Final Portfolio
MAT202, Winter 2024

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(Introduction) Instructions: Write a short introduction to your portfolio that reflects on your writing process this semester. Say which four questions you picked to revise, and explain your choices—why did you pick these four? What was it about those submissions that made you feel like you could still make a difference in them, and what kinds of improvements did you make for each one?

As you are going back and reading your assignments from earlier in the semester, how do you feel about the progress you've made?

As I go through my assignment submissions from a1 to a6, and finally my resubmissions for each of the chosen assignments, I realized how much my mathematical writing has impnroved. Prior to taking the course and during the beginning of MAT202, I used to struggle with providing too many details, having redundancy in my solutions, and at times, lacking a clear transition between arguments. Over the past few months, I've learned to make effective mathematical arguments while providing justifications that are direct, focused, and concise. I've gone from having long and wordy solutions with claims lacking enough justification to writing well-justified and strong solutions that maintain clarity and conciseness.

The 4 assignments I chose to revise are a1, a3, a5, and a6.

The first assignment we completed for this course was a1 and as such, I felt that this assignment could use some revision after reviewing it. I realized that the steps in for condition 2 were not explained, rather I would make a general statement along the lines of "and this gives us 4(4!)" without explaining the significance of multiplying 4 and 4!. Going over the assignment after a couple of months, I realized I was not able to fully understand my own solution to this part for this reason - the steps I performed were not fully explained. Further, I also had some errors where I added a ! in some numbers such as the final answer which I wrote as 30144! instead of 30144.

A3 was an assignment which consisted of lots of writing and making correct arguments. While my arguments were correct, I felt that there was much that could be improved in terms of writing. More specifically, there was some redundancy and unnecessary detail which made the proof too long and perhaps somewhat difficult for the reader to follow along. I would unnecessarily emphasize certain details, for example an unordered pair, assign it a letter name, then proceed to never use it. In essence, the writing could be made more concise and direct. Further, there was some inconsistencies in terms of notation. At times I used the binomial coefficient and the algebraic expression at others so I made the notation more consistent in my revised solution

After going through my a5 submission, I realized that there were some mathematical errors in my solution due to my limited understanding of congruence at the time. As an example, I thought that when dividing a congruence $ax \equiv b \pmod{n}$ by some number k, n should also be divided by k. It wasn't until later when I reviewed my notes in more detail that I realized we should check the $gcd(a, n)$ and divide by $gcd(a, n)$ instead so I decided to make this more explicit in the re submission. There are other improvements as well in terms of explaining why certain steps are performed such as the euclidean algorithm.

The reason I chose to revise my a6 submission is because I felt my mathematical writing and use of correct terminology could be significantly improved in this submission. To elaborate, I used words such as "pair", and "beside" to refer to adjacent vertices too often rather than the word "adjacent" itself. Further, as indicated by the TA's comment, I defined letters for vertices but still referred to them by their degree rather than the letter associated to each vertex which could make it difficult for the reader to follow along. In my revised submission, the writing is more direct and uses correct terminology.

(A1) Count the number of arrangements of the word BIJECTION such that:

- there are at least five other letters in between the two I's; and
- the word NICE does not appear.

For example, the arrangement BIJECTION is *not* valid since there are only four letters (J, E, C, T) in between the two I's.

Solution:

1. Count the number of arrangements of the word BIJECTION such that:

- there are at least five other letters in between the two I's; and
- the word NICE does not appear.

For example, the arrangement BIJECTION is *not* valid since there are only four letters (J, E, C, T) in between the two I's.

Solution: We begin by examining the first condition. Note that the word BIJECTION has a total of 9 letters. Since the condition states that there must be at least 5 letters in between the two I's and there are 9 letters in total, any valid rearrangement will have 5, 6 or 7 letters in between the two I's. As such, any rearrangement of BIJECTION satisfying the condition can be represented by one of the following cases:

Case 1: I _ _ _ _ _ _ _ I **(7 letters in between)**

Case 2: I _ _ _ _ _ _ I _ **(6 letters in between)**

Case 3: _ I _ _ _ _ _ _ I **(6 letters in between)**

Case 4: I _ _ _ _ _ I _ _ **(5 letters in between)**

Case 5: _ I _ _ _ _ _ I _ **(5 letters in between)**

Case 6: _ _ I _ _ _ _ _ I **(5 letters in between)**

Note that for the second condition, we can first look at the number of arrangements such that the word NICE does appear. In order for the word to appear, there must be an I in the arrangement such that there is at least one preceding string and two succeeding strings. As such, the word NICE can only appear in cases 3, 4, 5, and 6.

Since we have 6 cases and there are 7! ways to arrange the rest of the letters in BJECTON, there are a total of 6(7!) arrangements of BIJECTION such that they satisfy the first condition.

To satisfy the second condition—that the sequence 'NICE' does not appear—we need to calculate the arrangements where 'NICE' could potentially form and then subtract these from the total valid arrangements. It's important to note that 'NICE' can only appear in specific configurations, particularly in cases 3, 4, 5, and 6, due to the flexibility allowed by the positioning of the 'I's in these cases.

In these scenarios, to calculate the arrangements where 'NICE' appears as a contiguous block, we recognize that there are 4! ways to arrange the four letters of 'NICE'. Since we are considering the cases where 'NICE' forms as a block within the arrangement, and given that there are four specific configurations (cases 3, 4, 5, and 6) where this block can be inserted, we initially multiply the number of 'NICE' arrangements by 4, yielding $4(4!)$.

This calculation helps us determine the number of arrangements that satisfy the first condition but violate the second by including 'NICE'. To find the total number of valid arrangements that adhere to both conditions, we subtract these 'NICE' arrangements from the total arrangements that satisfy the first condition. Given that there are $6(7!)$ arrangements that meet the first condition (the arrangement of letters other than 'I' with sufficient spacing between the 'I's), using the complement, the total number of valid arrangements is calculated as $6(7!) - 4(4!)$.

(A2) From a standard deck of cards, count the number of four-card hands that satisfy the following condition:

- the number of distinct ranks represented in the hand must be equal to the number of distinct suits represented in the hand.

(For example, the hand $3\heartsuit$ $3\spadesuit$ $5\clubsuit$ $7\heartsuit$ is a valid hand since it contains exactly three different ranks and exactly three different suits. However, the hand $J\heartsuit$ $J\spadesuit$ $J\clubsuit$ $Q\clubsuit$ is invalid since it contains three different suits but only two distinct ranks.)

Solution:

(A3) Let $n > k > 3$ be integers. Write a combinatorial proof of the following identity:

$$\binom{n}{k}\left({}_kP_2 + {}_kP_3\right) = \binom{n}{2}\binom{n-2}{k-2}2! + \binom{n}{3}\binom{n-3}{k-3}3!.$$

Proof:

Consider a set $A = \{a_1, a_2, ..., a_n\}$ of n individuals. Our objective is to form a committee of size k from A and then select an ordered task force of either 2 or 3 members from this committee. We will show that both sides of the given identity count the number of ways to accomplish this task in different manners.

LHS: The term $\binom{n}{k}$ represents the number of ways to select a committee of size k from n individuals. After forming this committee, we consider two scenarios for the task force: forming an ordered pair or an ordered triplet. The term ${}_kP_2$ counts the permutations of selecting and ordering 2 members out of the k-member committee, and similarly, ${}_kP_3$ counts the permutations of selecting and ordering 3 members. Thus, the LHS captures the total number of ways to first choose a committee of k and then form an ordered task force of size 2 or 3 from this committee.

RHS: This side approaches the task by first selecting the task force and then forming the remainder of the committee.

Case 1 (Pair Selection): We begin by selecting 2 members from n to form a task force ($\binom{n}{2}$), and order this pair (2!). The remaining committee members $(k-2)$ are then chosen from the $n-2$ individuals not in the task force ($\binom{n-2}{k-2}$). This case represents the number of ways to form a committee with a specific ordered pair as part of it.

Case 2 (Triplet Selection): Similarly, we select 3 members from n to form a task force ($\binom{n}{3}$), order this triplet (3!), and then choose the remaining committee members $(k-3)$ from the $n-3$ individuals not in the task force ($\binom{n-3}{k-3}$). This case represents the number of ways to form a committee with a specific ordered triplet as part of it.

By combining Case 1 and Case 2 using the sum rule, the RHS gives the total count of forming a committee that includes either a specific ordered pair or a specific ordered triplet from the set A.

Since both the LHS and RHS count the total number of ways to form a committee of size k with an ordered task force of size 2 or 3, the given identity holds true.

(A4) Let $\mathcal{C}$ be a collection of finite sets of integers, $S_1, S_2, \ldots, S_n$, each set with at least two elements. Define the following relations:

- $R_{\mathcal{C}}$ is a relation on $\mathbb{Z}$, given by $(a, b) \in R_{\mathcal{C}}$ if and only if the largest set in $\mathcal{C}$ that contains a and the largest set in $\mathcal{C}$ that contains b have the same number of elements. (If there is no set containing an integer a then we will say that the largest set containing a has zero elements.)
- $\sim$ is a relation on $\mathcal{C}$, given by $S_i \sim S_j$ if and only if there exists an element in S_i that is in between the smallest and largest elements in S_j, inclusive.

Check whether or not each relation above satisfies each of the three properties of an equivalence relation for any collection $\mathcal{C}$, and justify your claims appropriately.

Solution:

(A5) Solve the congruence $156x \equiv 402 \pmod{570}$, and express your final answer in terms of congruence classes in $\mathbb{Z}_{570}$. Show and explain all steps in your calculation, including finding an inverse using the Euclidean algorithm.

Proof: To solve the congruence, we need to determine $\gcd(156, 570)$ which can be done using the Euclidean algorithm.

$$\begin{aligned}
570 &= 156 \cdot 3 + 102, \\
156 &= 102 \cdot 1 + 54, \\
102 &= 54 \cdot 1 + 48, \\
54 &= 48 \cdot 1 + 6, \\
48 &= 6 \cdot 8.
\end{aligned}$$

This gives us that $\gcd(156, 570) = 6$ which indicates that the congruence has at least one solution. We can rewrite $156x \equiv 402 \pmod{570}$ as $\frac{156}{6}x \equiv \frac{402}{6} \left(\text{mod } \frac{570}{gcd(156,570)}\right)$ which simplifies to $26x \equiv 67 \pmod{95}$.

Further, we can find $\gcd(26, 95)$ by applying the Euclidean algorithm once again:

$$\begin{aligned}
95 &= 26 \cdot 3 + 17, \\
26 &= 17 \cdot 1 + 9, \\
17 &= 9 \cdot 1 + 8, \\
9 &= 8 \cdot 1 + 1, \\
8 &= 1 \cdot 8.
\end{aligned}$$

Since the last non-zero remainder is 1, it tells us that $\gcd(26, 95) = 1$. Note that since $\gcd(26, 95) = 1$, we know that there exists a multiplicative inverse for $26 \pmod{95}$. We can find this multiplicative inverse using back substitution:

$$\begin{aligned}
1 &= 9 - 8 \cdot 1 \\
&= 9 - (17 - 9 \cdot 1) \\
&= 9 \cdot 2 - 17 \\
&= (26 - 17 \cdot 1) \cdot 2 - 17 \\
&= 26 \cdot 2 - 17 \cdot 3 \\
&= 26 \cdot 2 - (95 - 26 \cdot 3) \cdot 3 \\
&= 26 \cdot 11 - 95 \cdot 3.
\end{aligned}$$

Thus $26 \cdot 11 \equiv 1 \pmod{95}$ and so, 11 is the multiplicative inverse of 26 modulo 95.

Therefore, we can now multiply both sides of $26x \equiv 67 \pmod{95}$ by 11 to find the solutions for x:

$$\begin{aligned}
11 \cdot 26x &\equiv 11 \cdot 67 \pmod{95} \\
x &\equiv 737 \pmod{95} \\
x &\equiv 95 \cdot 7 + 72 \pmod{95} \\
x &\equiv 72 \pmod{95}.
\end{aligned}$$

To find the solutions in $\mathbb{Z}_{570}$, we calculate $72 + 95k$ for $k = 0, 1, 2, \ldots$ until we exceed 570. The solutions are:

$$\begin{aligned}
x &\equiv 72 \pmod{570} \\
x &\equiv 167 \pmod{570} \\
x &\equiv 262 \pmod{570} \\
x &\equiv 357 \pmod{570} \\
x &\equiv 452 \pmod{570} \\
x &\equiv 547 \pmod{570}.
\end{aligned}$$

Thus in $\mathbb{Z}_{570}$, solutions are [72], [167], [262], [357], [452], and [547].

(A6) For each sequence below, give a simple graph with that degree sequence, or argue that one doesn't exist.

(a) 6, 6, 5, 5, 3, 3, 1, 1

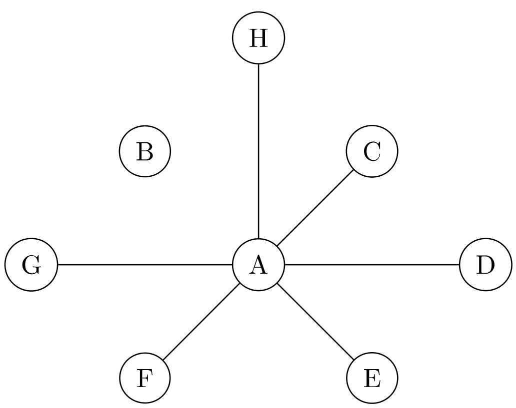
(b) 5, 5, 5, 5, 3, 3, 3, 3

Solution:

a) We begin by defining the following vertices A - H for each degree in the degree sequence as follows:

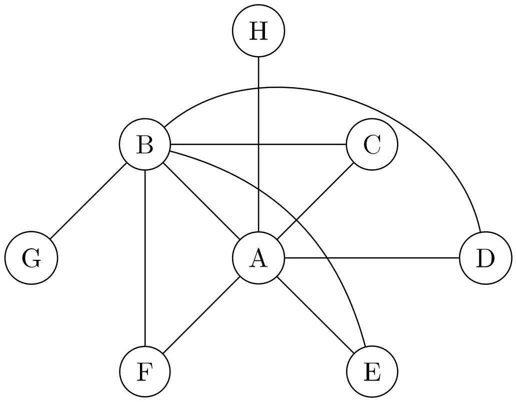
$$deg(A) : 6$$
$$deg(B) : 6$$
$$deg(C) : 5$$
$$deg(D) : 5$$
$$deg(E) : 3$$
$$deg(F) : 3$$
$$deg(G) : 1$$
$$deg(H) : 1$$

Consider vertices A, B, G, and H. We begin by creating edges for vertices A and B. Note that if vertex A is adjacent to both vertices G and H, then vertex B will not have enough remaining vertices to form 6 edges as shown below.

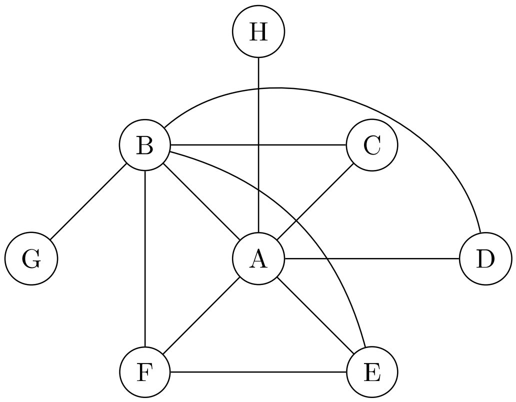


In the above graph, vertex B can only be adjacent to 5 other vertices, making it impossible for vertex B to satisfy its degree condition of 6. This problem occurs in any graph where either vertex of degree 6 is adjacent to both vertices of degree 1. As such, vertex A and vertex B can each only be adjacent to one of the vertices G or H.

We proceed by completing the edges for A, B, G, and H as follows:

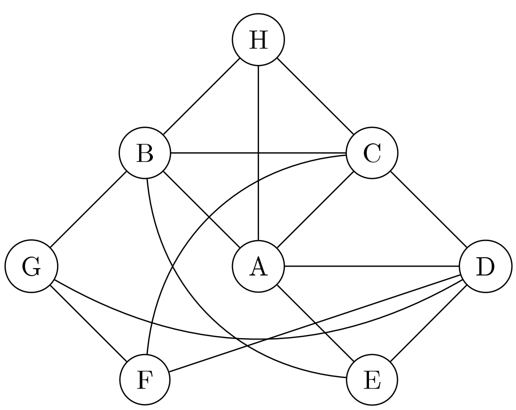


Note that vertices A, B, G, and H have reached their capacity for the number of adjacent vertices. Vertices C and D have 3 more adjacencies remaining, and vertices E and F each need one more adjacency added. We can satisfy the degree condition for vertices E and F by adding an edge between them.



However, it is now impossible to satisfy the degree conditions for vertices C and D, since each needs 3 additional adjacencies, but all other vertices have reached their adjacency limit. We could attempt to pair vertices E and F with vertices C and D, however, the same issue would arise as vertices C and D both require more than one additional adjacency, and no other vertices can accommodate further edges. Therefore, this degree sequence is impossible.

b) This degree sequence is possible as shown by the following graph:



Instructions: Wrap up your final portfolio by writing some concluding thoughts. You may also use some of these guide questions to aid your reflection:

- Over the semester, what have been the most significant changes to the way you write mathematics, if any?
- Were you satisfied with your performance in the course? With respect to writing? or the material?
- Have your beliefs on the importance of communication in mathematics changed? How did they change?
- How valuable was the TA feedback in your process of preparing this portfolio?
- How was your experience learning LaTeX?
- ...and anything else you'd like to mention!

During the semester, among the most significant of changes to my mathematical writing has been clarity. I used to think that writing long solutions and adding in extra jargon would make my proofs look impressive and strong. However, this instead made my writing confusing to follow along and remain focused. After reading some of the Professor's proofs and those in the textbook, I learned that the best proofs were those that were short, direct, and were easy to follow along. I realized that many of my proofs could be shortened if I just focused on the key details.

Another problem was that some of the statements and steps I performed were not well justified which would make it hard for me to understand my solution if I came back to it some time later. Some of these claims could be considered "hand-waving" due to the lack of correct justification. Through this course, I learned how to effectively connect my arguments together to reach a conclusion. Putting the first and second improvements together, overall, I learned to make my writing and arguments more effective by focusing on the key details, keeping the writing short, and removing unnecessary details. In come becoming a better writer, the feedback provided by my TAs was paramount as I realized areas where my writing was weak and could be improved.

I am satisfied with my performance in the course both with respect to writing and material. I've performed well in the correctness and writing components in my assignments and I've learned how to improve my writing further. When I read my proofs now compared to a few months ago, I realize how I've become better at making clear arguments that are easy for others to understand as well. Further, my own journey with writing has taught me that communication in mathematics is crucial. If a proof is easy to follow along, an individual can learn and understand that concept in a few minutes rather than spending much longer seeking clarification from videos, searching up the proof, or consulting others every single time.

One of the things I really appreciate about this course is that even though it is a Discrete Mathematics course where one would expect to learn math, students learn a lot more. The focus on writing, as well as the provision of ample resources for reading proofs and the Professor's solutions themselves help us develop not just mathematically, but also as a writer. And not just a mathematical writer, but even as a writer in other fields and contexts. In essence, I find it fascinating that MAT202 helps develop students as mathematicians **and** writers.