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Answer 1

a

The number of red candies must be 0 since we know that green ones are even and blue ones are odd . To reach 10 candies we must have odd numbers of red candies . Therefore, the required generating function that satisfies the given conditions is ;

$(x^0+x^2+x^4+...)$. $(x^5+x^7+x^9+x^{11}+...)$. $(x^1+x^3+x^5+...)$ since we have infinite source.

The paranthesis shows green, red and blue, respectively .To reach 10 candies , by multiplying the expressions of generating function , we see that the coefficient of x^{10} is 6 . Hence , there are 6 ways to choose 10 candies that satisfies the required conditions.

b

This question is pretty similar to the previous one . But generating function is little bit different. Since we have 5 of each types , the generating function is ;

$(x^0+x^2+x^4)$. $(x^1+x^3+x^5)$. x^5

When we multiply them , x^{10} has the coefficient 3 . Thus , there 3 ways to choose 10 candies for this question.

c

By using partial fraction ,we have ;

$$F(x) = 7x^2(\frac{2}{5} \cdot \frac{1}{(1-2x)} + \frac{3}{5} \cdot \frac{1}{(1+3x)})$$
$$= 7x^2(\frac{2}{5} \cdot \textstyle\sum_{i\geq 0} 2^i.x^i + \frac{3}{5} \cdot \sum_{i\geq 0} (-3)^i.x^i\ )$$
$$= 7x^2(1 - x + 7x^2 - 13x^3 + 55x^4 + ...\ )$$
$$= 7x^2 - 7x^3 + 49x^4 - 91x^5 + 385x^6 + ...$$

Hence , the sequence of the given generating function is ;

$\{0, 0, 7, -7, 49, -91, 385, ...\}$

d

Let's say $s_0 = 1$ and try this value on the recurrence relation .

$s_1 = 8.s_0 + 10^0 = 9$, so $s_0 = 1$ is a nice condition . Now we need a generating function for s_1,s_2,s_3... . Choose F(x) = $\sum_{n=0}^{\infty} s_n x^n$ as a generating function . According to this function , we have the recurrence relation $s_n x^n = 8s_{n-1}x^n + 10^{n-1}x^n$.

In order to start with n=1 to the function F , by changing the function simply , we obtain ;

F(x) $-$ 1 $= \sum_{n=1}^{\infty} s_n x^n = \sum_{n=1}^{\infty} (8s_{n-1}x^n + 10^{n-1}x^n)$

$= 8\sum_{n=1}^{\infty} s_{n-1}x^n + \sum_{n=1}^{\infty} 10^{n-1}x^n$

$=$ 8x$\sum_{n=0}^{\infty} s_n x^n$ + x$\sum_{n=0}^{\infty} 10^n x^n$

= 8xF(x) + x / (1 $-$ 10x)

When we solve it for F(x) , we have F(x) $= \dfrac{1-9x}{(1-8x)(1-10x)}$, from partial fraction ;

F(x) $= \dfrac{1}{2}\ (\dfrac{1}{1-8x} + \dfrac{1}{1-10x})$ which is equal to ;

F(x) $= \sum_{n=0}^{\infty} \dfrac{1}{2}\ (8^n + 10^n)\ x^n$.

Hence , $s_n = \dfrac{1}{2}\ (8^n + 10^n)$.

Answer 2

a

We can verify the given expression by using an example. Consider m=2 , k=4 and n=20 . We obviously know that 2 | 4 . Now , we have A_2 = (2,4,6,8,10,12,14,16,18,20] and A_4 = (4,8,12,16,20] since A_m is the set of numbers in the interval (m..n] that are divisible by m . According to the question , we need to see if $A_4 \subseteq A_2$ or not . We see the numbers in A_4 are 8,12,16,20 and we can clearly say that these numbers are included in A_2 which means $A_4 \subseteq A_2$. This expression is always true for some m and k if m | k . Hence , we conclude that if m | k , then $A_k \subseteq A_m$.

b

$C_n = \cup_{i=2}^{n-1} A_i = A_2 \cup A_3 \cup A_4 \cup ... \cup A_{44}$ and we have ;

$\cup_{primes p \leq \sqrt{n}} A_p = A_2 \cup A_3 \cup A_5$ for n = 45 .

According to the Theorem 2 of text book : If n is a composite integer , then n has a prime divisor less than or equal to $\sqrt{n}$.

When we consider this theorem , we should look 2 , 3 and 5 for n =45 since they are only primes less than or equal to $\sqrt{45}$. So we can say that $(A_2 \cup A_3 \cup A_5)$ includes $(A_2 \cup A_3 \cup A_4 \cup ... \cup A_{44})$ and $(A_2 \cup A_3 \cup A_4 \cup ... \cup A_{44})$ includes $(A_2 \cup A_3 \cup A_5)$.

Therefore , given two equations are equal .

c

We have the set of numbers such that ;

A_m = (m,2m,3m,4m,5m....n]

Since the first member is not included in A_m ,the sum of members of A_m is;

$| A_m | = \dfrac{n-2m}{m} + 1 = \dfrac{n}{m} - 1$ but if n/m is not an integer , we take the max. number that is less than the division n/m . Hence we have ;

$\mid A_m \mid = \lfloor$ n/m $\rfloor - 1$. We have reached the required equation . Hence , the given equation is true for m ≥ 2 .

d

For any relatively primes a,b $\leq$ n , to find the one number we can use some examples . Let's say a = 4 , b = 9 and n = 80 at first . Now we have ;

$A_4 \cap A_9 = \{36, 72\}$

$A_{36} = \{72\}$ (since first member is not included because of the open parenthesis)

So , the difference is 36 which is ab here .

Let's say a = 9 , b = 10 and n = 300 as another example . Now we have ;

$A_9 \cap A_{10} = \{90, 180, 270\}$

$A_{90} = \{180, 270\}$ (since first member is not included because of the open parenthesis again)

So , the difference is 90 which is ab here again .

Therefore , the one number in $(A_a \cap A_b) - A_{ab}$ is ab .

e

$\cap_{p \in P} A_p$ is the set of numbers that are divisible by least common multiple of primes up to n . Hence , the simple formula is the following :

$\mid \cap_{p \in P} A_p \mid = \lfloor \dfrac{n}{p_1 * p_2 * p_3 ... p_k} \rfloor$

f

By using the Inclusion-Exclusion principle we know $\mid C_{45} \mid = |A_2 \cup A_3 \cup A_5|$, for n = 45 , we obtain ;

$\mid C_{45} \mid = |A_2| + |A_3| + |A_5| - |A_2 \cap A_3| - |A_2 \cap A_5| - |A_3 \cap A_5| + |A_2 \cap A_3 \cap A_5|$.

g

To find the number of primes up to 45 , we need A_2 , A_3 , A_5 and their intersections . Now, we have ;

$|A_2| = 21$

$|A_3| = 14$

$|A_5| = 8$

$|A_2 \cap A_3| = 7$

$|A_2 \cap A_5| = 4$

$|A_3 \cap A_5| = 3$

$|A_2 \cap A_3 \cap A_5| = 1$, hence we reach ;

$\mid C_{45} \mid = |A_2| + |A_3| + |A_5| - |A_2 \cap A_3| - |A_2 \cap A_5| - |A_3 \cap A_5| + |A_2 \cap A_3 \cap A_5| = 21 + 14 + 8 - 7 - 4 - 3 + 1 = 30$. This the number of composite numbers up to 45 except 1 . Hence , the number of primes up to 45 is $44 - 30 = 14$.

Answer 3

a

We have to show if (a,b) $\ll$ (c,d) and (c,d) $\ll$ (e,f), then (a,b) $\ll$ (e,f) . By using given conditions ;

$((a<c) \lor ((a=c) \land (b \leq d))) \land ((c<e) \lor ((c=e) \land (d \leq f))) \rightarrow ((a<e) \lor ((a=e) \land (b \leq f)))$ which is actually the condition that we are trying to prove . Let's examine conditions in detail .

If $(a<c) \land (c<e)$, then $(a<e)$ is clearly true .

If $((a=c) \land (b \leq d)) \land (c<e)$, then $(a<e)$ is clearly true again .

If $(a<c) \land ((c=e) \land (d \leq f))$, then $(a<e)$ is clearly true again .

If $((a=c) \land (b \leq d)) \land ((c=e) \land (d \leq f))$, then $((a=e) \land (b \leq f))$ will clearly be true .

As we have showed , $\ll$ is transitive relation .

b

We need to check if $\propto$ is reflexive , symmetric and transitive . If it has all those features , then it is called equivalence relation .

Firstly , if it is symmetric , $f \propto g \rightarrow g \propto f$.

$f(x) = g(x) \rightarrow g(x) = f(x)$ for $x \geq k$. Thus , the relation $\propto$ is symmetric .

We know transitivity from part(a) . We need to show if $f \propto g$ and $g \propto h$,then $f \propto h$. We can easily say if $f(x) = g(x)$ and $g(x) = h(x)$, then $f(x) = h(x)$ for $x \geq k$. Hence , the relation is transitive .

If it is reflexive , it must satisfy $f \propto f$ which means $f(x) = f(x)$ for every $x \geq k$. It is clearly true . So , the relation is reflexive .

Therefore , $\propto$ is an equivalence relation .