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Answer 1

1.a

We can use $(f^{-1} \circ g^{-1})(C_0)$ instead of $f^{-1}(g^{-1}(\ C_0))$.

If we show that $(g \circ f) \circ (f^{-1} \circ g^{-1}) = \mathrm{I}$ and $(f^{-1} \circ g^{-1}) \circ (g \circ f) = \mathrm{I}$, it means the equation is true since the composition of a function and its inverse is equal to Identity.

For the first one:

$$\begin{aligned}(g \circ f) \circ (f^{-1} \circ g^{-1}) &= g \circ ((f \circ f^{-1}) \circ g^{-1}) \qquad \text{(from Associativity Law)}\\ &= g \circ (\mathrm{I} \circ g^{-1})\\ &= g \circ g^{-1}\\ &= \mathrm{I}\end{aligned}$$

For the second one:

$$\begin{aligned}(f^{-1} \circ g^{-1}) \circ (g \circ f) &= (\ f^{-1} \circ (g^{-1} \circ g)) \circ f \qquad \text{(from Associativity Law)}\\ &= (f^{-1} \circ \mathrm{I}) \circ f\\ &= f^{-1} \circ f\\ &= \mathrm{I}\end{aligned}$$

Hence , $(g \circ f)^{-1}\ (C_0) = f^{-1}(g^{-1}(\ C_0))$ for $C_0 \subseteq C$.

1.b

Let's say A = $\{a_1,a_2\}$, B = $\{b_1\}$ and C = $\{c_1\}$ to determine whether f is injective or not.

First assume that f is not 1-to-1 . It means $\exists$ two members $a_1 \neq a_2$ such that $f(a_1) = f(a_2) = b_1$ where $b_1 \in B$. We know $g \circ f(a_1)$ and $g \circ f(a_2)$ means $g(f(a_1))$ and $g(f(a_1))$.Then ,according to our assumption , $\exists\ g(f(a_1)) = g(f(a_2)) = c_1$ where $c_1 \in C$. Which means $g \circ f$ is not 1-to-1 and there is a contradiction . Because we know $g \circ f$ is injective(1-to-1) from the question. Therefore , f must be injective .

Now, let's assume A = $\{a_1,a_2\}$, B = $\{b_1,b_2,b_3\}$ and C = $\{c_1,c_2\}$ to determine whether g is injective or not.

There can be conditions such that :

$f(a_1) = b_1$
$f(a_2) = b_2$
$g(b_1) = c_1$
$g(b_2) = c_2$
$g(b_3) = c_2$

Then , $g \circ f(a_1) = c_1$ and $g \circ f(a_2) = c_2$. So , we'have seen that $g \circ f$ is injective . Therefore , the function g does not have to be injective .

1.c

Let's say A = $\{a_1,a_2\}$, B = $\{b_1,b_2,b_3\}$ and C = $\{c_1,c_2\}$ and use the conditions of previous question :

$f(a_1) = b_1$
$f(a_2) = b_2$
$g(b_1) = c_1$
$g(b_2) = c_2$
$g(b_3) = c_2$

It's clearly seen that f is not onto (surjective) in this example because of the member b_3. So , f does not have to be surjective. In addition , g is surjective in this assumption and if $g \circ f$ is surjective , there is no condition that can make function g non-surjective .Therefore , function g must be surjective .

Answer 2

2.a

If g is left inverse for f :

Now , we need to show that if $f(a) = f(b)$, then a = b .

If $f(a) = f(b)$, $g(f(a)) = g(f(b))$ .Then ,clearly $g(f(a)) = a$ and $g(f(b)) = b$ since g is left inverse for f. By comparing these two , we obtain $g(f(a)) = a = b = g(f(b))$. Therefore, f must be injective if it has a left inverse .

If h is right inverse for f :

Now , we need to show that f is onto , i.e. for any member m $\in$ B , $\exists$ n $\in$ A with $f(n) = m$.

If we choose $h(m) = n$, then we have $f(n) = f(h(m)) = m$ since h is right inverse for f. Thus , f must be surjective if it has a right inverse .

2.b

Let f: A $\rightarrow$ B and g,h : B $\rightarrow$ A with A = $\{1,2\}$, B = $\{$a,b,c$\}$.

Take $f(1) = a$, $f(2) = b$; $g(a) = 1$, $g(b) = 2$, $g(c) = 1$; $h(a) = 1$, $h(b) = 2$,$h(c) = 2$. It's clearly seen that $h \circ f = g \circ f = \iota_A$ but $g \neq h$. We've showed that there are two left inverses of f in this example . Hence , a function can have more than one left inverse .

Let's explain the question for right inverses by giving another example .

Say f : $\{$a,b$\}$ $\rightarrow$ $\{$c$\}$. In this case , $f(a) = f(b) = c$. Now we can define two different right inverses of f , say g and h . It is obvious that g : $\{$c$\}$ $\rightarrow$ $\{$a,b$\}$ and h : $\{$c$\}$ $\rightarrow$ $\{$a,b$\}$. There can be condition such that $g(c) = a$ and $h(c) = b$. These functions are both right inverse for f and they are different from each other . Thus , a function can have more than one right inverse .

2.c

Assume f has left and right inverse.

If f has left inverse g , then f is injective which is proven in part(a) . Similarly , if f has right inverse h , then f is surjective from part(a) again . Thus , as f is both injective and surjective , it is bijective . If f is a bijective function , its right and left inverses must be same . So , g = h = f^{-1} .

Answer 3

For the function f

We know f is defined on Z^+ . For the second parameter which is y , we can easily obtain any number we want on this region since it depends only on y , so the second parameter is surjective . Now , let's consider the conditions $y = 1$ and $y = 2$. For $y = 1$, our function becomes ;

$f(x\ ,\ 1) = (x + 1 - 1\ ,\ 1) = (x\ ,\ 1)$

And x $\in Z^+$. If $x = 1$, we have (1 , 1) ; if $x = 2$, we have (2 , 1) ; if $x = 3$, we have (3 , 1) and so on .

For $y = 2$, our function becomes ;

$f(x\ ,\ 2) = (x + 2 - 1\ ,\ 2) = (x + 1\ ,\ 2)$

If $x = 1$, we have (2 , 2) ; if $x = 2$, we have (3 , 2) ; if $x = 3$, we have (4 , 2) and so on .

The condition y $\leq$ x is satisfied and we can obtain numbers in Z^+ , hence first parameter is surjective , too . Since both parameters are surjective , f is surjective .

For the injectivity of f , the second parameter depends only on y , so there is only one value for each y and this parameter is clearly injective . Then , let's think some y is chosen which means first parameter depends only on x . So , there is only one value for each x , hence first parameter is injective ,too . Therefore , f is injective.

Since f is both surjective and injective , it is bijection .

For the function g

Let's try some values of x and y to determine whether it is surjective .

$g(1\ ,\ 1) = 1$
$g(2\ ,\ 1) = 2$
$g(2\ ,\ 2) = 3$
$g(3\ ,\ 1) = 4$

These values satisfy the condition y $\leq$ x and x,y $\in Z^+$. We can easily obtain all numbers $\in Z^+$. Hence , g is surjective . For the injectivity , we can compare some values of the function g .

$g(1\ ,\ 1) = 1$
$g(2\ ,\ 1) = 2$
$g(2\ ,\ 2) = 3$
$g(3\ ,\ 1) = 4$
$g(3\ ,\ 2) = 5$
$g(3\ ,\ 3) = 6$

These values satisfy the condition y $\leq$ x and x,y $\in Z^+$ again . We could think that values we've tried can give same images since they are close numbers . But all of them gave different images . While x and y are increasing , the image of g is increasing ,too . Hence , g is injective .

As g is both surjective and injective , it is bijection .

Answer 4

4.b

I couldn't prove the part (a) . But it says the set of algebraic numbers is countable. So , we can use this .

Let's say A for the set of algebraic numbers and B for the set of transcendental numbers . And R = A $\cup$ B . We know that R is uncountable . So one of A or B must be uncountable . A is countable from part(a) . Therefore, B is uncountable .

Answer 5

We can choose n for k . Now , we should show that $n\ln n$ is $O(n)$ and $n\ln n$ is $\Omega(n)$.

When n $\leq e$, | n ln$n$ | $\leq$ C | n | for C = 1 .

n $\geq$ e , | n ln$n$ | $\geq$ C | n | for C = 1 .

Therefore , n ln$n$ = $\Theta(k)$ for k = n .

Now , it's time to show that n = $\Theta(\frac{n}{lnn})$ because we are using n instead of k .

For n $\leq e$, | n | $\leq$ C | $\frac{n}{lnn}$ | for C = 1 .

n $\geq e$, | n | $\geq$ C | $\frac{n}{lnn}$ | for C = 1 .

We have showed that n = $\Theta(\frac{n}{lnn})$. Hence , n = $\Theta(\frac{k}{lnk})$.

Answer 6

6.a

Positive divisors of 6 which are other than itself are 1,2,3 . And 6 = 1+2+3 .

Positive divisors of 28 that are other than itself are 1,2,4,7,14 . And 28 = 1+2+4+7+28 .

Therefore , as they satisfied the definition of perfect number , 6 and 28 are perfect number .

6.b

Positive divisors of $2^{p-1}(2^p - 1$) other than itself are all the number with the type 2^a for 0 $\leq$ a $\leq$ p - 1 , and 2^b(2^p - 1) for 0 $\leq$ b $\leq$ p - 1 when 2^p - 1 is prime . And we can obtain $\sum_{a=0}^{n} 2^a = 2^{n+1} - 1$ by using the formula for $\sum_{j=0}^{n} ar^j$ from the text book (n $\in Z^+$) .

Let's take the sum of these positive divisors :

$$\begin{aligned}\sum_{a=0}^{p-1} 2^a + \sum_{b=0}^{p-2} 2^b(2^p-1) &= (2^p-1) + (2^p-1)\ (2^{p-1}-1) \\ &= (2^p-1)(1 + (2^{p-1}-1)) \\ &= (2^p-1)2^{p-1}\end{aligned}$$

We've reached our first expression . Hence , when (2^p-1) is prime , $2^{p-1}(2^p-1)$ is a perfect number .

Answer 7

7.a

From given expressions , we can say $x = a.m + c_1 = b.n + c_2$ where $a,b,m,n,\ c_1,c_2 \in$ Z with m $>$ 0 and n $>$ 0 . Then , we have :

$\rightarrow x - c_1 = a.m$
$\rightarrow x - c_2 = b.n$
Now , let's assume $gcd(m,n) = k$. And clearly ;
$\rightarrow k \mid (a.m)$ and $k \mid (b.n)$
$\rightarrow k \mid (x - c_1)$ and $k \mid (x - c_2)$
$\rightarrow (x - c_2) - (x - c_1) = (k.u) - (k.v)$ for some u,v $\in$ Z
$\rightarrow (c_1 - c_2) = k(u - v)$
$\rightarrow k \mid (c_1 - c_2)$
We proved the condition $gcd(m,n) \mid (c_1 - c_2)$.