CONDITIONAL PROBABILITY

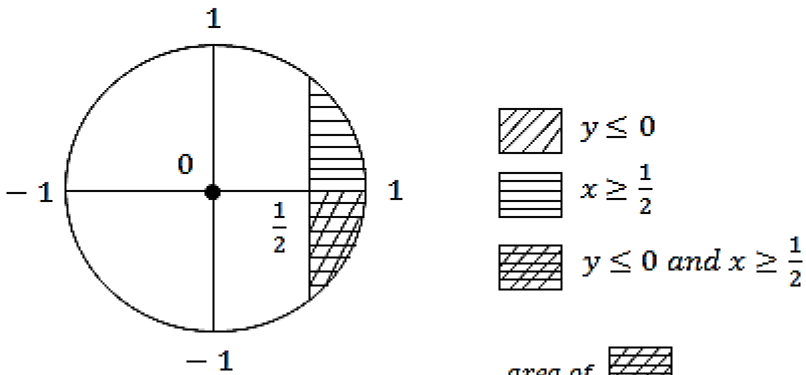
1. A point (x, y) is chosen from the circular region $D = \{(x, y)|x^2 + y^2 \leq 1\}$ in such a way that the probability that the point is chosen from any region in D is proportional to its area.

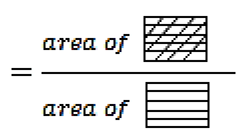
The conditional probability that $y \leq 0$ given that $x \geq {}^1/_2$

Solution:

$D = \{(x, y)|x^2 + y^2 \leq 1\}$ It forms a circle as $(0, 0)$ as center and radius ≤ 1.

$$P\left(y \leq 0 \middle| x \geq \frac{1}{2}\right) = \frac{P(y \leq 0 \text{ and } x \geq {}^1/_2)}{P(x \geq {}^1/_2)}$$





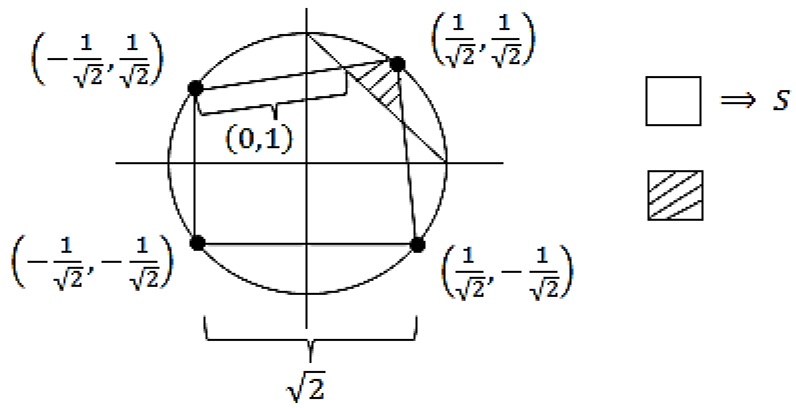
$= \frac{1}{2}$

2. A point (x, y) is chosen from the circular region $D = \{(x, y)|x^2 + y^2 \leq 1\}$, in such a way that the probability that the point is chosen from any region in D is proportional to its area.

The conditional probability that $x + y \geq 1$ given that (x, y) lies inside the square & given by

$\frac{-1}{\sqrt{2}} \leq x, y \leq \frac{1}{\sqrt{2}}$

$P(x + y \geq 1|(x, y) in\ S) =?$



$$P(x + y \geq 1 | (x, y) in\ S) = \frac{area\ of\ ▨}{area\ of\ the\ square}$$

$$= \frac{\frac{1}{2}(\sqrt{2}-1)(\sqrt{2}-1)}{\sqrt{2}\cdot\sqrt{2}}$$

$$= \frac{3-2\sqrt{2}}{4}$$

$$\approx 0.429$$

3. A test tube contains 25 bacteria, 5 of which are can stay alive for atleast 30 days, 10 of which will die in their second day. 10 of which are already dead.

Given that a randomly chosen bacterium for experiment is alive. What is the probability it will still be alive after one week?

(a) $^1/_3$

(b) $^2/_3$

(c) $^1/_5$

(d) $^4/_5$

Solution:

It is given that the bacteria is alive.

Let E be the event chosen one will live for 30 days.

Let F be the event that chosen one is already dead.

$$P\left(\frac{E}{F'}\right) = \frac{P(E \cap F')}{P(F')} = \frac{5/25}{15/25} = \frac{5}{15} = \frac{1}{3}$$

4. Four players Harry, Ron, Hermione and Ginny are playing a card game. A deck of 52 cards are dealt out equally. If Hermione and Ginny have a total of 8 spades among them, what is the probability that Harry has 3 of the remaining 5 spades?

(a) 0.669 (b) 0.339

(c) 0.331 (d) 0.661

Solution:

It is given that Harry and Ginny have a total of 8 spades among 26 cards.

∴ In the remaining 26 cards there are exactly 5 spades.

These 26 cards are distributed equally among Harry and Ron [13 each]

$$P(Harry\ has\ 3\ of\ 5\ spades) = \frac{(5_{C_3})(21_{C_{10}})}{(26_{C_{13}})} = 0.339$$

5. Prabha is working in a software company. Her manager is running a dinner for those employees having atleast one son. If Prabha is invited to the dinner and everyone knows she has two children. What is the probability that they are both boys?

(a) $\frac{1}{4}$ (b) $\frac{1}{3}$

(c) $\frac{1}{2}$ (d) $\frac{2}{3}$

Solution:

Total possibilities: $\{(b, b)$

(b, g)

(g, b)

$(g, g)\}$

It is given that she is invited to dinner.

∴ She has a boy.

P(She has two boys/ she is invited to dinner) $= \frac{1/4}{3/4} = \frac{1}{3}$

6. Elliot in undecided as to whether to take a Number theory course or a Network security course. He estimates that his probability of receiving on A grade would be ½ in Number theory, and 2/3 in Network security. If Elliot decides to base his decision of the flip of a unbiased coin what is the probability that she gets on A in Network security?

(a) $\frac{2}{3}$

(b) $\frac{1}{2}$

(c) $\frac{1}{3}$

(d) $\frac{3}{4}$

Solution:

Let P(C) denote probability of taking course.
P(CA) denotes probability that he gets A in the course C.

Let C: Network security

∴ P (Network security and getting A) = P(taking NS) × P(getting A in Network security)

P(NA) = P(N) × P(A/ N)

$P(NA) = \frac{1}{2}\times\frac{2}{3} = \frac{1}{3}$

7. Suppose that a bag contains 8 blue cubes and 4 green cubes. We draw 2 cubes from the bag without replacement. It is given that blue balls are of weight 1Kg and green balls are of weight 0.5 Kg. Suppose that the probability that a given cube in the bag is the next one selected is its weight divided by the sum of the weights of all cubes currently in the bag. What is the probability that both cubes are blue?

(a) $\frac{8}{12}$

(b) $\frac{14}{33}$

(c) $\frac{28}{45}$

(d) $\frac{14}{35}$

Solution:

Let B_i be the event that the i^{th} cube chosen is blue.

$P(B_1\,B_2) = P(B_1)\times P\left(\frac{B_2}{B_1}\right)$

If we number, the blue cubes

and let $C_i \;\; l = 1, 2, \dots \dots 8$ be the event that the first cube drawn is blue cube i

$$\therefore P(B_1) = \sum_{i=1}^{8} P(C_i)$$

$$\frac{Weight\ of\ the\ cube}{Total\ weight\ of\ all\ cubes} = 8\left(\frac{1}{8\times 1 + 4\times\frac{1}{2}}\right) = 8\left(\frac{1}{8+2}\right) = \frac{8}{10} = \frac{4}{5}$$

Give that the first cube is blue, now the bag contains 7 blue cubes and 4 green cubes.

$$\therefore P(B_1B_2) = P(B_1) \times P\left(\frac{B_2}{B_1}\right) = \frac{4}{5}\times\frac{7}{9} = \frac{28}{45}$$

8. An ordinary deck of 52 playing cards is randomly decoded into 4 piles of 13 cards each. The probability that each pile has exactly 1 ace is ________[Enter upto 3 decimals]

Solution:

Let Define events $E_i \;\; i = 1, 2, 3, 4$ as follows.

$E_1 = \{the\ ace\ of\ spades\ is\ in\ any\ of\ the\ piles\}$

$E_2 = \{the\ ace\ of\ spades\ and\ the\ ace\ of\ hears\ are\ in\ different\ pile\}$

$E_3 = \{the\ ace\ of\ spades, hearts\ and\ diamonds\ are\ all\ in\ different\ piles\}$

$E_4 = \{all\ 4\ aces\ in\ different\ piles\}$

The desired probability is $P(E_1E_2E_3E_4) = P(E_1)\cdot P(E_2|E_1)\cdot P(E_3|E_1E_2)\cdot P(E_1|E_1E_2E_3)$

$P(E_1) = 1$

$$P(E_2|E_1) = \frac{39}{51}$$

$$P(E_3|E_1E_2) = \frac{26}{50}$$

$P(E_1|E_1E_2E_3) = \frac{13}{49}$

$$P(E_1E_2E_3E_4) = 1\times\frac{39}{51}\times\frac{26}{50}\times\frac{13}{49} \approx 0.105$$

9. Hermione is taking her potions exam. Suppose the probability that she will finish the exam is less than h hours is $h/2$, for all $0 \leq h \leq 1$. Given that she is still waiting after 0.75 hours, what is the conditional probability that full hour is used?

Solution:

$$P(T > 1|T > 0.75) = \frac{P\big((T > 1)n(T > 0.75)\big)}{P(T > 0.75)}$$

$$= \frac{P(T>1)}{P(T>0.75)}$$

$$= \frac{1-P(T<1)}{1-P\left(T<\frac{3}{4}\right)}$$

$$= \frac{1-\frac{1}{2}}{1-\frac{3}{8}} = \frac{\frac{1}{2}}{\frac{5}{8}}$$

$$= \frac{1}{2} \times \frac{8}{5}$$

$$= \frac{4}{5}$$

10. A treasure has 6 similar gold coins and 9 silver coins. If 4 coins are to be randomly selected without replacement, the probability that the first 2 selected are gold and the last 2 silver _______ [Enter value up to 3 decimals].

Solution:

If G_i is the event that the i^{th} coin is gold

S_i is the event that the i^{th} coin is silver

$$P(W_1 \cap W_2 \cap B_3 \cap B_4) = P(W_1)P(W_2|W_1)P(B_3|W_1 \cap W_2)P(B_4|W_1 \cap W_2 \cap B_3)$$

$$= \frac{6}{15} \times \frac{5}{14} \times \frac{9}{13} \times \frac{8}{12}$$

$$= \frac{3\times2}{7\times13}$$

$$= \frac{6}{91} \approx 0.065$$

11. The King comes from a family of 2 children. What is the probability that the other child is his sister? [assume the countries with absolute primogeniture (first born)]

(a) $\frac{1}{3}$

(b) $\frac{2}{3}$

(c) $\frac{1}{2}$

(d) None of them

Solution:

Given that family has two children.

∴ There are 4 possibilities
$(b,g)\ (g,b)\ (b,b)\ (g,g)$

It is given that there is a King. So now there are
$(b,g)\ (g,b)\ \left(\underline{a},b\right)$

It is also given that the country is absolute primogeniture so King can't have older sister.

Now $(S)=(b,g)\ (b,b)$

∴ The probability that the other child is girl is $=\frac{1}{2}$

12. In the Hogwarts school of witchcraft and wizardry Harry Potter opted for 3 subjects in his 3rd year. The exam starts from June. If he passes the charms exam in June then he will take the second exam potions in July and if he also passes that one, then he will take the third exam Herbology in September.

If he fails an exam, then he is not allowed any other. The probability that he passes the Charms exam is 0.9. If he passes the Charms exam then the Conditional probability that he passes the Potions is 0.8, and if he passes the both Charms and Potions then the conditional probability that he passes the Herbology exam is 0.7.

(a) the probability that he passes all three exams ________ [up to 3 decimal digits]

Solution:

$$P(E_3)=P(E_1E_2E_3)=P(E_1)P(E_2|E_1)P(E_3|E_1E_2)$$

$$=0.9\times0.8\times0.7$$

$$=0.504$$

(b) Given that he did not pass all three exams, the conditional probability that he failed in Potions is _________ [up to 3 decimal digits]

Solution:

If he fails the Potions exam then he must have passed the Charms

$$P\left(E_1E_2' \middle| \underbrace{E_3'}\right)$$

Since he failed in 2^{nd}
he will fail in 3^{rd} *also*

$$P(E_1E_2'|E_3') = \frac{P(E_1E_2'E_3')}{P(E_3')}$$

$$= \frac{P(E_1E_2')}{P(E_3')} \{\because He\ can'twrite\ E_3\ if\ he\ failed\ in\ E_2\}$$

$$P(E_2|E_1) = 0.8$$

$$P(E_2'|E_1) = 0.2$$

$$P(E_1E_2') = P(E_2'|E_1) \times P(E_1)$$

$$= 0.2 \times 0.9 = 0.18$$

$$P(E_1E_2'|E_3') = \frac{0.18}{1 - P(E_3)}$$

$$= \frac{0.18}{1-0.504}$$

$$= \frac{0.18}{0.496}$$

$$= 0.3629$$

13. Sixteen players P_1, P_2, P_3 ... ... P_{16} play in a tournament. They are divided into eight pairs at random, from each pair a winner is decided on the basis of a game played between the two players of the pairs. Assuming that all the players are of equal strength, the probability that exactly one of the two players P_1 and P_2 is among the eight winners is ________.

(a) $\frac{11}{15}$

(b) $\frac{7}{15}$

(c) $\frac{8}{15}$

(d) $\frac{17}{30}$

Solution:

Let E_1 denote the event that P_1 and P_2 are paired
E_2 denote the event that P_1 and P_2 are not paired

Let A denote the event that one of two players P_1 and P_2 is among the winners.

Since, P_1 can be paired with any of the remaining 15 players

$$P(E_1) = \frac{1}{15}$$

$$P(E_2) = 1 - P(E_1) = 1 - \frac{1}{15} = \frac{14}{15}$$

In case E_1 occurs, it contains that one of P_1 and P_2 will be among the winners. In case E_2 occurs, the probability that exactly one of the P_1 and P_2 is among the winner is

$$P\left((P_1 \cap \bar{P}_2) \cup (\bar{P}_1 \cap P_2)\right) = P(P_1 \cap \bar{P}_2) + P(\bar{P}_1 \cap P_2)$$

$$= P(P_1)P(\bar{P}_2) + P(\bar{P}_1)P(P_2)$$

$$= \left(\tfrac{1}{2}\right)\left(1 - \tfrac{1}{2}\right) + \left(1 - \tfrac{1}{2}\right)\left(\tfrac{1}{2}\right)$$

$$= \tfrac{1}{4} + \tfrac{1}{4}$$

$$= \tfrac{1}{2}$$

$$\therefore P\left(\tfrac{A}{E_1}\right) = 1 \qquad P\left(\tfrac{A}{E_2}\right) = \tfrac{1}{2}$$

$$\therefore P(A) = P\left(\frac{A}{E_1}\right)P(E_1) + P\left(\frac{A}{E_2}\right)P(E_2)$$

$$= 1 \times \tfrac{1}{15} + \tfrac{1}{2}\left(\tfrac{14}{15}\right)$$

$$= \tfrac{1}{15}(1 + 7)$$

$$= \tfrac{8}{15}$$

14. You are in a game show! There are 10 closed doors, 0 leads to nothing and 1 leads to an expensive sports car. You are allowed to pick a door and earn the sports car if it's behind the door you choose. You choose a door and the host tells you he was preauthorized to make your chance of winning better! You have two options.

Option 1: Get the right to open two doors and win if the car is behind either of the ones you open.

Option 2: Have the host open 5 empty doors [None of them the one you had choose] and then get the right to switch if you want.

If you want to win the car, what should you do?

(a) Go with option 2 and switch

(b) You should be indifferent

(c) Go with option 2, then don't switch

(d) Go with option 1

Solution:

$\Longrightarrow$ If you take option 1, then you have 2 chances in 10 to get the car, for a probability of 1/5 of Success.

$\Longrightarrow$ If you take option 2, and don't, your chance of Success remains equal to 1/10 [not helpful]

$\Longrightarrow$ If you take option and switch

There is 9/10 probability that the car is not in the first door you choose and ¼ probability in the remaining 4 closed doors. [one you already choose a host open 5 doors]

∴ The probability you get the car is

$$\frac{9}{10} \times \frac{1}{4} = \frac{9}{40} > \frac{1}{5}\left(\frac{8}{40}\right)$$

$$= \frac{9}{40} > \frac{8}{40}$$

and thus, option 2 with switching is better than option 1 by 1/40.

15. Question 14, where in option two the host will open 4 doors.

Solution:

Same solution as above

Option 2 with switch will have

$$= \frac{9}{10} \times \frac{1}{5} = \frac{9}{50} < \frac{1}{5}\left(\frac{10}{50}\right)$$

and thus, option 1 is better than option 2 with switching by 1/50.

16. Suppose there are two doctors of surgeries that doctors perform. Doctor A has a higher success rate than Doctor B on first type of surgery. Doctor A also has a higher success rate than Doctor B on second type of surgery. Is it true that Doctor A necessarily has higher overall success rate than Doctor B?

(a) It must be false

(b) Yes, it must be true

(c) It can be true but not necessarily

Solution:

Though this might at first sight appeal to true, that is not necessarily false.

Suppose that the two types of surgeries are vastly different in difficulty, and Doctor A just performs many more of the difficult surgery than Doctor B, bringing his average down.

On the first surgery, Doctor A can perform it with 95% success and Doctor B achieves 90% success. However, on the second, Doctor A has a 50% success rate and Doctor B has 10% success rate.

Doctor A performs the first surgery 20% of the time.

$$Success\ rate = \frac{20}{100} \times \frac{95}{100} + \frac{80}{100} \times \frac{50}{100} = 59\%$$

However, Doctor B performs the first surgery 80% of the time.

$$Success\ rate = \frac{80}{100} \times \frac{90}{100} + \frac{20}{100} \times \frac{10}{100} = 74\%$$

17. Two fair dice are rolled and it is revealed that one of the numbers rolled was a 4. What is the probability that the other number rolled was a 6?

Note: You are not told which of the numbers rolled is a 4.

(a) $\frac{1}{18}$ (b) $\frac{1}{6}$

(c) $\frac{2}{11}$ (d) $\frac{1}{36}$

Solution:

Let E_6 be the event of other dice showing 6.

E_4 be the event of one dice showing 4.

$$P\left(\frac{E_4}{E_6}\right) = \frac{P(E_4 \cap E_6)}{P(E_6)}$$

$$= \frac{2/36}{11/36} \begin{array}{l} ((4,6),(6,4)) \\ ((1,4),(2,4),(3,4),(4,4),(5,4),(6,4),(4,1),(4,2),(4,3),(4,5),(4,6)) \end{array}$$

$$= \frac{2}{11}$$

18. One green ball, one blue ball and two red balls are placed in a bowl, one draw two simultaneously from the bowl and announce that atleast one of them is red. What is the chance that the other ball one have drawn out is also red?

(a) $\frac{1}{3}$ (b) $\frac{1}{6}$

(c) $\frac{1}{5}$ (d) $\frac{1}{4}$

Solution:

There are 6 possible pairings of the two balls draw [$^4C_2$]

$R_1, R_2,\quad R_1, B,\quad R_1, G$
$B, R_2,\quad B_1, G \times [It\ is\ given\ that\ one\ ball\ is\ Red]$
G, R_2

∴ Now only 5 possible combinations remaining

∴ Therefore, the chance that the Red1 & Red 2 pairing has been drawn are 1 in 5 = 1/5

Method 2:

$P(A) = P(drawing\ one\ red\ ball) = \frac{5}{6} [R_1R_2\ \ R_1B_1\ \ R_2B_1\ \ R_1G_1\ \ R_2G_1\ \ BG]$

$$P(A \cap B) = P(drawing\ both\ red\ balls) = \frac{1}{6}$$

$$P\left(\frac{B}{A}\right) = P(drawing\ 2^{nd}\ red\ ball\ given\ one\ ball\ is\ red)$$

$$= \frac{P(both\ red\ balls)}{P(one\ red\ ball)}$$

$$= \frac{P(A \cap B)}{P(A)}$$

$$= \frac{\frac{1}{6}}{\frac{5}{6}}$$

$$= \frac{1}{5}$$

19. On planet XZ101X, there are two types of creatures: small brain and big brain.

Small brain tells the truth 6/7 of the time and only 1/7 of the time, while big brain tells the truth 1/5 of the time and lie 4/5 of the time.

It is also known that there is 2/3 chance a creature from XZ101X is a small brain and a 1/3 chance that it is a big brain but there is no way of differentiating from these two types.

You are visiting XZ101X on a research trip. During your stay you come across a creature who states that it has found a one line proof format's last theorem.

Immediately after that, a second creature shows up and states that the first creatures statement was a true one.

If the probability that the first creature's statement was actually true is a/ b for some co-prime positive integer a, b. The value of $b - a =$ ________.

(a) 4489 (b) 1444

(c) 5933 (d) Can't determine

Solution:

Let A be the event the first creature's statement was truth
B be the event that the second creature says that the first creature's statement was true.

$$P\left(\frac{A}{B}\right) = \frac{P(A \cap B)}{P(B)}$$

In order to find $P(A \cap B)$ and $P(B)$ we will consider four cases based on the type of creatures.

<u>Case 1</u>: Consider the first one is tell the truth and 2nd one also telling the truth.

1st	2nd	Probability	
small	small	(2/3) (6/7) (2/3) (6/7)	2/3 : probability that the creature is small
small	big	(2/3) (6/7) (1/3) (1/5)	6/7 : it tells truth
big	small	(1/3) (1/5) (2/3) (6/7)	
big	big	(1/3) (1/5) (1/3) (1/5)	

$$\therefore P(A\cap B)=\frac{2}{3}\cdot\frac{6}{7}\cdot\frac{2}{3}\cdot\frac{6}{7}+\frac{2}{3}\cdot\frac{6}{7}\cdot\frac{1}{3}\cdot\frac{1}{5}+\frac{1}{3}\cdot\frac{1}{5}\cdot\frac{2}{3}\cdot\frac{6}{7}+\frac{1}{3}\cdot\frac{1}{5}\cdot\frac{1}{3}\cdot\frac{1}{5}$$

$$=\left(\frac{2}{3}\cdot\frac{6}{7}\right)^2+2\left(\frac{2}{3}\cdot\frac{6}{7}\cdot\frac{1}{3}\cdot\frac{1}{5}\right)+\left(\frac{1}{3}\cdot\frac{1}{5}\right)^2$$

$$=\left(\frac{2}{3}\cdot\frac{6}{7}+\frac{1}{3}\cdot\frac{1}{5}\right)^2$$

$$=\left(\frac{12}{21}+\frac{1}{15}\right)^2$$

$$=\left(\frac{67}{105}\right)^2$$

<u>Case 2</u>: First one tells truth second one tells lie [we don’t need to consider this case, it is already given that second one told first one’s statement was true]

<u>Case 3</u>: First one tells lie second one tells truth [for the same reason we will ignore this one]

<u>Case 4</u>: First one tells lie second one also tells lie

1st	2nd	Probability
small	small	$\left(\frac{2}{3}\right)\left(\frac{1}{7}\right)\left(\frac{2}{3}\right)\left(\frac{1}{7}\right)$
small	big	$\left(\frac{2}{3}\right)\left(\frac{1}{7}\right)\left(\frac{1}{3}\right)\left(\frac{4}{5}\right)$
big	small	$\left(\frac{1}{3}\right)\left(\frac{4}{5}\right)\left(\frac{2}{3}\right)\left(\frac{1}{7}\right)$
big	big	$\left(\frac{1}{3}\right)\left(\frac{4}{5}\right)\left(\frac{1}{3}\right)\left(\frac{4}{5}\right)$

$$Sum\ of\ Probabilities=\left(\frac{2}{3}\cdot\frac{1}{7}\right)^2+2\left(\frac{2}{3}\cdot\frac{1}{7}\cdot\frac{1}{3}\cdot\frac{4}{5}\right)+\left(\frac{1}{3}\cdot\frac{4}{5}\right)^2$$

$$= \left(\frac{2}{3} \cdot \frac{1}{7} + \frac{1}{3} \cdot \frac{4}{5}\right)^2$$

$$= \left(\frac{2}{21} + \frac{4}{15}\right)^2$$

$$= \left(\frac{38}{105}\right)^2$$

∴ P (2^{nd} creature telling that the 1^{st} creature's statement is true)

$$\therefore P(B) = P(case\ 1) + P(csae\ 2)$$

$$= \frac{(67)^2}{(105)^2} + \left(\frac{38}{105}\right)^2$$

$$P\left(\frac{A}{B}\right) = \frac{P(A \cap B)}{P(B)} = \frac{(67)^2}{(67)^2 + (38)^2}$$

$$\therefore a = (67)^2 \qquad b = (67)^2 + (38)^2$$

$$b - a = (38)^2 = 1444$$

20. A bag contains a blue ball, some red balls and some green balls, you reach into bag and pulls out three balls at random. The probability you pull out one of each color is exactly 3%. How many balls were initially in the bag, if the product of number of red balls and green balls given = 598.

(a) 49 (b) 50

(c) 51 (d) 52

Solution:

Let P(R, G, B) = the probability of choosing 3 different colors if there are R reds, G greens, B blues

$$= \frac{R_{C_1} \times G_{C_1} \times B_{C_1}}{R+G+3_{C_3}}$$

$$= \frac{R \times G \times B}{R+G+B_{C_3}}$$

Given B = 1

$$\therefore P(R, G, 1) = \frac{R \times G}{R+G+1_{C_3}}$$

$P(R, G, 1) = 3\%$

$\frac{RG}{R+G+1_{C_3}} = \frac{3}{100}$

$RG = \frac{3}{100} \times \frac{(R+G+1)(R+G)(R+G-1)}{6}$

$200RG = (R+G)((R+G)^2 - 1)$

$(R+G)^3 - (R+G) = 200RG$

If you go with the options and verify this equation,

$(49)^2 - 49 = 200 \times 21 \times 28$

$117600 = 117600$

$\therefore$ Sum of all balls equal to

$49 + 1 = 50$

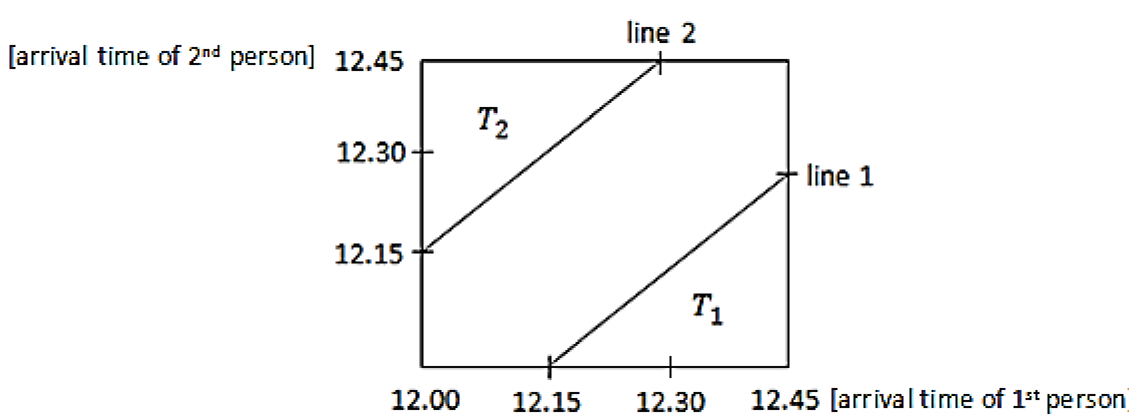
21. Two people have to spend exactly 15 consecutive minutes in a bar on a given day between 12.00 to 13.00. Assume uniformly arrival times, what is the probability that they will meet.

Solution:

Let's try to answer this question graphically.

$\Longrightarrow$ the two people can't arrive after 12.45 since they have to spend atleast 15min.

$\Longrightarrow$ they meet if their arrival times differ by less than 15min.



Some of the meeting points:

(12.00, 12.15) or (12.15, 12.00)
(12.30, 12.45) or (12.45, 12.30)

They meet if the point representing this arrival times is between the two lines (line 1, line 2).

∴ We just need to calculate the area in between the lines (line 1, line 2)

$$= total\ area - area(T_1) - area(T_2)$$

$$= 45 \times 45 - \frac{1}{2} \times 30 \times 30 - \frac{1}{2} \times 30 \times 30$$

$$= 45 \times 45 - 30 \times 30$$

$$= 2025 - 900$$

$$= 1125$$

$$\therefore Probability = \frac{area\ between\ the\ lines}{whole\ area\ of\ the\ square} = \frac{1125}{2025} = \frac{5}{9}$$

22. In a factory there are 100 units of certain product, 5 of which are defective. We pick three units from the 100 units at random what is the probability that none of them are defective? ________ [Enter up to 3 decimal digits]

Solution:

Let A_i be the event that the i^{th} chosen unit is not defective, for $i = 1, 2, 3$

By multiplication theorem,

$P(A_1 \cap A_2 \cap A_3) =?$

$$= P(A_1)P(A_2/A_1)P(A_3/A_1, A_2)$$

$$P(A_1) = \frac{95}{100}\ [It\ is\ given\ that\ 5\ are\ defective]$$

$$P(A_2/A_1) = \frac{94}{99}\left[\begin{array}{l} Since\ we\ have\ already\ selected\ one\ good\ one, the\ sample \\ space\ is\ (100-1) and\ number\ of\ non-defective\ (95-1) \end{array}\right]$$

$$P(A_3/A_1, A_2) = \frac{93}{98}$$

$$\therefore P(A_1 \cap A_2 \cap A_3) = \frac{95}{100} \times \frac{94}{99} \times \frac{93}{98} = 0.8650$$

23. I toss a coin repeatedly until J observes the first tails at which point J stop. Let X be the total number of coin tosses. $P(X = 5)$ is __________

(a) $\frac{1}{2}$

(b) $\frac{1}{5}$

(c) $\frac{1}{32}$

(d) $\frac{1}{64}$

Solution:

$P(X = 5)$ means that the first 4 coin tosses result in heads and the fifth one results in tails.

Thus, the probability of sequence $H\ H\ H\ H\ T$ when tossing a coin five times

$P(H\ H\ H\ H\ T) = P(H) \cdot P(H) \cdot P(H) \cdot P(H) \cdot P(T)$

$$= \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2}\ [all\ are\ independent\ event]$$

$$= \frac{1}{32}$$

24. Suppose that the probability of being killed in a single flight in $P_c = \frac{1}{4\times10^6}$ based or available statistics. Assume that different flights are independent if a man takes 5 flights per year the probability that he is killed in a plane crash within the next 4 years is _________ (approximately).

(a) $\frac{1}{10000}$

(b) $\frac{1}{5\times10^5}$

(c) $\frac{1}{4\times10^5}$

(d) $\frac{1}{2\times10^5}$

Solution:

Number of flights he will take in next 20 years $= 5\times4 = 20$

Let E be the event that the business man is killed in a plane crash within next 20 years

$$P(E) = 1 - P\ \underbrace{(he\ will\ survives\ in\ all\ 400\ flights)}_{Let\ F}$$

$$= 1 - P(F)$$

$P(F) = P_s \times P_s \ldots \ldots\ P_s\ (N\ times = 400\ times)$ as the flights are independent

$$= {P_s}^N$$

$= (1 - P_c)^N$

$\therefore P(E) = 1 - (1 - P_c)^N$

$= 1 - \left(1 - \frac{1}{4\times10^6}\right)^{20}$

$= 1 - (1 - 0.25\times10^{-6})$

$= 1 - 0.999995$

$= 0.000005$

$= \frac{5}{10^6}$

$= \frac{1}{2\times10^5}$

25. I have three bags that each contains 100 marbles:

Bag 1: 75 red and 25 blue
Bag 2: 60 red and 40 blue
Bag 3: 45 red and 56 blue

I choose one of the bag at random and then pick a marble from the chosen bag also at random.

What is the probability that the chosen marble is red?

(a) 0.60

(b) 0.75

(c) 0.45

(d) 0.50

Solution:

Let $R \longrightarrow$ $the\ event\ that\ the\ chosen\ marble\ is\ red$

$B_i \longrightarrow the\ event\ that\ choose\ Bag\ i$

$P(R/B_1) = \frac{75}{100}$

$P(R/B_2) = \frac{60}{100}$

$P(R/B_3) = \frac{45}{100}$

$P(R) = P(R/B_1)\ P(B_1) + P(R/B_2)\ P(B_2) + P(R/B_3)\ P(B_3)$

$$= 0.75\times\frac{1}{3} + 0.60\times\frac{1}{3} + 0.45\times\frac{1}{3} = 0.60$$

26. A box contains three coins: two regular coins and one take two-headed coin $(P(H) = 1)$.

You pick a coin at random and toss it. What is the probability that it lands heads up?

(a) $\frac{1}{3}$ (b) $\frac{1}{4}$

(c) $\frac{2}{3}$ (d) $\frac{1}{2}$

Solution:

Let C_1 be the event that you choose a regular coin

Let C_2 be the event that you choose the two-headed coin

$$P(H/C_1) = 1/2$$

$$P(H/C_2) = 1$$

$$\therefore P(H) = P(H/C_1)\,P(C_1) + P(H/C_2)\,P(C_2)$$

$$= \frac{1}{2}\times\frac{2}{3} + 1\times\frac{1}{3}$$

$$= \frac{1}{3} + \frac{1}{3}$$

$$= \frac{2}{3}$$

27. For three events A, B and C, we know that

- A and C are independent
- B and C are independent
- A and B are disjoint
- $P(A \cup C) = \frac{2}{3} \quad P(B \cup C) = \frac{3}{4} \quad P(A \cup B \cup C) = \frac{11}{12}$

$P(A) =$ ________

(a) $\frac{1}{2}$ (b) $\frac{1}{3}$

(c) $\frac{12}{17}$ (d) $\frac{2}{3}$

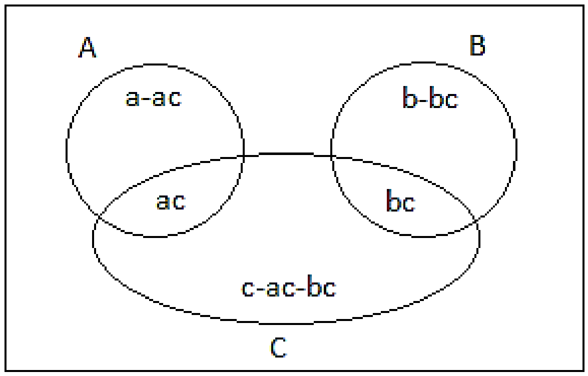
Solution:

Let's assume $P(A) = a$

$P(B) = b$

$P(C) = c$

Let's visualize these events



It's given

$$P(A \cup C) = a + c - ac = \frac{2}{3}$$

$$P(B \cup C) = b + c - bc = \frac{3}{4}$$

$$P(A \cup B \cup C) = a + b + c - ac - bc = \frac{11}{12}$$

$$a + b + c - ac - bc - a - c + ac - b - c + bc = \frac{11}{12} - \frac{2}{3} - \frac{3}{4}$$

$$-c = \frac{11}{12} - \frac{17}{12}$$

$$\boxed{c = \frac{1}{2}}$$

$$a + \frac{1}{2} - a\left(\frac{1}{2}\right) = \frac{2}{3}$$

$$\frac{a}{2} + \frac{1}{2} = \frac{2}{3}$$

$$\frac{a}{2} = \frac{2}{3} - \frac{1}{2} = \frac{4-3}{6}$$

$$\frac{a}{2} = \frac{1}{6}$$

$\boxed{a = \frac{1}{3}}$

28. Elliot bought a computer, the manual states that the life time T of the product [in years], the computer works properly until it breaks down, satisfies

$P(T \geq t) = e^{-t/5}\ , for\ all\ t \geq 0$

He purchases the computer and uses it for two years without any problems. What is the probability that it breaks down in the third year?

(a) $E_1 \rightarrow breaks\ down\ in\ third\ year$
$E_2 \rightarrow does't\ breaks\ down\ in\ first\ two\ year$

we have to find $P(E_1/E_2)$

$P(E_2) = P(T \geq 2) = e^{-2/5}$

$P(E_1) = P(2 \leq T \leq 3)$

$= P(T \geq 2) - P(T \geq 3)$

$= e^{-2/5} - e^{-3/5}$

$P\left(\frac{E_1}{E_2}\right) = \frac{P(E_1 \cap E_2)}{P(E_1)} \quad E_1 \subset E_2$

$= \frac{P(E_1)}{P(E_2)}$

$= \frac{e^{-\frac{2}{5}} - e^{-\frac{3}{5}}}{e^{-\frac{2}{5}}}$

$= 0.1813$

29. A machine produce parts that are either good (90%), slightly defective (2%) or obviously defective (8%) produced parts get passed through an automatic inspection machine, which is able to detect any part that obviously defective and discard it. What is the quality of the parts that make it through the inspection machine and get shipped?

(a) 0.978

(b) 0.987

(c) 0.990

(d) 0.988

Solution:

Let G be the event that a randomly chosen shipped part is good.

SD ⟶ slightly defective
OD ⟶ obviously defective

Given, P(G) = 0.9
P(SD) = 0.02
P(OD) = 0.08

We need to find the probability that a part is given that it is passed through inspection machine

$$P\left(\frac{G}{OD'}\right) = \frac{P(G \cap OD')}{P(OD')}$$

$$= \frac{P(G)}{P(OD')}$$

$$= \frac{0.90}{1-0.08}$$

$$= \frac{90}{92}$$

$$= 0.978$$

30. Your neighbor has two children; you learn that he has a son Joe. What is the probability that Joe's sibling is a brother?

(a) $\frac{1}{2}$ (b) $\frac{1}{3}$

(c) $\frac{1}{4}$ (d) $\frac{3}{4}$

Solution:

Let E be the Event that the neighbor has a son.

$$P(E) = \frac{3}{4}\ [E = \{(b, g)\ (g, b)\ (b, b)]$$

$$P(F) = P(Joe\ has\ a\ brother) = \frac{1}{4}[E = \{(b, b)\}]$$

$$P(F/E) = \frac{P(F \cap E)}{P(E)}$$

$$= \frac{P(F)}{P(E)}$$

$$= \frac{\frac{1}{4}}{\frac{3}{4}}$$

$$= \frac{1}{3}$$

Linked Question to 30

31. Your neighbor has 2 children. He picks one of them at random and comes by your house; he brings a boy named Elliot. What is the probability that Elliot's sibling is a brother?

(a) $\frac{1}{2}$ (b) $\frac{1}{3}$

(c) $\frac{1}{4}$ (d) $\frac{3}{4}$

Solution:

In the above Example had event E that "your neighbor has a son". Let's consider E' that "your neighbor randomly chooses one of his 2 children, and that chosen one is a son".

$\therefore E' \subset E$

$\therefore E'$ happening implies that event E happens.

It does not go the other way [Just because he has a son doesn't mean that he choose that son at random]

$$P(F/E') = \frac{P(F \cap E')}{P(E')}$$

$$= \frac{P(\{b,b\})}{P(E'/\{b,b\})P(\{b,b\}) + P(E'/\{bg\})\,P(\{bg\}) + P(E'/\{gb\})P(\{gb\}) + P(E'/\{gg\})P(\{gg\})}$$

$$= \frac{1/4}{1\times\frac{1}{4}+\frac{1}{2}\times\frac{1}{4}+\frac{1}{2}\times\frac{1}{4}+0\times\frac{1}{4}}$$

Selecting one boy from two son's

Selecting one boy from two daughters

$$= \frac{1/4}{1/4+1/4}$$

$$= \frac{1/4}{2/4}$$

$$= \frac{1}{2}$$

Linked question with 29: Consider the problem 29 again, but now assume that a one year given warranty is given for the parts that are shipped to the customer. Suppose that a good part fails within first year with probability 0.01, while a slightly defective part fails.

30. Within the first year with probability 0.10. What is the probability that a customer receives a part that fails within the first year and therefore entitled to a warranty replacement?

(a) 0.021 (b) 0.120

(c) 0.25 (d) 0.012

Solution:

We know that $P(G) = \frac{90}{92}$

$$P(SD) = \frac{2}{92}$$

Let E be the event that a randomly selected customer's part fails in the first year.

We have given that

$P(E/G) = 0.01$

$P(E/SD) = 0.10$

$P(E) = P(E/G)P(G) + P(E/SD)P(SD)$

$= 0.01\times\frac{90}{92} + 0.10\times\frac{2}{92}$

$= 0.012$

33. If six cards are selected at random (without replacement) from a standard deck of 52 cards, what is the probability there will be no pairs? (two cards of same denomination)

(a) (b)

(c) (d)

Solution:

Let E_i be the event that the first i cards have no pairs among them. Then we want to compute $P(E_6) = P(E_1 \cap E_2 \ldots\ldots\ldots E_6)$ $E_6 \subset E_5 \subset E_4 \ldots\ldots\ldots \subset E_1$

$P(E_1 \cap E_2 \ldots\ldots\ldots E_6) = P(E_1)P(E_2/E_1)P(E_3/E_1\,E_2)\ldots\ldots\ldots$

$$= \frac{52}{52}\times\frac{48}{51}\times\frac{44}{50}\times\frac{40}{49}\times\frac{36}{35}\times\frac{32}{47}$$

$$= \frac{44\times4\times12\times32}{17\times5\times49\times47}$$

34. Let A and B be independent events $P(A) = \frac{1}{4}, P(A \cup B) = 2P(B) - P(A), P(B'/A) =$ ________

(a) $\frac{2}{5}$ (b) $\frac{3}{5}$

(c) $\frac{1}{4}$ (d) $\frac{3}{4}$

Solution:

$$P(B' \cap A) = \frac{P(B' \cap A)}{P(A)}$$

$$= \frac{P(A) - P(B \cap A)}{P(A)}$$

$$= \frac{\frac{1}{4} - \left(\frac{2}{5}\right)\left(\frac{1}{4}\right)}{\frac{1}{4}}$$

$$= \frac{3}{5}$$

$$\because P(A \cup B) = P(A) + P(B) - P(A \cup B)$$

$$= \frac{1}{4} + P(B) - \frac{1}{4}P(B)$$

$$= \frac{1}{4} + P(B) - \frac{1}{4}P(B)$$

$$= 2P(B) - \frac{1}{4}$$

$$= \boxed{P(B) = 2/5}$$

35. Consider independent trails consisting of rolling a pair of fair dice, over and over. What is the probability that a sum of 5 appears before sum of 7?

(a) $\frac{1}{6}$

(b) $\frac{1}{5}$

(c) $\frac{2}{9}$

(d) None of them

Solution:

Let E be the event we see a sum 5 before a sum of 7.
Let F be the event the first roll is a 5.
Let G be the event that the first roll is a 7.
Let H be the event that the first roll is a sum other than 5 or 7.

$$P(F) = \frac{4}{36} \quad [(1,4)\ (4,1)\ (3,2)\ (2,3)]$$

$$P(G) = \frac{6}{36} \quad [(1,6)\ (6,1)\ (2,5)\ (5,2)\ (3,4)\ (4,3)]$$

$$P(H) = \frac{26}{36} \quad [1 - P(F) - P(G)]$$

$$P(E) = P(E/F)P(F) + P(E/G)P(G) + P(E/H)P(H)$$

$\underline{P(E/F)}$

Given that the first roll is 5, probability getting a 5 before 7 is 1.

$\underline{P(E/G)}$

Given that the first roll is 7, probability getting a 5 before 7 is 0.

$\underline{P(E/G)}$

Given that the first roll's sum is neither 5 nor 7, we can think of the process starts all over again.

The chance we get a 5 before 7 is just like it was $P(E)$ before we started rolling.

$$\boxed{P(E/H) = P(E)}$$

$$\therefore P(E) = 1\times\frac{4}{36} + 0\times\frac{6}{36} + \frac{26}{36}\times P(E)$$

$$P(E) = \frac{4}{36} + \frac{26}{36}P(E)$$

$$P(E)\frac{10}{36} = \frac{4}{36}$$

$$\boxed{P(E) = 2/5}$$

36. Given $P(A) = 0.9$ and $P(B) = 0.8$ the $\frac{maximum}{minimum}$ $value\ of\ P(A/B) =$ ______ [Enter upto 3 decimal values]

Solution:

<u>Minimum</u>:

$$P(A/B) = \frac{P(A \cap B)}{P(B)}$$

For minimum value of $P(A \cap B)$ we should have minimum of $P(A \cap B)$.

$$\therefore Min\ P(A \cap B) = P(A) + P(B) - P(A \cup B)$$

To minimize $P(A \cap B) = 1$

$P(A \cap B) = 0.8 + 0.9 - 1$

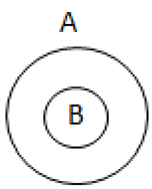
$\boxed{P(A \cap B) = 0.7}$

$\therefore min\ P(A \cap B) = \frac{0.7}{0.8} = \frac{7}{8} = 0.875$

<u>Maximum</u>:

$P(A/B) = \frac{P(A \cap B)}{P(B)}$

$max\big(P(A \cap B)\big)\ happens\ when\ B \subset A$



$\therefore P(A \cap B) = 0.8$

$$\therefore \big(P(0.8/0.8)\big) = \frac{0.8}{0.8} = 1$$