_{a-y}^{\sqrt{a^2-y^2}} y dx dy$ and then evaluate it. $\left[\text{Ans: } \frac{a^3}{6} \right]$

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13. Change the order of integration in $\int_0^{2a} \int_{x^2/4a}^a (x + y) dy dx$ and hence evaluate it. $\left[\text{Ans: } \frac{9a^3}{5} \right]$

14. Evaluate $\int_1^2 \int_0^{4/x} xy dy dx$ by changing the order of integration. $\left[\text{Ans: } 8 \log 2 \right]$

15. Change the order of integration in $\int_0^1 \int_y^{2-y} xy dx dy$ and then evaluate it. $\left[\text{Ans: } \frac{1}{3} \right]$

16. Change the order of integration in $\int_1^2 \int_1^{x^2} \frac{x^2}{y} dy dx$ and hence evaluate the same. $\left[\text{Ans: } \frac{2}{9} \log 2 - 7 \right]$

17. Change the order of integration in $\int_0^a \int_{y^2/a}^y \frac{y dy dx}{(a-x)\sqrt{ax-y^2}}$ $\left[\text{Ans: } \frac{\pi a}{2} \right]$

18. Evaluate $\int_3^5 \int_0^{4/x} xy dx dy$ after reversing the order of integration. $\left[\text{Ans: } 8 \log \frac{5}{3} \right]$

19. Change the order of integration in $\int_0^\infty \int_0^\infty e^{-x^2(1+t^2)} x dx dt$ and then evaluate it. $\left[\text{Ans: } \frac{\pi}{4} \right]$

20. Change the order of integration in each of the double integrals $\int_0^1 \int_1^2 \frac{dx dy}{x^2 + y^2}$ and $\int_1^2 \int_y^2 \frac{dx dy}{x^2 + y^2}$ and hence express

their sum as one double integral and evaluate it. [Ans: $\frac{\pi}{4} \log 2$]

21. Evaluate $\int_0^1 \int_{\sqrt{x}}^1 e^{x/y} dx dy$ by interchanging the order of integration. [Ans: 1/2]

1.3 Plane Area as Double Integral

Plane area enclosed by one or more curves can be expressed as a double integral. Let R be the plane region, the area of which is required. Let us divide the area into a large number of elemental areas $ABCD$, by dividing lines parallel to the y -axis of intervals Δx and the lines parallel to the x -axis, at intervals of Δy .

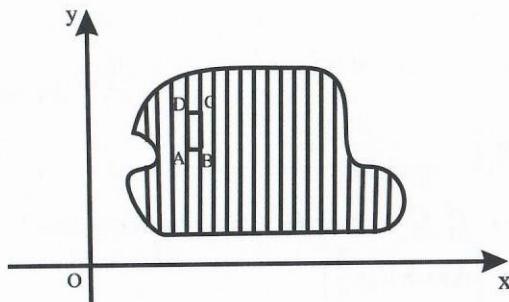


Fig. 18

$$\text{Area of the rectangle} = \Delta x \Delta y$$

$$\text{Rectangle of area } \Delta = \lim_{\Delta x \rightarrow 0} [\sum \sum \Delta x \Delta y] = \int \int_R dx dy$$

Note: In polar system $A = \int \int_D r dr d\theta$

Example 1.35. Show, by double integration, that the area between the parabolas $y^2 = 4ax$ and $x^2 = 4ay$ is $\frac{16}{3}a^2$.

Solution: The region by integration is as shown.

Solving $y^2 = 4ax$ and $x^2 = 4ay$ are set $(0, 0), (4a, 4a)$. Take a strip parallel to y axis implies limits for $y = \frac{x^2}{4a}$ and $y = 2\sqrt{ax}$ and then x varies from 0 to $4a$.

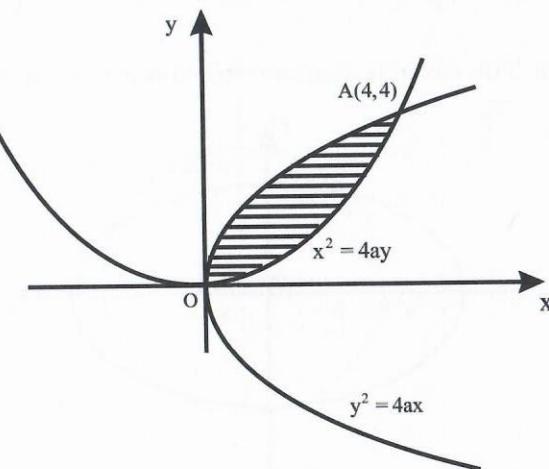


Fig. 19

$$\therefore \text{Area} = \int_0^{4a} \int_{x^2/4a}^{2\sqrt{ax}} dx = \int_0^{4a} [y]_{x^2/4a}^{2\sqrt{ax}} dx$$

$$= \int_0^{4a} \left(2\sqrt{ax}^{1/2} - \frac{x^2}{4a} \right) dx = \left[2\sqrt{a} \frac{x^{1/2} + 1}{\frac{1}{2} + 1} - \frac{1}{4a} \frac{x^3}{3} \right]_0^{4a}$$

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$$\begin{aligned}
 &= \left[\frac{4}{3} \sqrt{a} x^{3/2} - \frac{1}{12a} x^3 \right]_0^{4a} = \frac{4}{3} \sqrt{a} (4a^{3/2}) - \frac{1}{12a} (4a)^3 \\
 &= \frac{4}{3} a^2 (4 \times 2) - \frac{1}{12a} \times 64a^3 \\
 &= \frac{32}{3} a^2 - \frac{16}{3} a^2 = \frac{16}{3} a^2 \text{ Sq.units.}
 \end{aligned}$$

Example 1.36. Find by double integration, the area enclosed by the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

Solution: The curve is symmetrical about both axes.

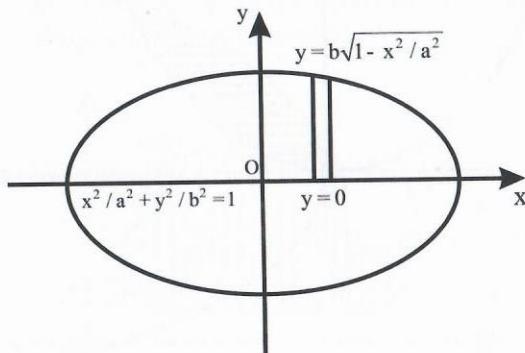


Fig. 20

\therefore Area = 4 area in I quadrant.

$$\begin{aligned}
 &= 4 \int \int_A dy dx = 4 \int_0^a \int_0^{b\sqrt{1-x^2/a^2}} dy dx \\
 &= 4 \int_0^a [y]_0^{b\sqrt{1-x^2/a^2}} = \frac{4b}{a} \int_0^a \sqrt{a^2 - x^2} dx
 \end{aligned}$$

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$$\begin{aligned}
 &= \frac{4b}{a} \left[\frac{x}{2} \sqrt{a^2 - x^2} + \frac{a^2}{2} \sin^{-1} \frac{x}{a} \right]_0^a \\
 &= \frac{4b}{a} \times \frac{a^2}{2} \times \frac{\pi}{2} = \pi ab \text{ Sq.units.}
 \end{aligned}$$

Example 1.37. Find by double integration, smaller of the areas bounded by the circle $x^2 + y^2 = 9$ and $x + y = 3$.

Solution: The region of integration is as shown y varies from $3 - x$ to $\sqrt{9 - x^2}$ and x varies from 0 to 3.

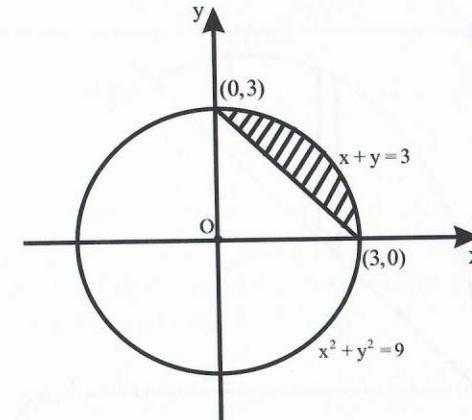


Fig. 21

$$\text{Area} = \int \int dy dx$$

$$\begin{aligned}
 &= \int_0^3 \int_{3-x}^{\sqrt{9-x^2}} dy dx = \int_0^3 [\sqrt{9-x^2} - (3-x)] dx \\
 &= \left[\frac{x}{2} \sqrt{9-x^2} + \frac{9}{2} \sin^{-1} \frac{x}{3} \right]_0^3 - 3[x]_0^3 + \left[\frac{x^2}{2} \right]_0^3
 \end{aligned}$$

$$= \frac{9\pi}{2} - 9 + \frac{9}{2} = \frac{9\pi}{4} - \frac{9}{2} = \frac{9}{4}(\pi - 2) \text{ sq.units.}$$

Example 1.38. Find the area between the parabola $y = 4x - x^2$ and the line $y = x$.

Solution:

The two curves intersect at points whose abscissa are given by $4x - x^2 = x$ or $x^2 - 3x = 0$ (i.e.) $x = 0, 3$. Using vertical strips, the required area lies between $x = 0, x = 3$ and $y = x, y = 4x - x^2$.

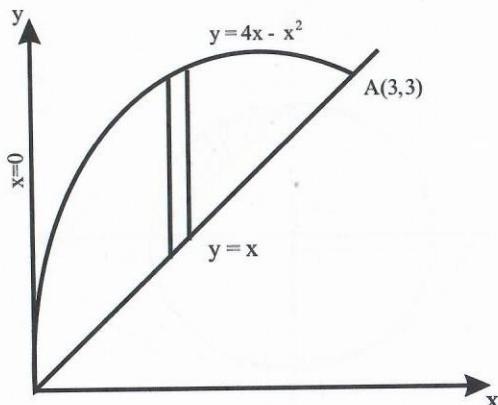


Fig. 22

$$\begin{aligned}\therefore \text{Required area} &= \int_0^3 \int_x^{4x-x^2} dy dx = \int_0^3 [y]_x^{4x-x^2} dx \\ &= \int_0^3 (4x - x^2 - x) dx = \int_0^3 (3x - x^2) dx \\ &= \left[\frac{3x^2}{2} - \frac{x^3}{3} \right]_0^3 = 4.5 \text{ Sq.units.}\end{aligned}$$

Example 1.39. Find by double integration, the area bounded by the parabola $y = x^2$ and the line $y = 2x + 3$.

Solution: The region of integration is as shown.

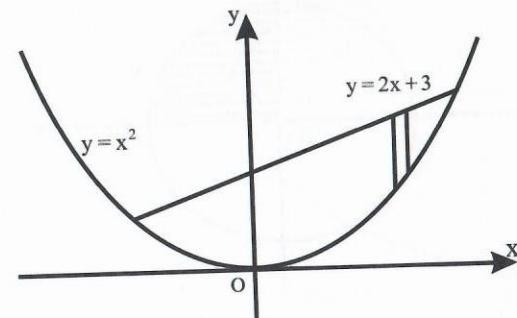


Fig. 23

Solving $y = 2x + 3$ we get $x^2 - 2x - 3 = 0$. (i.e.) $x = 3, -1$.

Required area = $\int \int dy dx$ where y varies from $y = x^2$ and $y = 2x + 3$. Further x varies from -1 to 3.

$$\therefore \text{Required area} = \int_{-1}^3 \int_{x^2}^{2x+3} dy dx = \int_{-1}^3 [y]_{x^2}^{2x+3} dx$$

$$= \int_{-1}^3 (2x + 3 - x^2) dx = \left[2\left(\frac{x^2}{2}\right) + 3x - \frac{x^3}{3} \right]_{-1}^3$$

$$= \frac{32}{3} = 10\frac{2}{3} \text{ Sq.units.}$$

Example 1.40. Find the area of the circle using double integral.

Solution: Area of the circle $A = 4 \int \int_D dx dy$ where D is the

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bounded region in the I quadrant of xy -plane. In D , x varies from 0 to r , and for fixed x , y varies from 0 to $\sqrt{r^2 - x^2}$.

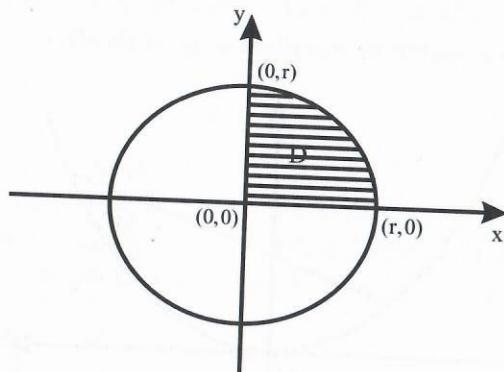


Fig. 24

$$A = 4 \int_0^r \int_0^{\sqrt{r^2 - x^2}} dy dx = 4 \int_0^r \sqrt{r^2 - x^2} \text{ put } x = r \sin \theta$$

$$= 4r^2 \int_0^{\pi/2} \cos^2 \theta d\theta = 4r^2 \int_0^{\pi/2} \frac{1 + \cos 2\theta}{2} d\theta$$

$$= 2r^2 \left[\theta + \frac{\sin 2\theta}{2} \right]_0^{\pi/2} = \pi r^2 \text{ as } \sin \pi = 0, \sin 0 = 0$$

Example 1.41. Find the area of the region D bounded by the parabolas $y = x^2$ and $x = y^2$.

Solution: The point of intersection of the curves are given by $(0,0)$ and $(1,1)$. The required region D is as shown below.

$$\text{The required area } A = \int \int_D dx dy = \int_0^1 \int_{x^2}^{\sqrt{x}} dy dx$$

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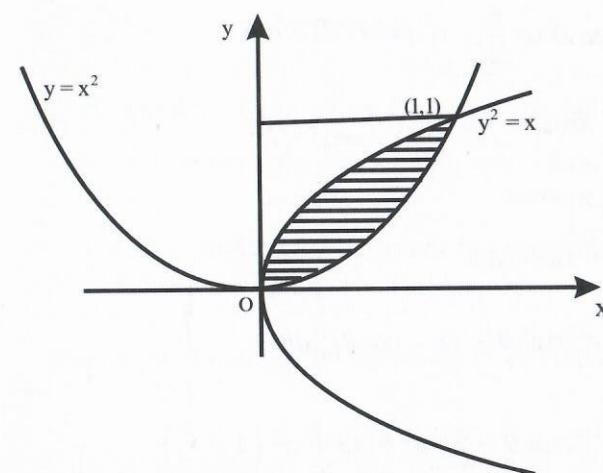


Fig. 25

$$= \int_0^1 [y]_{x^2}^{\sqrt{x}} dx = \int_0^1 (\sqrt{x} - x^2) dx$$

$$= \left[2 \frac{x^{3/2}}{3} - \frac{x^3}{3} \right]_0^1 = \frac{1}{3}$$

Example 1.42. Find by double integration, the area lying inside the circle $r = a \sin \theta$ and outside the cardioid $r = a(1 - \cos \theta)$.

Solution: Eliminating r between the equations of two curves $\sin \theta = 1 - \cos \theta$ or $\sin \theta + \cos \theta = 1$.

Squaring $1 + \sin 2\theta = 1$ or $\sin 2\theta = 0 \therefore 2\theta = 0$ or π

$$\text{(i.e.) } \theta = 0 \text{ or } \frac{\pi}{2}.$$

For the required area, r varies from $a(1 - \cos \theta)$ to $a \sin \theta$ and θ

varies from 0 to $\frac{\pi}{2}$.

$$\begin{aligned}\text{Required area} &= \int_0^{\pi/2} \int_{a(1-\cos\theta)}^{a\sin\theta} r dr d\theta \\ &= \int_0^{\pi/2} \left[\frac{r^2}{2} \right]_{a(1-\cos\theta)}^{a\sin\theta} d\theta \\ &= \frac{1}{2} \int_0^{\pi/2} a^2 [\sin^2\theta - (1 - \cos\theta)^2] d\theta \\ &= \frac{a^2}{2} \int_0^{\pi/2} (2\cos\theta - 2\cos^2\theta) d\theta = a^2 \left(1 - \frac{\pi}{4} \right)\end{aligned}$$

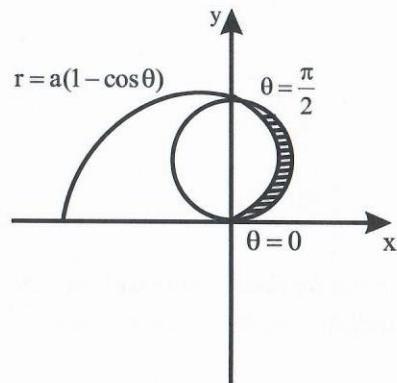


Fig. 26

EXERCISE

- Find the smaller of the areas bounded by the ellipse $4x^2 + 9y^2 = 36$ and the straight line $2x + 3y = 6$. [Ans: $\frac{3}{2}(\pi - 2)$]
- Find by double integration, the area bounded by the parabolas $y^2 = 4 - x$ and $y^2 = 4 - 4x$ [Ans: 8]
- Find the area bounded by the circle $r = 2 \sin\theta$ and $r = 4 \sin\theta$ [Ans: 3π]
- Find the area enclosed by the curve $r^2 = a^2 \cos 2\theta$, by double integration. [Ans: a^2]
- Find the area bounded by the parabola $y^2 = 4 - x$ and $y^2 = x$ by double integration. [Ans: $\frac{16}{3}\sqrt{2}$]
- Find the area common to the parabola $y^2 = x$ and $x^2 + y^2 = 2$ by double integration. [Ans: $\frac{\pi}{2} + \frac{1}{3}$]
- Find the area that lies outside the circle $r = a \cos\theta$ and inside the circle $r = 2a \cos\theta$ [Ans: $\frac{3}{4}\pi a^2$]
- Find the area common to the two circles $r = a$ and $r = 2a \cos\theta$. [Ans: $a^2 \left(\frac{2\pi}{3} - \frac{\sqrt{3}}{2} \right)$]

1.4 Change of Variable from Cartesian to Polar Co-ordinates

Let $\int \int_R f(x, y) dx dy$ be the double integral. $x = r \cos \theta, y = r \sin \theta$ is the transformation from Cartesian to Polar coordinates.

Then $dx dy = |J| dr d\theta$ where $J = \frac{\partial(x, y)}{\partial(r, \theta)}$ is the Jacobian of the transformation and

$$J = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos \theta & -r \sin \theta \\ \sin \theta & r \cos \theta \end{vmatrix}$$

$$= r(\cos^2 \theta + \sin^2 \theta) = r$$

$$\therefore \int \int_R f(x, y) dx dy = \int \int_R f(r, \theta) r dr d\theta$$

Example 1.43. Change to polar coordinates and evaluate $\int_0^a \int_y^a x dx dy$

Solution:

The region of integration is $x = y, x = a, y = 0, y = a$.

(i.e.) The triangle OAB putting $x = r \cos \theta, y = r \sin \theta$, the line $x = y$ becomes $r \cos \theta = r \sin \theta$

$$(i.e.) \tan \theta = 1 \Rightarrow \theta = \frac{\pi}{4}$$

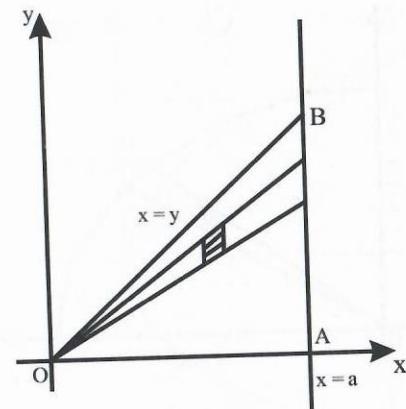


Fig. 27

$$\text{Hence in polar form } I = \int_0^{\frac{\pi}{4}} \int_0^{a/\cos \theta} r^2 \cos \theta dr d\theta$$

$$\begin{aligned} &= \int_0^{\frac{\pi}{4}} \cos \theta \left(\frac{r^3}{3} \right)_0^{a/\cos \theta} d\theta = \frac{a^3}{3} \int_0^{\frac{\pi}{4}} \sec^2 \theta d\theta \\ &= \frac{a^3}{3} [\tan \theta]_0^{\frac{\pi}{4}} = \frac{a^3}{3} \end{aligned}$$

Example 1.44. Change to polar coordinates and evaluate

$$\int_0^a \int_0^{\sqrt{a^2 - x^2}} e^{-(x^2 + y^2)} dx dy$$

Solution:

Putting $x = r \cos \theta, y = r \sin \theta$, the given limit $y^2 = a^2 - x^2$.

(i.e.) The circle $x^2 + y^2 = a^2$ changes to $r = a$ and $y = 0$.

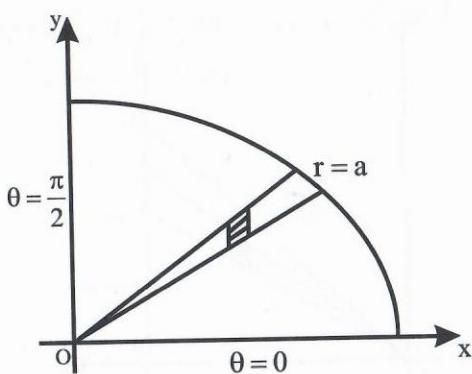


Fig. 28

(i.e.) The x-axis changes to initial line $\theta = 0$. Hence, in the given region r changes from 0 to a and θ changes from 0 to $\frac{\pi}{2}$.

$$\begin{aligned} I &= \int_0^{\frac{\pi}{2}} \int_0^a e^{-r^2} r dr d\theta \\ &= \int_0^{\frac{\pi}{2}} \left(-\frac{1}{2} e^{-r^2} \right)_0^a d\theta \\ &= -\frac{1}{2} \int_0^{\frac{\pi}{2}} (e^{-a^2} - 1) d\theta = \frac{\pi}{4} (1 - e^{-a^2}) \end{aligned}$$

Example 1.45. Evaluate $\int_0^{4a} \int_{y^2/4a}^y dy dx$ by changing to polar coordinates.

Solution:

The region of integration is bounded by the parabola $x = y^2/4a$.

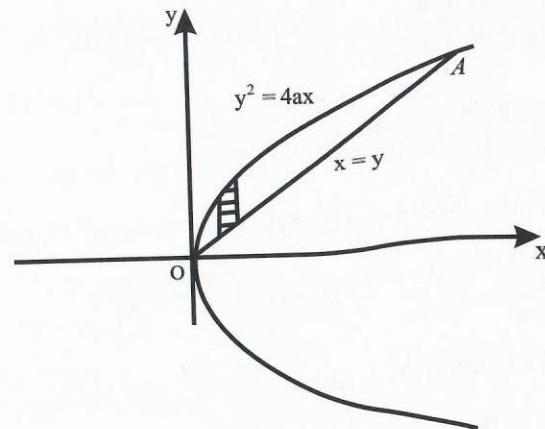


Fig. 29

(i.e.) $y^2 = 4ax$ and the line $x = y$.

By putting $x = r \cos \theta, y = r \sin \theta$, the parabola becomes $r^2 \sin^2 \theta = 4ar \cos \theta$.

(i.e.) $r = \frac{4a \cos \theta}{\sin^2 \theta}$ and the line becomes $x \cos \theta = r \sin \theta$

(i.e.) $\theta = \frac{\pi}{4}$

Hence r varies from 0 to $\frac{4a \cos \theta}{\sin^2 \theta}$ and θ varies from $\frac{\pi}{4}$ to $\frac{\pi}{2}$.

$$I = \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \int_0^{\frac{4a \cos \theta}{\sin^2 \theta}} r d\theta dr = \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \left(\frac{r^2}{2} \right)_0^{\frac{4a \cos \theta}{\sin^2 \theta}}$$

$$\begin{aligned}
 &= \frac{1}{2} \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \left[\frac{16a^2 \cos^2 \theta}{\sin^4 \theta} \right] d\theta = 8a^2 \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \cot^2 \theta \csc^2 \theta d\theta \\
 &= 8a^2 \left[-\frac{\cot^3 \theta}{3} \right]_{\frac{\pi}{4}}^{\frac{\pi}{2}} = -\frac{8a^2}{3}[0 - 1] = \frac{8a^2}{3}
 \end{aligned}$$

Example 1.46. Express the following integral in polar coordinates and evaluate $\int_0^a \int_{\sqrt{ax-x^2}}^{\sqrt{a^2-x^2}} \frac{dxdy}{\sqrt{a^2-x^2-y^2}}$

Solution: The limits of y are $\sqrt{ax-x^2}$ and $\sqrt{a^2-x^2}$.

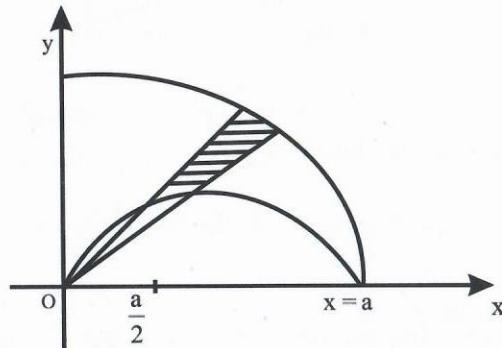


Fig. 30

(i.e.) Upper half of the circles i) $x^2 + y^2 - ax = 0$; $\left(x - \frac{a}{2}\right)^2 + (y - 0)^2 = \left(\frac{a}{2}\right)^2$ and ii) $x^2 + y^2 = a^2$

To change the given integral to polar coordinates, we put $x = r \cos \theta$, $y = r \sin \theta$ and $dxdy = rdrd\theta$.

The equations of the circles now become

i) $r^2 - ar \cos \theta = 0$ (i.e.) $r = a \cos \theta$

ii) $r^2 = a^2$ (i.e.) $r = a$

Hence r changes from $r = a \cos \theta$ to a and θ changes from 0 to $\frac{\pi}{2}$

$$\begin{aligned}
 \therefore I &= \int_0^{\frac{\pi}{2}} \int_{a \cos \theta}^a \frac{rdrd\theta}{\sqrt{a^2 - r^2}} = \int_0^{\frac{\pi}{2}} [-\sqrt{a^2 - r^2}]_{a \cos \theta}^a d\theta \\
 &= \int_0^{\frac{\pi}{2}} a \sin \theta d\theta = [a \cos \theta]_0^{\frac{\pi}{2}} = a
 \end{aligned}$$

Example 1.47. Evaluate $\int \int_R \frac{xydxdy}{\sqrt{x^2+y^2}}$ where R is the region in the first quadrant enclosed by the circles $x^2 + y^2 = a^2$ and $x^2 + y^2 = 4a^2$, by converting into polar coordinates.

Solution: We put $x = r \cos \theta$, $y = r \sin \theta$ $\therefore dxdy = rdrd\theta$

r varies from $r = a$ to $r = 2a$ and θ varies from 0 to $\frac{\pi}{2}$

$$\begin{aligned}
 \text{Hence } I &= \int_0^{\frac{\pi}{2}} \int_a^{2a} \frac{r^3 \sin \theta \cos \theta drd\theta}{\sqrt{r^2}} \\
 &= \left(\int_a^{2a} r^2 dr \right) \left(\frac{1}{2} \int_0^{\frac{\pi}{2}} \sin 2\theta d\theta \right) \\
 &= \frac{1}{2} \left(\frac{r^3}{3} \right)_0^{2a} \left(\frac{-\cos 2\theta}{2} \right)_0^{\frac{\pi}{2}} \\
 &= -\frac{1}{12}(8a^3 - a^3)(\cos \pi - \cos 0)
 \end{aligned}$$

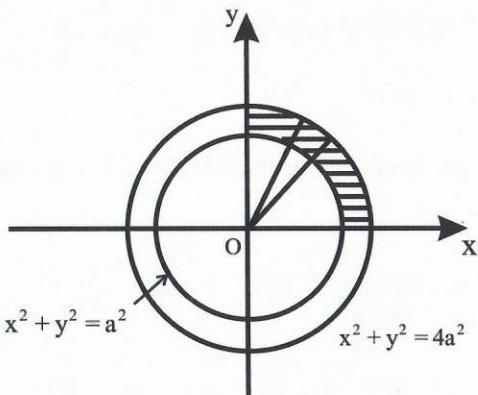


Fig. 31

$$= -\frac{1}{12}(7a^3)(-2) = \frac{7}{6}a^3$$

Example 1.48. Evaluate $\int_0^\infty \int_0^\infty e^{-(x^2+y^2)} dx dy$ by using polar coordinates and hence evaluate $\int_0^\infty e^{-(x^2)} dx$.

Solution:

$\int_0^\infty \int_0^\infty e^{-(x^2+y^2)} dx dy = \int \int_A e^{-(x^2+y^2)} dA$, where A is the positive quadrant.

Put $x = r \cos \theta, y = r \sin \theta \therefore dx dy = r dr d\theta$

In the positive quadrant A, given the limits for r varies from 0 to ∞ and θ varies from 0 to $\frac{\pi}{2}$

$$I = \int_0^\infty \int_0^\infty e^{-(x^2+y^2)} dx dy = \int_0^{\frac{\pi}{2}} \int_0^\infty e^{-(r^2)} r dr d\theta$$

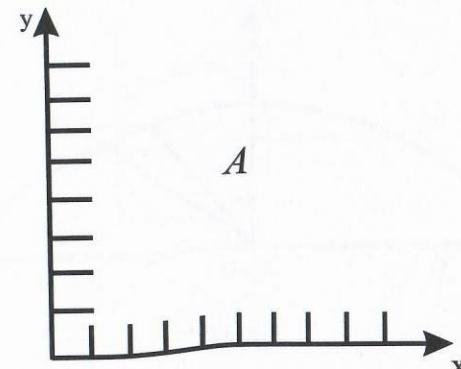


Fig. 32

$$\text{Put } r^2 = R \Rightarrow 2rdr = dR$$

$$\therefore I = \int_0^{\frac{\pi}{2}} \frac{1}{2} \left(\frac{e^{-R}}{-1} \right)_0^\infty d\theta = \frac{1}{2} \int_0^{\frac{\pi}{2}} d\theta = \frac{1}{2} \times \frac{\pi}{2} = \frac{\pi}{4}$$

$$\text{Since } \int_0^\infty \int_0^\infty e^{-(x^2+y^2)} dx dy = \int_0^\infty e^{-x^2} dx * \int_0^\infty e^{-y^2} dy$$

$$= I \times I = I^2$$

$$I^2 = \frac{\pi}{4}$$

$$\therefore I = \int_0^\infty e^{-x^2} dx = \frac{\sqrt{\pi}}{2}$$

Example 1.49. Evaluate $\int_{-a}^a \int_0^{\sqrt{a^2-x^2}} (x^2+y^2) dx dy$ by changing to polar coordinates.

Solution:

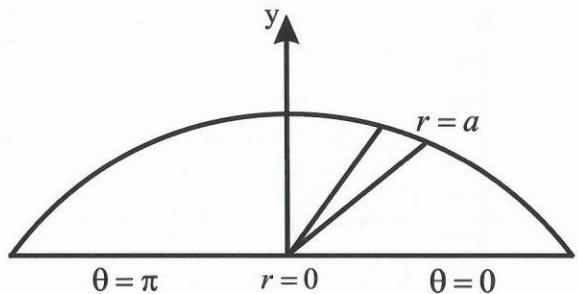


Fig. 33

Put $x = r \cos \theta, y = r \sin \theta \therefore dxdy = rdrd\theta$

$$I = \int_{-a}^a \int_0^{\sqrt{a^2 - x^2}} (x^2 + y^2) dxdy$$

$$= \int_0^\pi \int_0^a r^2 \cdot r dr d\theta = \left(\frac{r^4}{4} \right)_0^a [\theta]_0^\pi = \frac{\pi a^4}{4}$$

Example 1.50. Evaluate $\int_0^{2a} \int_0^{\sqrt{2ax - x^2}} (x^2 + y^2) dy dx$.

Solution:

Here y varies from 0 to $\sqrt{2ax - x^2}$

(i.e.) $y = 0$ to $x^2 + y^2 = 2ax$ which is a circle centre $(a, 0)$ and radius a . Also the circle passes through $(0, 0)$. Since x varies from 0 to $2a$, the area of integration is the area above the x-axis and inside the circle. The polar equation of the circle is $x = 2a \cos \theta$.

Changing to polar coordinates $x = r \cos \theta, y = r \sin \theta$

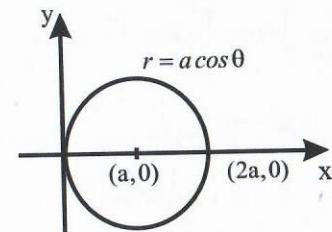


Fig. 34

$$\therefore dxdy = rdrd\theta$$

$$I = \int_0^{\pi/2} \int_0^{2a \cos \theta} r^2 r dr d\theta = \int_0^{\pi/2} \left(\frac{r^4}{4} \right)_0^{2a \cos \theta} d\theta$$

$$= 4a^4 \int_0^{\pi/2} \cos^4 \theta d\theta = 4a^4 \frac{3}{4} \cdot \frac{1}{2} \cdot \frac{\pi}{2} = \frac{3}{4} \pi a^4$$

$$\left[\int_0^{\pi/2} \cos^n \theta d\theta = \begin{cases} \frac{n-1}{n} \times \frac{n-3}{n-2} \times \dots \times \frac{\pi}{2}, & n \text{ is even} \\ \frac{n-1}{n} \times \frac{n-3}{n-2} \times \dots \times 1, & n \text{ is odd} \end{cases} \right]$$

EXERCISE

Changing to polar coordinates evaluate the following:

$$1. \int \int x^2 y^2 dxdy \text{ over the circle } x^2 + y^2 = a^2 \quad \left[\text{Ans: } \frac{\pi}{24} a^6 \right]$$

$$2. \int_0^a \int_y^a \frac{x dx dy}{x^2 + y^2} \quad \left[\text{Ans: } \frac{\pi a}{4} \right]$$

3. $\int_0^a \int_0^x \frac{x^3 dx dy}{\sqrt{x^2 + y^2}}$ [Ans: $\frac{a^4}{4} \log(1 + \sqrt{2})$]

4. $\int_0^a \int_0^{\sqrt{a^2 - x^2}} y^2 \sqrt{x^2 + y^2} dx dy$ over the region bounded by
 $y = 0, x = y, x = 1$ [Ans: $\frac{1}{20} \pi a^5$]

5. $\iint \frac{dxdy}{(x^2 + y^2)^{3/2}}$ over the region bounded by $y = 0, x = y, x = 1$ [Ans: $\frac{\pi}{12}$]

1.5 Volume as a Triple Integral

Triple integrals can be used to evaluate volume V of a finite region D in space.

The volume $V = \iiint_D dxdydz$

Example 1.51. Find the volume of the ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \text{ by triple integrals.}$$

Solution:

Volume = 8 × volume in the first octant.

$$\begin{aligned} V &= 8 \times \int_0^a \int_0^{b\sqrt{1-\frac{x^2}{a^2}}} \int_0^{c\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}} dz dy dx \\ &= 8 \times \int_0^a \int_0^{b\sqrt{1-\frac{x^2}{a^2}}} [z]_0^{c\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}} dy dx \end{aligned}$$

$$= 8c \int_0^a \int_0^{b\sqrt{1-\frac{x^2}{a^2}}} \left[\sqrt{1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}} \right] dy dx$$

$$\text{Put } r^2 = \left(1 - \frac{x^2}{a^2} \right) b^2$$

$$= \frac{8c}{b} \int_0^a \int_0^r \sqrt{r^2 - y^2} dy dx$$

$$= \frac{8c}{b} \int_0^a \left[\frac{r^2}{2} \sin^{-1} \frac{y}{r} + \frac{y}{2} \sqrt{r^2 - y^2} \right]_0^r dr$$

$$= \frac{2c\pi}{b} \int_0^a r^2 dx = \frac{2c\pi}{b} \int_0^a \left(1 - \frac{x^2}{a^2} \right) b^2 dx$$

$$= 2cb\pi \left(x - \frac{x^3}{3a^2} \right)_0^a = \frac{4\pi}{3} abc$$

Example 1.52. Find the volume bounded by the cylinder $x^2 + y^2 = 4$ and the planes $y + z = 4$ and $z = 0$.

Solution:

z varies from $z = 0$ to $z = 4 - y$ and x, y varies over all points of the circle $x^2 + y^2 = 4$.

$$\begin{aligned} \text{Volume } V &= \int_{-2}^2 \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \int_0^{4-y} dz dy dx \\ &= \int_{-2}^2 \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} [z]_0^{4-y} dy dx = \int_{-2}^2 \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} (4-y) dy dx \end{aligned}$$

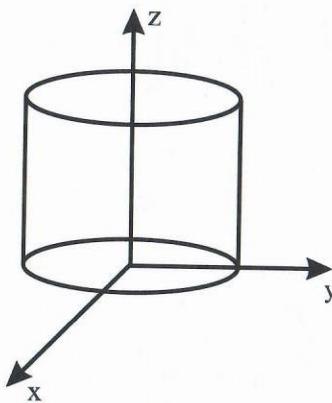


Fig. 35

$$= \int_{-2}^2 \left[4y - \frac{y^2}{2} \right]_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} dx = 8 \times 2 \int_0^2 \sqrt{4-x^2} dx$$

$$V = 16 \left[\frac{x}{2} \sqrt{4-x^2} + \frac{4}{2} \sin^{-1} \frac{x}{2} \right] = 16\pi$$

Example 1.53. Find the volume of the tetrahedron bounded by the coordinate planes and $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$

Solution:

Volume required = $\iiint dxdydz$ with limits.

$$= \int_0^a \int_0^{b(1-\frac{x}{a})} \int_0^{c(1-\frac{x}{a}-\frac{y}{b})} dz dy dx$$

$$= c \int_0^a \int_0^{b(1-\frac{x}{a})} \left(1 - \frac{x}{a} - \frac{y}{b} \right) dy dx$$

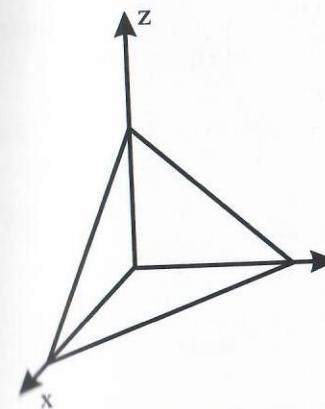


Fig. 36

$$\begin{aligned} &= c \int_0^a \left[\left(1 - \frac{x}{a} \right) y - \frac{y^2}{2b} \right]_0^{b(1-\frac{x}{a})} dx \\ &= c \int_0^a \left[b \left(1 - \frac{x}{a} \right)^2 - \frac{b^2}{2b} \left(1 - \frac{x}{a} \right)^2 \right] dx \\ &= \frac{bc}{2} \int_0^a \left(1 - \frac{x}{a} \right)^2 dx = \frac{bc}{2} \left[\frac{\left(1 - \frac{x}{a} \right)^2}{3} \times \left(\frac{-a}{1} \right) \right] \end{aligned}$$

$$= \frac{abc}{6}$$

Example 1.54. Find the volume of the sphere $x^2 + y^2 + z^2 = a^2$ using the triple integrals.

Solution:

1 MULTIPLE INTEGRALS

Since the sphere $x^2 + y^2 + z^2 = a^2$ is symmetric about the coordinate planes, the volume of the sphere = $8 \times$ volume of the first octant.

$$= 8 \times \int \int \int_V dx dy dz$$

$$= 8 \int_0^a \int_0^{\sqrt{a^2-x^2}} \int_0^{\sqrt{a^2-x^2-y^2}} dz dy dx$$

$$= 8 \int_0^a \int_0^{\sqrt{a^2-x^2}} [z]_0^{\sqrt{a^2-x^2-y^2}} dy dx$$

$$= 8 \int_0^a \int_0^{\sqrt{a^2-x^2}} \sqrt{a^2 - x^2 - y^2} dy dx$$

$$\therefore \left[\int \sqrt{(a^2 - x^2)} dx = \frac{x}{2} \sqrt{(a^2 - x^2)} + \frac{a^2}{2} \sin^{-1} \frac{x}{2} \right]$$

$$= 8 \int_0^a \left[\frac{y}{2} \sqrt{(a^2 - x^2) - y^2} + \frac{a^2 - x^2}{2} \sin^{-1} \left(\frac{y}{\sqrt{a^2 - x^2}} \right) \right]_0^{\sqrt{a^2-x^2}} dx$$

$$= 2\pi \int_0^a (a^2 - x^2) dx$$

$$= 2\pi \left[a^2 x - \frac{x^3}{3} \right] = \frac{4\pi}{3} a^3$$

EXERCISE

1. Find the volume bounded by xy -plane, the cylinder $x^2 + y^2 = 4$ and the plane $x + y + z = 3$ $\left[\text{Ans: } \frac{1}{3\pi} (9\pi - 4) \right]$

2. Find the volume in the positive octant bounded by the

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plane $x + 2y + 3z = 4$ and the coordinate planes. $\left[\text{Ans: } \frac{16}{9} \right]$

3. Evaluate $\int \int \int_V dx dy dz$, where V is the volume enclosed by the cylinder $x^2 + y^2 = 1$ and the planes $z = 0, z = 2 - x$ $\left[\text{Ans: } 2\pi - \frac{4}{3} \right]$