Suppose T is a linear operator on V, a vector space over the field F. If p is a polynomial over F, then $p(T)$ is again a linear operator on V. If q is another polynomial over F, then

$$(p+q)(T) = p(T) + q(T)$$
$$(pq)(T) = p(T)q(T)$$

Therefore, the collection of polynomials p which annihilate T, in the sense that p(T) = 0, is an ideal in the polynomial algebra $F[x]$. It may be the zero ideal, i.e., it may be that T is not annihilated by any non-zero polynomial. But, that cannot happen if the space V is finite-dimensional.

Suppose T is a linear operator on the n-dimensional space V. Look at the first (n^2+1) powers of T:

$$I, T, T^2, \ldots, T^{n^2}.$$

`This is a sequence of $n^2 +1$ operators in $L(V,V)$, the space of linear operators on $V$.`

$$c_0 I + c_1 T + \ldots + c_{n^2} T^{n^2} = 0$$

for some scalars c_i, not all zero. So, the ideal of polynomials which annihilate T contains a non-zero polynomial of degree n^2 or less.

According to Theorem 5 of Chapter 4, every polynomial ideal consists of all multiples of some fixed monic polynomial, the generator of the ideal. Thus, there corresponds to the operator T a monic polynomial p with this property: If f is a polynomial over F, then $f(T) = 0$ if and only if $f = pg$, where g is some polynomial over F.

Definition. Let T be a linear operator on a finite-dimensional vector space V over the field F. The **minimal polynomial** for T is the (unique) monic generator of the ideal of polynomials over F which annihilate T.

The name ‘minimal polynomial’ stems from the fact that the generator of a polynomial ideal is characterized by being the monic polynomial of minimum degree in the ideal. That means that the minimal polynomial p for the linear operator T is uniquely determined by these three properties:

(1) p is a monic polynomial over the scalar field F.
(2) $p(T) = 0$.
(3) No polynomial over F which annihilates T has smaller degree than p has.

If A is an $n \times n$ matrix over F, we define the **minimal polynomial** for A in an analogous way, as the unique monic generator of the ideal of all polynomials over F which annihilate A.