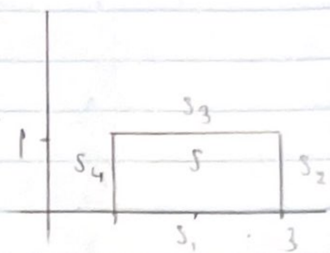
Jut 11

1 (a) Let $S = \{(u,v) : 1 \le u \le 3, \ 0 \le v \le 1\}$.

$T(u,v) = (x(u,v), y(u,v))$ be the transformation from the uv-plane to the xy-plane defined by

$x = x(u,v) = \frac{1}{2}(u+v)$ and $y = y(u,v) = \frac{1}{2}(u-v)$. Sketch the image of S under the transformation.

along S_1: $1 \le u \le 3$, $v = 0$



$x(u,0) = \frac{1}{2}(u+0) = \frac{1}{2}u$

$y = (u,0) = \frac{1}{2}(u-0) = \frac{1}{2}u$

$\Rightarrow y = x$, $\boxed{\frac{1}{2} \le x \le \frac{3}{2}}$

$x = \frac{1}{2}u$, $\boxed{1 \le u \le 3 \rightarrow \frac{1}{2} \le \frac{u}{2} \le \frac{3}{2} \rightarrow \frac{1}{2} \le x \le \frac{3}{2}}$

along S_2: $u = 3$, $0 \le v \le 1$

$x(3,v) = \frac{1}{2}(3+v)$

$y(3,v) = \frac{1}{2}(3-v)$

$\Rightarrow x = \frac{1}{2}(3+v)$, $y = \frac{1}{2}(3-v)$ $\rightarrow$ $2x = 3+v$, $+\ 2y = 3-v$, $2x+2y = 6 \rightarrow x+y=3$

$$\boxed{\begin{aligned} 0 &\le v \le 1 \\ 3 &\le v+3 \le 4 \\ \frac{3}{2} &\le \frac{v+3}{2} \le 2 \\ \frac{3}{2} &\le x \le 2 \end{aligned}}$$

So, $y = -x+3$, $\boxed{\frac{3}{2} \le x \le 2}$.

along S_3: $1 \le u \le 3$, $v = 1$,

$x(u,1) = \frac{1}{2}(u+1)$

$y(u,1) = \frac{1}{2}(u-1)$

$\rightarrow$ $2x = u+1$, $-\ 2y = u-1$, $2x - 2y = 2 \rightarrow x - y = 1$

$$\boxed{\begin{aligned} 1 &\le u \le 3 \\ 2 &\le u+1 \le 4 \\ 1 &\le \frac{u+1}{2} \le 2 \\ 1 &\le x \le 2 \end{aligned}}$$

So, $y = x-1$, $\boxed{1 \le x \le 2}$

along S_4: $u=1$, $0 \le v \le 1$

$$\boxed{\begin{aligned} 0 &\le v \le 1 \\ 1 &\le v+1 \le 2 \\ \tfrac{1}{2} &\le \tfrac{v+1}{2} \le 1 \\ \tfrac{1}{2} &\le x \le 1 \end{aligned}}$$

$x(1,v) = \frac{1}{2}(1+v)$ $\Rightarrow$ $2x = 1+v$

$y(1,v) = \frac{1}{2}(1-v)$ $+$ $2y = 1-v$

$2x + 2y = 2 \rightarrow x + y = 1$

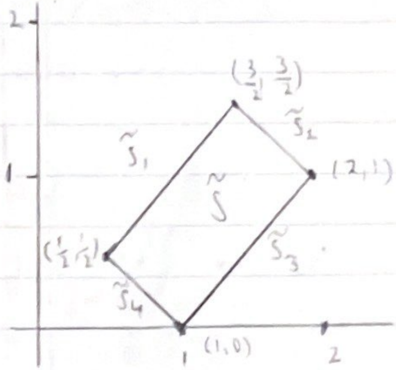
So, $y = -x + 1$, $\boxed{\frac{1}{2} \le x \le 1}$

Thus, $\tilde{S}_1$: $y = x$, $\frac{1}{2} \le x \le \frac{3}{2}$ $(\frac{1}{2}, \frac{1}{2})$ $(\frac{3}{2}, \frac{3}{2})$

$\tilde{S}_2$: $y = -x + 3$, $\frac{3}{2} \le x \le 2$ $(\frac{3}{2}, \frac{3}{2})$ $(2,1)$

$\tilde{S}_3$: $y = x - 1$, $1 \le x \le 2$ $(1,0)$ $(2,1)$

$\tilde{S}_4$: $y = -x + 1$, $\frac{1}{2} \le x \le 1$ $(\frac{1}{2}, \frac{1}{2})$ $(1,0)$

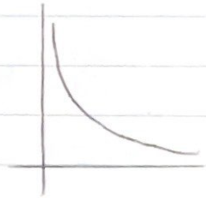


1(b) Let $S = \{(u,v) : 1 \le uv \le 2,\ 2u^2 \le v \le 3u^2\}$. Sketch the region S. Find the transformation

$T(u,v) = (x(u,v),\ y(u,v))$ that will take S into a rectangle. Justify your answer.

(reciprocal functions)

from S, $1 = uv$, and $uv = 2$ have this
$v = \frac{1}{u}$ $v = \frac{2}{u}$ Shape (in general)



and

$v = 2u^2$ and $v = 3u^2$ have this general shape
(parabolas)

now, we find their intersection points:

① $\begin{cases} v = \frac{1}{u} \\ v = 2u^2 \end{cases} \rightarrow 2u^2 = \frac{1}{u}$

$u^3 = \frac{1}{2}$

$u = \sqrt[3]{\frac{1}{2}} \approx 0.8$

so, $v = \frac{1}{\sqrt[3]{\frac{1}{2}}}$

$v = \sqrt[3]{2} \approx 1.3$

point $(x, y) = \left(\sqrt[3]{\frac{1}{2}}, \sqrt[3]{2}\right) \approx (0.8, 1.3)$

② $\begin{cases} v = \frac{1}{u} \\ v = 3u^2 \end{cases} \rightarrow 3u^2 = \frac{1}{u}$

$u^3 = \frac{1}{3}$

$u = \sqrt[3]{\frac{1}{3}} \approx 0.7$

so, $v = \frac{1}{\sqrt[3]{\frac{1}{3}}}$

$v = \sqrt[3]{3} \approx 1.4$

point $(x, y) = \left(\sqrt[3]{\frac{1}{3}}, \sqrt[3]{3}\right) \approx (0.7, 1.4)$

③ $\begin{cases} v = \frac{2}{u} \\ v = 2u^2 \end{cases} \rightarrow 2u^2 = \frac{2}{u}$

$u^3 = 1$

$u = 1$

so, $v = \frac{2}{1}$

$v = 2$

point $(x, y) = (1, 2)$

④ $\begin{cases} V = \frac{2}{u} \\ V = 3u^2 \end{cases} \rightarrow 3u^2 = \frac{2}{u}$

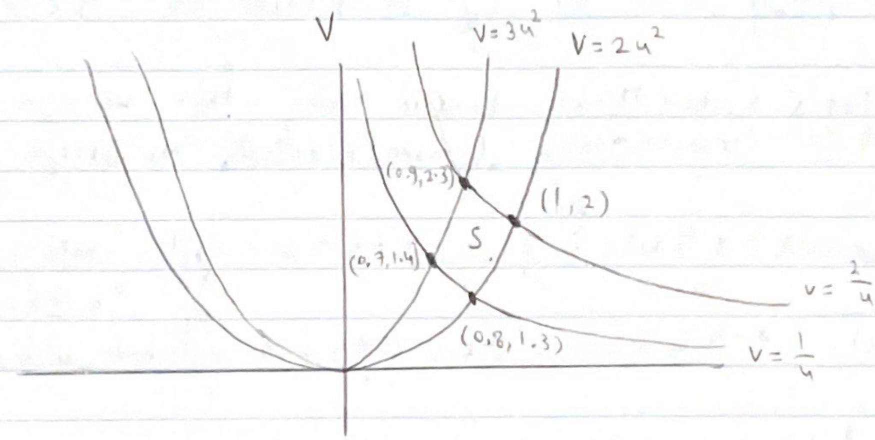
$u^3 = \frac{2}{3}$

$u = \sqrt[3]{\frac{2}{3}} \approx 0.9$

so, $v = \frac{2}{\sqrt[3]{\frac{2}{3}}}$

$V = 2^{\frac{2}{3}} \sqrt[3]{3} \approx 2.3$

point $(x, y) = \left(\sqrt[3]{\frac{2}{3}}, 2^{\frac{2}{3}} \sqrt[3]{3}\right) \approx (0.9, 2.3)$



now, find the transformation $T(u, v) = (x(u, v), y(u, v))$

from set S, $1 \leq uv \leq 2$ and $2u^2 \leq V \leq 3u^2$

$\downarrow$

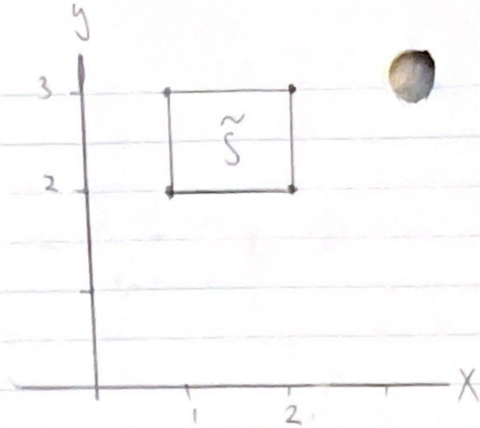
let $x = uv$ and $y = \frac{v}{u^2}$ $\qquad 2 \leq \frac{V}{u^2} \leq 3$

so, $1 \le uv \le 2 \to 1 \le x \le 2$

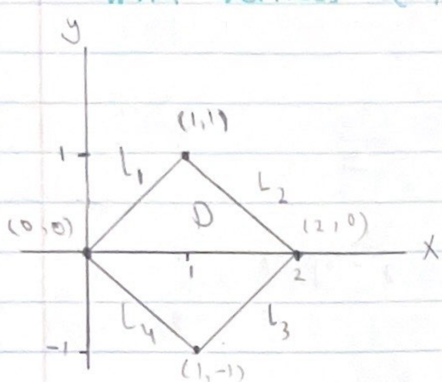
$2 \le \frac{v}{u^2} \le 3 \to 2 \le y \le 3$.

so $T(u,v) = (x,y) = \left(uv, \frac{v}{u^2}\right)$

and the image of S is $\longrightarrow$
(the image of S under T)



2. Evaluate $\iint_D (x+y)\sin(x-y)\,dA$ where D is the parallelogram with vertices $(0,0)$, $(1,1)$, $(2,0)$ and $(1,-1)$



find equation for each line L_1, L_2, L_3, L_4

L_1: $(0,0), (1,1) \to y = x$
L_2: $(1,1), (2,0) \to y = -x+2$
L_3: $(2,0), (1,-1) \to y = x-2$
L_4: $(1,-1), (0,0) \to y = -x$

let's write these lines in a way that we can get the transformation (more precisely the inverse transformation)

L_1: $x - y = 0$ L_2: $x + y = 2$
L_3: $x - y = 2$ L_4: $x + y = 0$

note that $0 \le x - y \le 2$ and $0 \le x + y \le 2$.

you could have let $u = x - y$ and $v = x + y$.

$\rightarrow$ let $u = x + y$ and $v = x - y$ so, $0 \le u \le 2$, $0 \le v \le 2$.

note that $T^{-1}(x,y) = (u,v) = (x+y, x-y)$ is the inverse transformation,

now, let's find the transformation T.

$$+\quad \begin{aligned} u &= x+y \\ v &= x-y \end{aligned} \qquad\qquad -\quad \begin{aligned} u &= x+y \\ v &= x-y \end{aligned}$$

$$u+v = 2x \rightarrow x = \tfrac{1}{2}(u+v) \qquad\qquad u-v = 2y \rightarrow y = \tfrac{1}{2}(u-v)$$

So, the transformation T is $T(u,v) = \left(\frac{1}{2}(u+v), \frac{1}{2}(u-v)\right)$

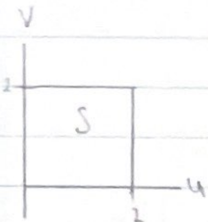
let's find the Jacobian

$$J = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} \qquad\qquad \begin{aligned} x &= \tfrac{1}{2}(u+v) \\ y &= \tfrac{1}{2}(u-v) \end{aligned}$$

$$J = \begin{vmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{vmatrix} = \left(\frac{1}{2}\right)\left(-\frac{1}{2}\right) - \frac{1}{2}\left(\frac{1}{2}\right) = -\frac{1}{4} - \frac{1}{4} = -\frac{1}{2}$$

So, the integral,

$$\iint_D (x+y)\sin(x-y)\, dA = \int_0^2 \int_0^2 u \sin(v) \left|-\frac{1}{2}\right| dv\, du$$



$$= \int_0^2 \int_0^2 \frac{1}{2} u \sin(v)\, dv\, du = \frac{1}{2} \int_0^2 u\, du \int_0^2 \sin(v)\, dv$$

$$= \frac{1}{2} \left[\frac{u^2}{2}\Big|_0^2\right]\left[-\cos(v)\Big|_0^2\right] = -\frac{1}{4}\left(2^2 - 0^2\right)\left(\cos(2) - \cos(0)\right)$$

$$= -\frac{1}{4}\, 4\left(\cos(2) - 1\right) = 1 - \cos(2)$$

3(a) Let D be the region $D = \{(x,y) : 1 \le x^2+y^2 \le 2,\ y \ge 0\}$. Compute the volume of the region below $f(x,y) = e^{x^2+y^2}$ and above D.

from set D, $y \geq 0$, and $x^2+y^2=1 \rightarrow$ semi-circle with radius of 1

and $y \geq 0$ and $x^2+y^2=2 \rightarrow$ semicircle with radius of $\sqrt{2}$

using polar coordinates,

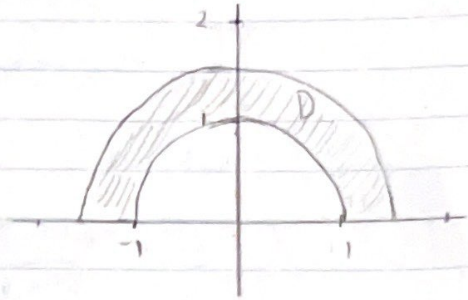
$1 \leq x^2+y^2 \leq 2$

$1 \leq r^2 \leq 2$

$1 \leq r \leq \sqrt{2}$

here $y \geq 0$

so, $0 \leq \theta \leq \pi$



$$\iint_D e^{x^2+y^2}\, dA = \int_0^{\pi} \int_1^{\sqrt{2}} e^{r^2}\, \underline{\underline{r}}\, dr\, d\theta$$

(note that there is r and this due to change of variable. you can find r by computing the Jacobian. for $x = r\cos(\theta)$, $y = r\sin(\theta)$)

$$= \int_0^{\pi} \int_1^{\sqrt{2}} e^{r^2}\, r\, dr\, d\theta$$

let $u = r^2$

$du = 2r\, dr$

$$= \int_0^{\pi} \int_1^{2} \frac{1}{2} e^{u}\, du\, d\theta$$

if $r = 1 \quad u = 1$

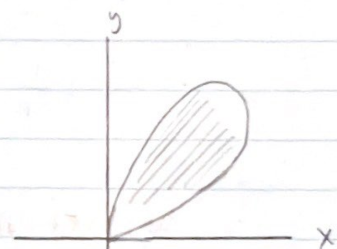
$r = \sqrt{2} \quad u = 2$

$$= \int_0^{\pi} \frac{1}{2} e^{u} \Big|_1^2\, d\theta$$

$$= \frac{1}{2}\int_0^{\pi} (e^2 - e^1)\, d\theta = \frac{1}{2}(e^2 - e^1)\, \theta \Big|_0^{\pi}$$

$$= \frac{1}{2}(e^2 - e^1)(\pi - 0) = \frac{\pi}{2}(e^2 - e^1)$$

3(b) Find the area enclosed by one loop of the 3-petalled rose $r = \sin(3\theta)$



one loop is formed from $r=0$ to $r=0$

let $r = 0$, $0 = \sin(3\theta)$

$\arcsin(0) = 3\theta$

$3\theta = 0, \pi, 2\pi, \ldots$

$\theta = 0, \frac{\pi}{3}, \frac{2\pi}{3}, \ldots$

(we want first 2 values to form the loop) so, $0 \le \theta \le \frac{\pi}{3}$ and $0 \le r \le \sin(3\theta)$

$$\iint_D 1 \, dA = \int_0^{\frac{\pi}{3}} \int_0^{\sin(3\theta)} 1 \, r \, dr \, d\theta$$ (using polar coordinate)

$$= \int_0^{\frac{\pi}{3}} \frac{r^2}{2} \Big|_0^{\sin(3\theta)} d\theta = \frac{1}{2}\int_0^{\frac{\pi}{3}} \left(\sin^2(3\theta) - 0^2\right) d\theta$$

$$= \frac{1}{2}\int_0^{\frac{\pi}{3}} \sin^2(3\theta)\, d\theta = \frac{1}{2}\int_0^{\frac{\pi}{3}} \frac{1-\cos(6\theta)}{2}\, d\theta$$

$$= \frac{1}{4}\int_0^{\frac{\pi}{3}} (1-\cos(6\theta))\,d\theta = \frac{1}{4}\left(\theta - \frac{\sin(6\theta)}{6}\right)\Big|_0^{\frac{\pi}{3}}$$

$$= \frac{1}{4}\left[\left(\frac{\pi}{3} - 0\right) - \frac{1}{6}\left(\sin\left(\frac{6\pi}{3}\right) - \sin(0)\right)\right]$$

$$= \frac{1}{4}\left(\frac{\pi}{3} - \frac{1}{6}(0-0)\right) = \frac{1}{4}\cdot\frac{\pi}{3} = \frac{\pi}{12}.$$

4 (a)

Let $f(x,y) = \frac{x^3}{y}$, and let C denotes the segment of the curve $y = \frac{x^2}{2}$ from the point $(0,0)$ to $(2,2)$.

compute $\int_C f(x,y)\, ds$.

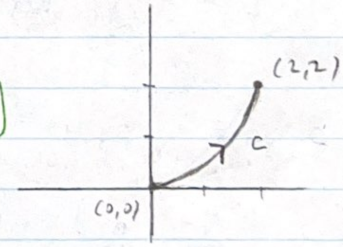
$$\int_C f(x,y)\, ds = \int_a^b f(x(t), y(t))\, |\vec{r}\,'(t)|\, dt$$

line integral for scalar function $f(x,y)$

if $(0,0)$
$x=0$, $y=0$
$0=t$
$0=\frac{t^2}{2} \to t=0$
if $(2,2)$
$x=2$, $y=2$
$2=t$
$2=\frac{t^2}{2} \to t=2$

start with parametrizing $y = \frac{x^2}{2}$

let $x = t$ $\quad y = \frac{t^2}{2}$ $\quad \boxed{0 \le t \le 2}$



so, $\vec{r}(t) = \langle t, \frac{t^2}{2} \rangle$

$$\vec{r}\,'(t) = \langle 1, \frac{2t}{2} \rangle = \langle 1, t \rangle$$

$$|\vec{r}\,'(t)| = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = \sqrt{1^2 + t^2} = \sqrt{1+t^2}$$

so, $\int_C f(x,y)\,ds = \int_0^2 \frac{t^3}{\left(\frac{t^2}{2}\right)} \sqrt{(1)^2 + (t^2)}\ dt$

$= \int_0^2 2t\sqrt{1+t^2}\ dt$ let $u = 1+t^2$

$du = 2t\ dt \rightarrow \frac{du}{2t} = dt$

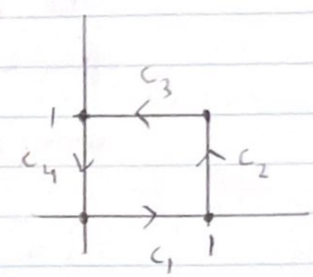
$= \int_1^5 \cancel{2t}\sqrt{u}\ \frac{du}{\cancel{2t}}$ if $t=0,\quad u=1$

$t = 2 \quad u = 1+2^2 = 5$

$= \int_1^5 u^{\frac{1}{2}}\ du = \left.\frac{u^{\frac{1}{2}+1}}{\frac{1}{2}+1}\right|_1^5 = \left.\frac{2}{3}u^{\frac{3}{2}}\right|_1^5$

$= \frac{2}{3}\left(5^{\frac{2}{3}} - 1^{\frac{2}{3}}\right) = \frac{2}{3}\left(5^{\frac{2}{3}} - 1\right).$

(b) Evaluate $\int_C (x^2-y^2)\,dx + (x^2+y^2)\,dy$ where C is the square with vertices $(0,0)$, $(1,0)$, $(0,1)$ and $(1,1)$ traced Counterclockwise.



$\int_C P(x,y)\,dx + Q(x,y)\,dy$

$= \int_{C_1} P\,dx + Q\,dy + \int_{C_2} P\,dx + Q\,dy$

$+ \int_{C_3} P\,dx + Q\,dy + \int_{C_4} P\,dx + Q\,dy$

as you can see, we can breakdown the integral into 4 integrals. and integrate along each line segment; and then add up. the results now, Parameterize each <u>line</u> <u>Segment</u>

$\underline{C_1}$: $\vec{r}(t) = \langle 0,0\rangle + t(\langle 1,0\rangle - \langle 0,0\rangle)$
$= \langle t, 0\rangle \quad , \quad 0 \le t \le 1$

$\vec{r}\,'(t) = \langle 1, 0\rangle$

$\underline{C_2}$: $\vec{r}(t) = \langle 1,0\rangle + t(\langle 1,1\rangle - \langle 1,0\rangle)$
$= \langle 1,0\rangle + t\langle 0,1\rangle = \langle 1, t\rangle \quad 0 \le t \le 1$

$\vec{r}\,'(t) = \langle 0, 1\rangle$

$\underline{C_3}$: $\vec{r}(t) = \langle 1,1\rangle + t(\langle 0,1\rangle - \langle 1,1\rangle)$
$= \langle 1,1\rangle + t\langle -1,0\rangle = \langle 1-t, 1\rangle \quad 0 \le t \le 1$

$\vec{r}\,'(t) = \langle -1, 0\rangle$

$\underline{C_4}$: $\vec{r}(t) = \langle 0,1\rangle + t(\langle 0,0\rangle - \langle 0,1\rangle)$
$= \langle 0,1\rangle + t\langle 0,-1\rangle = \langle 0, 1-t\rangle \quad 0 \le t \le 1$

$\vec{r}\,'(t) = \langle 0, -1\rangle$

<u>Note</u>: $\vec{r}(t) = \langle x(t), y(t)\rangle$ (the components of $\vec{r}(t)$ are $x(t)$, and $y(t)$)

$\vec{r}\,'(t) = \langle \underbrace{x'(t)\,dt}_{dx}, \underbrace{y'(t)\,dt}_{dy}\rangle$ (the components of $\vec{r}\,'(t)$ are derivative of $x(t)$ and $y(t)$)

So, the integral is

$$\int_C (x^2-y^2)\,dx + (x^2+y^2)\,dy$$

$$= \int_0^1 [(t^2-0^2)(1) + (t^2+0^2)(0)]\,dt$$

$$+ \int_0^1 [(1^2-t^2)(0) + (1^2+t^2)(1)]\,dt$$

$$+ \int_0^1 \left[\left((1-t)^2 - 1^2\right)(-1) + \left((1-t)^2 + 1^2\right)(0) \right] dt$$

$$+ \int_0^1 \left[\left(0^2 - (1-t)^2\right)(0) + \left(0^2 + (1-t)^2\right)(-1) \right] dt$$

$$= \int_0^1 t^2 \, dt + \int_0^1 1 + t^2 \, dt - \int_0^1 (1-t)^2 - 1 \, dt - \int_0^1 (1-t)^2 \, dt$$

$$= \frac{t^3}{3}\Big|_0^1 + t\Big|_0^1 + \frac{t^3}{3}\Big|_0^1 - \int_0^1 1 - 2t + t^2 - 1 \, dt - \int_0^1 1 - 2t + t^2 \, dt$$

$$= \frac{1}{3}(1^3 - 0^3) + (1 - 0) + \frac{1}{3}(1^3 - 0^3) - \left((-2)\frac{t^2}{2}\Big|_0^1 + \frac{t^3}{3}\Big|_0^1 \right)$$

$$- \left(t\Big|_0^1 - 2\frac{t^2}{2}\Big|_0^1 + \frac{t^3}{3}\Big|_0^1 \right)$$

$$= \frac{1}{3} + 1 + \frac{1}{3} - \left(-(1^2 - 0) + \frac{1}{3}(1^3 - 0) \right) - \left((1 - 0) - (1^2 - 0) + \frac{1}{3}(1^3 - 0) \right)$$

$$= \frac{2}{3} + 1 + 1 - \frac{1}{3} - 1 + 1 - \frac{1}{3} = 2$$