UNIT III
LAPLACE TRANSFORMS

Department of Mathematics,
SRMIST, Kattankulathur.

INTRODUCTION

- Basic Definitions
- Transforms of Simple Functions
- Basic Operational properties
- Transforms of Derivatives and Integrals
- Initial and Final Value Theorems
- Laplace Transform of Periodic Functions
- Inverse Transforms
- Convolution Theorem
- Applications of Laplace Transforms for solving First and Second Order Linear Ordinary Differential Equation

LAPLACE TRANSFORM

Transformation: An operation which converts a mathematical expression to a different but equivalent form.

E**XAMPLE** : $\int e^x dx = e^x + c$

Laplace Transform: A Function f(t) be continuous and defined for all positive values of t. The Laplace Transform of f(t) associates a function S defined by the equation

$$F(S) = L[f(t)] = \int_0^\infty e^{-st} f(t)\,dt, t > 0$$

SUFFICIENT CONDITION FOR THE EXISTENCE OF LAPLACE TRANSFORM

1) f(t) should be either piecewise continuous or continuous function in closed interval [a,b].

2) Function should possess exponential order.

Piecewise continuous function: A function f(t) is said to be piecewise continuous in the closed interval if it is defined in that interval and in such a way that interval is divided into a finite number of subintervals in each of which f(t) is continuous.

Example:

$$f(x) = \begin{cases} x^2 + 4x + 3, & x < -3 \\ x + 3, & -3 \le x < 1 \\ -2, & x = 1 \end{cases}$$

Exponential Order: A function f(t) is said to be of exponential order if

$$\underset{t\to\infty}{lt}\, e^{-st} f(t) = a \text{ finite quantity}$$

$$Example: t^2,\ x^n$$

$$\underset{t\to\infty}{lt}\, e^{-st} t^2 = \underset{t\to\infty}{lt}\left[\frac{t^2}{e^{st}}\right] = \left[\frac{\infty}{\infty}\right] (L\ HOSPITAL'S\ Rule)$$

$$= \underset{t\to\infty}{lt}\left[\frac{2t}{se^{st}}\right] = \left[\frac{\infty}{\infty}\right] (L\ HOSPITAL'S\ Rule)$$

$$= \underset{t\to\infty}{lt}\left[\frac{2}{s^2 e^{st}}\right] = \left[\frac{2}{\infty}\right] = 0$$

APPLICATIONS:

1) Laplace Transform is used to solve linear DE,ODE as well as partial.

2) It is also used to solve boundary value problems without finding general solution but need to find the values of arbitrary constants.

IMPORTANT RESULTS

1) $L\left[e^{at}\right]=\dfrac{1}{s-a}, s>a.$

Proof :

By definition, $L[f(t)]=\int_0^\infty e^{-st} f(t)dt,\ t>0$

$$L\left[e^{at}\right]=\int_0^\infty e^{-st}e^{at}dt$$

$$=\int_0^\infty e^{-(s-a)t}dt$$

$$=\left[\frac{e^{-(s-a)t}}{s-a}\right]_0^\infty$$

$$=\left[\frac{1}{s-a}\right]$$

$$\text{Prove that } L\left[e^{-at}\right] = \left[\frac{1}{s+a}\right], s > -a$$

$$\text{Proof :}$$

$$By\,definition,\ L[f(t)] = \int_0^\infty e^{-st} f(t)dt,\ t > 0$$

$$L\left[e^{-at}\right] = \int_0^\infty e^{-st} e^{-at} dt$$

$$= \int_0^\infty e^{-(s+a)t} dt$$

$$= \left[\frac{e^{-(s+a)t}}{s+a}\right]_0^\infty$$

$$= \left[\frac{1}{s+a}\right]$$

$$\text{Prove that } L[\cos at] = \frac{s}{s^2 + a^2}$$

Proof :

$$\text{By definition, } L[f(t)] = \int_0^\infty e^{-st} f(t)\,dt,\ t > 0$$

$$L[\cos at] = \int_0^\infty e^{-st} \cos at\,dt$$

$$= \left[\frac{e^{-st}}{s^2 + a^2} \{-s\cos at + a\sin at\} \right]_0^\infty$$

$$= \left[-\frac{1}{s^2 + a^2} \{-s + 0\} \right]$$

$$= \frac{s}{s^2 + a^2}$$

Prove that $L\left[\sin at\right] = \dfrac{a}{s^2 + a^2}$

Proof:

By definition, $L[f(t)] = \int_0^\infty e^{-st} f(t)dt, t > 0$

we know that $e^{iat} = \cos at + i\sin at$ and $\sin at$ = imaginary part of e^{iat}

$$L[\sin at] = \int_0^\infty e^{-st} \sin at dt$$

$$= \text{imaginary part of} \int_0^\infty e^{-st} e^{iat} dt$$

$$= \text{imaginary part of } \mathrm{L}\left[e^{iat}\right]$$

$$= \text{imaginary part of } \left(\frac{1}{s - ia}\right)$$

$$= \text{imaginary part of } \left\{\frac{s + ia}{s^2 + a^2}\right\} \text{ (by taking conjuagate)}$$

$$\mathrm{L[sinat]} = \frac{a}{s^2 + a^2}$$

Prove that $L[\cosh at] = \dfrac{s}{s^2 - a^2}$

Proof:

We know that $\cosh at = \dfrac{e^{at} + e^{-at}}{2}$

$$L[\cosh at] = L\left[\frac{e^{at} + e^{-at}}{2}\right]$$

$$= \frac{1}{2}\left\{L[e^{at}] + L[e^{-at}]\right\}$$

$$= \frac{1}{2}\left\{\frac{1}{s-a} + \frac{1}{s+a}\right\}$$

$$= \frac{1}{2}\left\{\frac{2s}{s^2 - a^2}\right\}$$

$$\mathrm{L}[\cosh at] = \frac{s}{s^2 - a^2}$$

Similarily, $\mathrm{L}[\mathrm{sinhat}] = \dfrac{s}{s^2 + a^2}$

Prove that $L[1] = \frac{1}{s}, s > 0$

Proof:

$$L[f(t)] = \int_0^\infty e^{-st} f(t)dt, t > 0$$

$$L[1] = \int_0^\infty e^{-st} e^{ot} dt$$

$$= \int_0^\infty e^{-st} dt$$

$$= \left[\frac{e^{-st}}{-s}\right]_0^\infty$$

$$= \left[0 + \frac{1}{s}\right]$$

$$L[1] = \left[\frac{1}{s}\right]$$

LAPLACE TRANSFORM OF DERIVATIVES

$$L[f(t)] = sL[f(t)] - f(0)$$

Proof:

By definiton, $L[f(t)] = \int_0^\infty e^{-st} f(t)dt$

$$L[f'(t)] = \int_0^\infty e^{-st} f'(t)dt$$

$$= \int_0^\infty e^{-st} d[f(t)]$$

$$= \left[e^{-st} f(t)\right]_0^\infty - \int_0^\infty f(t)[-e^{-st} sdt\ ]$$

$$= -f(0) + s\int_0^\infty e^{-st} f(t)dt$$

$$L[f'(t)] = sL[f(t)] - f(0)$$

similarily, $L[f''(t)] = s^2 L[f(t)] - sf(0) - f'(0)$

$$L[f'''(t)] = s^3 L[f(t)] - s^2 f(0) - sf'(0) - f''(0)$$

LINEARITY PROPERTY

If c_1 and c_2 are constants and $f_1(t)$ and $f_2(t)$ are given functions then

$$[c_1 f_1(t)+c_2 f_2(t)]=c_1 L[f_1(t)]+c_2 L[f_2(t)].$$

Proof:

$$L[f(t)]=\int_0^\infty e^{-st} f(t)dt$$

$$L[c_1 f_1(t)+c_2 f_2(t)]=\int_0^\infty e^{-st}[c_1 f_1(t)+c_2 f_2(t)]dt$$

$$=\int_0^\infty e^{-st} c_1 f_1(t)dt+\int_0^\infty e^{-st} c_2 f_2(t)dt$$

$$=c_1\int_0^\infty e^{-st} f_1(t)dt+c_2\int_0^\infty e^{-st} f_2(t)dt$$

$$L[c_1 f_1(t)+c_2 f_2(t)] = c_1 L[f_1(t)]+c_2 L[f_2(t)]$$

Problems

1)Find $L[e^{2t}+3e^{-5t}]$

Solution:

$$L[e^{2t}+3e^{-5t}]=L[e^{2t}]+L[3e^{-5t}]$$

$$=\frac{1}{s-2}+3L\left[e^{-5t}\right]$$

$$=\frac{1}{s-2}+\frac{3}{s+5}$$

2) Find $L[\sinh 6t+3e^{-5t}+\cos 5t]$

solution:

$$L[\sinh 6t+3e^{-5t}+\cos 5t]=L[\sinh 6t]+L[3e^{-5t}]+L[\cos 5t]$$

$$=\frac{6}{s^2-6^2}+\frac{3}{s+5}+\frac{s}{s^2+5^2}$$

Find $L[\sin^3 2t]$

Solution:

we know that $\sin^3 A = \dfrac{3\sin A - \sin 3A}{4}$

$$L[\sin^3 2t] = L\left[\frac{3\sin 2t - \sin 6t}{4}\right]$$

$$= \frac{3}{4}L[\sin 2t] - \frac{1}{4}L[\sin 6t]$$

$$= \frac{3}{4} * \frac{2}{s^2+4} - \frac{1}{4} * \frac{6}{s^2+36}$$

$$= \frac{3}{2}\left[\frac{1}{s^2+4} - \frac{1}{s^2+36}\right]$$

Find $L[\sin(\omega t+\alpha)], \alpha$ -constant

Solution:

we know that $\sin(\omega t+\alpha) = \sin\omega t\cos\alpha + \cos\omega t\sin\alpha$

$$L[\sin(\omega t+\alpha)] = L[\sin\omega t\cos\alpha + \cos\omega t\sin\alpha]$$

$$= L[\sin\omega t\cos\alpha] + L[\cos\omega t\sin\alpha]$$

$$= \cos\alpha\left[\frac{\omega}{s^2+\omega^2}\right] + \sin\alpha\left[\frac{s}{s^2+\omega^2}\right]$$

$$L[\sin(\omega t+\alpha)] = \frac{1}{s^2+\omega^2}\,[\omega\cos\alpha + s\sin\alpha]$$

$$Find\ \ L[t^n] = \frac{\Gamma(n+1)}{s^{n+1}}$$

Proof:

$$L[f(t)] = \int_0^\infty e^{-st} f(t)\,dt$$

$$L[t^n] = \int_0^\infty e^{-st} t^n\,dt$$

$$Let\ \ st = x \Rightarrow t = \frac{x}{s}$$

$$dt = \frac{dx}{s}$$

When $t = 0 \Rightarrow x = 0$

When $t = \infty \Rightarrow x = \infty$

$$\int_0^\infty e^{-x}\left(\frac{x}{s}\right)^n \frac{dx}{s} = \int_0^\infty e^{-x}\left(\frac{x^n}{s^n}\right)\frac{dx}{s}$$

$$= \frac{1}{s^{n+1}}\int_0^\infty e^{-x} x^n\,dx$$

$$= \frac{1}{s^{n+1}}\Gamma(n+1)$$

Find L[sin$\sqrt{t}$]

solution:

$$\sin\sqrt{t} = \frac{\sqrt{t}}{1!} - \frac{\left(\sqrt{t}\right)^3}{3!} + \frac{\left(\sqrt{t}\right)^5}{5!} - ....$$

$$L\left[\sin\sqrt{t}\right] = L\left[t^{\frac{1}{2}}\right] - \frac{1}{3!}L\left[t^{\frac{3}{2}}\right] + \frac{1}{5!}L\left[t^{\frac{5}{2}}\right] - ...$$

$$= \frac{\Gamma\left(\frac{1}{2}+1\right)}{s^{\frac{1}{2}+1}} - \frac{1}{3!}\frac{\Gamma\left(\frac{3}{2}+1\right)}{s^{\frac{3}{2}+1}} + \frac{1}{5!}\frac{\Gamma\left(\frac{5}{2}+1\right)}{s^{\frac{5}{2}+1}} - ...$$

$$= \frac{\Gamma\left(\frac{3}{2}\right)}{s^{\frac{3}{2}}} - \frac{1}{3!}\left(\frac{\frac{3}{2}*\frac{1}{2}\Gamma\left(\frac{1}{2}\right)}{s^{\frac{5}{2}}}\right) + \frac{1}{5!}\left(\frac{\frac{5}{2}*\frac{3}{2}*\frac{1}{2}\Gamma\left(\frac{1}{2}\right)}{s^{\frac{7}{2}}}\right) - ....$$

$$= \frac{\frac{1}{2}\Gamma\left(\frac{1}{2}\right)}{s^{\frac{3}{2}}} - \frac{1}{3!}\left(\frac{\frac{3}{2}*\frac{1}{2}\Gamma\left(\frac{1}{2}\right)}{s^{\frac{5}{2}}}\right) + \frac{1}{5!}\left(\frac{\frac{5}{2}*\frac{3}{2}*\frac{1}{2}\Gamma\left(\frac{1}{2}\right)}{s^{\frac{7}{2}}}\right) - .....$$

$$= \frac{\frac{1}{2}\Gamma\left(\frac{1}{2}\right)}{s^{\frac{3}{2}}}\left[1 - \frac{1}{3!}\left(\frac{3/2}{s}\right) + \frac{1}{5!}\left(\frac{15/4}{s^2}\right) - ....\right] = \frac{\sqrt{\pi}}{2s^{3/2}}\left[1 - \frac{1}{1!}\left(\frac{1}{4s}\right) + \frac{1}{2!}\left(\frac{1}{4s}\right)^2 - .....\right]$$

$$= \frac{\sqrt{\pi}}{2s^{3/2}}\left[e^{\frac{-1}{4s}}\right]$$

Find L[1]

Solution :

$$L\left[t^{n}\right]=\frac{n!}{s^{n+1}}-----------(1)$$

$$L[1]=L[t^{0}]=\frac{0!}{s^{0+1}}=\frac{1}{s}$$

Note:

Put n=1 in eqn.(1), $L\left[t^{1}\right]=\frac{1!}{s^{1+1}}=\frac{1}{s^{2}}$

Put n=2 in eqn.(1), $L\left[t^{2}\right]=\frac{2!}{s^{2+1}}=\frac{2}{s^{3}}$

Find $L\left[\sqrt{t}\right]$

Solution :

$$L[t^n] = \frac{\Gamma(n+1)}{s^{n+1}}$$

Put $n = {}^{1}\!/_{2}$

$$L[t^{1/2}] = \frac{\Gamma\left(1/2 + 1\right)}{s^{1/2+1}}$$

$$= \frac{1}{s^{3/2}}\Gamma\left(\frac{3}{2}\right)$$

$$= \frac{1}{s^{3/2}}\frac{1}{2}\Gamma\left(\frac{1}{2}\right)$$

$$= \frac{\sqrt{\pi}}{2s^{3/2}}$$

FIRST SHIFTING THEOREM

If L[f(t)]=F(s),then $L[e^{at} f(t)] = F(s-a)$

Pr*oof* :

$$L[f(t)] = \int_0^\infty e^{-st} f(t)dt = F(s)$$

$$L[e^{at} f(t)] = \int_0^\infty e^{-st} e^{at} f(t)dt$$

$$= \int_0^\infty e^{-(s-a)t} f(t)dt$$

$$L[e^{at} f(t)] = F(s-a)$$

similarily, $L[e^{-at} f(t)] = F(s+a)$

UNIT STEP FUNCTION (OR) HEAVISIDE FUNCTION

The function is denoted by H(t) and is defined as

$$H(t)=\begin{cases} 1, t \geq 0 \\ 0, t < 0 \end{cases} \quad \text{and also } H(t-a)=\begin{cases} 1, \text{if } t>a \\ 0, \text{if } t \leq a \end{cases} \quad \text{where } a>0$$

SECOND SHIFTING THEOREM (OR) SECOND TRANSLATION

If $L[f(t)]=F(s)$ and $G(t)=\begin{cases} f(t-a), t>a \\ 0, t<a \end{cases}$ then $L[G(t)]=e^{-as}F(s)$

Proof:

$$L[f(t)]=\int_0^\infty e^{-st} f(t)dt$$

$$L[G(t)]=\int_0^\infty e^{-st}G(t)dt$$

$$=\int_0^a e^{-st}G(t)dt + \int_a^\infty e^{-st}G(t)dt$$

$$=\int_a^\infty e^{-st} f(t-a)dt$$

put t-a=U

t=U+a $\Rightarrow$ dt=du

when t=a,U=0

when t=∞,U=∞

$$L[G(t)]=L[G(U+a)]$$

$$=\int_0^\infty e^{-s(u+a)} f(u)du$$

$$=e^{-as}\int_0^\infty e^{-su} f(u)du$$

$$=e^{-as}\int_0^\infty e^{-st} f(t)dt$$

$$=e^{-as}F(s)$$

Find $\mathrm{L}\left[e^{-3t}\sin^2 t\right]$

solution:

$$\mathrm{L}\left[\sin^2 t\right]=\mathrm{L}\left[\frac{1-\cos 2t}{2}\right]$$

$$=\frac{1}{2}\left\{L[1]-L[\cos 2t]\right\}$$

$$\mathrm{L}\left[\sin^2 t\right]=\frac{1}{2}\left(\frac{1}{s}-\frac{s}{s^2+4}\right)$$

By first shifting theorem, s $\rightarrow$ s+3

$$\mathrm{L}\left[e^{-3t}\sin^2 t\right]=\frac{1}{2}\left[\frac{1}{s+3}-\frac{s+3}{(s+3)^2+4}\right]$$

Find $\mathrm{L}\left[\cosh t \sin 2t\right]$

solution:

$$\mathrm{L}\left[\cosh t \sin 2t\right] = L\left[\left(\frac{e^{t}+e^{-t}}{2}\right)\sin 2t\right]$$

$$= \frac{1}{2}\mathrm{L}\left[e^{t}\sin 2t\right] + \frac{1}{2}\mathrm{L}\left[e^{-t}\sin 2t\right]$$

we know that $\mathrm{L}\left[\sin 2t\right] = \dfrac{2}{s^{2}+4}$

$$\mathrm{L}\left[\cosh t \sin 2t\right] = \frac{1}{2}\mathrm{L}\left[e^{t}\sin 2t\right] + \frac{1}{2}\mathrm{L}\left[e^{-t}\sin 2t\right]$$

$$= \frac{1}{2}\left[\frac{2}{(s-1)^{2}+4} + \frac{2}{(s+1)^{2}+4}\right] \text{(By First Shifting Theorem)}$$

$$= \left[\frac{1}{(s-1)^{2}+4} + \frac{1}{(s+1)^{2}+4}\right]$$

Find $\mathcal{L}\left[t^2 e^{-2t}\right]$

solution:

$$\mathcal{L}\left[t^2\right] = \frac{2}{s^3}$$

$$\mathcal{L}\left[t^2 e^{-2t}\right] = \mathcal{L}\left[e^{-2t} * \frac{2}{s^3}\right]$$

$$\mathcal{L}\left[t^2 e^{-2t}\right] = \frac{2}{(s+2)^3}$$

Find $\mathcal{L}\left[e^{-t}\left(3\sinh 2t - 5\cosh 2t\right)\right]$

solution:

$$\mathcal{L}[3\sinh 2t] = 3\left[\frac{2}{s^2-4}\right]$$

$$\mathcal{L}[5\cosh 2t] = 5\left[\frac{s}{s^2-4}\right]$$

$$\mathcal{L}\left[e^{-t}\left(3\sinh 2t - 5\cosh 2t\right)\right] = \left[\frac{6}{(s+1)^2-4} - \frac{5(s+1)}{(s+1)^2-4}\right]$$

Prove that $L\left[H(t)\right]=\dfrac{2\left(1-e^{-\pi s}\right)}{s^2+4}$

where H(t)= $\begin{cases}\sin 2t, 0<t<\pi \\ 0, t>\pi\end{cases}$

Proof:

L[f(t)]= $\int_0^\infty e^{-st} f(t)dt$

L[H(t)]= $\int_0^\infty e^{-st} H(t)dt$

$$=\int_0^\pi e^{-st} H(t)dt + \int_\pi^\infty e^{-st} H(t)dt$$

L[H(t)]= $\int_0^\pi e^{-st} H(t)dt$

$$=\frac{e^{-\pi s}}{s^2+4}\left[\left(-s\sin 2\pi-2\cos 2\pi\right)-\left(-s\sin 0-2\right)\right]$$

$$=\frac{1}{s^2+4}\left[e^{-\pi s}(-2)+2\right]$$

$$=\frac{2\left(1-e^{-\pi s}\right)}{s^2+4}$$

Find the laplace transform of G(t) where

$$G(t)=\begin{cases} \cos\left(t-\frac{2\pi}{3}\right), t>\frac{2\pi}{3} \\ 0, t<\frac{2\pi}{3} \end{cases}$$

Solution:

According to second shifting theorem

L[f(t)]=F(s) and $G\left(t\right)=\begin{cases} f(t-a), t>a \\ 0, t<a \end{cases}$

$$L[G(t)]=e^{-as}F(s)$$

$f(t)=\cos t$ and a=$\frac{2\pi}{3}$

$$L[f(t)]=L[\cos t]=\frac{s}{s^2+1}=F(s)$$

$$L[G(t)]=e^{\frac{-2\pi s}{3}}\left(\frac{s}{s^2+1}\right)$$

CHANGE OF SCALE OF PROPERTY:

If L[f(t)]=F(s),then L[f(at)]= $\frac{1}{a}F\left(\frac{s}{a}\right)$

Pr*oof* :

$$L[f(t)] = \int_0^\infty e^{-st} f(t)dt$$

$$L[f(at)] = \int_0^\infty e^{-st} f(at)dt$$

Let at=x; t= $\frac{x}{a} \Rightarrow$ dt= $\frac{dx}{a}$

when x=0;t=0

when x=∞;t=∞

$$L[f(at)] = L[f(x)] = \int_0^\infty e^{-s\left(\frac{x}{a}\right)} f(x)\frac{dx}{a}$$

$$= \frac{1}{a}\int_0^\infty e^{-\left(\frac{s}{a}\right)t} f(t)dt$$

$$L[f(at)] = \frac{1}{a}\ \mathrm{F}\left(\frac{s}{a}\right)$$

If $L[f(t)]=F(s)$, then $L[t\ f(t)]= -\dfrac{d}{ds}F(s)$

Proof:

$F(s)=L[f(t)]$

Differentiating

$$\frac{d}{ds}F(s)=\frac{d}{ds}\int_0^{\infty} e^{-st} f(t)dt$$

By Leibnitz rule if the limits are constants and integrating wrt differentiation then total differentiation is taken as partial differentiation

$$=\int_0^{\infty}\frac{\partial}{\partial s}\, e^{-st} f(t)dt$$

$$= \int_0^{\infty} e^{-st} (-t)f(t)dt$$

$$= -\int_0^{\infty} e^{-st} f(t)\ t\ dt$$

$$= -\ L[t\ f(t)]$$

$$L[t\ f(t)]= -\frac{d}{ds}F(s)$$

$similarily$, $L[t^2\ f(t)]= \dfrac{d^2}{ds^2}F(s)$

....

$$L[t^n\ f(t)]=(-1)^n \frac{d^n}{ds^n}F(s)$$

Find L[t sin2t]

solution:

f(t)=sin2t

L[f(t)]=sin2t= $\frac{2}{s^2+4} = F(s)$

We know that, L[t f(t)]= $-\frac{d}{ds}F(s)$

$$= -\frac{d}{ds}\left[\frac{2}{s^2+4}\right]$$

$$= -\left(-\frac{2s}{\left(s^2+4\right)^2}\right) = \frac{4s}{\left(s^2+4\right)^2}$$

Find $L\left[te^{-t}\cosh t\right]$

Solution:

$$L[\cosh t]=\frac{s}{s^2-1}$$

$$L[e^{-t}\cosh t]=\frac{s+1}{(s+1)^2-1}$$

$$L[te^{-t}\cosh t]= -\frac{d}{ds}\left[\frac{(s+1)}{(s+1)^2-1}\right]$$

$$= -\left\{\frac{\left[(s+1)^2-1\right]-\left[(s+1)2(s+1)\right]}{\left[(s+1)^2-1\right]^2}\right\}$$

$$= -\left\{\frac{-(s+1)^2-1}{\left[(s+1)^2-1\right]^2}\right\}$$

$$= \left\{\frac{1+(s+1)^2}{\left[(s+1)^2-1\right]^2}\right\}$$

If L[f(t)]=F(s) and if $\frac{f(t)}{t}$ has a limit $t \to 0$ then $L\left[\frac{f(t)}{t}\right] = \int_s^\infty F(s)ds$

Proof :

$$F(s)=L[f(t)]=\int_0^\infty e^{-st} f(t)dt$$

$$\int_s^\infty F(s)ds = \int_s^\infty \int_0^\infty e^{-st} f(t)dtds$$

$$= \int_s^\infty ds \int_0^\infty e^{-st} f(t)dt$$

$\because$ t and s are independent variables, we can change the order of integration

$$= \int_0^\infty dt \int_s^\infty e^{-st} f(t)ds$$

$$= \int_0^\infty f(t)dt \int_s^\infty e^{-st} ds$$

$$= \int_0^\infty f(t)dt \left[\frac{-e^{-st}}{-t}\right]_s^\infty$$

$$= \int_0^\infty f(t)dt \left[\frac{e^{-st}}{t}\right]$$

$$= \int_0^\infty \frac{f(t)}{t} \mathrm{e}^{-st} dt$$

$$= \int_0^\infty \mathrm{e}^{-st} \frac{f(t)}{t} dt$$

$$= L\left[\frac{f(t)}{t}\right]$$

Find $L\left[\frac{\sin at}{t}\right]$ and hence show that $\int_0^\infty \frac{\sin t}{t} dt = \frac{\pi}{2}$

solution:

$$L[\sin at] = \frac{a}{s^2 + a^2} = F(s)$$

$$L\left[\frac{\sin at}{t}\right] = \int_s^\infty \frac{a}{s^2 + a^2} ds$$

$$= a \int_s^\infty \frac{ds}{s^2 + a^2}$$

$$= a\left[\frac{1}{a} \tan^{-1}\left(\frac{s}{a}\right)\right]$$

$$= \tan^{-1}(\infty) - \tan^{-1}\left(\frac{s}{a}\right)$$

$$= \frac{\pi}{2} - \tan^{-1}\left(\frac{s}{a}\right)$$

$$= \cot^{-1}\left(\frac{s}{a}\right)$$

$$L\left[\frac{\sin at}{t}\right] = \tan^{-1}\left(\frac{a}{s}\right)$$

II PART:

$$L[f(t)] = \int_0^\infty e^{-st} f(t)\,dt$$

$$f(t) = \frac{\sin at}{t}$$

$$L\left[\frac{\sin at}{t}\right] = \int_0^\infty e^{-st} \frac{\sin at}{t} dt$$

By equation (1), $\tan^{-1}\left(\frac{a}{s}\right) = L\left[\frac{\sin at}{t}\right] = \int_0^\infty e^{-st} \frac{\sin at}{t} dt$

Let s = 0 and a = 1

$$\tan^{-1}\left(\frac{1}{0}\right) = L\left[\frac{\sin t}{t}\right] = \int_0^\infty \frac{\sin t}{t} dt$$

$$\frac{\pi}{2} = \int_0^\infty \frac{\sin t}{t} dt$$

Result :

If $\mathrm{L}\left[f(t)\right]=F(s)$, then $\mathrm{L}\left[\dfrac{f(t)}{t}\right]=\int_0^\infty F(u)\,du$

provided $\lim\limits_{t\to 0}\dfrac{f(t)}{t}$ exists

Find $L\left[\frac{1-e^{t}}{t}\right]$

Solution:

$$L\left[\frac{1-e^{t}}{t}\right] = \int_{s}^{\infty} L[1-e^{t}]\,ds$$

$$= \int_{s}^{\infty} \left\{L[1]-L[e^{t}]\right\} ds$$

$$= \int_{s}^{\infty} L[1]ds - \int_{s}^{\infty} L[e^{t}]ds$$

$$= \int_{s}^{\infty} \frac{ds}{s} - \int_{s}^{\infty} \frac{ds}{s-1}$$

$$= \left[\log(s)-\log(s-1)\right]_{s}^{\infty}$$

$$= \log\left(\frac{s}{s-1}\right)_{s}^{\infty}$$

$$= \log\left(\frac{1}{1-1/\infty}\right) - \log\left(\frac{1}{1-1/s}\right)$$

$$= 0 - \log\left(\frac{1}{1-1/s}\right) = \log\left(\frac{1}{1-1/s}\right)^{-1}$$

$$= \log\left(\frac{s}{s-1}\right)^{-1}$$

$$L\left[\frac{1-e^{t}}{t}\right] = \log\left(\frac{s-1}{s}\right)$$

Find $L\left[\frac{\sin^2 t}{t}\right]$

Solution:

$$L\left[\frac{\sin^2 t}{t}\right] = L\left[\frac{1-\cos 2t}{2t}\right]$$

$$= \frac{1}{2} L\left[\frac{1-\cos 2t}{t}\right]$$

$$= \frac{1}{2}\int_s^\infty L\left[1-\cos 2t\right]ds$$

$$= \frac{1}{2}\int_s^\infty \left\{L\left[1\right]- L\left[\cos 2t\right]\right\}ds$$

$$= \frac{1}{2}\int_s^\infty \left(\frac{1}{s}-\frac{s}{s^2+4}\right)ds$$

$$= \frac{1}{2}\int_s^\infty \frac{ds}{s} - \frac{1}{2}\int_s^\infty \frac{2sds}{s^2+4}$$

$$= \frac{1}{2}\left\{(\log s)_s^\infty - \frac{1}{2}\left[\log\left(s^2+4\right)\right]_s^\infty\right\}$$

$$= \frac{1}{2}\left\{(\log s)_s^\infty - \left[\log\sqrt{\left(s^2+4\right)}\right]_s^\infty\right\}$$

$$= \frac{1}{2}\log\left(\frac{s}{\sqrt{\left(s^2+4\right)}}\right)_s^\infty$$

$$L[\sin^2 t] = \frac{1}{2}\log\left(\frac{s}{s\sqrt{1+4/s^2}}\right)$$

$$= \frac{1}{2}\left\{\log 1 - \log\left(\frac{1}{\sqrt{1+4/s^2}}\right)\right\}$$

$$= \frac{1}{2}\left[0 - \log\frac{1}{\sqrt{\left(s^2+1/s^2\right)}}\right]$$

$$= \frac{1}{2}\log\left[\frac{s}{\sqrt{s^2+4}}\right]^{-1}$$

$$= \frac{1}{2}\log\left[\frac{\sqrt{s^2+4}}{s}\right]$$

Find $\mathrm{L}\left[\dfrac{\sin 3t\cos t}{t}\right]$

solution:

$$\sin 3t\cos t = \frac{\sin(3t+t)+\sin(3t-t)}{2}$$

$$\mathrm{L}\left[\frac{\sin 3t\cos t}{t}\right] = \frac{1}{2}\mathrm{L}\left[\frac{\sin 4t+\sin 2t}{t}\right]$$

$$= \frac{1}{2}\int_s^{\infty}\{\mathrm{L}[\sin 4t]+\mathrm{L}[\sin 2t]\}ds$$

$$= \frac{1}{2}\int_s^{\infty}\left\{\frac{4}{s^2+16}+\frac{2}{s^2+4}\right\}ds$$

$$= \frac{1}{2}\left\{4\int_s^{\infty}\frac{ds}{s^2+16}+2\int_s^{\infty}\frac{ds}{s^2+4}\right\}$$

$$= \frac{1}{2}\left\{4\left[\frac{1}{4}\tan^{-1}\left(\frac{s}{4}\right)\right]_s^{\infty}+2\left[\frac{1}{2}\tan^{-1}\left(\frac{s}{2}\right)\right]_s^{\infty}\right\}$$

$$\mathrm{L}\left[\frac{\sin 3t\cos t}{t}\right]=\frac{1}{2}\left[\tan^{-1}\left(\frac{s}{4}\right)+\tan^{-1}\left(\frac{s}{2}\right)\right]_s^{\infty}$$

$$=\frac{1}{2}\left[\tan^{-1}(\infty)+\tan^{-1}(\infty)-\tan^{-1}\left(\frac{s}{4}\right)-\tan^{-1}\left(\frac{s}{2}\right)\right]$$

$$=\frac{1}{2}\left[\frac{\pi}{2}+\frac{\pi}{2}-\tan^{-1}\left(\frac{s}{4}\right)-\tan^{-1}\left(\frac{s}{2}\right)\right]$$

$$\mathrm{L}\left[\frac{\sin 3t\cos t}{t}\right]=\frac{1}{2}\left[\pi-\tan^{-1}\left(\frac{s}{4}\right)-\tan^{-1}\left(\frac{s}{2}\right)\right]$$

RESULT:

$$\mathrm{L}\left[\int_0^t f(x)dx\right]=\frac{1}{s}\mathrm{L}\left[f(t)\right]$$

Find $\mathrm{L}\left[e^{-t}\int_0^t t\cos t\,dt \right]$

Solution:

$$\mathrm{L}\left[\int_0^t t\cos t\,dt \right] = \frac{1}{s}\mathrm{L}\left[\mathrm{tcost}\right]$$

$$= \frac{1}{s}\left(-\frac{d}{ds}\mathrm{L}\left[\mathrm{tcost}\right] \right)$$

$$= \frac{1}{s}\left[-\frac{d}{ds}\left(\frac{s}{s^2+1} \right) \right]$$

$$= -\frac{1}{s}\left[\frac{\left(s^2+1\right)-2s}{\left(s^2+1\right)^2} \right]$$

$$= \frac{\left(s^2-1\right)}{s\left(s^2+1\right)^2}$$

$$\mathrm{L}\left[e^{-t}\int_0^t t\cos t\,dt \right] = \frac{\left(\left(s+1\right)^2-1\right)}{\left(s+1\right)\left(\left(s+1\right)^2+1\right)^2}$$

Find $L\left[e^{-t}\int_0^t \frac{\sin t}{t}\,dt\right]$

Solution:

$$L\left[\int_0^t \frac{\sin t}{t}\,dt\right] = \frac{1}{s} L\left[\frac{\sin t}{t}\right]$$

$$= \frac{1}{s}\int_s^\infty L[\sin t]\,ds$$

$$= \frac{1}{s}\int_s^\infty \frac{1}{s^2+1}\,ds$$

$$= \frac{1}{s}\left[\tan^{-1}(s)_s^\infty\right]$$

$$= \frac{1}{s}\left[\tan^{-1}\infty - \tan^{-1}(s)\right]$$

$$= \frac{1}{s}\left[\frac{\pi}{2} - \tan^{-1}(s)\right]$$

$$= \frac{1}{s}\cot^{-1}(s)$$

$$L\left[e^{-t}\int_0^t \frac{\sin t}{t}\,dt\right] = \frac{\cot^{-1}(s+1)}{(s+1)}$$

INITIAL VALUE THEOREM

If $L\left[f(t)\right] = F(s)$, then $\lim_{t\to 0} f(t) = \lim_{s\to\infty} sF(s)$

Proof:

$$L\left[f'(t)\right] = sL\left[f(t)\right] - f(0)$$

$$= sF(s) - f(0)$$

$$sF(s) - f(0) = L\left[f'(t)\right]$$

$$sF(s) - f(0) = \int_0^\infty e^{-st} f'(t)\,dt$$

$$\lim_{s\to\infty}\left[sF(s) - f(0)\right] = \lim_{s\to\infty}\int_0^\infty e^{-st} f'(t)\,dt$$

$$\lim_{s\to\infty}\left[sF(s) - f(0)\right] = 0$$

$$\lim_{s\to\infty}\left[sF(s)\right] = f(0)$$

$$\lim_{s\to\infty}\left[sF(s)\right] = \lim_{t\to 0} f(t)$$

Verify Initial value theorem for f(t)=ae^{-bt}

Proof:

We know that, $\lim_{t\to 0} f(t) = \lim_{s\to\infty} sF(s)$

LHS:

$$\lim_{t\to 0} f(t) = \lim_{t\to 0} \mathrm{a}e^{-bt} = a ------------ (1)$$

$$\mathrm{L}\left[f(t)\right] = L\left[\mathrm{a}e^{-bt}\right] = \mathrm{a}L\left[e^{-bt}\right]$$

$$= \mathrm{a}\left(\frac{1}{s+b}\right) = F(s)$$

$$\lim_{s\to\infty} sF(s) = \lim_{s\to\infty}\left[s\left(\frac{a}{s+b}\right)\right]$$

$$= \lim_{s\to\infty}\left[s\left(\frac{a}{s\left(1+b/s\right)}\right)\right]$$

$$= \mathrm{a} -- (2)$$

From (1) and (2)

$$\lim_{t\to 0} f(t) = \lim_{s\to\infty} sF(s)$$

Hence Initial Value theorem is verified

FINAL VALUE THEOREM

If $L\left[f(t)\right] = F(s)$, then $\lim_{t\to\infty} f(t) = \lim_{s\to 0} sF(s)$

Proof:

$$L\left[f'(t)\right] = sL\left[f(t)\right] - f(0)$$

$$= sF(s) - f(0)$$

$$sF(s) - f(0) = L\left[f'(t)\right]$$

$$sF(s) - f(0) = \int_0^\infty e^{-st} f'(t)\,dt$$

$$\lim_{s\to 0}\left[sF(s) - f(0)\right] = \lim_{s\to 0}\int_0^\infty e^{-st} f'(t)\,dt$$

$$\lim_{s\to 0}\left[sF(s) - f(0)\right] = \int_0^\infty f'(t)\,dt$$

$$\lim_{s\to\infty}\left[sF(s)\right] = \int_0^\infty d\left[f(t)\right]$$

$$= \left[f(t)\right]_0^\infty$$

$$\lim_{s\to 0}\left[sF(s)\right] - f(0) = f(\infty) - f(0)$$

$$\lim_{s\to 0}\left[sF(s)\right] = f(\infty) = \lim_{t\to\infty}\left[f(t)\right]$$

Verify Final value theorem,

$$f(t)=1+e^{-t}\left[\sin t+\cos t\right]$$

Proof :

We Know that $\lim\limits_{t\to\infty} f(t)=\lim\limits_{s\to 0} sF(s)$

$$\lim_{t\to\infty} f(t)=\lim_{t\to\infty}\left[1+e^{-t}\left[\sin t+\cos t\right]\right]$$

$$\lim_{t\to\infty} f(t) = 1\text{---}(1)$$

$$L[f(t)]=L\left[1+e^{-t}\left[\sin t+\cos t\right]\right]$$

$$= L[1]+L\left[e^{-t}\sin t\right]+L\left[e^{-t}\cos t\right]$$

$$=\frac{1}{s}+\frac{1}{\left((s+1)^2+1\right)}+\frac{(s+1)}{\left((s+1)^2+1\right)}$$

$$\mathrm{sF(s)=s}\left[\frac{1}{s}+\frac{1}{\left((s+1)^2+1\right)}+\frac{(s+1)}{\left((s+1)^2+1\right)}\right]$$

$$\lim_{s\to 0}\mathrm{sF}(\mathrm{s})=\lim_{s\to 0}\left[1+\frac{s}{s^2+2s+2}+\frac{s(s+1)}{s^2+2s+2}\right]$$

$$=1+0+0$$

$$\lim_{s\to 0}\mathrm{sF}(\mathrm{s})=1--------------------(2)$$

From (1) and (2)

$$\lim_{\mathrm{t}\to\infty}\mathrm{f(t)}=\lim_{s\to 0}\mathrm{sF}(\mathrm{s})$$

Hence final value theorem is verified.

1) Find $L\left[2\cos 4t - 3\sin 4t\right]$

2) Find $L\left[t^2 \cos at\right]$

3) Find $L\left[\sin 2t \sin 3t\right]$

4) Evaluate $L\left[e^{-t}\left(3\sinh 2t - 5\cosh 2t\right)\right]$

5) Show that $\int_0^\infty te^{-3t} \sin t\,dt = \frac{3}{50}$

6) Evaluate $\int_0^\infty \frac{\cos 6t - \cos 4t}{t} dt$

7) Find the Laplace transforms of $\frac{\sin 3t \sin t}{t}$

8) Using the Laplace transform of the derivatives find $L\left[t \sinh at\right]$

9) Evaluate $L\left[\int_0^t te^{-t} dt\right]$

10) Evaluate $L\left[\int_0^t \frac{1-e^{-t}}{t} dt\right]$

11) Verify the Initial Value Theorem for t+sin3t

12) Verify the Final Value Theorem for $1+e^{-t}\left(\sin t + \cos t\right)$

PERIODIC FUNCTIONS

A Function f(t) is said to have a period T if for all t,

$f(T + t) = f(t)$ where T is a positive constant . The least value of $T>0$ is

called period of f(t).

Eg: Consider $f(t) = \sin t$

$$f(t + 2\pi) = \sin t = f(t)$$

$$f(t + 4\pi) = \sin t = f(t)$$

…………………

…………………..

Therefore, Sin t is a periodic function with period 2π.

LAPLACE TRANSFORM OF PERIODIC FUNCTIONS

If f (t) is a piecewise continuous periodic functions

with period T then

$$L[f(t)] = \frac{1}{1-e^{-Ts}} \int_0^t e^{-sT} f(t) dt$$

By definition of Laplace Transform

$$L\{f(t)\} = \int_0^\infty e^{-st} f(t) dt$$

$$= \int_0^T e^{-st} f(t) dt + \int_T^{2T} e^{-st} f(t) dt + \int_{2T}^{3T} e^{-st} f(t) dt + \ldots\ldots\ldots\ldots\infty$$

$$put\ t = u + T\ in\ the\ \mathrm{sec}ond\ \mathrm{int}egral,$$

$$\Rightarrow dt = du$$

$$\therefore \int_{T}^{2T} e^{-st} f(t)dt = \int_{u=0}^{u=T} e^{-s(u+T)} f(u+T)du$$

$$= e^{-sT} \int_{0}^{T} e^{-su} f(u)du, (\because f(u+T) = f(u))$$

put $t = u + 2T$ *in the third* int*egral*

$$\Rightarrow u = t - 2T$$

$$\Rightarrow dt = du$$

$$\therefore \int_{2T}^{3T} e^{-st} f(t)dt = \int_{u=0}^{u=T} e^{-s(u+2T)} f(u+2T) f(u+2T)du$$

$$= e^{-2sT} \int_0^T e^{-su} f(u)du, \ (\because \ f(u+2T) = f(u))$$

$$\therefore L\{f(t)\} = \int_0^T e^{-su} f(u)du + e^{-sT} \int_0^T e^{-su} f(u)du +$$

$$e^{-2sT} \int_0^T e^{-su} f(u)du + \ldots\infty$$

$$L\{f(t)\} = (1 + e^{-sT} + e^{-2sT} + \ldots.\infty)\int_0^T e^{-su} f(u)du$$

$$= (1 - e^{-sT})^{-1}\int_0^T e^{-su} f(u)du, \quad (\because (1-x)^{-1} = 1 + x + x^2 + \ldots)$$

$$= \frac{1}{1-e^{-sT}}\int_0^T e^{-su} f(u)du$$

$$L\{f(t)\} = \frac{1}{1-e^{-sT}}\int_0^T e^{-sT} f(t)dt$$

PROBLEMS

1) *Find the Laplace Transform of the rectangular wave for the given function,*

$$f(t) = \begin{cases} 1, & 0 < t < b \\ -1, & b < t < 2b \end{cases}$$

Solution:

WKT $L\{f(t)\} = \dfrac{1}{1-e^{-sT}} \int_0^T e^{-st} f(t)\,dt, \quad 0 < t < T$

Here $T = 2b$

$$L\{f(t)\} = \frac{1}{1-e^{-2bs}} \int_0^{2b} e^{-st} f(t)\,dt$$

$$= \frac{1}{1-e^{-2bs}} \left\{ \int_0^{b} e^{-st}(1)\,dt + \int_b^{2b} e^{-st}(-1)\,dt \right\}$$

$$= \frac{1}{1-e^{-2bs}} \left\{ \left(\frac{e^{-st}}{-s} \right)_0^b - \left(\frac{-e^{-st}}{s} \right)_b^{2b} \right\}$$

$$L\{f(t)\} = \frac{1}{1-e^{-2bs}}\left[\frac{e^{-bs}}{-s} + \frac{1}{s} + \frac{e^{-2bs}}{s} - \frac{e^{-bs}}{s}\right]$$

$$= \frac{1}{1-e^{-2bs}}\,\frac{1}{s}\left[e^{-2bs} - 2e^{-bs} + 1\right]$$

$$L\{f(t)\} = \frac{1}{1-e^{-2bs}}\,\frac{1}{s}\left(1-e^{-bs}\right)^{2}$$

$$L\{f(t)\} = \frac{1}{\left(1+e^{-bs}\right)\left(1-e^{-bs}\right)} \frac{1}{s}\left(1-e^{-bs}\right)^2$$

$$= \frac{\left(1-e^{-bs}\right)}{s\left(1+e^{-bs}\right)}$$

$$L\{f(t)\} = \frac{1}{s}\left(\frac{e^{-\frac{bs}{2}} - e^{-\frac{bs}{2}}}{e^{-\frac{bs}{2}} + e^{-\frac{bs}{2}}}\right) = \frac{1}{s} \tan h \frac{bs}{2}$$

PROBLEM 2 :

Find the laplace transform of the function

$$f(t) = \begin{cases} t, & 0 < t < \pi \\ 2\pi - t, & \pi < t < 2\pi \end{cases}$$

$$\text{where } f(t+2\pi) = f(t)$$

SOLUTION:

$$\text{WKT } L\{f(t)\} = \frac{1}{1-e^{-sT}} \int_0^T e^{-sT} f(t)\,dt, \qquad 0 < t < T$$

$$\text{where } T = 2\pi$$

$$L\{f(t)\}=\frac{1}{1-e^{-2\pi s}}\left\{\int_0^{\pi} te^{-st}dt+\int_{\pi}^{2\pi}(2\pi-t)e^{-st}dt\right\}$$

$$=\frac{1}{1-e^{-2\pi s}}\left\{\left[(t)\left(\frac{e^{-st}}{-s}\right)-(1)\left(\frac{e^{-st}}{s^2}\right)\right]_0^{\pi}+\left[(2\pi-t)\left(\frac{e^{-st}}{-s}\right)-(-1)\left(\frac{e^{-st}}{s^2}\right)\right]_0^{2\pi}\right\}$$

$$=\frac{1}{1-e^{-2\pi s}}\left[\frac{-\pi e^{-s\pi}}{s}-\frac{e^{-s\pi}}{s^2}+\frac{1}{s^2}+\frac{\pi}{s}e^{-s\pi}+\frac{e^{-2s\pi}}{s^2}-\frac{e^{-s\pi}}{s^2}\right]$$

$$=\frac{1}{1-e^{-2s\pi}}\left[\frac{1}{s^2}\left(1-2e^{-s\pi}+e^{-2s\pi}\right)\right]$$

$$= \frac{\left(1-e^{-s\pi}\right)^2}{s^2\left(1-e^{-2s\pi}\right)} = \frac{\left(1-e^{-s\pi}\right)^2}{s^2\left(1-e^{-s\pi}\right)\left(1+e^{-s\pi}\right)}$$

$$L\{f(t)\} = \frac{\left(1-e^{-s\pi}\right)}{s^2\left(1+e^{-s\pi}\right)}$$

$$= \frac{1}{s^2}\left(\frac{e^{-\frac{s\pi}{2}} - e^{-\frac{s\pi}{2}}}{e^{-\frac{s\pi}{2}} + e^{-\frac{s\pi}{2}}}\right)$$

$$L\{f(t)\} = \frac{1}{s^2}\tanh\left(\frac{\pi s}{2}\right)$$

PROBLEM 3:

Find the Laplace Transform of the periodic function given by

$$f(t) = \begin{cases} \sin \omega t, \; 0 < t < \dfrac{\pi}{\omega} \\[2ex] 0, \qquad \dfrac{\pi}{\omega} < t < \dfrac{2\pi}{\omega} \end{cases}$$

and its period is $\dfrac{2\pi}{\omega}$.

SOLUTION:

$$WKT \; \mathbf{L}\{\mathbf{f}(\mathbf{t})\} = \frac{\mathbf{1}}{\mathbf{1}-\mathbf{e}^{-\mathbf{sT}}} \int_0^T \mathbf{e}^{-\mathbf{st}} \mathbf{f}(\mathbf{t})\mathbf{dt}, \qquad \mathbf{0} < \mathbf{t} < \mathbf{T}$$

$$where \; T = \frac{2\pi}{\omega}$$

$$L\{f(t)\}=\frac{1}{1-e^{-s2\pi/\omega}}\int_0^{\frac{2\pi}{\omega}} e^{-st} f(t)dt$$

$$=\frac{1}{1-e^{-s2\pi/\omega}}\left\{\int_0^{\frac{\pi}{\omega}} e^{-st}\sin\omega t+\int_{\pi/\omega}^{2\pi/\omega} e^{-st}\,0\;dt\right\}$$

$$=\frac{1}{1-e^{-2s\pi/\omega}}\left[\frac{e^{-st}}{s^2+\omega^2}\left(-s\sin\omega t-\omega\cos\omega t\right)\right]_0^{\pi/\omega}$$

$$L\{f(t)\} = \frac{1}{1-e^{-2s\pi/\omega}}\left[\frac{e^{-s\pi/\omega}}{s^2+\omega^2}(-\omega\cos\pi) - \frac{1}{s^2+\omega^2}(-\omega\cos 0)\right]$$

$$= \frac{1}{1-e^{-2s\pi/\omega}} \cdot \frac{\omega}{s^2+\omega^2}\left(1+e^{-s\pi/\omega}\right)$$

$$L\{f(t)\} = \frac{\omega}{s^2+\omega^2}\,\frac{1}{\left(1-e^{-s\pi/\omega}\right)}$$

TASK :

1) *Find the Laplace Transform of the function*

$$f(t) = \begin{cases} t & for \quad 0 < t < \pi \\ \pi - t & for \quad \pi < t < 2\pi \end{cases}$$

2) *Find the Laplace Transform of the function*

$$f(t) = \begin{cases} t - 1; & 1 < t < 2 \\ 0; & Otherwise \end{cases}$$

3) *Find the Laplace Transform of the function*

$$f(t) = \frac{2t}{3}; 0 \le t \le 3$$

INVERSE LAPLACE TRANSFORM

If the Laplace Transform of the function $f(t)$ *is* $F(s)$ $(i.e).,$ $L\{f(t)\} = F(s)$ *then* $f(t)$ *is called Inverse Laplace Transform and is denoted by* $L^{-1}\{F(s)\}.$

IMPORTANT RESULTS

1) $L^{-1}\left[\dfrac{1}{s-a}\right] = e^{at}$

2) $L^{-1}\left[\dfrac{1}{s+a}\right] = e^{-at}$

3) $L^{-1}\left[\dfrac{s}{s^2+a^2}\right] = \cos at$

4) $L^{-1}\left[\dfrac{a}{s^2+a^2}\right] = \sin at$

$$5)\ L^{-1}\left[\frac{s}{s^2-a^2}\right]=\cosh at$$

$$6)\ L^{-1}\left[\frac{a}{s^2+a^2}\right]=\sinh at$$

$$7)\ L^{-1}\left[\frac{1}{s}\right]=1$$

$$8)\ L^{-1}\left[\frac{1}{s^2}\right]=t$$

$$9)\ L^{-1}\left[\frac{1}{\left(s^2-a^2\right)}\right]=te^{at}$$

$$10)\ L^{-1}\left[\frac{n!}{s^{n+1}}\right]=t^n$$

LINEARITY PROPERTY

$If\ F_1(s)\ and\ F_2(s)\ are\ Laplace\ Transform\ of\ f_1(t)\ and\ f_2(t)$

$respectively,\ then$

$$\mathbf{L^{-1}\left[C_1F_1(s)+C_2F_2(s)\right]=C_1L^{-1}\left[F_1(s)\right]+C_2L^{-1}\left[F_2(s)\right]}$$

PROBLEM 1

$$Find\ L^{-1}\left[\frac{1}{s-3}+\frac{1}{s}+\frac{s}{s^2-4}\right]$$

Solution:

$$L^{-1}\left[\frac{1}{s-3}+\frac{1}{s}+\frac{s}{s^2-4}\right]=L^{-1}\left[\frac{1}{s-3}\right]+L^{-1}\left[\frac{1}{s}\right]+L^{-1}\left[\frac{s}{s^2-2^2}\right]$$

$$=\mathbf{e^{3t}+1+\cosh 2t}$$

PROBLEM 2

$$Find\ L^{-1}\left[\frac{1}{s^2}+\frac{1}{s+4}+\frac{1}{s^2+4}+\frac{s}{s^2-9}\right]$$

Solution:

$$L^{-1}\left[\frac{1}{s^2}+\frac{1}{s+4}+\frac{1}{s^2+4}+\frac{s}{s^2-9}\right]$$

$$=L^{-1}\left[\frac{1}{s^2}\right]+L^{-1}\left[\frac{1}{s+4}\right]+L^{-1}\left[\frac{1}{s^2+4}\right]+L^{-1}\left[\frac{s}{s^2-3^2}\right]$$

$$=L^{-1}\left[\frac{1}{s^2}\right]+L^{-1}\left[\frac{1}{s+4}\right]+\frac{1}{2}L^{-1}\left[\frac{2}{s^2+2^2}\right]+L^{-1}\left[\frac{s}{s^2-3^2}\right]$$

$$=\mathbf{t}+\mathbf{e}^{-4t}+\frac{1}{2}\mathbf{sin2t}+\mathbf{cosh3t}$$

FIRST SHIFTING PROPERTY:

$$If\ \ L\{f(t)\} = F(s),\ then\ L\left[e^{-at} f(t)\right] = F(s+a).$$

$$Hence\ L^{-1}[F(s+a)] = e^{-at} f(t)$$

$$\Rightarrow \mathbf{L^{-1}[F(s+a)] = e^{-at} L^{-1}[F(s)]}$$

PROBLEM 1

$$Find\ \ L^{-1}\left[\frac{1}{(s+1)^2}\right]$$

Solution:

$$L^{-1}\left[\frac{1}{(s+1)^2}\right] = e^{-t} L^{-1}\left[\frac{1}{s^2}\right] = \mathbf{e^{-t} t}$$

PROBLEM 2: $Find\ L^{-1}\left[\frac{(s-3)}{(s-3)^2+4}\right]$

Solution:

$$L^{-1}\left[\frac{(s-3)}{(s-3)^2+4}\right] = e^{3t}L^{-1}\left[\frac{s}{s^2+2^2}\right] = \mathbf{e^{3t}\cos 2t}$$

PROBLEM 3: $Find\ L^{-1}\left[\frac{s}{(s-b)^2+a^2}\right]$

Solution:

$$L^{-1}\left[\frac{s}{(s-b)^2+a^2}\right] = L^{-1}\left[\frac{s-b+b}{(s-b)^2+a^2}\right]$$

$$L^{-1}\left[\frac{s}{(s-b)^2+a^2}\right] = L^{-1}\left[\frac{s-b}{(s-b)^2+a^2}\right]+L^{-1}\left[\frac{b}{(s-b)^2+a^2}\right]$$

$$= e^{bt}\,L^{-1}\left[\frac{s}{s^2+a^2}\right]+\frac{b}{a}e^{bt}L^{-1}\left[\frac{a}{s^2+a^2}\right]$$

$$\therefore \mathbf{L}^{-1}\left[\frac{\mathbf{s}}{(\mathbf{s}-\mathbf{b})^2+\mathbf{a}^2}\right] = \mathbf{e}^{\mathbf{bt}}\ \mathbf{cos\ at} + \frac{\mathbf{b}}{\mathbf{a}}\mathbf{e}^{\mathbf{bt}}\mathbf{sin\ at.}$$

RESULT 1: $\mathbf{L}^{-1}[\mathbf{F}'(\mathbf{s})] = -\mathbf{t}\left(\mathbf{L}^{-1}[\mathbf{F}(\mathbf{s})]\right)$

PROBLEM 1 $Find \quad L^{-1}\left[\frac{s}{\left(s^2+a^2\right)^2}\right]$

Solution:

$$F'(s) = \frac{s}{\left(s^2+a^2\right)^2}$$

$$F(s) = \int \frac{s}{\left(s^2+a^2\right)^2} ds$$

$$Put \quad \left(s^2+a^2\right) = t$$

$$2sds = dt$$

$$\int \frac{s}{\left(s^2+a^2\right)^2}\,ds = \int \frac{dt}{2t^2}$$

$$= \frac{1}{2}\left(-\frac{1}{t}\right) = \frac{1}{2}\left[-\frac{1}{s^2+a^2}\right]$$

$$\int \frac{s}{\left(s^2+a^2\right)^2}\,ds = \frac{-1}{2\left(s^2+a^2\right)}$$

WKT $\mathbf{L^{-1}[F'(s)] = -t\,L^{-1}[F(s)]}$

$$= -t\,L^{-1}\left[\frac{-1}{2\left(s^2+a^2\right)}\right]$$

$$L^{-1}\left[\frac{s}{\left(s^2+a^2\right)^2}\right]=\frac{t}{2}\;L^{-1}\left[\frac{1}{\left(s^2+a^2\right)}\right]$$

$$=\frac{t}{2}\cdot\frac{1}{a}\;L^{-1}\left[\frac{a}{\left(s^2+a^2\right)}\right]$$

$$\mathbf{L}^{-1}\left[\frac{\mathbf{s}}{\left(\mathbf{s}^2+\mathbf{a}^2\right)^2}\right]=\frac{\mathbf{t}}{\mathbf{2a}}\;\mathbf{sinat}$$

PROBLEM 2 $\quad Find \;\; L^{-1}\left[\dfrac{s+3}{\left(s^2+6s+13\right)^2}\right]$

Solution:

$$F'(s) = \frac{s+3}{\left(s^2+6s+13\right)^2}$$

$$F(s) = \int \frac{(s+3)ds}{\left(s^2+6s+13\right)^2}$$

$$Put\ \ s^2+6s+13=t$$

$$(2s+6)ds = dt$$

$$(s+3)ds = \frac{dt}{2}$$

$$\int \frac{(s+3)ds}{\left(s^2+6s+13\right)^2} = \int \frac{dt}{2t^2} = \frac{1}{2}\left(\frac{-1}{t}\right)$$

$$\int \frac{(s+3)ds}{\left(s^2+6s+13\right)^2} = \frac{1}{2}\left[\frac{-1}{s^2+6s+13}\right]$$

$$WKT\ \mathbf{L}^{-1}\left[\mathbf{F}^{1}(\mathbf{s})\right] = -\mathbf{t}\,\mathbf{L}^{-1}[\mathbf{F}(\mathbf{s})]$$

$$= -t\,L^{-1}\left[\frac{-1}{2\left(s^2+6s+13\right)}\right]$$

$$= \frac{t}{2}\,L^{-1}\left[\frac{1}{\left(s^2+6s+13\right)}\right]$$

$$= \frac{t}{2}\,L^{-1}\left[\frac{1}{(s+3)^2+4}\right]$$

$$= \frac{t}{2}e^{-3t}\,\frac{1}{2}\,L^{-1}\left[\frac{2}{s^2+2^2}\right]$$

$$= \frac{t}{4}e^{-3t}\,\sin 2t$$

RESULT 2: $\mathbf{L^{-1}[s\,F(s)] = \dfrac{d}{dt} L^{-1}[F(s)]}$

PROBLEM 1 $Find\ \ L^{-1}\left[\dfrac{s}{(s+2)^2+4}\right]$

Solution:

$$F(s) = \frac{s}{(s+2)^2+4}$$

WKT $\mathbf{L^{-1}[s\,F(s)] = \dfrac{d}{dt} L^{-1}[F(s)]}$

$$L^{-1}\left[s.\,\frac{1}{(s+2)^2+4}\right] = \frac{d}{dt}\,L^{-1}\left[\frac{1}{(s+2)^2+4}\right]$$

$$= \frac{d}{dt}\, e^{-2t}\ L^{-1}\left[\frac{1}{s^2+2^2}\right]$$

$$= \frac{d}{dt}\, e^{-2t}\ \frac{1}{2}\ L^{-1}\left[\frac{2}{s^2+2^2}\right]$$

$$= \frac{d}{dt} e^{-2t}\ \frac{1}{2}\ \sin 2t$$

$$= \frac{1}{2}\ \frac{d}{dt}\left(e^{-2t}\sin 2t\right)$$

$$= \frac{1}{2}\left[e^{-2t}\, 2\cos 2t + \sin 2t.\left(-2e^{-2t}\right)\right]$$

$$\therefore L^{-1}\left[\frac{s}{(s+2)^2+4}\right] = e^{-2t}\left[\cos 2t - \sin 2t\right]$$

PROBLEM 2 *Find* $L^{-1}\left[\frac{s^2}{\left(s^2+a^2\right)^2}\right]$

Solution:

$$L^{-1}\left[\frac{s^2}{\left(s^2+a^2\right)^2}\right] = L^{-1}\left[s.\frac{s}{\left(s^2+a^2\right)^2}\right]$$

$$= L^{-1}\left[s.\ F(s)\right]$$

$$where\ \ F(s) = \frac{s}{\left(s^2 + a^2\right)^2}$$

WKT $\mathbf{L^{-1}\left[s\ F(s)\right] = \frac{d}{dt} L^{-1}\left[F(s)\right]}$

$$= \frac{d}{dt} L^{-1}\left[\frac{s}{\left(s^2 + a^2\right)^2}\right] \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

$$Consider\ \ L^{-1}\left[\frac{s}{\left(s^2 + a^2\right)^2}\right]$$

$$F^1(s) = \frac{s}{\left(s^2 + a^2\right)^2}$$

$$F(s) = \int \frac{s}{\left(s^2 + a^2\right)^2} ds$$

$$Put \quad \left(s^2 + a^2\right) = t$$

$$2s \; ds = dt \Rightarrow s \, ds = \frac{dt}{2}$$

$$\int \frac{s}{\left(s^2 + a^2\right)^2} ds = \int \frac{dt}{2t^2}$$

$$= \frac{1}{2}\left(-\frac{1}{t}\right)$$

$$= (-) \; \frac{1}{2}\left(\frac{1}{s^2 + a^2}\right)$$

WKT $L^{-1}\left[F'(s)\right] = -t\,L^{-1}\left[F(s)\right]$

$$= -t\; L^{-1}\left[\frac{-1}{2\left(s^2 + a^2\right)}\right]$$

$$= \frac{t}{2}\; L^{-1}\left[\frac{1}{\left(s^2 + a^2\right)}\right]$$

$$= \frac{t}{2}.\frac{1}{a} L^{-1}\left[\frac{a}{\left(s^2 + a^2\right)}\right]$$

$$= \frac{t}{2a}\sin\, at \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2)$$

substituting (2) *in* (1),

$$\therefore L^{-1}\left[\frac{s^2}{\left(s^2+a^2\right)^2}\right]=\frac{d}{dt}\left(\frac{t}{2a}\sin at\right)$$

$$=\frac{1}{2a}\frac{d}{dt}\left(t\sin at\right)$$

$$=\frac{1}{2a}\left(a\,t\cos at+\sin at\right)$$

RESULT 3 $$L^{-1}\left[\frac{F(s)}{s}\right] = \int_0^t L^{-1}\,[F(s)]\,dt$$

PROBLEM 1 *Find* $L^{-1}\left[\frac{1}{s(s^2+a^2)}\right]$

Solution:

$$WKT \quad L^{-1}\left[\frac{F(s)}{s}\right] = \int_0^t L^{-1}\,[F(s)]\,dt$$

$$F(s) = \frac{1}{s^2+a^2}$$

$$L^{-1}\left[\frac{1}{s}.\frac{1}{s^2+a^2}\right] = \int_0^t L^{-1}\left[\frac{1}{s^2+a^2}\right]dt$$

$$= \frac{1}{a}\int_0^t L^{-1}\left[\frac{a}{s^2+a^2}\right]dt$$

$$= \frac{1}{a}\int_0^t \sin\, at\, dt$$

$$= \frac{1}{a}\left[-\frac{\cos at}{a}\right]_0^t$$

$$\therefore\ L^{-1}\left[\frac{1}{s\left(s^2+a^2\right)}\right] = \frac{1}{a^2}\left[1-\cos at\right]$$

PROBLEM 2 *Find* $L^{-1}\left[\frac{1}{s^2(s+a)}\right]$

Solution:

$$\text{WKT } L^{-1}\left[\frac{F(s)}{s}\right] = \int_0^t L^{-1}[F(s)]\,dt$$

$$L^{-1}\left[\frac{1}{s^2(s+a)}\right] = L^{-1}\left[\frac{1}{s}.\frac{1}{s(s+a)}\right]$$

$$where\ F(s) = \frac{1}{s(s+a)}$$

$$L^{-1}\left[\frac{1}{s}.\frac{1}{s+a}\right] = \int_0^t L^{-1}\left[\frac{1}{s(s+a)}\right] dt \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

$$L^{-1}\left[\frac{1}{s(s+a)}\right] = \int_0^t L^{-1}\left[\frac{1}{s+a}\right] dt$$

$$= \int_0^t e^{-at} \; dt$$

$$= \left[\frac{e^{-at}}{-a}\right]_0^t$$

$$= \frac{1}{a}\left[1 - e^{-at}\right] \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2)$$

substitute (2) *in* (1),

$$L^{-1}\left[\frac{1}{s^2(s+a)}\right]=\int_0^t \frac{1}{a}\left[1-e^{-at}\right]dt$$

$$=\frac{1}{a}\int_0^t \left[1-e^{-at}\right]dt$$

$$=\frac{1}{a}\left\{[t]_0^t-\left[\frac{e^{-at}}{-a}\right]_0^t\right\}$$

$$L^{-1}\left[\frac{1}{s^2(s+a)}\right]=\frac{1}{a}\left\{t+\frac{e^{-at}}{a}-\frac{1}{a}\right\}$$

PROBLEM 3: *Find* $L^{-1}\left[\dfrac{1}{s(s^2-2s+5)}\right]$

Solution:

$$\text{WKT } L^{-1}\left[\frac{F(s)}{s}\right]=\int_0^t L^{-1}\,[F(s)]\,dt$$

$$L^{-1}\left[\frac{1}{s(s^2-2s+5)}\right]=\int_0^t L^{-1}\left[\frac{1}{s(s^2-2s+5)}\right]dt$$

$$=\int_0^t L^{-1}\left[\frac{1}{s^2-2s+1+5-1}\right]dt$$

$$=\int_0^t L^{-1}\left[\frac{1}{(s-1)^2+2^2}\right]dt$$

$$=\int_0^t e^t\; L^{-1}\left[\frac{1}{s^2+2^2}\right]dt$$

$$L^{-1}\left[\frac{1}{s\left(s^2-2s+5\right)}\right] = \int_0^t e^t \ \frac{1}{2} \ L^{-1}\left[\frac{2}{s^2+2^2}\right] dt$$

$$= \frac{1}{2}\int_0^t e^t \sin 2t \ dt$$

$$WKT \quad \int \mathbf{e}^{\mathbf{ax}} \ \mathbf{sinbx\,dx} = \frac{\mathbf{e}^{\mathbf{ax}}}{\mathbf{a}^2+\mathbf{b}^2}\left[\mathbf{asinbx}-\mathbf{bcosbx}\right]$$

$$= \frac{1}{2}\left[\frac{e^t}{5}\left(\sin 2t - 2\cos 2t\right)\right]_0^t$$

$$\mathbf{L}^{-1}\left[\frac{\mathbf{1}}{\mathbf{s}\left(\mathbf{s}^2-\mathbf{2s}+\mathbf{5}\right)}\right] = \frac{\mathbf{1}}{\mathbf{10}}\left\{\mathbf{e}^{\mathbf{t}}\left[\mathbf{sin2t}-\mathbf{2cos2t}\right]+\mathbf{2}\right\}$$

PROBLEM 1: $Find \quad L^{-1}\left[\log\left(\frac{s+b}{s+a}\right)\right]$

Solution :

$$Let \;\; F(s) = \log\left(\frac{s+b}{s+a}\right)$$

$$F(s) = \log(s+b) - \log(s+a)$$

$$\Rightarrow \qquad F'(s) = \frac{1}{s+b} - \frac{1}{s+a}$$

$$L^{-1}[F'(s)] = e^{-bt} - e^{-at}$$

By the result, $\mathbf{L^{-1}[F(s)] = -\frac{1}{t}L^{-1}[F'(s)]}$

$$L^{-1}\left[\log\left(\frac{s+b}{s+a}\right)\right] = -\frac{1}{t}\left(e^{-bt} - e^{-at}\right)$$

$$\mathbf{L^{-1}\left[\log\left(\frac{s+b}{s+a}\right)\right] = \frac{e^{-at} - e^{-bt}}{t}}$$

PROBLEM 2: *Evaluate* $L^{-1}\left(\log\left(\frac{s^2+a^2}{s^2+b^2}\right)\right)$

Solution:

WKT $\mathbf{L^{-1}[F(s)] = -\frac{1}{t}L^{-1}[F^1(s)]}$

$$Let\ F(s) = \log\left(\frac{s^2+a^2}{s^2+b^2}\right)$$

$$= \log(s^2+a^2) - \log(s^2+b^2)$$

$$F'(s) = \frac{2s}{s^2+a^2} - \frac{2s}{s^2+b^2}$$

$$\Rightarrow L^{-1}[F'(s)] = 2\cos at - 2\cos bt$$

$$\mathbf{L^{-1}\left[\log\left(\frac{s^2+a^2}{s^2+b^2}\right)\right] = \frac{2}{t}(\cos bt - \cos at)}$$

TASK :

1) $Find\ \ L^{-1}\left[\dfrac{s}{(s+2)^2}\right]$

2) $Find\ \ L^{-1}\left[\dfrac{1}{(s-4)^5}+\dfrac{5}{(s-2)^2+5^2}+\dfrac{(s+3)}{(s+3)^2+6^2}\right]$

3) $Find\ \ L^{-1}\left[\dfrac{2s^3-1}{(s+2)^{18}}\right]$

4) $Find\ \ L^{-1}\left[\dfrac{2(s+1)}{(s^2+2s+2)^2}\right]$

$$5)\ Find\ \ L^{-1}\left[\frac{1}{s(s+3)}\right]$$

$$6)\ Find\ \ L^{-1}\left[\frac{s+2}{(s+2)^2+\omega^2}\right]$$

$$7)\ Find\ \ L^{-1}\left[\log\left(\frac{1+s}{s^2}\right)\right]$$

$$8)\ Find\ \ L^{-1}\left[\log\left(1+\frac{\omega^2}{s^2}\right)\right]$$

RESULT 4: $L^{-1}\left[e^{-as}\ F(s)\right] = f(t-a)\ H(t-a)$

$$= f(t)_{t \to t-a}\ H(t-a)$$

$$\mathbf{L^{-1}\left[e^{-as}\ F(s)\right] = L^{-1}[F(s)]_{t \to t-a}\, H(t-a)}$$

PROBLEM 1: *Find the Inverse Laplace Transform of*

$$F(s) = \tan^{-1}\left(\frac{a}{s}\right) + \cot^{-1}\left(\frac{s}{b}\right)$$

Solution:

$$F(s) = \tan^{-1}\left(\frac{a}{s}\right) + \cot^{-1}\left(\frac{s}{b}\right)$$

$$F'(s) = \frac{1}{1+\frac{a^2}{s^2}}\left(-\frac{a}{s^2}\right) - \frac{1}{1+\frac{s^2}{b^2}}\left(-\frac{1}{b}\right)$$

$$\frac{d}{ds}\,F(s) = -\left[\frac{a}{s^2+a^2} + \frac{b}{s^2+b^2}\right]$$

$$WKT\;\; \mathbf{L}[\mathbf{t}\;\mathbf{f}(\mathbf{t})] = -\frac{\mathbf{d}}{\mathbf{ds}}\mathbf{F}(\mathbf{s})$$

$$-L[t\;f(t)] = -\left[\frac{a}{s^2+a^2} + \frac{b}{s^2+b^2}\right]$$

$$t\ f(t) = \sin at + \sin bt$$

$$f(t) = \frac{1}{t}[\sin at + \sin bt]$$

$$L^{-1}[F(s)] = \frac{1}{t}[\sin at + \sin bt]$$

$$L^{-1}\left[\tan^{-1}\left(\frac{a}{s}\right) + \cot^{-1}\left(\frac{s}{b}\right)\right] = \frac{1}{t}[\sin at + \sin bt]$$

PROBLEM 2: $If\ \ L\left[f(t)\right]=e^{-3s}\tan^{-1}(s).\ \ Find\ \ f(0).$

Solution:

$$Given \quad L\left[f(t)\right]=e^{-3s}\tan^{-1}(s)$$

$$\Rightarrow \ f(t)=L^{-1}\left[e^{-3s}\tan^{-1}(s)\right] \qquad \dots\dots\dots\dots\dots\dots\dots\dots(1)$$

$$=L^{-1}\left[\tan^{-1}(s)\right]_{t\,\to\,t-3}H(t-3) \quad \left[u\sin g\ \ result\ 4\right]$$

$$Find: \ L^{-1}\left[\tan^{-1}(s)\right]$$

$$Let \ \ L^{-1}\left[\tan^{-1}(s)\right] = g(t)$$

$$\therefore \ L\left[g(t)\right] = \tan^{-1}(s)$$

$$\Rightarrow G(s) = \tan^{-1}(s)$$

$$WKT \quad L[t\ \ g(t)] = \frac{d}{ds}\,G(s)$$

$$= -\frac{d}{ds}\,L\left[\,g(t)\,\right]$$

$$= -\frac{d}{ds}\left(\tan^{-1}(s)\right)$$

$$L[t\ \ g(t)] = -\left(\frac{1}{1+s^2}\right)$$

$$t\ g(t) = L^{-1}\left[-\frac{1}{1+s^2}\right]$$

$$= -\sin t$$

$$\Rightarrow g(t) = -\frac{\sin t}{t}$$

$$\therefore L^{-1}\left[\tan^{-1}(s)\right] = -\frac{\sin t}{t}$$

$$\therefore \quad (1) \Rightarrow \quad f(t) = \left(-\frac{\sin t}{t}\right)_{t \to t-3} H(t-3)$$

$$= \left(-\frac{\sin(t-3)}{(t-3)}\right) H(t-3)$$

$$f(t) \quad = \begin{cases} \left(-\dfrac{\sin(t-3)}{(t-3)}\right), & t > 3 \\ 0 \qquad , & t < 3 \end{cases}$$

$$\therefore \qquad \mathbf{f(0) = 0}$$

METHOD OF PARTIAL FRACTIONS :

TYPE I:

$$\frac{fn}{(s)(s+a)} = \frac{A}{s} + \frac{B}{s+a}$$

TYPE II:

$$\frac{fn}{(s)\left(s^2+2as+a^2\right)} = \frac{A}{s} + \frac{Bs+C}{\left(s^2+2as+a^2\right)}$$

TYPE III:

$$\frac{fn}{(s+a)^3} = \frac{A}{(s+a)} + \frac{B}{(s+a)^2} + \frac{C}{(s+a)^3}$$

TYPE IV:

$$\frac{fn}{\left(s^2+a^2\right)^2\left(s^2+b^2\right)} = \frac{As+B}{\left(s^2+a^2\right)} + \frac{Cs+D}{\left(s^2+a^2\right)^2} + \frac{Es+F}{\left(s^2+b^2\right)}$$

PROBLEM 1: *Find* $L^{-1}\left[\frac{1}{s(s+1)(s+2)}\right]$

Solution :

$$F(s) = \frac{1}{s(s+1)(s+2)}$$

$$\frac{1}{s(s+1)(s+2)} = \frac{A}{s} + \frac{B}{s+1} + \frac{C}{s+2} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(\mathbf{1})$$

$$\frac{1}{s(s+1)(s+2)} = \frac{A(s+1)(s+2)+B\,s(s+2)+C\,s(s+1)}{s(s+1)(s+2)}$$

$$1 = A(s+1)(s+2)+B\,s(s+2)+C\,s(s+1)$$

From the above equation we get

$$\mathbf{A} = \frac{\mathbf{1}}{\mathbf{2}};\quad \mathbf{B} = -\mathbf{1};\quad \mathbf{C} = \frac{\mathbf{1}}{\mathbf{2}}$$

Substitute the values of A, B *and* C *in* (1),

$$\frac{1}{s(s+1)(s+2)} = \frac{(1/2)}{s} - \frac{1}{s+1} + \frac{(1/2)}{s+2}$$

Taking L^{-1} *on both sides,*

$$L^{-1}\left[\frac{1}{s(s+1)(s+2)}\right] = \frac{1}{2} L^{-1}\left(\frac{1}{s}\right) - L^{-1}\left(\frac{1}{s+1}\right) + \frac{1}{2} L^{-1}\left(\frac{1}{s+2}\right)$$

$$\mathbf{L^{-1}\left[\frac{1}{s(s+1)(s+2)}\right] = \frac{1}{2}(1) - e^{-t} + \frac{1}{2} e^{-2t}}$$

PROBLEM 2 : *Find* $L^{-1}\left[\frac{1-s}{(s+1)(s^2+4s+13)}\right]$

Solution :

$$F(s) = \frac{1-s}{(s+1)(s^2+4s+13)}$$

$$\frac{1-s}{(s+1)(s^2+4s+13)} = \frac{A}{(s+1)} + \frac{Bs+C}{(s^2+4s+13)} \quad \text{......................}(\mathbf{1})$$

$$\frac{1-s}{(s+1)(s^2+4s+13)} = \frac{A(s^2+4s+13)+(Bs+C)(s+1)}{(s+1)(s^2+4s+13)}$$

$$1-s = A(s^2+4s+13)+(Bs+C)(s+1)$$

From the above equation we get ,

$$\mathbf{A} = \frac{\mathbf{1}}{\mathbf{5}}; \quad \mathbf{B} = -\frac{\mathbf{1}}{\mathbf{5}}; \quad \mathbf{C} = -\frac{\mathbf{8}}{\mathbf{5}}$$

Substituting the values of A, B *and* C *in* (1), *we get*

$$\frac{1-s}{(s+1)(s^2+4s+13)} = \frac{1}{5(s+1)} + \frac{\left(-\frac{1}{5}\right)s - \frac{8}{5}}{(s^2+4s+13)}$$

$$= \frac{1}{5(s+1)} - \frac{s}{5(s^2+4s+13)} - \frac{8}{5(s^2+4s+13)}$$

$$= \frac{1}{5(s+1)} - \frac{s}{5\left[(s+2)^2+9\right]} - \frac{8}{5\left[(s+2)^2+9\right]}$$

Taking L^{-1} *on both sides*,

$$L^{-1}\left(\frac{1-s}{(s+1)(s^2+4s+13)}\right) = \frac{1}{5}L^{-1}\left(\frac{1}{(s+1)}\right) - \frac{1}{5}L^{-1}\left(\frac{s}{\left[(s+2)^2+9\right]}\right)$$

$$-\frac{8}{5}L^{-1}\left(\frac{1}{\left[(s+2)^2+9\right]}\right)$$

$$L^{-1}\left[\frac{1-s}{(s+1)(s^2+4s+13)}\right]=\frac{1}{5}e^{-t}-\frac{1}{5}L^{-1}\left[\frac{(s+2)-2}{(s+2)^2+3^2}\right]-\frac{8}{5}L^{-1}\left[\frac{1}{(s+2)^2+3^2}\right]$$

$$=\frac{1}{5}e^{-t}-\frac{1}{5}\left\{L^{-1}\left[\frac{(s+2)}{(s+2)^2+3^2}\right]-2L^{-1}\left[\frac{1}{(s+2)^2+3^2}\right]\right\}-\frac{8}{5}L^{-1}\left[\frac{1}{(s+2)^2+3^2}\right]$$

$$=\frac{1}{5}e^{-t}-\frac{1}{5}\left\{e^{-2t}L^{-1}\left[\frac{s}{s^2+3^2}\right]-2e^{-2t}L^{-1}\left[\frac{1}{s^2+3^2}\right]\right\}-\frac{8}{5}e^{-2t}L^{-1}\left[\frac{1}{s^2+3^2}\right]$$

$$L^{-1}\left[\frac{1-s}{(s+1)(s^2+4s+13)}\right]$$

$$=\frac{1}{5}e^{-t}-\frac{1}{5}e^{-2t}L^{-1}\left[\frac{s}{s^2+3^2}\right]+\frac{2}{5}\,\frac{e^{-2t}}{3}\,L^{-1}\left[\frac{3}{s^2+3^2}\right]-\frac{8}{5}\,\frac{e^{-2t}}{3}L^{-1}\left[\frac{3}{s^2+3^2}\right.$$

$$=\frac{1}{5}e^{-t}-\frac{1}{5}e^{-2t}\cos 3t+\frac{2}{15}e^{-2t}\sin 3t-\frac{8}{15}e^{-2t}\sin 3t$$

$$L^{-1}\left[\frac{1-s}{(s+1)(s^2+4s+13)}\right]=\frac{1}{5}e^{-t}-\frac{1}{5}e^{-2t}\cos 3t-\frac{3}{5}e^{-2t}\sin 3t$$

PROBLEM 3: $Find \quad L^{-1}\left[\frac{5s^2-15s-11}{(s+1)(s-2)^3}\right]$

Solution:

$$F(s)=\frac{5s^2-15s-11}{(s+1)(s-2)^3}$$

$$\frac{5s^2-15s-11}{(s+1)(s-2)^3}=\frac{A}{(s+1)}+\frac{B}{(s-2)}+\frac{C}{(s-2)^2}+\frac{D}{(s-2)^3} \quad \ldots\ldots\ldots\ldots\ldots(\mathbf{1})$$

$$\frac{5s^2-15s-11}{(s+1)(s-2)^3}=\frac{A(s-2)^3+B(s+1)(s-2)^2+C(s+1)(s-2)+D(s+1)}{(s+1)(s-2)^3}$$

$$5s^2-15s-11=A(s-2)^3+B(s+1)(s-2)^2+C(s+1)(s-2)+D(s+1)$$

From the above equation we get,

$$\mathbf{A} = -\frac{\mathbf{1}}{\mathbf{3}};\ \ \mathbf{B} = \frac{\mathbf{1}}{\mathbf{3}};\ \ \mathbf{C} = \mathbf{4};\ \ \mathbf{D} = -\mathbf{7}$$

Substituting the values in (1),

$$\frac{5s^2 - 15s - 11}{(s+1)(s-2)^3} = \frac{\left(-\frac{1}{3}\right)}{(s+1)} + \frac{\left(\frac{1}{3}\right)}{(s-2)} + \frac{4}{(s-2)^2} - \frac{7}{(s-2)^3}$$

Taking L^{-1} *on both sides,*

$$L^{-1}\left(\frac{5s^2 - 15s - 11}{(s+1)(s-2)^3}\right) = L^{-1}\left(\frac{\left(-\frac{1}{3}\right)}{(s+1)} + \frac{\left(\frac{1}{3}\right)}{(s-2)} + \frac{4}{(s-2)^2} - \frac{7}{(s-2)^3}\right)$$

$$= \left(-\frac{1}{3}\right)e^{-t} + \left(\frac{1}{3}\right)e^{2t} + 4e^{2t}\ L^{-1}\left(\frac{1}{s^2}\right) - 7e^{2t}\ L^{-1}\left(\frac{1}{s^3}\right)$$

$$L^{-1}\left[\frac{5s^2 - 15s - 11}{(s+1)(s-2)^3}\right] = \left(-\frac{1}{3}\right)e^{-t} + \left(\frac{1}{3}\right)e^{2t} + 4e^{2t}.t - 7e^{2t}\ \frac{t^2}{2}$$

PROBLEM 4: *Find* $L^{-1}\left[\frac{1}{s^2(s^2+1)(s^2+9)}\right]$

Solution :

$$F(s) = \frac{1}{s^2(s^2+1)(s^2+9)}$$

Put $s^2 = u$ *in above equation*

$$\frac{1}{s^2(s^2+1)(s^2+9)} = \frac{1}{u(u+1)(u+9)}$$

$$\text{Consider} \quad \frac{1}{u(u+1)(u+9)} = \frac{A}{u} + \frac{B}{u+1} + \frac{C}{u+9} \ldots\ldots\ldots\ldots\ldots\ldots(1)$$

$$\frac{1}{u(u+1)(u+9)} = \frac{A(u+1)(u+9)+Bu(u+9)+Cu(u+1)}{u(u+1)(u+9)}$$

$$1 = A(u+1)(u+9)+Bu(u+9)+Cu(u+1)$$

From the above equation we get,

$$\mathbf{A} = \frac{\mathbf{1}}{\mathbf{9}}; \quad \mathbf{B} = -\frac{\mathbf{1}}{\mathbf{8}}; \quad \mathbf{C} = \frac{\mathbf{1}}{\mathbf{72}}$$

Substituting the values in (1) *we get,*

$$\frac{1}{u(u+1)(u+9)} = \frac{\frac{1}{9}}{u} + \frac{\left(-\frac{1}{8}\right)}{u+1} + \frac{\frac{1}{72}}{u+9}$$

$$\frac{1}{u(u+1)(u+9)} = \frac{\frac{1}{9}}{u} + \frac{\left(-\frac{1}{8}\right)}{u+1} + \frac{\frac{1}{72}}{u+9}$$

Taking L^{-1} *on both sides,*

$$L^{-1}\left(\frac{1}{u(u+1)(u+9)}\right) = L^{-1}\left(\frac{\frac{1}{9}}{u}\right) + L^{-1}\left(\frac{\left(-\frac{1}{8}\right)}{u+1}\right) + L^{-1}\left(\frac{\frac{1}{72}}{u+9}\right)$$

$$= \frac{1}{9}L^{-1}\left(\frac{1}{u}\right) - \frac{1}{8}L^{-1}\left(\frac{1}{u+1}\right) + \frac{1}{72}L^{-1}\left(\frac{1}{u+3^2}\right)$$

$$= \frac{1}{9}t - \frac{1}{8}\sin t + \frac{1}{72}.\left(\frac{\sin 3t}{3}\right)$$

$$\mathbf{L^{-1}\left(\frac{1}{s(s^2+1)(s^2+9)}\right) = \frac{1}{9}t - \frac{1}{8}\sin t + \frac{1}{72}.\left(\frac{\sin 3t}{3}\right)}$$

TASK

1) *Find the inverse transform of* $\dfrac{1}{(s-2)(s^2+1)}$

2) *Find the inverse transform of* $\dfrac{s}{(s^2+a^2)(s^2+b^2)}$

3) *Find the inverse transform of* $\left(\dfrac{4s+15}{16s^2-25}\right)$

4) *Find* $L^{-1}\left(\dfrac{a^2}{s(s+a)^3}\right)$

5) *Find* $L^{-1}\left[\dfrac{2s^2-6s+5}{(s-1)(s-2)(s-3)}\right]$

CONVOLUTION:

If $f(t)$ and $g(t)$ are given functions then the convolution of $f(t)$ and $g(t)$ is defined as $\int_0^t f(u)g(t-u)\,du$.

It is denoted by, $f(t)*g(t)$

$(i.e)\ f(t)*g(t)=\int_0^t f(u)g(t-u)\,du.$

CONVOLUTION THEOREM:

If $f(t)$ and $g(t)$ are functions defined for $t \geq 0$

then $L[f(t)*g(t)]=L[f(t)].L[g(t)]$

$(i.e)\ L[f(t)*g(t)]=F(s).G(s)$

Proof :

$$L\left[f(t)\right]=\int_0^\infty e^{-st} f(t)dt$$

$$L\left[f(t)*g(t)\right]=\int_0^\infty e^{-st} f(t)*g(t)dt$$

$$=\int_0^\infty e^{-st}\left\{\int_0^t f(u)g(t-u)du\right\}\text{dt}$$

$$=\int_0^\infty\int_0^t e^{-st} f(u)\ \text{g}(t-u)\text{dudt}\text{---------------------}(1)$$

Limits of u = 0 to t; t = 0 to ∞

After changing the order of integration,

limits of u = 0 to ∞; t = u to ∞

(1) becomes,

$$L\left[f(t)*g(t)\right]=\int_0^\infty\int_u^\infty e^{-st} f(u)\ \text{g}(t-u)\text{dtdu}$$

$$=\int_0^\infty f(u)\left\{\int_u^\infty e^{-st} g(t-u)dt\right\}\text{du}$$

Let $t-u = \mathrm{v}$

$\mathrm{dt} = \mathrm{dv}$

when t = u, then v=0; t = ∞,then v = ∞

$$L\left[f(t)*g(t)\right]=\int_0^\infty f(u)\left\{e^{-s(u+v)}g(v)\,dv\right\}\mathrm{du}$$

$$=\int_0^\infty f(u)\left\{\int_0^t e^{-su}e^{-sv}g(v)\,dv\right\}\mathrm{du}$$

$$=\int_0^\infty e^{-st} f(u)du\left\{\int_0^\infty e^{-sv}g(v)\,dv\right\}$$

$$=\int_0^\infty e^{-st} f(t)dt\left\{\int_0^\infty e^{-sv}g(t)\,dt\right\}$$

$$=\mathrm{L}\left[f(t)\right]\ \mathrm{L}\left[g(t)\right]$$

$$L\left[f(t)*g(t)\right] = \mathrm{F}(s)\ \mathrm{G}(s)$$

Hence, Convolution Theorem is proved

COROLLARY:

$$\mathrm{L}\left[f(t)*g(t)\right]=\mathrm{F}(s).\mathrm{G}(s)$$

$$f(t)*g(t)=\mathrm{L}^{-1}\left[\mathrm{F}(s)\right].\mathrm{L}^{-1}\left[\mathrm{G}(s)\right]$$

$$\mathrm{L}^{-1}\left[\mathrm{F}(s)\right]*\mathrm{L}^{-1}\left[\mathrm{G}(s)\right]=\mathrm{L}^{-1}\left[\mathrm{F}(s)\right].\mathrm{L}^{-1}\left[\mathrm{G}(s)\right]$$

NOTE:

$$f(t)*g(t)=g(t)*f(t)$$

PROBLEMS:

1) Using Convolution Theorem find $\mathrm{L}^{-1}\left[\dfrac{1}{s\left(s^2+1\right)}\right]$

Solution:

$$\mathrm{L}^{-1}\left[\frac{1}{s\left(s^2+1\right)}\right]=\mathrm{L}^{-1}\left[\frac{1}{s}\right].\mathrm{L}^{-1}\left[\frac{1}{s^2+1}\right]$$

$$=1*\sin t$$

$$=\int_0^t \sin(t-u)\,du$$

$$\mathrm{L}^{-1}\left[\frac{1}{s\left(s^2+1\right)}\right]=1-\cos t$$

2) Using Convolution Theorem find $L^{-1}\left[\frac{s}{\left(s^2+a^2\right)^2}\right]$

Solution:

$$L^{-1}\left[\frac{s}{\left(s^2+a^2\right)^2}\right]=L^{-1}\left[\frac{s}{\left(s^2+a^2\right)}\right]L^{-1}\left[\frac{1}{\left(s^2+a^2\right)}\right]$$

$$=L^{-1}\left[\frac{s}{\left(s^2+a^2\right)}\right].\frac{1}{a}L^{-1}\left[\frac{a}{\left(s^2+a^2\right)}\right]$$

$$=\cos at*\frac{1}{a}\sin at$$

$$=\frac{1}{a}\int_0^t \cos au.\sin a(t-u)du$$

$$=\frac{1}{2a}\int_0^t\left\{\sin a(u+t-u)-\sin a(u-t-u)\right\}du$$

$$=\frac{1}{2a}t\sin at$$

3) Find $1*e^t$

Solution:

$$1*e^t=\int_0^t 1.e^{t-u}du$$

$$=\left[\frac{e^{t-u}}{-1}\right]_0^t$$

$$=e^t-1$$

4) Find $t*e^t$

Solution:

$$t*e^t=\int_0^t ue^{t-u}du$$

$$=\left[\frac{ue^{t-u}}{-1}\right]_0^t-\left[\frac{e^{t-u}}{1}\right]_0^t$$

$$=e^t-t-1$$

Solving linear second order ordinary differential equations with constant coefficient

1)Solve $\frac{d^2y}{dt^2}+4\frac{dy}{dt}-5y=5, y(t)=0, \frac{dy}{dt}=2\, when\, t=0;$

$y(0)=0; y^1(0)=2; y(t)=0, y^1(t)=0$

Solution:

$$y^{11}(t)+4y^1(t)-5y(t)=5$$

$$L\left[y^{11}(t)\right]+4L\left[y^1(t)\right]-5L[y(t)]=5L[1]$$

$$s^2L[y(t)]-sy(0)-y^1(0)+4\left[sL[y(t)]-y(0)\right]-5L\left[y(t)\right]=5L[1]$$

$$s^2L[y(t)]-s(0)-2+4\left[sL[y(t)]-y(0)\right]-5L\left[y(t)\right]=5\left(\frac{1}{s}\right)$$

$$L[y(t)]\left[s^2+4s-5\right]-2=\frac{5}{s}$$

$$y(t)=L^{-1}\left[\frac{5+2s}{s\left[s^2+4s-5\right]}\right]\text{...}(1)$$

$$\frac{5+2s}{s\left[s^2+4s-5\right]}=\frac{5+2s}{s(s+5)(s-1)}=\frac{A}{s}+\frac{B}{s+5}+\frac{C}{s-1}\text{...............}(2)$$

$$5+2s = A(s+5)(s-1) + Bs(s-1) + cs(s+5)..............(3)$$

We get A = -1, B = $-\frac{1}{6}$, C = $\frac{7}{6}$

Substituting in (2),

$$\frac{5+2s}{s(s+5)(s-1)} = \frac{-1}{s} - \frac{1}{6(s+5)} + \frac{7}{6(s-1)}$$

$$L^{-1}\left[\frac{5+2s}{s(s+5)(s-1)}\right] = L^{-1}\left[\frac{-1}{s}\right] - \frac{1}{6}L^{-1}\left[\frac{1}{(s+5)}\right] + \frac{7}{6}L^{-1}\left[\frac{1}{(s-1)}\right]$$

$$= \frac{7}{6}e^{t} - \frac{1}{6}e^{-5t} - 1....................................(4)$$

substituting (4) *in* (1)

$$\therefore y(t) = \frac{7}{6}e^{t} - \frac{1}{6}e^{-5t} - 1$$

TASK:

1) Solve $y''(t) - 3y'(t) + 2y(t) = e^{2t}$

2) Solve $y''(t) + 2y'(t) - 3y(t) = \sin t,\ y(0) = 0, y'(0) = 0$

$when\, t = 0, y(t) = 0, y'(t) = 0$