Tut. 12

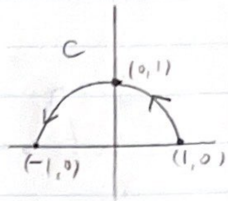
1. done. (please see tutorial 11)

2. (a) Let $\vec{F}$ be the vector field $\vec{F}(x,y) = \langle x^2+y^2, xy \rangle$ and let C denote the unit semi-circle in the upper half plane traced counter-clockwise.

compute $\int_C \vec{F} \cdot d\vec{r}$.

$$\int_C \vec{F} \cdot d\vec{r} = \int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)\, dt, \quad a \leq t \leq b.$$

let's parametrize the unit semi-circle with $\vec{r}(t) = \langle \cos(t), \sin(t) \rangle$, $t \in [0, \pi]$



$\vec{r}\,'(t) = \langle -\sin(t), \cos(t) \rangle$

now, write $\vec{F}(x,y)$ in terms of $\vec{r}(t)$.

$$\vec{F}(\vec{r}(t)) = \langle \cos^2(t) + \sin^2(t), \cos(t)\sin(t) \rangle$$
$$= \langle 1, \cos(t)\sin(t) \rangle$$

so, $$\int_C \vec{F} \cdot d\vec{r} = \int_0^\pi \langle 1, \cos(t)\sin(t) \rangle \cdot \langle -\sin(t), \cos(t) \rangle dt$$

$$= \int_0^\pi \left(-\sin(t) + \cos^2(t)\sin(t)\right) dt$$

$$= \cos(t) \Big|_0^\pi + \int_0^\pi \cos^2(t)\sin(t)\, dt$$

$u = \cos(t)$
$du = -\sin(t)dt$

$$= \cos(\pi) - \cos(0) + \int_1^{-1} -u^2\, du$$

if $t = 0$ $u = 1$
$t = \pi$ $u = -1$

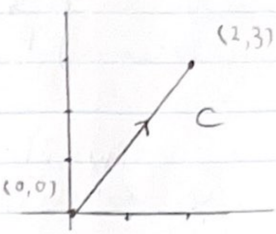
$$= -1 - 1 - \frac{u^3}{3} \Big|_1^{-1}$$

$= -2 - \frac{1}{3}\left(((-1)^3 - 1^3\right) = -2 - \frac{1}{3}(-2)$

$= -2 + \frac{2}{3} = -\frac{4}{3}.$

2(b) Let $\vec{F}$ be the vector field $\vec{F} = (x^2+y)\vec{i} + y^2\vec{j}$. Let C denotes the line segment connecting the origin to the point $(2,3)$. Evaluate $\int_C \vec{F} \cdot d\vec{r}$.

first parametrize the line segment.



$\vec{r}(t) = \langle 0,0 \rangle + t(\langle 2,3 \rangle - \langle 0,0 \rangle)$

$= \langle 0,0 \rangle + t\langle 2,3 \rangle$

$= \langle 2t, 3t \rangle \quad , \ t \in [0,1]$

$\vec{r}\,'(t) = \langle 2, 3 \rangle$

now, write $F(x,y)$ in terms of $\vec{r}(t)$.

$\vec{F}(\vec{r}(t)) = ((2t)^2 + 3t)\vec{i} + (3t)^2\vec{j} = (4t^2+3t)\vec{i} + 9t^2\vec{j}$

$= \langle 4t^2+3t, 9t^2 \rangle$

so, $\int_C \vec{F} \cdot dr = \int_0^1 \langle 4t^2+3t, 9t^2 \rangle \cdot \langle 2,3 \rangle \, dt$

$= \int_0^1 2(4t^2+3t) + 3(9t^2)\, dt = \int_0^1 8t^2 + 6t + 27t^2 \, dt$

$= \int_0^1 (35t^2 + 6t)\, dt = \frac{35}{3}t^3 \Big|_0^1 + \frac{6t^2}{2}\Big|_0^1$

$= \frac{35}{3}(1^3 - 0^3) + 3((1)^2 - 0^2)$

$= \frac{35}{3} + 3 = \frac{35}{3} + \frac{9}{3} = \frac{44}{3}$

3. Let $\vec{F}(x,y) = \langle \sin(x) - y\sin(xy), \cos(y) - x\sin(xy) \rangle$.
Let C denote the curve parametrized by $\vec{r}(t) = \langle t, t^2 \rangle$ from $t = 0$ to $t = \pi$.

Evaluate $\int_C \vec{F} \cdot d\vec{r}$

Evaluating $\int_C \vec{F} \cdot d\vec{r}$ directly with a parametrization can be a challenging task. However, if we can show that $F(x,y)$ is a gradient vector field then, we can use then the fundamental theorem of line integrals.

Showing $F(x,y)$ is a gradient vector field is not enough. We need to find a scalar function $f(x,y)$ such that $F = \nabla f$.

$$\vec{F}(x,y) = \langle \underbrace{\sin(x) - y\sin(xy)}_{f_x}, \underbrace{\cos(y) - x\sin(xy)}_{f_y} \rangle$$

$$f(x,y) = \int f_x \, dx = \int (\sin(x) - y\sin(xy)) \, dx$$

$$= -\cos(x) + \cos(xy) + g(y)$$

take the derivative w.r.t. y $\hookrightarrow$ $f_y(x,y) = -x\sin(xy) + g'(y)$

Set them equal.

now sub, $f_y(x,y) = \cos(y) - x\sin(xy)$

$$-x\sin(xy) + g'(y) = \cos(y) - x\sin(xy)$$

$$g'(y) = \cos(y)$$

$$\int g'(y) \, dy = \int \cos(y) \, dy$$

$$g(y) = \sin(y) + C$$

So, $f(x,y) = -\cos(x) + \cos(xy) + \sin(y) + C.$

to compute $\int_C \vec{F} \cdot d\vec{r}$. first, write $\vec{F} = \nabla f$ and then use the fundamental theorem of line integral

$$\int_C \vec{F} \cdot d\vec{r} = \int_C \nabla f \cdot d\vec{r}$$

$$= f(\vec{r}(t_2)) - f(\vec{r}(t_1))$$

in this problem, $t_1 = 0$, $t_2 = \pi$

So, $\int_C \vec{F} \cdot d\vec{r} = f(\vec{r}(\pi)) - f(\vec{r}(0))$

$$f(\vec{r}(\pi)) = f(\pi, \pi^2) = -\cos(\pi) + \cos(\pi \cdot \pi^2) + \sin(\pi^2) + C$$
$$= -(-1) + \cos(\pi^3) + \sin(\pi^2) + C$$
$$= 1 + \cos(\pi^3) + \sin(\pi^2) + C$$

$$f(\vec{r}(0)) = f(0, 0^2) = f(0,0) = -\cos(0) + \cos(0 \cdot 0) + \sin(0) + C$$
$$= -1 + 1 + C = C$$

So,

$$\int_C \vec{F} \cdot d\vec{r} = f(\vec{r}(\pi)) - f(\vec{r}(0))$$
$$= 1 + \cos(\pi^3) + \sin(\pi^2) + C - C$$
$$= 1 + \cos(\pi^3) + \sin(\pi^2)$$

therefore the line integral is

$$\int_C \vec{F} \cdot d\vec{r} = 1 + \cos(\pi^3) + \sin(\pi^2)$$

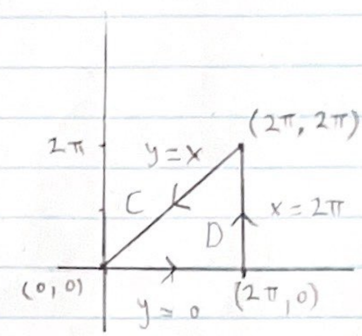
4. Let C denote the triangle with vertices $(0,0)$, $(2\pi, 0)$ and $(2\pi, 2\pi)$ traversed counter-clockwise.
Let $\vec{F}$ be the vector field

$$\vec{F}(x,y) = \langle 22y + 2x\sin(y) + 17\sin(x),\ x^2\cos(y) + 13y^{200\sin(y)} \rangle$$

Compute $\int_C \vec{F} \cdot d\vec{r}$

C is enclosed Path, and $\vec{F}$ is not conservative

Evaluating this line integral directly can be challenging. However, we can use Green theorem.



$$\int_C \vec{F} \cdot d\vec{r} = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$$

$D = \{(x,y) \mid 0 \le x \le 2\pi,\ 0 \le y \le x\}$

$$\frac{\partial Q}{\partial x} = 2x\cos(y) \qquad \frac{\partial P}{\partial y} = 22 + 2x\cos(y)$$

$$\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA = \int_0^{2\pi}\int_0^{x} 2x\cos(y) - (22 + 2x\cos(y))\, dy\, dx$$

$$= \int_0^{2\pi}\int_0^{x} (2x\cos(y) - 22 - 2x\cos(y))\, dy\, dx$$

$$= \int_0^{2\pi}\int_0^{x} -22\, dy\, dx = \int_0^{2\pi} -22y \Big|_0^{x} dx$$

$$= \int_0^{2\pi} -22(x-0)\, dx = -22\int_0^{2\pi} x\, dx = -22\frac{x^2}{2}\Big|_0^{2\pi}$$

$$= -11x^2\Big|_0^{2\pi} = -11((2\pi)^2 - 0^2) = -11(4\pi^2) = -44\pi^2$$