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Answer 1

1.1

We have $a_n - a_{n-1} = n^2$, let's say *f(n)* $= n^2$.
We can use a formula for these types of situations :

$a_n = a_0 + \sum_{i=0}^{n}$ *f(i)* where *f(n)* $= n^2$.

To find a_0 , we can use $a_1 = 0$. For n = 1 ,we get $a_1 = a_0 + 1 = 1 \rightarrow a_0 = 0$. Then we obtain ;

$a_n = \sum_{i=0}^{n} i^2 = \frac{n(n+1)(2n+1)}{6} \rightarrow a_n = \frac{2n^3+3n^2+n}{6}$.

1.2

This is a linear nonhomogeneous recurrence relation. We should first solve the homogeneous part which is $a_n = 2a_{n-1}$.

By using the formula $r^2 + c_1 r - c_2 = 0$ where our $c_1 = 2$ and we have no c_2 . So we have $r^2 - 2r = 0$ which has two roots (0 and 2) . 0 does not affect the equation , hence $a_n^h = \alpha.2^n$.

For the nonhomogeneous part we could say a_n^p = C.2^n but in the homogeneous part we have same root (2^n) . So , let's say a_n^p = C.2^n.n and then we have the equation :

C.2^n.n = 2.C.2^{n-1}.(n-1) + 2^n

C.2^n.n = 2^n.C.(n-1) + 2^n

C.2^n.n = 2^n.(Cn - C + 1)

Cn = Cn - C + 1 $\rightarrow$ C = 1 . So , a_n^p = 2^n.n is a particular solution . By Theorem 5 from the textbook, all solutions are of the form ;

$a_n = \alpha.2^n + 2^n$.n = $2^n(\alpha$ + n) .

Now we can find α by using $a_0 = 1$. For n=0 , we have $\alpha.1 + 1.0 = 1 \rightarrow \alpha = 1$.

The solution is $a_n = 2^n$(n + 1) .

Answer 2

Let *P(n)* be the proposition that shows *f(n)* $\leq$ *g(n)* .

Basis Step : *P(1)* is true since $21 \leq 21$.

Inductive Step : Assume *P(k)* holds for an arbitrary positive integer k . We must show that *P(k+1)* holds .

If $P(k)$ holds , we have $k^2 + 15k + 5 \leq 21k^2$
$f(k+1) = k^2 + 17k + 21 = f(k) + 2k + 16$ then ,we can write this equation for $P(k+1)$;
$f(k+1) = f(k) + 2k + 16 \leq 21k^2 + 2k + 16 < 21k^2 + 42k + 21 = 21(k+1)^2 = g(k+1)$

We have showed that $P(k+1)$ holds . Therefore , $f(n) \leq g(n)$ for all positive integers n .

Answer 3

3.2

3.2.a

Let's use h(T) for height of the tree . We can recursively define this height as :
Basis Step : The height of an arbitrary binary tree with single vertex is h(T) = 0 and the height of an empty tree is h(T) = -1 .
Recursive Step : We can think T_1 and T_2 as left and right arbitrary trees such that T = T_1 . T_2 . Then our arbitrary binary tree has height h(T) = 1 + max(h(T_1),h(T_2)) where h(T_1) and h(T_2) are heights of those arbitrary trees .

3.2.b

Let's use f(223-tree) as max. number of vertices , g(223-tree) as min. number of vertices and h(T) as height of the tree .

For f(223-tree) :
Basis Step : The max. number of vertices of the 223-tree which has only a root is
f(223-tree) = 1 .
Recursive Step : We can think T_1 and T_2 as left and right subtrees such that 223-tree = T_1 . T_2 . Then the 223-tree has the number of vertices f(223-tree) = 1 + f(T_1) + f(T_2) .

For g(223-tree) :
Basis Step : The min. number of vertices of the 223-tree which has only a root is
g(223-tree) = 1 .
Recursive Step : We can think T_1 and T_2 as left and right subtrees such that 223-tree = T_1 . T_2 . Then the 223-tree has the number of vertices g(223-tree) = 1 + g(T_1) + g(T_2) .

3.2.c

To reach max. number of vertices , we need a full binary tree instead of a tree with other type and we know the formula from the lecture slides for number of the vertices f(T) = $2^{h(T)+1}-1$ where h(T) is the height of the tree .
Proof for f(223-tree) :
Basis Step : The result is satisfied for a binary tree which has only a root ,i.e. f(223-tree) = 1 and h(223-tree) = 0 . $1 = 2^{0+1}-1$.

Recursive Step : We can think T_1 and T_2 as left and right subtrees of 223-tree . Assume f(T_1) = $2^{h(T_1)+1}-1$ and f(T_2) = $2^{h(T_2)+1}-1$. And we already know the recursive formula of n(T) . We will use f(T) instead of n(T) since f(T) indicates the minimum number of vertices .

$$\begin{aligned} \text{f(223-tree)} &= 1 + \text{f}(T_1) + \text{f}(T_2) \\ &= 1 + (2^{h(T_1)+1}-1) + (2^{h(T_2)+1}-1) \\ &= 2 \ . \ \max(2^{h(T_1)+1}, 2^{h(T_2)+1}) -1 \\ &= 2 \ . \ 2^{max(h(T_1),h(T_2))} -1 \\ &= 2 \ . \ 2^{h(223-tree)} -1 \\ &= 2^{h(223-tree)+1} -1 \ . \end{aligned}$$

To reach min. number of vertices , we need a tree with the property that for every vertex the absolute value of the difference of heights of its left subtree and right subtree is at most 2 instead of a full binary tree . Since full binary tree contains more vertices than the tree which has difference between left and right subtrees .

For a tree with h(223-tree) = 3 and difference between the height of the left and right subtrees is 2 , min number of vertices g(223-tree) = 5 .

For a tree with h(223-tree) = 4 and difference between the height of the left and right subtrees is 2 , min number of vertices g(223-tree) = 7 .

For a tree with h(223-tree) = 5 and difference between the height of the left and right subtrees is 2 , min number of vertices g(223-tree) = 9 .

By using these values ,we obtain the formula for the minimum number of vertices g(T) = 2.h(T) -1 .

Proof for g(223-tree) :

Basis Step : The result is satisfied for a binary tree which has a right subtree with height 2 and left subtree with height 0 , g(223-tree) = 2 and h(223-tree) = 2 ,i.e. 2 = 2 . 2 -1.

Recursive Step : We can think T_1 and T_2 as left and right subtrees of 223-tree . Assume g(T_1) = 2h(T_1) -1 and g(T_2) = 2h(T_2) -1 . And we already know the recursive formula of n(T) . We will use g(T) instead of n(T) since g(T) indicates the minimum number of vertices .

$$\begin{aligned} \text{g(223-tree)} &= 1 + \text{g}(T_1) + \text{g}(T_2) \\ &= 1 + (2\text{h}(T_1) -1) + (2\text{h}(T_2) -1) \\ &= \max(2\text{h}(T_1), 2\text{h}(T_2)) -1 \\ &= 2.\max(\text{h}(T_1), \text{h}(T_2)) -1 \\ &= 2.\text{h(223-tree)} -1 \ . \end{aligned}$$

Answer 4

4.1

We know the formula for combination with repetition from the text book . The Theorem 2 in the text book says there are C(n + r − 1 , r) r-combinations from a set with n elements when repetition of elements is allowed .

4.1.a

In this question ,we take n as the working number of the first loop which is n . And we take r as the number of nested loops . To find a , we use r = 2 and to find b ,we use r = 3 since they are at different layers .

a = 2 . C(n + 2 − 1 , 2) = 2 . C(n + 1 , 2) = (n + 1) . n = $n^2 + n$

We multiplied the combination by 2 since a is increasing 2 by 2 .

b = C(n + 3 − 1 , 3) = C(n + 2 , 3) = $\frac{(n+2).(n+1).n}{3!} = \frac{n^3+3n^2+2n}{6}$

We did not do any multiplication for this combination since b is increasing 1 by 1 .

4.1.b

If a = b , we need solve an equation for the values that we found in part (a) .

$n^2 + n = \frac{n^3+3n^2+2n}{6}$

$6n^2 + 6n = n^3 + 3n^2 + 2n$

$n^3 - 3n^2 - 4n = 0$

By taking n parenthesis ;

$n(n^2 - 3n - 4) = 0$

$n(n-4)(n+1) = 0$

We have n = 0 , n = -1 and n = 4 but we know n ≥ 1 since n is the working number of the first loop which means we just use the value n = 4 . Therefore , n = 4 .

4.2

4.2.a

C(10,2) = $\frac{10!}{2!.8!}$ = 45

C(8,2) = $\frac{8!}{2!.6!}$ = 28

C(6,2) = $\frac{6!}{2!.4!}$ = 15

$\binom{10}{2} . \binom{8}{2} . \binom{6}{2}$ = 45 . 28 . 15 = 18900 ways to distribute 10 different fruits into 3 distinguishable plates, each plate has exactly 2 fruits .

4.2.b

C(10,1) = $\frac{10!}{1!.9!}$ = 10

C(9,2) = $\frac{9!}{2!.7!}$ = 36

C(7,3) = $\frac{7!}{3!.4!}$ = 35

C(4,4) = $\frac{4!}{4!.0!}$ = 1

$\binom{10}{1}$. $\binom{9}{2}$. $\binom{7}{3}$.$\binom{4}{4}$ = 10 . 36 . 35 . 1 = 12600 ways .

4.2.c

If we use 1 plate , we can distribute fruits by using only $\binom{6}{6}$ = 1 way .

For 2 plates , we can distribute fruits like 5-1 , 4-2 , 3-3 . Then , we have :

$\binom{6}{5}$. $\binom{1}{1}$ = 6

$\binom{6}{4}$. $\binom{2}{2}$ = 15

$\binom{6}{3}$. $\binom{3}{3}$ = 20

So , there are 41 ways .

For 3 plates , we can distribute fruits like 4-1-1 , 3-2-1 , 2-2-2 . Then , we have :

$\binom{6}{4}$. $\binom{2}{1}$. $\binom{1}{1}$ = 30

$\binom{6}{3}$. $\binom{3}{2}$. $\binom{1}{1}$ = 60

$\binom{6}{2}$. $\binom{4}{2}$. $\binom{2}{2}$ = 90

So ,there are 180 ways .

For 4 plates , we can distribute fruits like 3-1-1-1 , 2-2-1-1 . Then , we have :

$\binom{6}{3}$. $\binom{3}{1}$. $\binom{2}{1}$. $\binom{1}{1}$ = 120

$\binom{6}{2}$. $\binom{4}{2}$. $\binom{2}{1}$. $\binom{1}{1}$ = 180

So ,there are 300 ways .

Therefore, there are 1 + 41 + 180 + 300 = 522 ways to distribute 6 different fruits into 4 indistinguishable plates .

4.2.d

We don't have to use all of the dragons . So , we can use formula for using no dragon , 1 dragon , 2 dragons , 3 dragons and so on . There are C(n + r - 1 , n - 1) ways to place r indistinguishable objects into n distinguishable boxes from the lecture slides .

For 0 dragon fruit, r = 0 and n = 4 :
C(4 + 0 - 1 , 4 - 1) = 1
For 1 dragon fruit , r = 1 and n = 4 :
C(4,3) = 4
For 2 dragon fruits , r = 2 and n = 4 :
C(5,3) = 10
For 3 dragon fruits , r = 3 and n = 4 :
C(6,3) = 20
For 4 dragon fruits , r = 4 and n = 4 :
C(7,3) = 35
For 5 dragon fruits , r = 5 and n = 4 :
C(8,3) = 56
For 6 dragon fruits , r = 6 and n = 4 :
C(9,3) = 84
Hence there are 1 + 4 + 10 + 20 + 35 + 56 + 84 = 210 ways .