Most of the discussion in the previous chapters were about the relations between categories, where we mostly talked about constructing various categories, how they relate through functors and conditions (like size and concreteness) imposed on said categories. The following discussion in different, as it relates to the internal structure and data of categories themselves.

First, we provide an alternate definition for a category to be concrete:

Definition 1. A functor $\mathrm{F} : \mathrm{C} \to \mathrm{D}$ between categories is called *faithful* if for any pair of objects $c, c' \in \mathrm{C}$, the induced function

$$\mathrm{F}_{c,c'} : \mathrm{Hom}(c, c') \to \mathrm{Hom}(\mathrm{F}c, \mathrm{F}c')$$

is injective.

Definition 2. A category C is said to be *concrete* if there exists a faithful functor $\mathrm{U} : \mathrm{C} \to \mathsf{Set}$.

Notice that the mapping induced by the functor on objects need not be injective. If it was the case (as it is with small categories) we consider the category C to be a subcategory of Set, which is exactly what we meant when we said that every small category is concrete.

Now, concreteness of a category is helpful for relating some terminology, which we describe below.

Definition 3. Given a morphism $f : a \to b$ in a category C, we say that f is

- a monomorphism if for any $x, y : c \to a$, $fx = fy$ implies that $x = y$,
- an epimorphism if for any $z, w : b \to d$, $zf = wf$ implies that $z = w$.

Example 1. We consider a couple of examples here. The monomorphisms in Set are exactly the injective functions, and the epimorphisms are exactly the surjective functions. Similarly, the monomorphisms in Grp are exactly the injective group homomorphisms, and the epimorphisms are the surjective homs.

As an exercise, either show the following proposition, or at least convince yourself that it is true:

Proposition 1. Given a concrete category C, a morphism $f : a \to b$ in C is a monomorphism (dually, an epimorphism) if and only if its underlying function $\mathrm{U}f : \mathrm{U}a \to \mathrm{U}b$ is injective (surjective).

Given any discussion with monomorphisms and epimorphisms, we also wish to consider the isomorphisms of the category. One can see that any monic and epic morphism in Set is also an isomorphism, and in a similar fashion, any injective and surjective group homomorphism is also a group isomorphism. Thus, would it be the case that any mono/epi in a category is an isomorphism?

Example 2 (Counter-example 1)**.** We show this isn't the case with some topology. Consider the following maps:

$$[0,1) \overset{i}{\hookrightarrow} [0,1] \overset{q}{\twoheadrightarrow} S^1$$

where i is the inclusion map, and q is the quotient map obtained by the obvious quotient construction. Now, notice that the composition qi is a bijective continuous map, which can more explicitly be written as $qi(t) := e^{2\pi i t}$. Now, by the fact that

$$0 = \pi_1([0,1)) \to \pi_1([0,1]) \to \pi_1(S^1) \cong \mathbb{Z}$$

is not an isomorphism, we see that the above map cannot be a homeomorphism.

Example 3 (Counter-example 2)**.** A purely categorical example is as follows: Given a poset $(P, \leq)$, we construct the category $\mathbb{P}$. Now, notice that for any morphism $f : a \to b$ in $\mathbb{P}$, and any pair $x, y : c \to a$, we have that $fx = fy$ by transitivity and the uniqueness of such maps. Also by uniqueness, we have $x = y$ vacuously. Thus, $fx = fy \implies x = y$ always holds, and thus any morphism in $\mathbb{P}$ is monic (and epic by the same argument).

Now, notice that any supposed isomorphism $f : a \to b$ has an inverse $g : b \to a$, with $fg = 1_b$ and $gf = 1_a$, but the condition is actually stronger than that: by antisymmetry, in such a case we have $a = b$. Thus, even though every morphism is both monic and epic, the only isomorphisms are the identity morphisms.

Although not every mono/epi is an iso, the converse is always true, and moreover, in any concrete category, the underlying function of an isomorphism is always a bijection.

Question 1. What is a sufficient condition for a mono/epi to ALWAYS be an isomorphism? (Hint: Check the exercises of Chapter 2 in Awodey.)

Now, we turn our attention to more internal concepts, which will both act as constructing blocks for later concepts and also as specialized forms of more general concepts (this will make more sense when we discuss (co)limits).

Definition 4. An object $t \in C$ is called an initial (terminal) object if for any object $c \in C$, there is a unique morphism $t \to c$ ($c \to t$).

Proposition 2. Given a category C, any two initial objects in C are isomorphic. Dually, any two terminal objects are isomorphic.

Proof. We prove this for one case, the dual argument follows immediately. Let t, t' be two initial objects in a category C. Then, notice that we have the following unique morphisms:

$$f : t \to t',\ g : t' \to t,\ 1_t : t \to t,\ 1_{t'} : t' \to t'$$

Notice that the uniqueness enforces the last two maps to be identities. Then, further notice that the compositions $fg : t' \to t'$ and $gf : t \to t$ further have to satisfy the uniqueness condition, and thus are equal to the respective identity morphisms. Thus, they are isomorphisms and $t \cong t'$. □

Here are some examples for initial and terminal objects:

(i) The empty set $\emptyset$ is initial in Set, and the singleton sets $\{*\}$ are terminal.
(ii) The empty category 0 is initial and $\mathbb{1}$ is terminal in Cat.
(iii) For the category Grp (and by extension, $\mathsf{Mod_R}$) has the group 1 as both its initial and terminal object.
(iv) The category Ring of unital rings has no terminal ring, but the ring $\mathbb{Z}$ is initial, as any unital ring hom must have $0 \mapsto 0$ and $1 \mapsto 1$.
(v) For (co)slice categories $\mathbf{C}/X$ ($X/\mathbf{C}$), the identity $1_X : X \to X$ is terminal (initial).

We will very quickly notice that the next few definitions of objects will always satisfy a certain condition, and will be the final objects (either initial or terminal) of ALL objects of its category that also satisfy the same afforementioned condition. We are intentionally keeping the definitions here vague, but these constructions will be the stepping stones for almost all the discussion we will have with (co)limits.

Definition 5. Given $f, g : c \to c'$ in a category C, the equalizer of f and g is an object e, along with a morphism $e : \varepsilon \to c$ such that $fe = ge$, and moreover for any $z : \zeta \to c$ with $fz = gz$, we have a unique $u : \zeta \to \varepsilon$ such that $eu = z$. Diagrammatically,

$$\begin{tikzcd}
\varepsilon \arrow[r, "e"'] & a \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & b \\
\zeta \arrow[u, "\exists! u"] \arrow[ur, "z"'] & &
\end{tikzcd}$$

A coequalizer of maps is an equalizer in C^{op}.

Do you see how the definition splits into two? There's the equalizing condition, and then there is the finality condition, stating that any other morphism equalizing f and g, there has to be a unique morphism u into ε that commutes. This is the essence of what we will lead up to with (co)limits; they will very literally be limiting objects (along with the respective morphisms).

Question 2. Show that every equalizer is monic, and every coequalizer is epic.

Here is a sketch diagram for the equalizer case:

$$\begin{tikzcd}
\varepsilon \arrow[r, "e"] & a \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & b \\
\zeta \arrow[u, bend left=60, "\exists! u"] \arrow[u, shift right, "x"] \arrow[u, shift left, "y"'] \arrow[ur, shift left, "ex"] \arrow[ur, shift right, "ey"'] & &
\end{tikzcd}$$

We conclude by giving definitions for pullbacks; as an exercise, define pushouts (dual pullbacks) explicitly as it is below, and observe the definining condition and the finality conditions.

Definition 6. Given the following structure (often called a *cospan*) $a \xrightarrow{f} c \xleftarrow{g} b$ in C, the pullback of the cospan is an object $a \times_c b$, along with morphisms λ_a, λ_b such that the following diagram commutes:

$$\begin{tikzcd}
a \times_c b \arrow[r, "\lambda_b"'] \arrow[d, "\lambda_a"] & b \arrow[d, "g"'] \\
a \arrow[r, "f"] & c
\end{tikzcd}$$

and for any other object p with morphisms π_a, π_b commuting its respective square, there is a unique morphism $u : p \to a \times_c b$ such that the following diagram commutes:

$$\begin{tikzcd}
p \arrow[dr, dashed, "\exists! u"] \arrow[drr, bend left, "\pi_b"] \arrow[ddr, bend right, "\pi_a"'] & & \\
& a \times_c b \arrow[r, "\lambda_b"'] \arrow[d, "\lambda_a"] & b \arrow[d, "g"'] \\
& a \arrow[r, "f"] & c
\end{tikzcd}$$

Question 3. How are equalizers/coequalizers related to kernels/cokernels in Ab? How are pushouts related to CW complexes in Top?