Proofs of Laplace Transform Properties

Utku Çolak

03.08.2020

Before we start, I think it is good to remind Integration By Parts formula here which we will use a lot while proving those theorems

$$\int u \cdot v' = u \cdot v - \int v \mathrm{d}u$$

Theorem 1:

Where **a** is an arbitrary constant value, laplace transform of **a** is

$$\mathscr{L}\{\ a\} = \frac{a}{s}$$

Proof:

$$\int_0^\infty a \cdot e^{-st}\mathrm{d}t = a \cdot \int_0^\infty e^{-st} dt = a \cdot \left.\frac{e^{-st}}{-s}\right|_0^\infty$$

$$a \cdot (0 - (\frac{1}{-s})) = \frac{a}{s}$$

Theorem 2:

$$\mathscr{L}\{\ t\} = \frac{1}{s^2}$$

Proof:

$$\mathscr{L}\{\ t\} = \int_0^\infty t \cdot e^{-st}\mathrm{d}t$$

By using Integration By Parts techniques, **u** function can be set as **t** and **dv** function can be set as e^{-st}. Thus, **v** function can be found as $\frac{e^{-st}}{-s}$ After all that, equation can be obtained as

$$= t \cdot \left.\frac{e^{-st}}{-s}\right|_0^\infty - \int_0^\infty e^{-st}\mathrm{d}t \quad (1)$$

$$= 0 - \left.\frac{e^{-st}}{-s}\right|_0^\infty \quad (2)$$

$$= 0 - \frac{1}{-s^2} \quad (3)$$

$$= \frac{1}{s^2} \quad (4)$$

Theorem 3:

$$\mathscr{L}\{\ t^n\} = \frac{n!}{s^{n+1}}$$

Proof: While proving that theorem, Integration By Parts rule will be used as it has been already used at Theorem 2.

$$\mathscr{L}\{\ t^n\} = \int_0^\infty t^n \cdot e^{-st}\mathrm{d}t \tag{5}$$

$$= t^n \cdot \frac{e^{-st}}{-s}\Big|_0^\infty - \frac{n}{-s}\int_0^\infty e^{-st}\cdot t^{n-1}\mathrm{d}t \tag{6}$$

Since first part of equation is zero, which can be confirmed by L'Hospital Rule, we shall apply another Integration By Parts rule to the second part of the equation

$$= t^n \cdot \frac{e^{-st}}{-s}\Big|_0^\infty - \frac{n}{-s}\int_0^\infty e^{-st}\cdot t^{n-1}\mathrm{d}t \tag{7}$$

$$= \frac{n}{s}\cdot t^{n-1}\cdot \frac{e^{-st}}{-s}\Big|_0^\infty - \frac{n-1}{-s}\int_0^\infty e^{-st}\cdot t^{n-2}\mathrm{d}t \tag{8}$$

$$= \frac{n}{s}\cdot\frac{n-1}{s}\int_0^\infty e^{-st}\cdot t^{n-2}\mathrm{d}t \tag{9}$$

$$= \frac{n}{s}\cdot\frac{n-1}{s}\cdot\frac{n-2}{s}...(t\cdot\frac{e^{-st}}{-s}\Big|_0^\infty) - \frac{1}{-s}\int_0^\infty e^{-st}\mathrm{d}t \tag{10}$$

Since integral of $e^{-st}\mathrm{d}t$ is equal to $\frac{1}{s}$, last term of the equation will become

$$= \frac{n}{s}\cdot\frac{n-1}{s}\cdot\frac{n-2}{s}...(t\cdot\frac{e^{-st}}{-s}\Big|_0^\infty) - \frac{1}{-s}\cdot\frac{1}{s} \tag{11}$$

$$= \frac{n!}{s^{n+1}} \tag{12}$$

Theorem 4:

$$\mathscr{L}\{\ sin(at)\} = \frac{a}{s^2+a^2}$$

Proof:

$$Let\ \mathscr{L}\{\ sin(at)\} = \int_0^\infty e^{-st}\cdot sin(at)\mathrm{d}t = y \tag{13}$$

$$\int_0^\infty e^{-st}\cdot sin(at)\mathrm{d}t = sin(at)\cdot\frac{e^{-st}}{-s}\Big|_0^\infty - \int_0^\infty e^{-st}\cdot\frac{a}{-s}\cdot cos(at)\mathrm{d}t \tag{14}$$

Once again, first part of the equation is zero so we can ignore it, also due to double minus sign we can ignore also minus sign and just put a plus sign there, let's simplify that equation and we can see we need to apply another Integration By Parts rule there...

$$y = \frac{a}{s} \cdot \int_0^\infty e^{-st} \cdot cos(at)\mathrm{d}t \quad = \frac{a}{s} \cdot [cos(at) \cdot \frac{e^{-st}}{-s}\Big|_0^\infty - \frac{a}{s}\int_0^\infty e^{-st} \cdot sin(at)\mathrm{d}t] \tag{15}$$

Since $cos(at) \cdot \frac{e^{-st}}{-s}\Big|_0^\infty$ is equal to $\frac{1}{s}$ and very last part of the equation, $\int_0^\infty e^{-st} \cdot sin(at)\mathrm{d}t$ is equal to **y** as we defined before. So last form the equation is,

$$y = \frac{a}{s} \cdot \frac{1}{s} - (\frac{a}{s} \cdot y) \tag{16}$$

$$\frac{a}{s^2} = y \cdot (1 + \frac{a^2}{s^2}) \tag{17}$$

$$y \cdot \frac{a^2 + s^2}{s^2} = \frac{a}{s^2} \tag{18}$$

$$y = \frac{a}{a^2 + s^2} \tag{19}$$

Theorem 5:

$$\mathscr{L}\{\ cos(at)\} = \frac{s}{s^2 + a^2}$$

Proof:

$$Let\ \mathscr{L}\{\ cos(at)\} = \int_0^\infty e^{-st} \cdot cos(at)\mathrm{d}t = y \tag{20}$$

$$\int_0^\infty e^{-st} \cdot cos(at)\mathrm{d}t = cos(at) \cdot \frac{e^{-st}}{-s}\Big|_0^\infty - \frac{a}{s}\int_0^\infty e^{-st} \cdot sin(at)\mathrm{d}t \tag{21}$$

$$y = \frac{1}{s} - \frac{a}{s} \cdot [sin(at) \cdot \frac{e^{-st}}{-s} - \frac{a}{-s} \cdot \int_0^\infty e^{-st} \cdot cos(at)\mathrm{d}t \tag{22}$$

$$y = \frac{1}{s} - \frac{a^2 y}{s^2} \tag{23}$$

$$y(\frac{s^2 + a^2}{s^2}) = \frac{1}{s} \tag{24}$$

$$y = \frac{s}{s^2 + a^2} \tag{25}$$