Zeromode-Fockspace

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1 Introduction

2 Energy Cutoff - Finite 'n'

Ground state for the system is given by

$$|0\rangle = \sqrt{1-\zeta^2}\sum_{n=0}^{\infty}\zeta^n |n\rangle_1 \otimes |n\rangle_2 \tag{1}$$

If we introduce a cut-off $[n_{max} = N]$ for the high energy states given by $|n\rangle_1 \otimes |n\rangle_2$, the form of entanglement entropy becomes

$$S(\zeta) = -\sum_{n=0}^{N} p_n \ln p_n = -\sum_{n=0}^{N}(1-\zeta^2)\zeta^{2n} \times \ln\{(1-\zeta^2)\zeta^{2n}\} \tag{2}$$

Now we plot $S(\zeta)$ for various values of N

We find that the divergence in entanglement entropy dissapears for finite 'N' values.

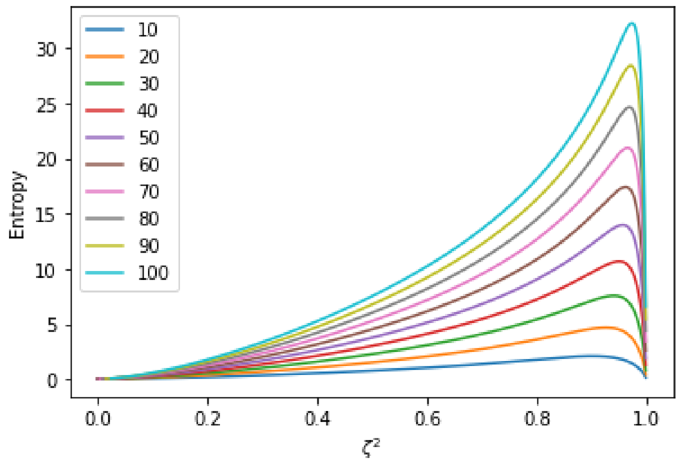


Figure 1: Entropy v/s ζ^2 for N=10 to N=100

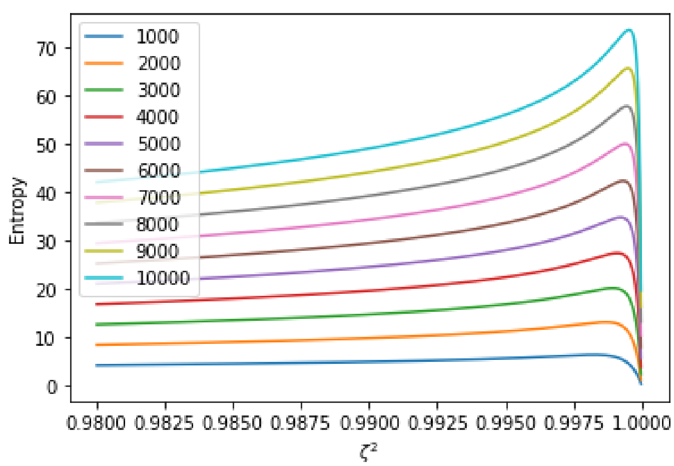


Figure 2: Entropy v/s ζ^2 for N=100 to N=10000

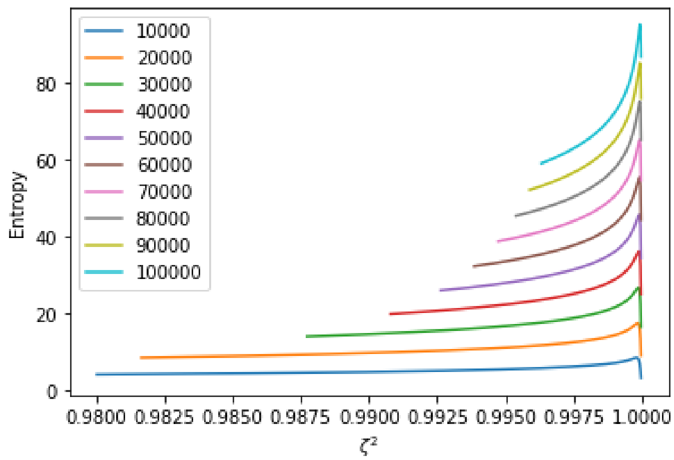


Figure 3: Entropy v/s ζ^2 for N=10000 to N=100000

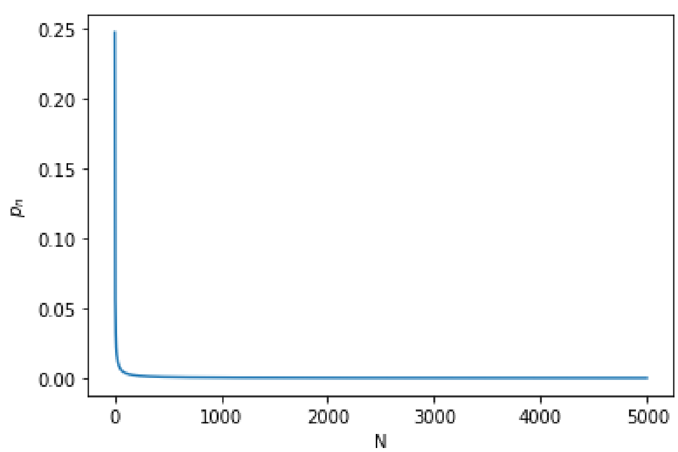


Figure 4: $p_n\, v/s\, N$ using E_n form

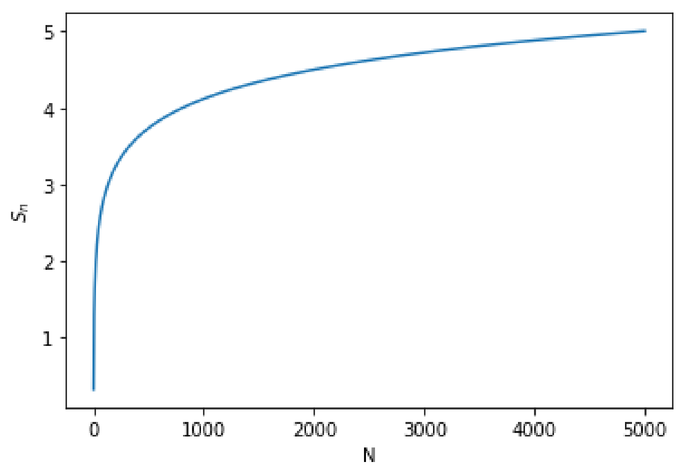


Figure 5: $S_n \, v/s \, N$ with constant E_n

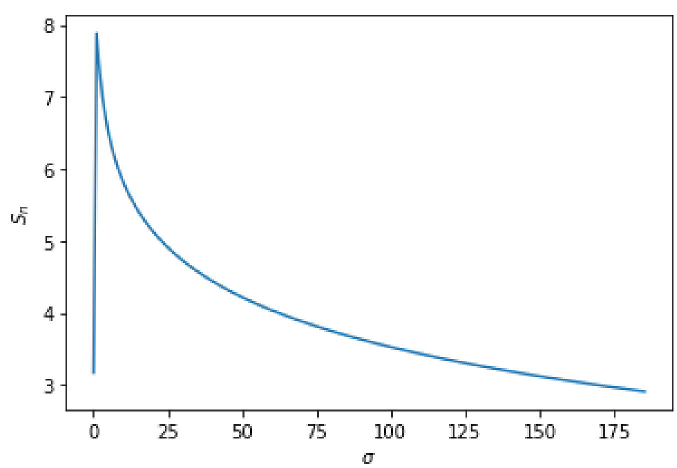


Figure 6: $S_n \, v/s \, E_n$ with constant n

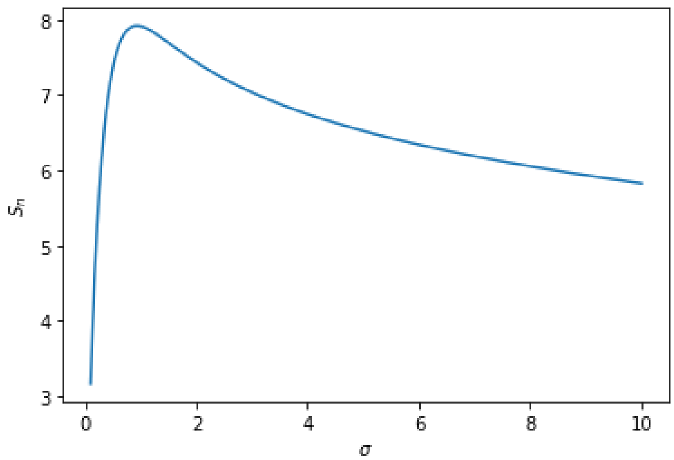


Figure 7: $S_n\, v/s\, E_n$ with constant n, domain [0.1,10]

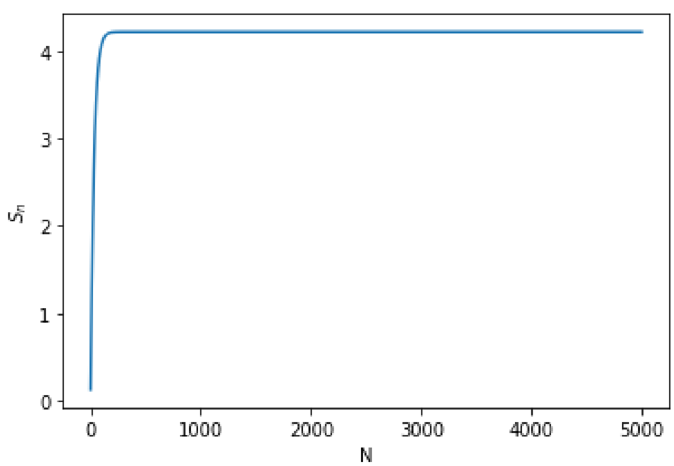


Figure 8: $S_n\, v/s\, N$ with constant δ

3 Hubble Radius

Here we use the Hubble radius as an upper limit on the wavelength of the harmonic oscillator.

The energy of n^{th} mode in a quantum harmonic oscillator is given by $E_n = \frac{1}{2}m\omega^2 A^2$, where A is the amplitude of the harmonic oscillator. For a dispersion-less system, we can consider the phase velocity, which is given by $v_n = \frac{\omega_n}{k}$. Here k is the wavenumber. For a harmonic oscillator, the relation $v_n = \omega_n A_n$also holds true. Thus $k = \frac{\omega_n}{v_n} = \frac{1}{A_n}$.

$$therefore \frac{2\pi}{\lambda} = \sqrt{\frac{m\omega^2}{2E_n}} = \sqrt{\frac{m\omega^2}{2(n\hbar\omega)}} = \sqrt{\frac{m\omega}{2n\hbar}} \tag{3}$$

$$\omega_{min} = \frac{8\pi^2\hbar}{m}\frac{n}{\lambda^2} \tag{4}$$

$$\lambda^2 = c_0 \frac{n}{\omega} \tag{5}$$

[Comment: Generally frequency is defined by the perameters of the source, while wavelength is dependent on the parameters of the medium of propagation. Here we are fixing a wavelegth cut-off and thus have variables ω and n] Assuming IR cut-off for $\lambda = \lambda_{\max}$, we have:

$$\lambda_{\max}^2 = c_0 \frac{N}{\omega} \tag{6}$$

Given a value of $\lambda_{\max}$, we can have several value of N and ω that can satisfy the above condition. N corresponds to the maximum number of energy eigen value that the system can have.

For the fock space representation,

$$\Omega = m\sqrt{\omega_+\omega_-} \tag{7}$$

Take $\sqrt{\omega_+} = \beta$. Set $\Omega = \omega$ for a given $\lambda_{\max}$.

$$\sqrt{\omega_-} = \frac{c_0}{m}\frac{N}{\lambda_{\max}^2\beta} \tag{8}$$

From earlier discussion we have

$$p_n = (1-\zeta^2)\zeta^{2n} = \frac{4\sqrt{\omega_-\omega_+}}{(\sqrt{\omega_-}+\sqrt{\omega_+})^2} \times \left(\frac{\sqrt{\omega_-}-\sqrt{\omega_+}}{\sqrt{\omega_-}+\sqrt{\omega_+}}\right)^{2n} \tag{9}$$

$\zeta = \left(\frac{\sqrt{\omega_-}-\sqrt{\omega_+}}{\sqrt{\omega_-}+\sqrt{\omega_+}}\right)$. Thus for the cut-off N, the above expression becomes

$$p_N = \frac{4N\lambda_{\max}^2\beta^2}{(N+\lambda_{\max}^2\beta^2)^2} \times \left(\frac{N-\lambda_{\max}^2\beta^2}{N+\lambda_{\max}^2\beta^2}\right)^{2N} \tag{10}$$

We can rewrite the above expression as

$$p_N = \frac{4\Delta}{(\Delta+1)^2} \times \left(\frac{\Delta-1}{\Delta+1}\right)^{2N} \qquad \Delta = \frac{N}{\lambda_{\text{max}}^2 \beta^2} \tag{11}$$

Define

Consider a cut-off N on the number of energy states that we add in the entanglement entropy calculation as in Eq(2).
$S(\zeta) = -\sum_{n=0}^{N} p_n \ln p_n$
This N gives us a minima on the angular frequency ω allowed for the system and hence the range on ζ allowed.

For each N, take the frequency ω to be this minimum frequency, and plot the entanglement entropy v/s ζ.

Note that the minimum frequency allowed for each state, increases with the state quantum number N. Thus for $N \to \infty$, $\omega_{min} \to \infty$. Thus if we include the higher energy states, the system can only have infinite frequency modes, which is unphysical.

Consider the relation of N in terms of λ and ζ.

$$\zeta = \frac{1-\Delta}{1+\Delta} \tag{12}$$

Since $\Delta = \frac{N}{\lambda_{max}^2 \beta^2}$ Implies,

$$\zeta = \frac{\lambda_{max}^2 - N}{\lambda_{max}^2 + N} \tag{13}$$

Therefore,

$$N = \lambda_{max}^2 \beta^2 \frac{1-\zeta}{1+\zeta} \tag{14}$$

So we are changing the cut-off energy by changing the Δ in Fig. 12 but summing over all energy levels.

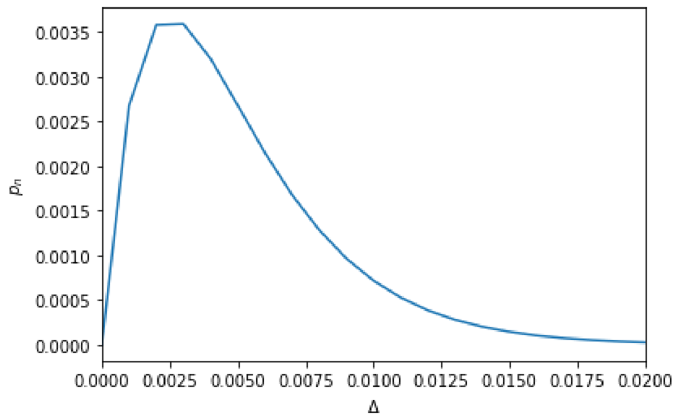


Figure 9: $p_N\ v/s\ \Delta$

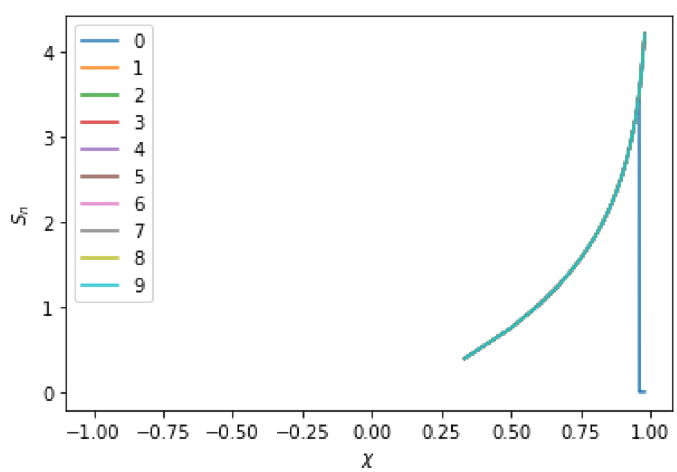


Figure 10: $S_n\ v/s\ \zeta$

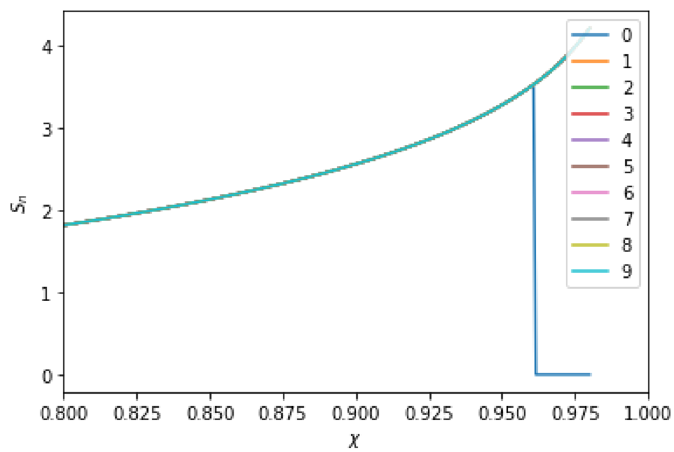


Figure 11: $S_n\, v/s\, \zeta$

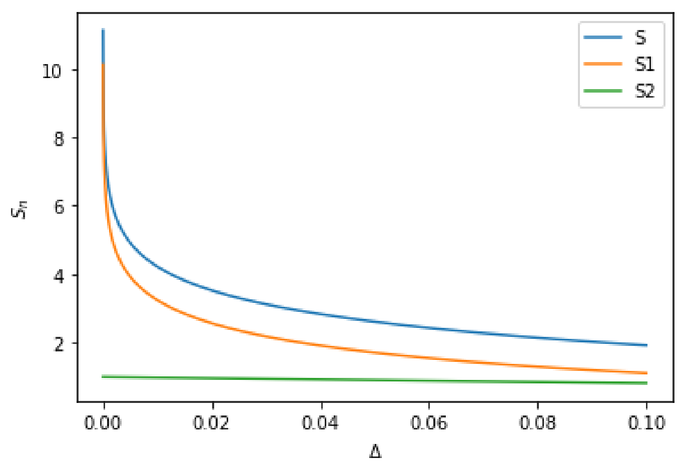


Figure 12: $S_n\, v/s\, \Delta$
$S = -\ln(1-\zeta^2) - \frac{\zeta^2}{1-\zeta^2}\ln(\zeta^2)$
$S_1 = -\ln(1-\zeta^2)$
$S_2 = -\frac{\zeta^2}{1-\zeta^2}\ln(\zeta^2)$

3.0.1 Convergence of Entanglement Entropy

Consider the complete summation form of entanglement entropy,

$$S_n(\zeta) = -\ln(1-\zeta^2) - \frac{\zeta^2}{1-\zeta^2}\ln\zeta^2 \tag{15}$$

(Why considering the infinite series summation?) here n represents the n^{th} energy level considered and hence defines a cut-off on frequency. To study the convergence of this entropy, we can look at how the difference $S_{N+1}-S_N$ evolves with N. The following plot shows the required result.

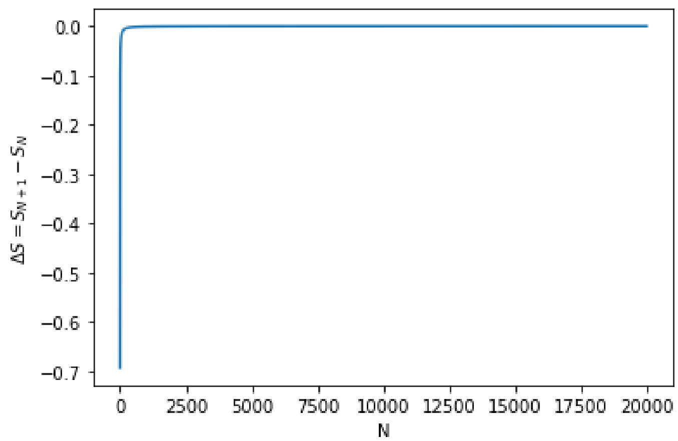


Figure 13: $\Delta S\, v/s\, N$ for N going to 20000. The saturation is clearly visible

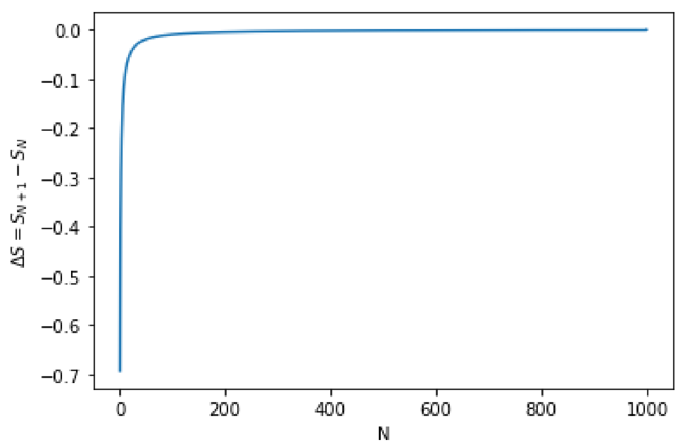


Figure 14: $\Delta S\, v/sN$ for N going till 1000

Take the log of log N on the X-axis. We find a linear relationship between $\ln(\ln N)$ and ΔS. From ref 1 we find that the IR divergence is logarithmic just like we find in the Fock space description. Thus the UV, IR and Fock space description of entanglement entropy diverges as $\ln x$.

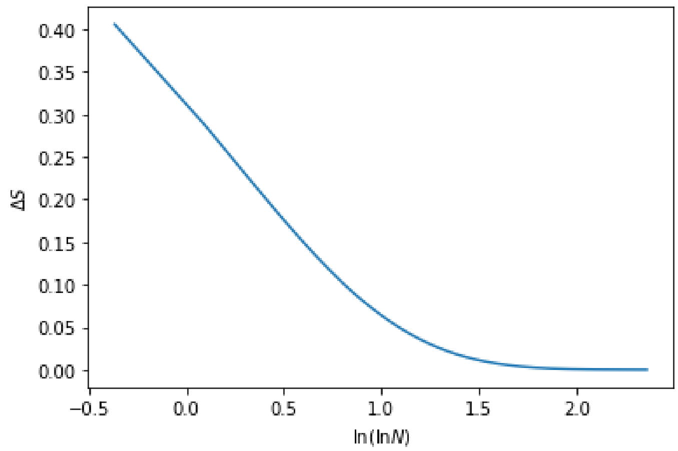


Figure 15: $\Delta S\, v/s\, \ln\ln N$

3.0.2 Connection between IR, UV and Fock space log divergence

IR cut-off is defined with a length parameter, in our case teh Hubble radius, and thus a frequency cut-off is applied. This frequency cut-off translates to a number N for the frequencies considered in the Fock space description, thus providing the connection.

$$\omega_{min} = \frac{8\pi^2\hbar}{m\lambda_{max}^2} n \tag{16}$$

Put $\lambda_{max} = R_H = \frac{c}{H_0} = 14.4 \times 10^9$ light years= 1.36×10^{22} m (Hubble Radius)

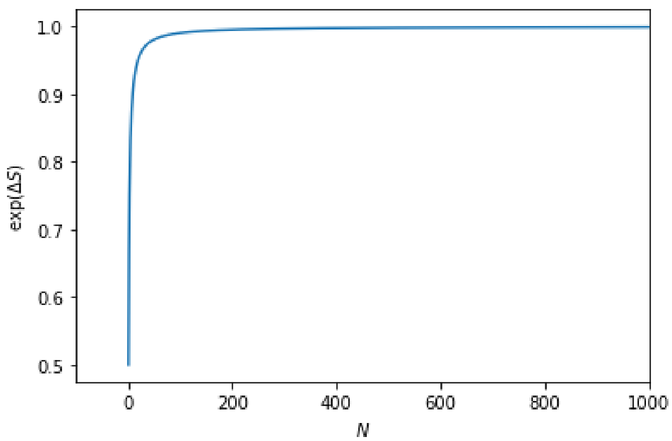
Therefore, $\omega_{min_{R_H}} = 4.5 \times 10^{-77} \times \frac{n}{m}\ s^{-1}$

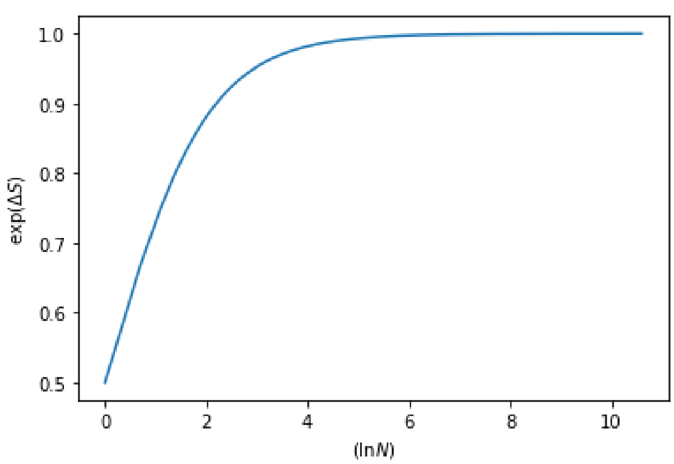
3.0.3 Cut-off in Curved Space-times

Use EUP modified commutation relations to get the general cut-off for entanglement entropy in curved space-times.

3.1 log divergence

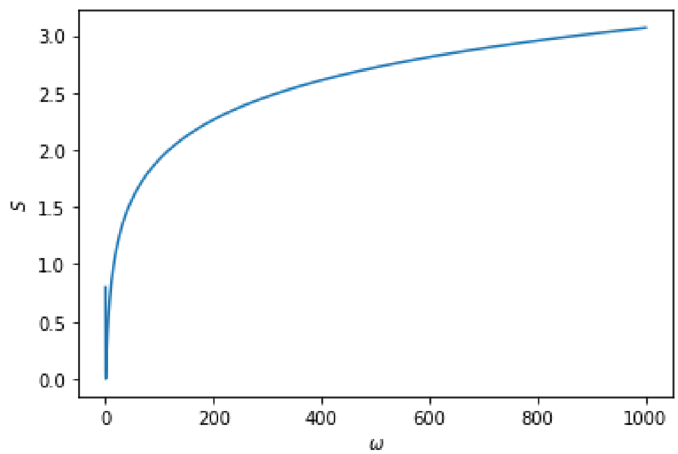
Consider the Boltzmann definition of entropy as $\exp(\Delta S)$. Beyond a certain value of N, we find saturation in ΔS, so beyond a certain N_0, S remains almost constant as shown in Fig 15. The S used here is over an infinite set of eigenvalues.

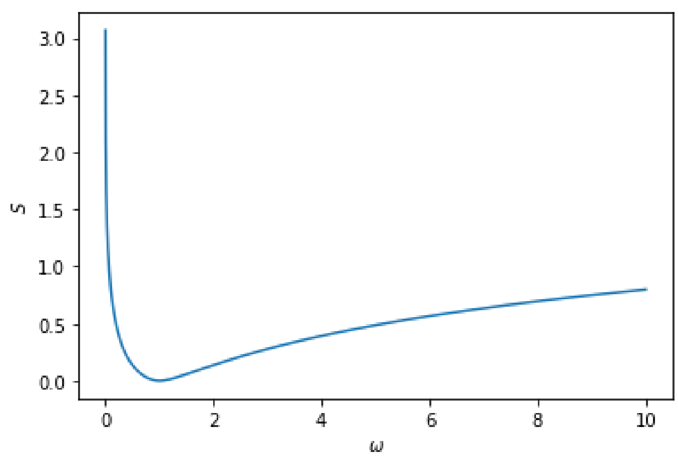

1.0
0.9
0.8
0.7
0.6
0.5
exp(ΔS)
0
200
400
600
800
1000
N


1.0
0.9
0.8
0.7
0.6
0.5
exp(ΔS)
0
2
4
6
8
10
(ln$N$)

3.2 ΔS v/s ω

Consider the entropy with respect to ω_-(or $\omega = m\sqrt{\omega_-\omega_+}$)
We observe a minima for $\omega_- = 1$ and a divergence in entropy as $\omega \to 0$.



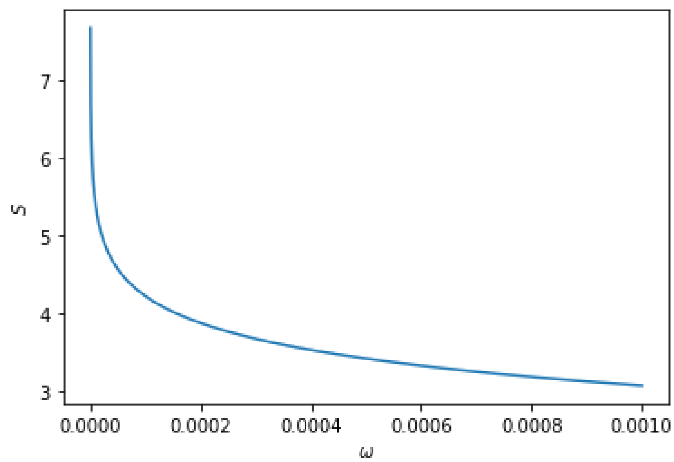


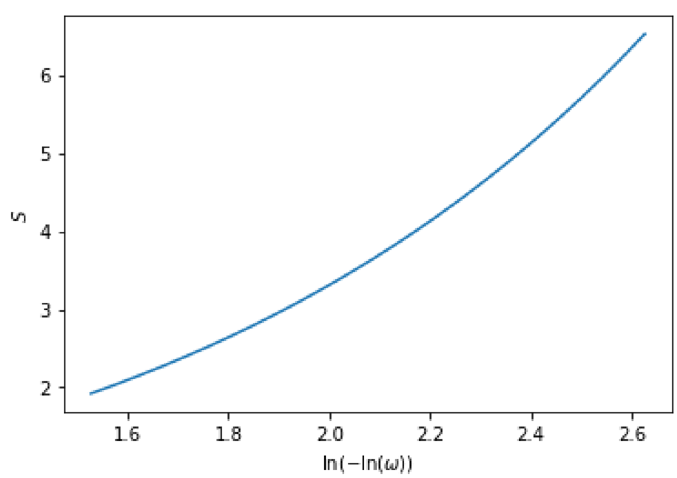
We find that S saturates with ω as ω increases. To confirm this we plot $\Delta S\, v/s\, \omega$.

To understand the variation of ΔS with ω, we plot $\Delta S\, v/s \ln(-\ln \omega)$

We observe that S {The sum over all Fock state eigenvalues}[ΔS] converges after a certain $\omega[N]$. This means that the entanglement entropy S becomes independent of $\omega[N]$. Thus, after a certain threshold(cut-off) on $\omega[N]$, the entanglement entropy is constant.

Earlier, we found a cut-off on ω from the value of the Hubble constant.


S
ω
7
6
5
4
3
0.0000
0.0002
0.0004
0.0006
0.0008
0.0010


S
ln(−ln(ω))
6
5
4
3
2
1.6
1.8
2.0
2.2
2.4
2.6

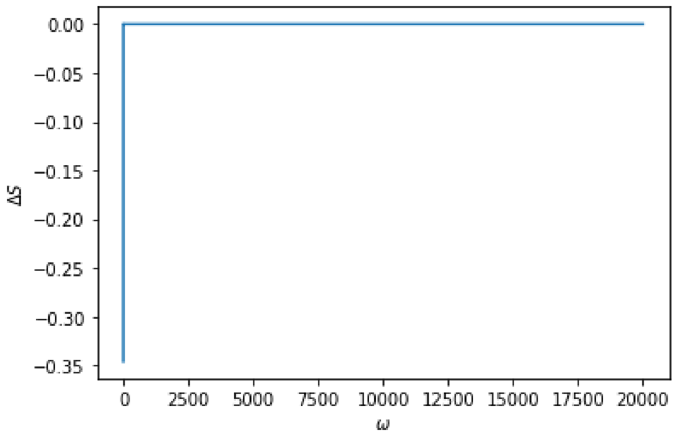


Figure 16: $\Delta S\, v/s\, \omega$

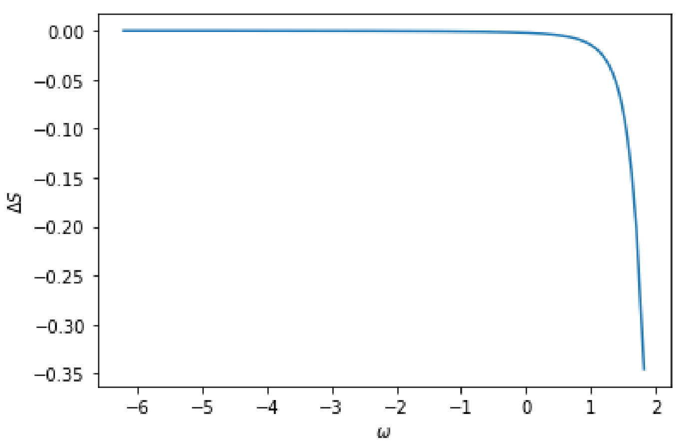


Figure 17: $\Delta S\, v/s\ \ln(-\ln\omega)$

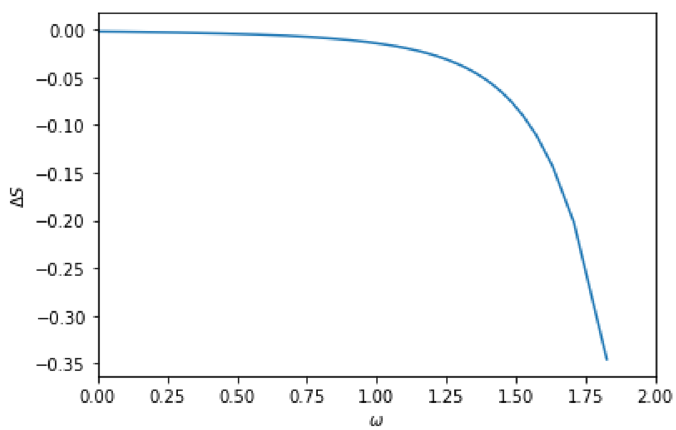


Figure 18: $\Delta S\, v/s\ \ln(-\ln \omega)$