計算科学・量子計算における情報圧縮

Data compression in computational science and quantum computing

2023.01.19

#13: 量子誤り訂正とテンソルネットワーク

Quantum Error Correction and Tensor Network

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Today's topic

1. Computational science, quantum computing, and data compression
2. Review of linear algebra
3. Singular value decomposition
4. Application of SVD and generalization to tensors
5. Entanglement of information and matrix product states
6. Application of MPS to eigenvalue problems
7. Tensor network representation
8. Data compression in tensor network
9. Tensor network renormalization
10. Quantum mechanics and quantum computation
11. Simulation of quantum computers
12. Quantum-classical hybrid algorithms and tensor network
13. Quantum error correction and tensor network

Outline

- About Report No.2
- Why we need error correction?
  - error corrections in classical computer
  - errors in quantum computer
- Classical error correction
- Simple quantum error correction
  - three-qubit code
  - Shor code
- Stabilizer and the surface (toric) code
  - sufrace code
- Tensor network simulation of surface code

Report No.2

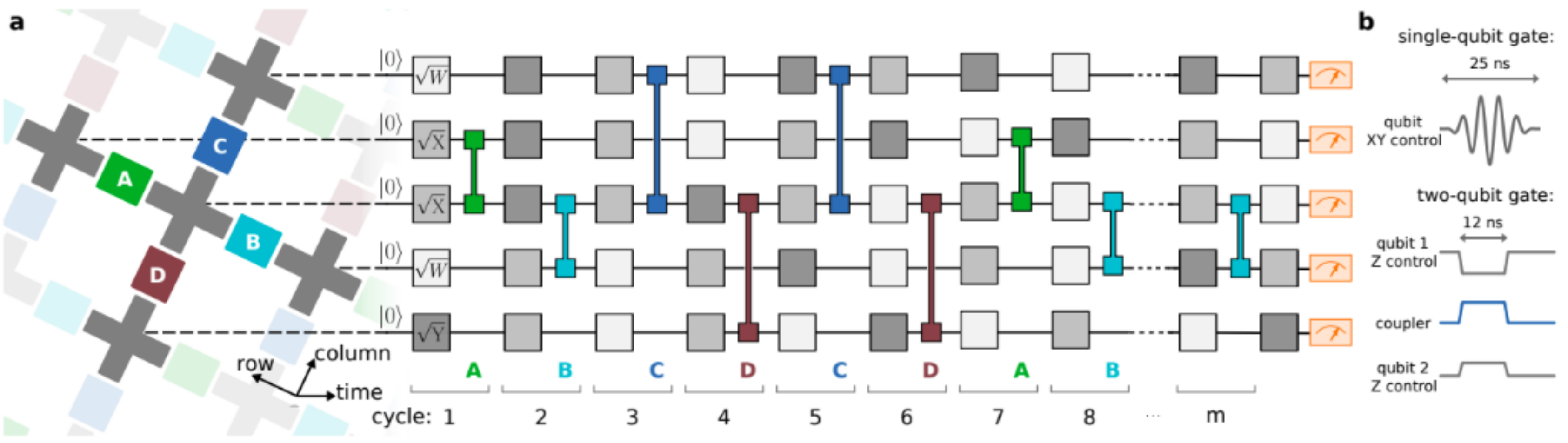
- Deadline: 2023/02/02 23:59
- Submission: via ITC-LMS

- tensornetwork.ipynb
  - simulation of quantum circuit using TensorNetwork library
    - problem 1 (mandatory): probability of states 11 and 01 of Bell state
    - problem 2 (mandatory): GHZ state
    - problem 3 (optional): entangled state of n qubits
- grover.ipynb
  - simulation of Grover’s algorithm using Qiskit and TensorNetwork library
    - problem 1 (mandatory): running Qiskit simulation of Grover’s algorithm
    - problem 2 (mandatory): running TensorNetwork simulation
    - problem 3(mandatory): scaling of computation time

Why we need error correction?

How can we check reliability of the computer output?

- We usually use computers to calculate things we cannot calculate.
  - to ensure the result, we need to understand its operation principle
- Actual devices do not work as the ideal system, expected in algorithms.
  - deviation from an ideal qubit
    - effect of excitation states
  - errors in operations
  - environments
    - thermal noise
    - dissipation
    - radiation



Arute, F., Arya, K., Babbush, R. et al. Quantum supremacy using a programmable superconducting processor. Nature 574, 505–510 (2019). https://doi.org/10.1038/s41586-019-1666-5

Error corrections in classical computer

- During daily use of classical computers, we do not need to take care the deviation from the ideal (digital) computer
- In classical computers, a lot of error correction mechanisms are implemented
  - repetition code
  - parity check bit
  - ECC memory
- Users can deal with classical computer as if it is an ideal device

- Similarly, by correcting quantum errors, we can run various quantum algorithms as in text books

Typical errors in quantum computer

- Bit flip
  - flips the states of a qubit from $|0\rangle$ to $|1\rangle$ or vice versa

$$X|0\rangle = |1\rangle$$
$$X|1\rangle = |0\rangle$$
$$\Rightarrow \quad X(\alpha|0\rangle + \beta|1\rangle) = \alpha|1\rangle + \beta|0\rangle$$

- Phase flip
  - changes the phase of the states of a qubit

$$Z|0\rangle = |0\rangle$$
$$Z|1\rangle = -|1\rangle$$
$$\Rightarrow \quad Z(\alpha|0\rangle + \beta|1\rangle) = \alpha|0\rangle - \beta|1\rangle$$

- Depolarization
  - the states of a qubit is replaced by the completely mixed state
- Amplitude damping
  - energy dissipation to the environment

Noise operations

- Bit flip error (with error rate p)

$$X|0\rangle = |1\rangle \qquad X|1\rangle = |0\rangle \quad \Rightarrow \quad X(\alpha|0\rangle + \beta|1\rangle) = \alpha|1\rangle + \beta|0\rangle$$

- How to express noice effects as operations

$$|\Psi'\rangle = \sqrt{1-p}|\Psi\rangle + \sqrt{p}X|\Psi\rangle?$$

  - $\sqrt{1-p}\,\mathbf{1} + \sqrt{p}X$ is not a unitary operator

- Noice is represented by a superoperator

Operator-sum representation

$$\rho' = (1-p)\rho + pX\rho X^\dagger$$

$$\mathcal{U}(\rho) = \sum_i E_i \rho E_i^\dagger$$

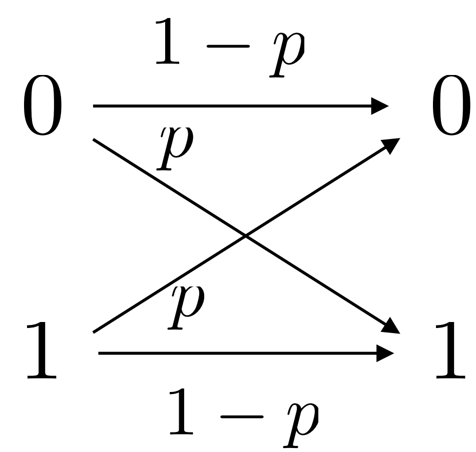
  - a pure state is transformed into a mixed state

$$|0\rangle\langle 0| \to (1-p)|0\rangle\langle 0| + p|1\rangle\langle 1|$$

Classical error correction

Classical bit flip error

- Suppose a bit flip error occurs with a probability p



  - If we transfer a classical information (bit) through this channel, the receiver will get incorrect bit with a probability p.
- How do we detect the error and protect our information?
  - strategy: encode the information using more than one bit, and recover (decode) after we receive the information

Three-bit encoding based on repetition

- Encode 0 and 1 into three bit

$$0 \to 000$$
$$1 \to 111$$

  - 000 and 111 are considered as logical 0 and logical 1, respectively
- The receiver may get 000, 001, 010, 011, 100, 101, 110, or 111
  - decode using the majority rule

$$\begin{matrix} 000 \\ 001 \\ 010 \\ 100 \end{matrix} \to 0, \qquad \begin{matrix} 111 \\ 110 \\ 101 \\ 011 \end{matrix} \to 1$$

- Note: to decode the state, measurement of all bits is needed

Error in three-bit encoding

$$\begin{matrix} 000 \\ 001 \\ 010 \\ 100 \end{matrix} \to 0, \qquad \begin{matrix} 111 \\ 110 \\ 101 \\ 011 \end{matrix} \to 1$$

- If only a single bit flip happens, we can alway recover the state
- Probability of multiple errors

$$p_3 = \underbrace{p^3}_{\text{3 bits flipped}} + \underbrace{3p^2(1-p)}_{\text{2 bits flipped}}$$

  - p_3 is smaller than p, if $p < 1/2$
- By using more than one bit, we can reduce the error rate

Error syndrome

- Position of error can be identified by measuring the product (parity) of bit pairs (1,2) and (1,3) only (called “error syndrome”)

	P_{12}	P_{13}	error
000	1	1	none
001	1	-1	3
010	-1	1	2
100	-1	-1	1

	P_{12}	P_{13}	error
111	1	1	none
110	1	-1	3
101	-1	1	2
011	-1	-1	1

- same table can be used for both cases

Simple quantum error correction

Quantum error correction?

- Similar approach to the classical error correction?
- At a glance, it seems difficulties to apply the classical repetition idea.
  - we cannot copy a quantum state
    - it is impossible to duplicate quantum state as $|\Psi\rangle \rightarrow |\Psi\rangle \otimes |\Psi\rangle \otimes |\Psi\rangle$
    - “no-cloning theorem” (lecture #10)
  - measurements at decoding procedure may destroy the superposition
  - state (and error) of the qubit is continuous

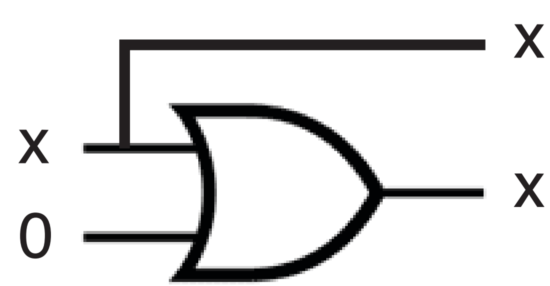
$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle \rightarrow \alpha'|0\rangle + \beta'|1\rangle$$

- Let’s apply repetition idea to the basis, $|0\rangle$ and $|1\rangle$, instead of $|\Psi\rangle$!

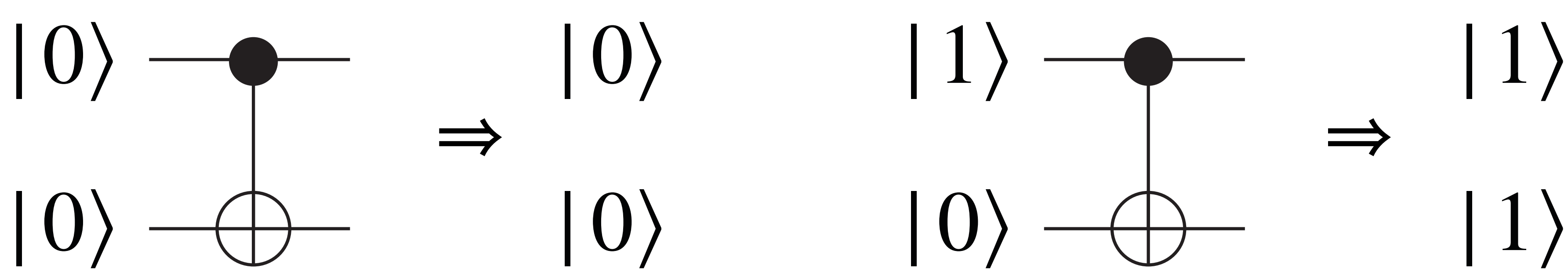
$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle \rightarrow \alpha|000\rangle + \beta|111\rangle$$

COPY gate

- We can swap two quantum states
  - can we copy a quantum state?
- In classical logic circuits, it is possible to use an OR gate (or XOR gate)



  - how about using CX gate?



COPY gate

- However, if we put the superposition state in the first qubit

$$|\psi 0\rangle = (\alpha|0\rangle + \beta|1\rangle)|0\rangle = \alpha|00\rangle + \beta|10\rangle$$
$$\Rightarrow \alpha|00\rangle + \beta|11\rangle$$

- On the other hand, the output we wanted was

$$|\psi\psi\rangle = (\alpha|0\rangle + \beta|1\rangle)(\alpha|0\rangle + \beta|1\rangle)$$
$$= \alpha^2|00\rangle + \alpha\beta|01\rangle + \beta\alpha|10\rangle + \beta^2|11\rangle$$

- No cloning theorem: "it is impossible to create an independent and identical copy of an arbitrary unknown quantum state"
  - difficulties in quantum algorithm design
  - (though it contains all the classical algorithms)

Quantum error correction?

- Similar approach to the classical error correction?
- At a glance, it seems difficulties to apply the classical repetition idea.
  - we cannot copy a quantum state
    - it is impossible to duplicate quantum state as $|\Psi\rangle \to |\Psi\rangle \otimes |\Psi\rangle \otimes |\Psi\rangle$
    - "no-cloning theorem" (lecture #10)
  - measurements at decoding procedure may destroy the superposition
  - state (and error) of the qubit is continuous

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle \to \alpha'|0\rangle + \beta'|1\rangle$$

- Let's apply repetition idea to the basis, $|0\rangle$ and $|1\rangle$, instead of $|\Psi\rangle$!

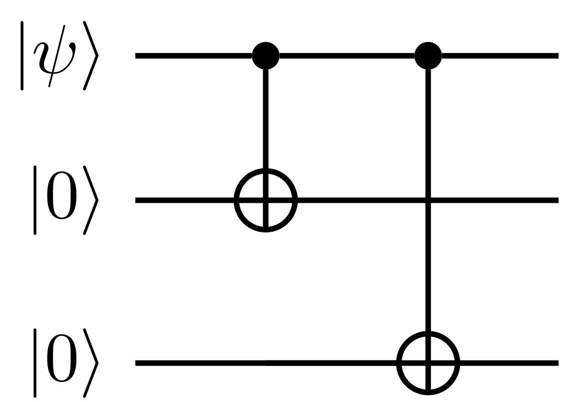
$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle \to \alpha|000\rangle + \beta|111\rangle$$

Quantum circuit for encoding

- Encoding to 3-qubit state

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle \to \alpha|000\rangle + \beta|111\rangle$$

  - can be done by using



  - it is easy to check

$$|0\rangle \to |000\rangle$$
$$|1\rangle \to |111\rangle$$

  - and it is sufficient for the superposed case

Three-qubit code for bit flip

- Encoded state $\alpha|000\rangle + \beta|111\rangle$ is in the space spanned by $|000\rangle$ and $|111\rangle$ (code space)
- Bit-flip error
  - bit flip X is operated with probability p, independently for each three qubits
- What happens after the single bit flip, e.g., for the first qubit?

$$\alpha|000\rangle + \beta|111\rangle \rightarrow \alpha|100\rangle + \beta|011\rangle$$

  - state goes outside of the code space, vector space spanned by $|000\rangle$ and $|111\rangle$
  - different subspaces (orthogonal with each other) depending on the position of error

No bit flips:	$\mathrm{Span}\{|000\rangle, |111\rangle\}$
Bit flip on qubit 1:	$\mathrm{Span}\{|100\rangle, |011\rangle\}$
Bit flip on qubit 2:	$\mathrm{Span}\{|010\rangle, |101\rangle\}$
Bit flip on qubit 3:	$\mathrm{Span}\{|001\rangle, |110\rangle\}$

Detecting the error: syndrome measurement

No bit flips:	$\mathrm{Span}\{\|000\rangle, \|111\rangle\}$
Bit flip on qubit 1:	$\mathrm{Span}\{\|100\rangle, \|011\rangle\}$
Bit flip on qubit 2:	$\mathrm{Span}\{\|010\rangle, \|101\rangle\}$
Bit flip on qubit 3:	$\mathrm{Span}\{\|001\rangle, \|110\rangle\}$

- Projective measurement (syndrome measurement) can distinguish errors

$$M = \sum_{i=0}^{3} m P_m$$

$$P_0 = |000\rangle\langle 000| + |111\rangle\langle 111|$$
$$P_1 = |100\rangle\langle 100| + |011\rangle\langle 011|$$
$$P_2 = |010\rangle\langle 010| + |101\rangle\langle 101|$$
$$P_3 = |001\rangle\langle 001| + |110\rangle\langle 110|$$

- Note: if we consider projectors on, e.g., $|000\rangle\langle 000|$, it may destroy a quantum superposition. On the other hand, syndrome measurement does not.

Recovering from errors

- Depending on the result of the syndrome measurement $P_M = \langle\psi|M|\psi\rangle$

$$P_0 = |000\rangle\langle 000| + |111\rangle\langle 111|$$
$$P_1 = |100\rangle\langle 100| + |011\rangle\langle 011|$$
$$P_2 = |010\rangle\langle 010| + |101\rangle\langle 101|$$
$$P_3 = |001\rangle\langle 001| + |110\rangle\langle 110|$$

- $P_M = 0$: no error
  - Do nothing
- $P_M = 1$: bit flip error on the qubit 1
  - Apply X to the qubit 1.
- $P_M = 2$: bit flip error on the qubit 2
  - Apply X to the qubit 2.
- $P_M = 3$: bit flip error on the qubit 3
  - Apply X to the qubit 3.

- For the cases where the bit flip occurs only one (or zero) qubit, this procedure can recover the original state perfectly

Phase flip error

- Bit flip errors are almost similar to the classical errors
  - Can we correct other errors?
- Phase flip
  - bit flip Z is operated with probability p, independently for each three qubits

$$Z(\alpha|0\rangle + \beta|1\rangle) = \alpha|0\rangle - \beta|1\rangle$$

  - bit flip error can be corrected as bit flip error by realizing Z as bit flip for |+> and |->

$$Z|+\rangle = |-\rangle$$
$$Z|-\rangle = |+\rangle$$

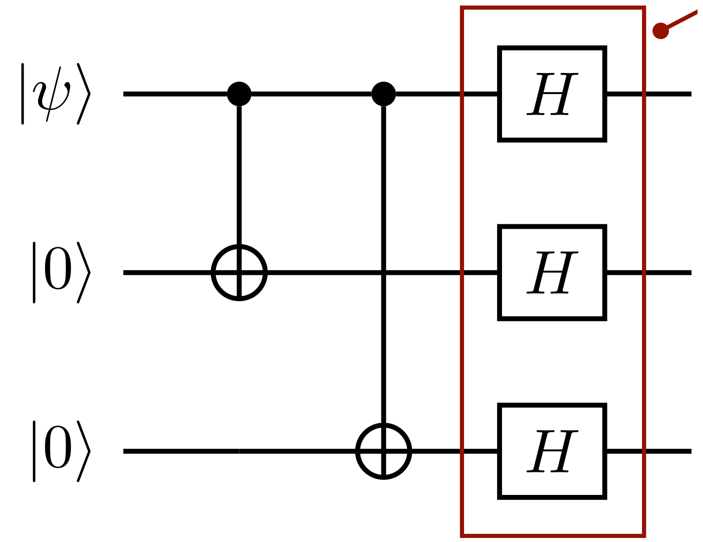
$$|+\rangle \equiv \frac{|0\rangle + |1\rangle}{\sqrt{2}}, |-\rangle \equiv \frac{|0\rangle - |1\rangle}{\sqrt{2}}$$

Three qubit code for phase flip errors

- Encode the state as

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle \rightarrow \alpha|+++\rangle + \beta|---\rangle$$

  - encoding can be done by using



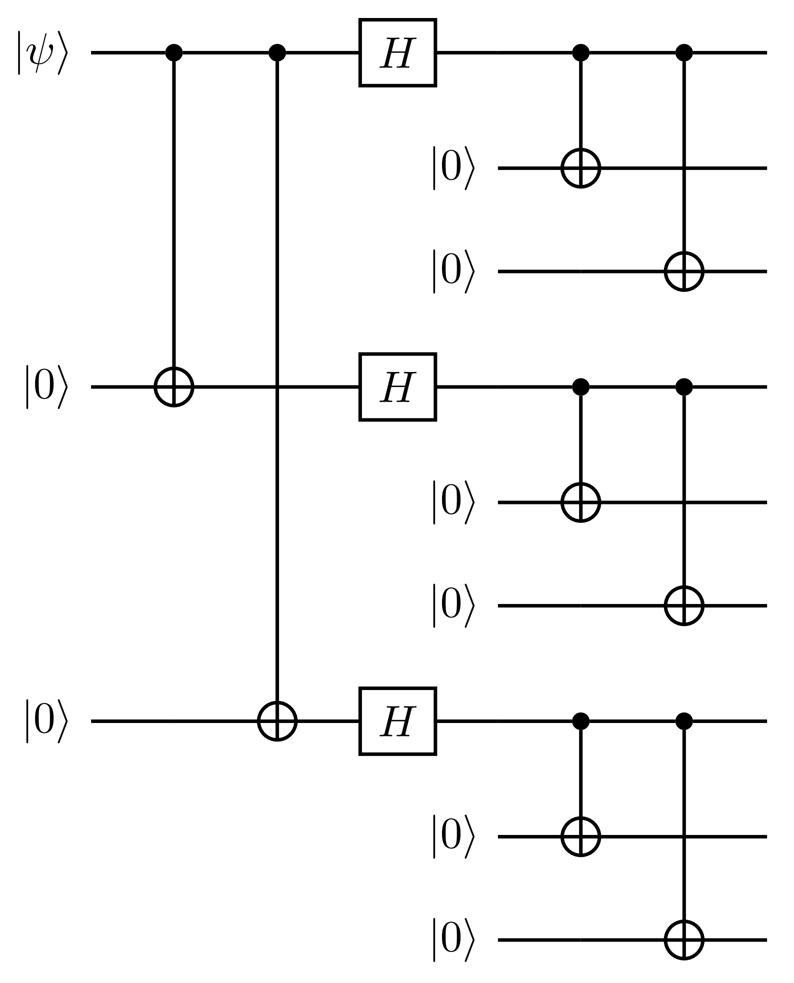
- Syndrome measurements and error correction:
  - replace ($|0\rangle$, $|1\rangle$, X) for bit flip error by ($|+\rangle$, $|-\rangle$, Z)

The Shor code

- Error correction code for an arbitrary single qubit error
  - combination of three qubit codes for bit flip and phase flip

$$|0\rangle \to |+++\rangle \quad \to \frac{1}{2\sqrt{2}}(|000\rangle + |111\rangle) \otimes (|000\rangle + |111\rangle) \otimes (|000\rangle + |111\rangle)$$

$$|1\rangle \to |---\rangle \quad \to \frac{1}{2\sqrt{2}}(|000\rangle - |111\rangle) \otimes (|000\rangle - |111\rangle) \otimes (|000\rangle - |111\rangle)$$



Stabilizer and the surface (toric) code

Another view of the three qubit codes

- Three qubit bit flip code

No bit flips:	$\mathrm{Span}\{\|000\rangle, \|111\rangle\}$
Bit flip on qubit 1:	$\mathrm{Span}\{\|100\rangle, \|011\rangle\}$
Bit flip on qubit 2:	$\mathrm{Span}\{\|010\rangle, \|101\rangle\}$
Bit flip on qubit 3:	$\mathrm{Span}\{\|001\rangle, \|110\rangle\}$

- Define the code states as a set of three qubit states satisfying

$$Z_1 Z_2 |\phi\rangle = |\phi\rangle$$
$$Z_2 Z_3 |\phi\rangle = |\phi\rangle$$

Simultaneous eigenstate of $Z_1 Z_2$ and $Z_2 Z_3$ with the eigenvalue 1.

- Errors can also be detected by Z_1Z_2 and Z_2Z_3 measurement

$$(Z_1 Z_2, Z_2 Z_3) = \underline{(1,1)}, \underline{(1,-1)}, \underline{(-1,1)}, \underline{(-1,-1)}$$

No bit flips | Bit flip on 3 | Bit flip on 1 | Bit flip on2

  - note: Z_1Z_2 and Z_2Z_3 do not destroy superposition of state

Pauli group and stabilizer

n-qubit Pauli group G_n: $\{\pm 1, \pm i\} \times \{I, X, Y, Z\}^{\otimes n}$

1-qubit G_1: $\{\pm X, \pm iI, \pm X, \pm iX, \pm Y, \pm iY, \pm Z, \pm iZ\}$

2-qubit G_2:

$$\{\pm 1, \pm i\} \times \{II, XI, IX, YI, IY, ZI, IZ, XY, YX, YZ, ZY, ZX, XZ, XX, YY, ZZ\}$$

- Stabilized vector subspace

Suppose S is a subgroup of G_n. We define a vector subspace V_s as a set of n-qubit states *stabilized* by all elements of S. And S is called the stabilizer.

- e.g. three qubit bit flip code: $S = \{I, Z_1Z_2, Z_2Z_3, Z_3Z_1\}$
- note: Z_1Z_2 and Z_2Z_3 (generators) are sufficient to define S

$$S = \langle Z_1Z_2, Z_2Z_3 \rangle$$

Stabilizer codes for quantum error correction

Stabilizer code:

We encode quantum states into the vector space stabilized by stabilizer S with $-I \notin S$.

*More precisely, we define a $[n, k]$ stabilizer code with S which has $n-k$ *independent* and *commuting* generators. The dimension of the $[n, k]$ stabilizer code is 2^{n-k}.

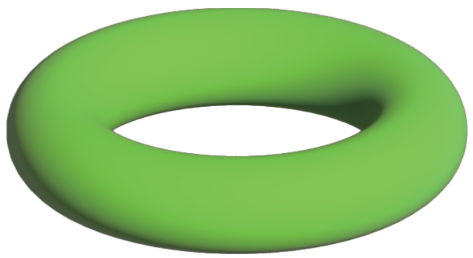
- Stabilizer code
  - syndrome measurements correspond to measurements by generators
- Examples:
  - three qubit bit flip code = a [3,2] stabilizer code with $S = \langle Z_1Z_2, Z_2Z_3 \rangle$
  - Three qubit phase flip code = a [3,2] stabilizer code with $S = \langle X_1X_2, X_2X_3 \rangle$
  - Shor code = a [9,8] stabilizer code with

$$S = \langle Z_1Z_2, Z_2Z_3, Z_4Z_5, Z_5Z_6, Z_7Z_8, Z_8Z_9, X_1X_2X_3X_4X_5X_6, X_4X_5X_6X_7X_8X_9 \rangle$$

Surface code

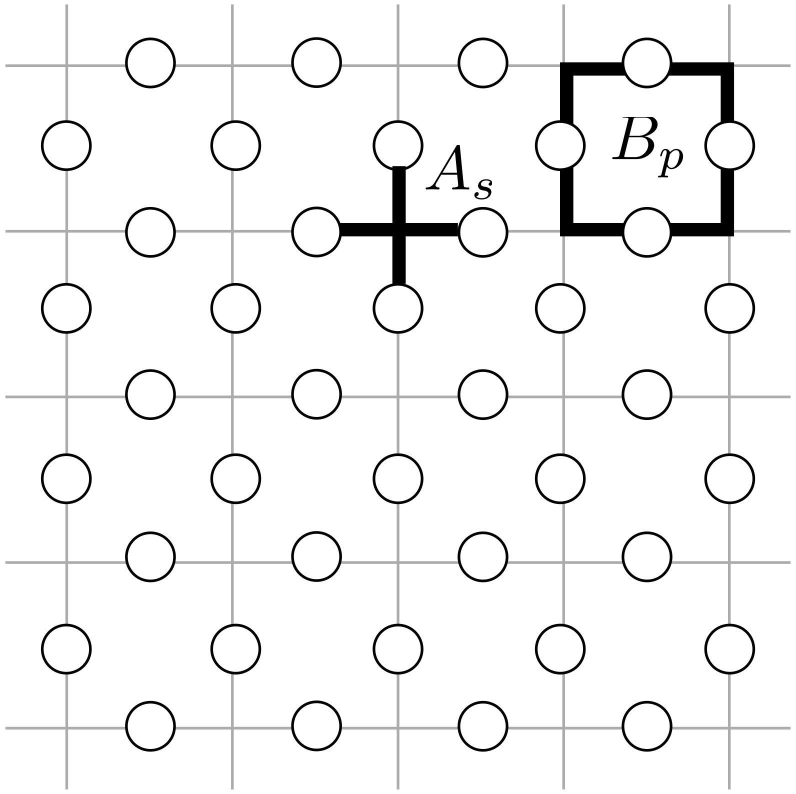
- Surface code: a topological error correcting code
  - consider stabilizers for all vertices s and plaquettes p on the torus

$$A_s = \prod_{j \in \mathrm{star}(s)} X_j \qquad B_p = \prod_{j \in \partial p} Z_j$$



  - all stabilizers commute with each other
- Quantum state $|\psi\rangle$ within the surface code

$$A_s|\psi\rangle = |\psi\rangle$$
$$B_p|\psi\rangle = |\psi\rangle$$



  - for all s and p

A. Kitaev, arXiv:quant-ph/9707021; Ann. Phys. 303, 2 (2003).

Logical qubits in the surface code

- # of qubits in $L \times L$ torus: $2L^2$
- # of vertices (stars): L^2
- # of plaquettes: L^2
- # of independent stabilizers: $2L^2 - 2$
  - Due to the relations

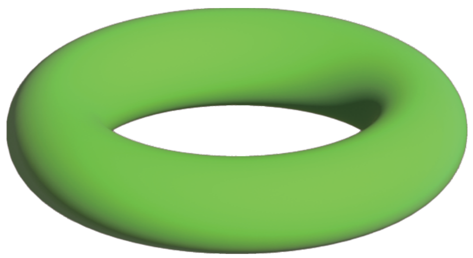
$$\prod_s A_s = I$$

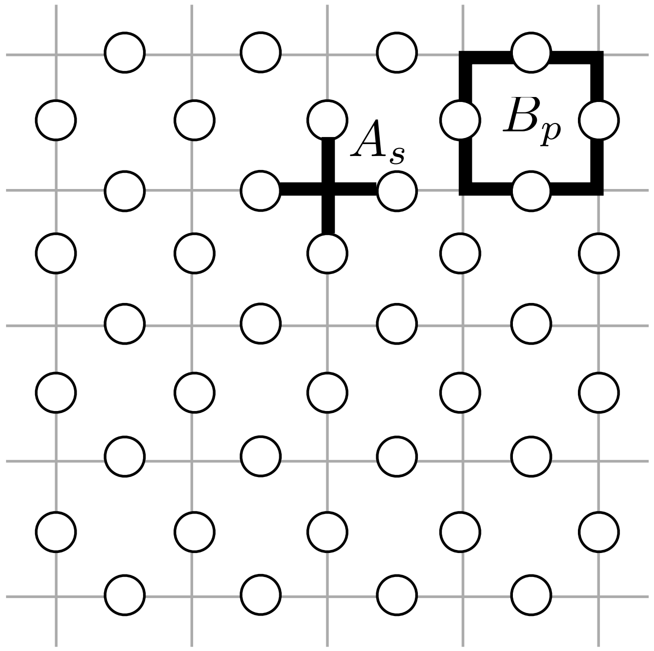
$$\prod_p B_p = I$$

  - two of the $2L^2$ starbilizers are not independent
- Dimension of the surface code space is

$$2^{2L^2-(2L^2-2)} = 2^2$$

  - → surface code encodes a two qubits state.

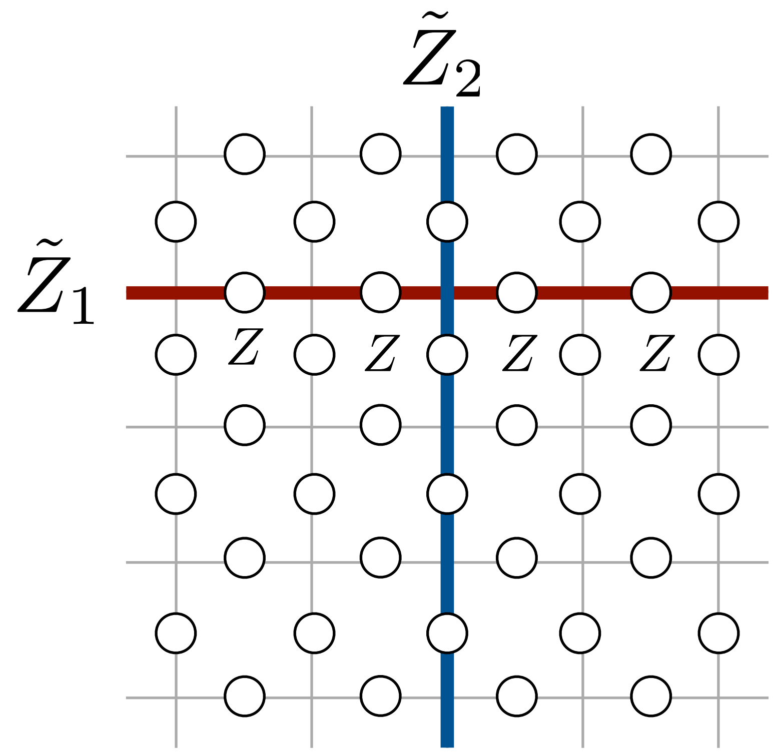


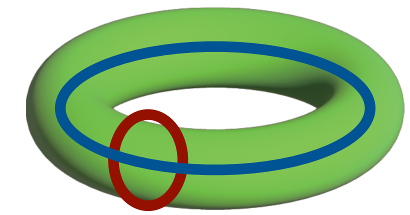


Logical qubits in the surface code

- The logical qubit states are characterized by non-trivial loop operators

$$\tilde{Z}_i = \prod_{j \in \text{non-trivial loop}} Z_j$$

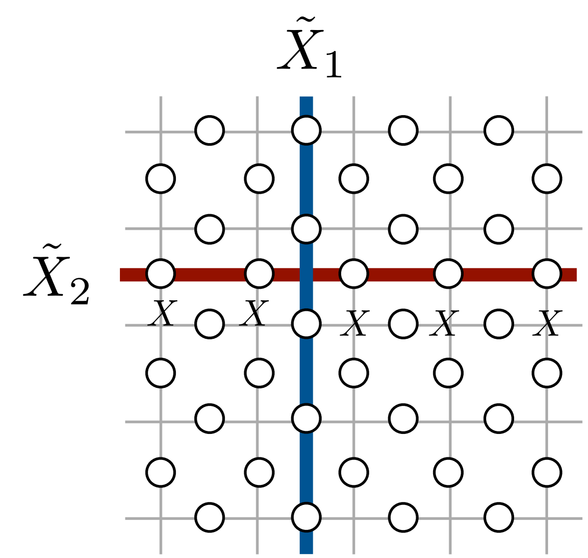




- We can define logical states, e.g.,

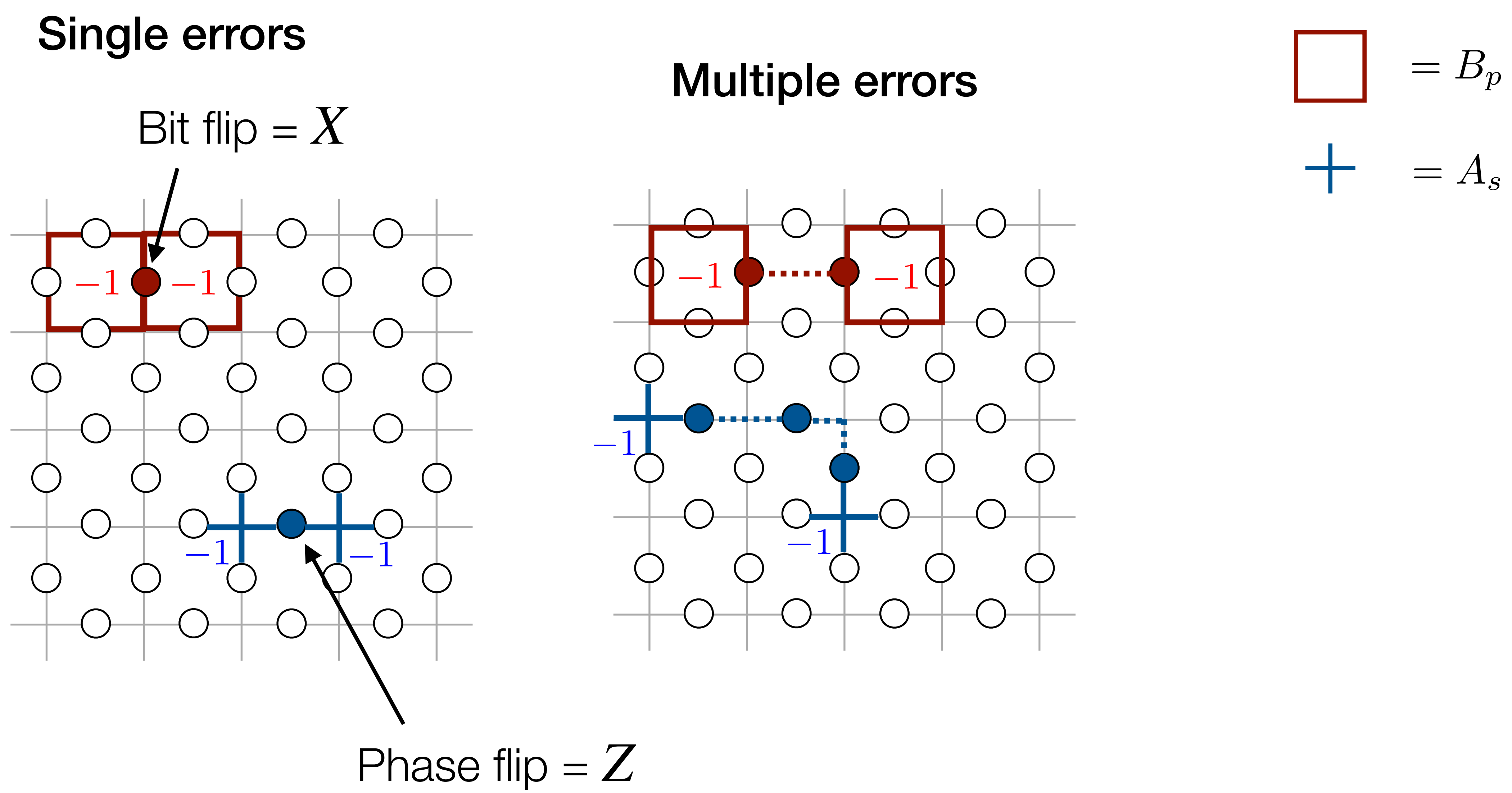
$$\begin{aligned}
\tilde{Z}_1|00\rangle_L &= |00\rangle_L \\
\tilde{Z}_2|00\rangle_L &= |00\rangle_L \\
\tilde{Z}_1|10\rangle_L &= -|10\rangle_L \\
\tilde{Z}_2|10\rangle_L &= |10\rangle_L \\
&\vdots
\end{aligned}$$

- We can also define corresponding X operators →



Syndrome measurements

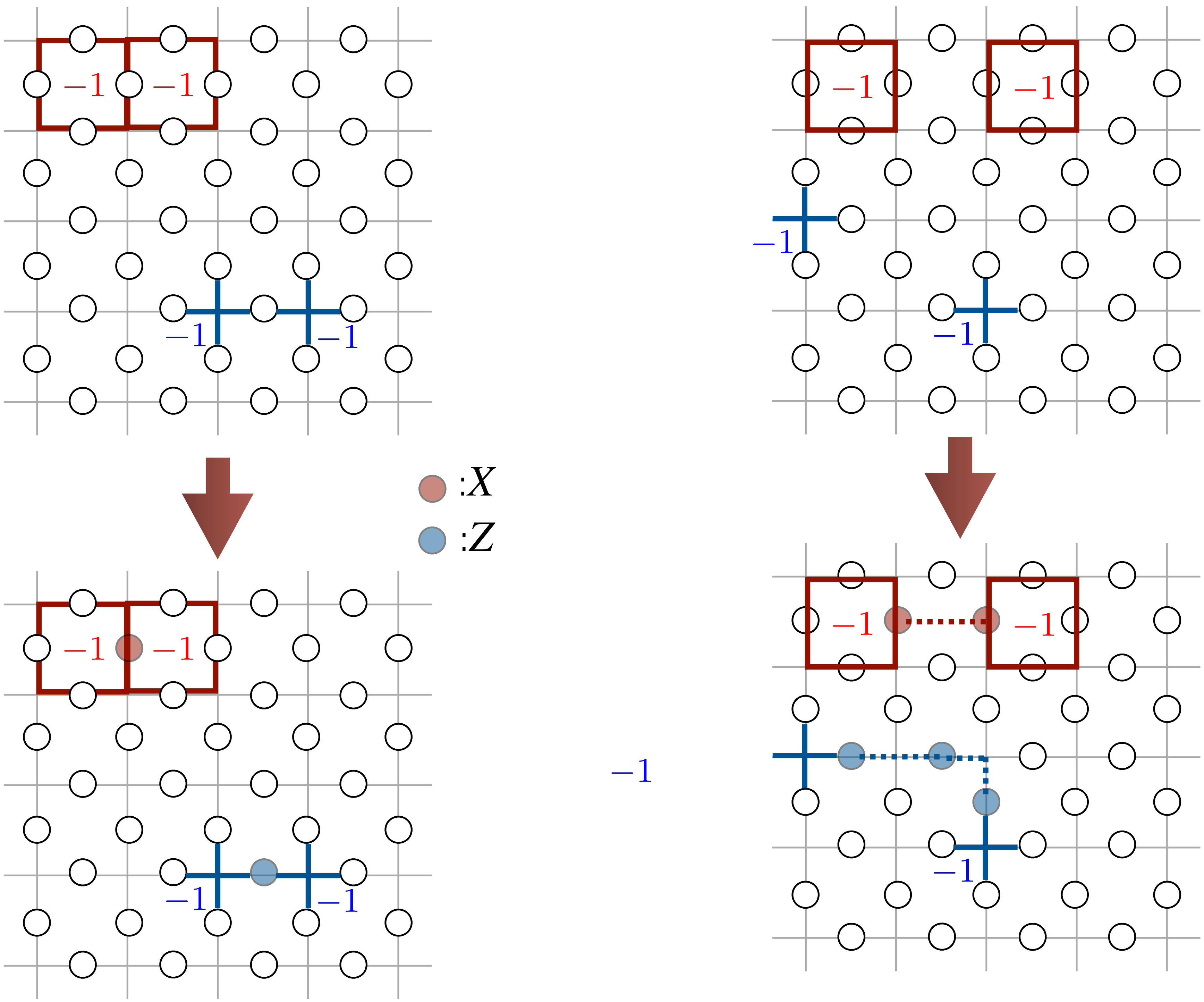
- Errors in qubits can be detected by the measurements of A_s and B_p



- Generally, error syndromes appear at the edges (boundaries) of error chain

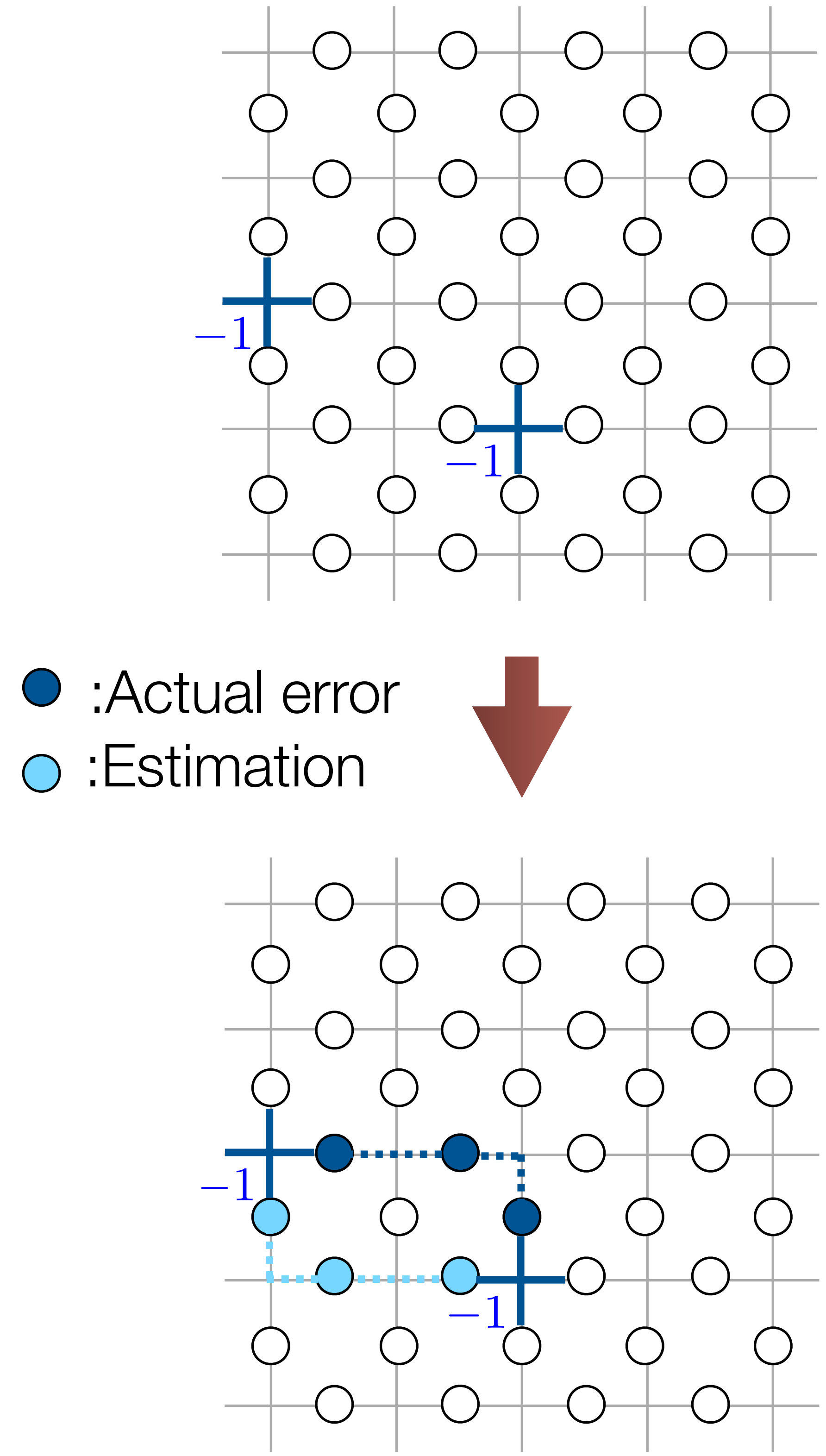
Error correction

- From error syndromes, we can estimate error position



Failure of error correction

- In some cases, we may estimate different positions from the actual errors
  - Even in such cases, error correction works well as the actual and estimated errors form a (trivial) loop corresponding to a stabilizer
- When errors are too dense → loop may become non-trivial
  - → error correction procedure changes the logical state
- There is a threshold in the error density for the successful error correction



Tensor network simulation of surface code

Simulation of quantum error correction

- Classical simulation of quantum error correction code
  - for evaluating quantum error correction codes
  - error threshold, etc
- Quantum error correction code simulation → quantum circuit simulation
  - Schrödinger simulation; Feynman simulation; Feynman sampling
  - tensor network simulation
- In surface code
  - code state is represented by a simple tensor network

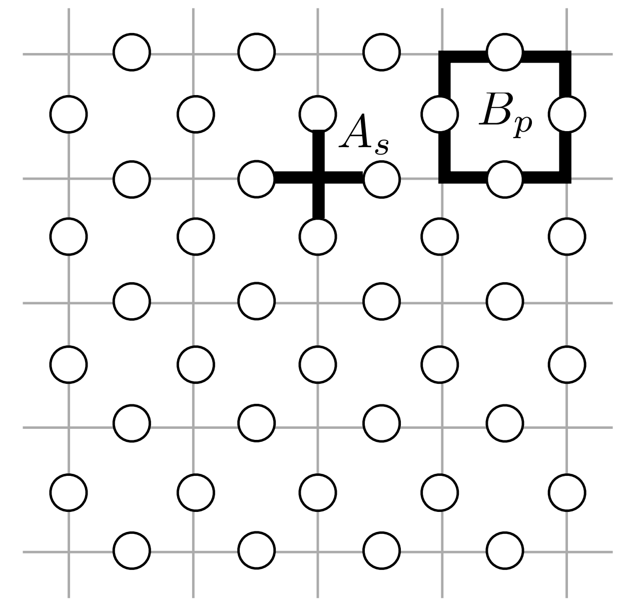
Tensor network representation of surface code

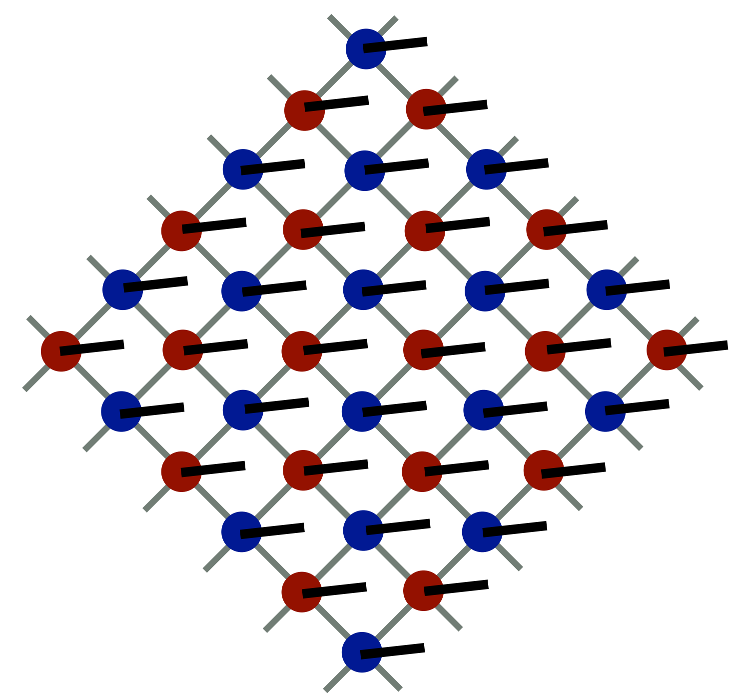
- Surface code state
  - simultaneous eigenstate (with eigenvalues 1) of

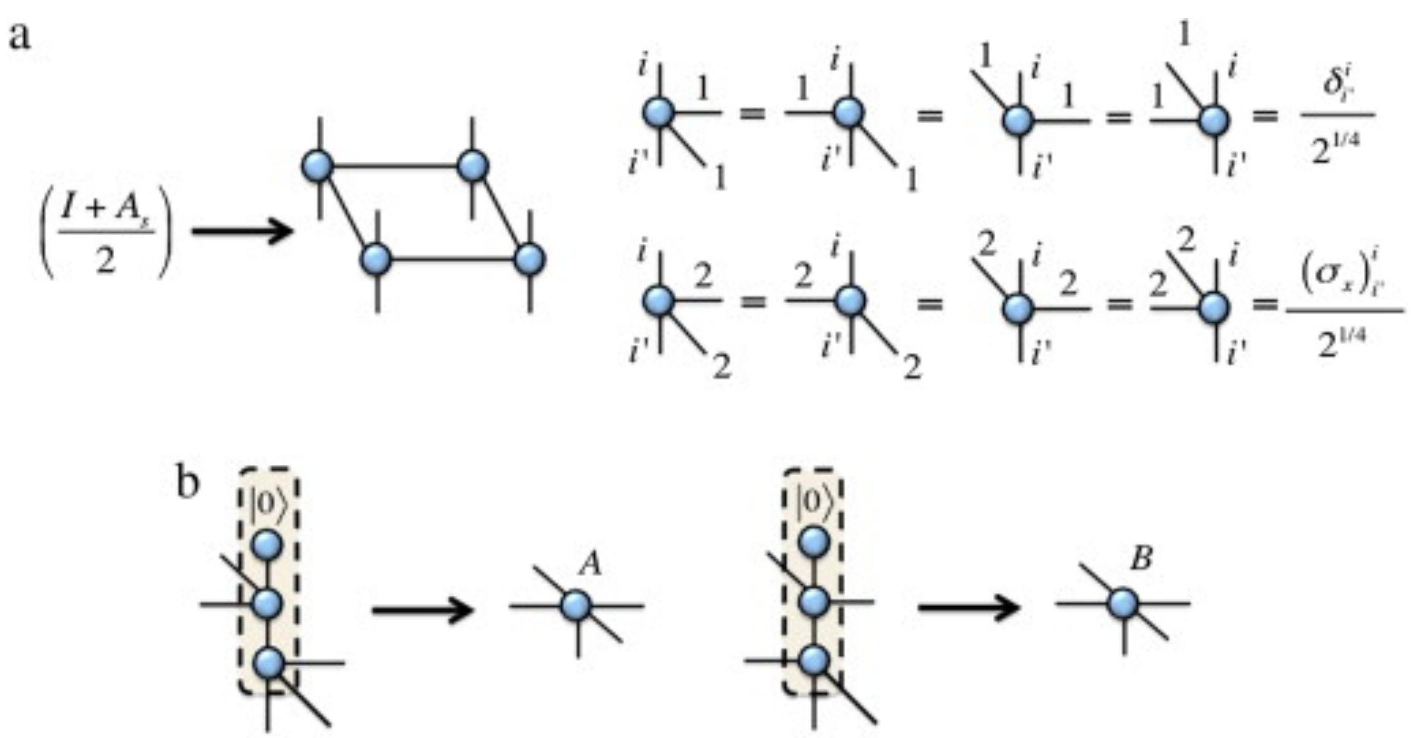
$$A_s = \prod_{j \in \mathrm{star}(s)} X_j \qquad B_p = \prod_{j \in \partial p} Z_j$$

  - tensor network representation

$$|\Psi_{TC}\rangle = \prod_s \frac{(\mathbb{I} + A_s)}{2} \prod_p \frac{(\mathbb{I} + B_p)}{2} |0\rangle^{\otimes N \to \infty} = \prod_s \frac{(\mathbb{I} + A_s)}{2} |0\rangle^{\otimes N \to \infty},$$





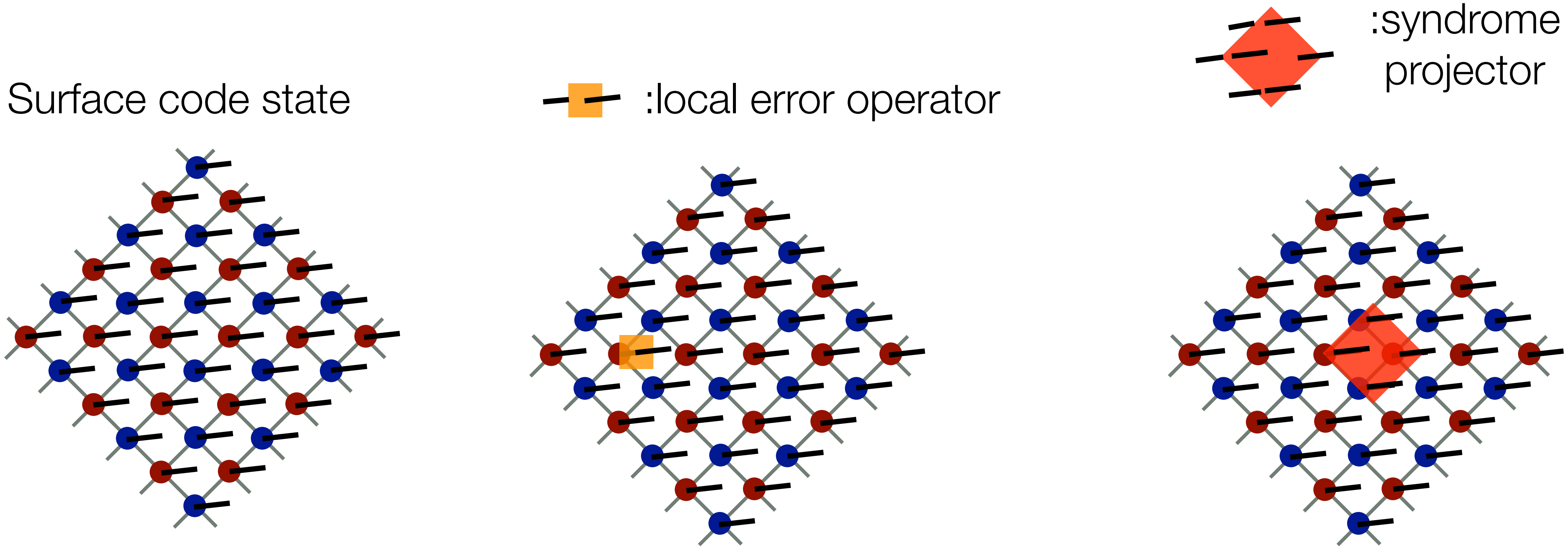


F. Verstraete, et al, Phys. Rev. Lett. 96, 220601 (2006)

R. Orús, Ann. of Phys. 349, 117 (2014)

Simulation of the surface code

- Tensor network representation
  - code state, error operations, syndrome measurements

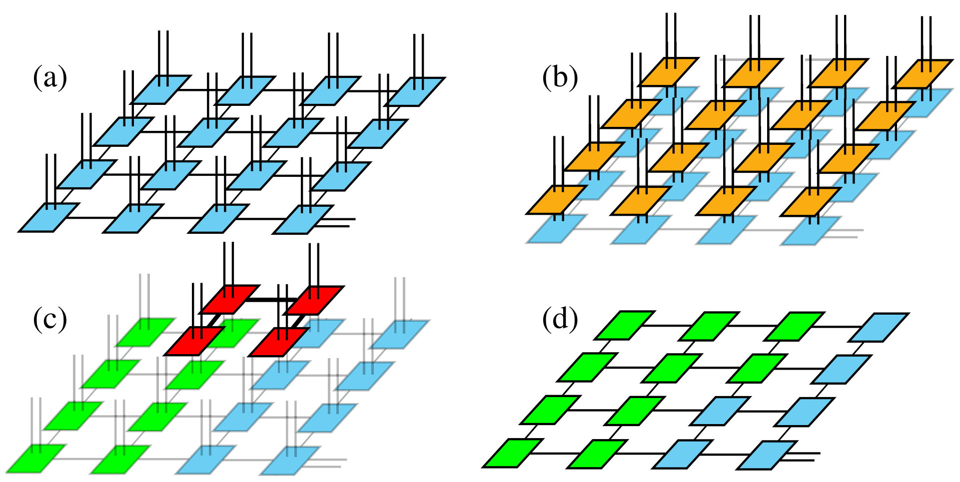


- Note: in the case of noise simulation, we need to consider mixed states instead of pure states

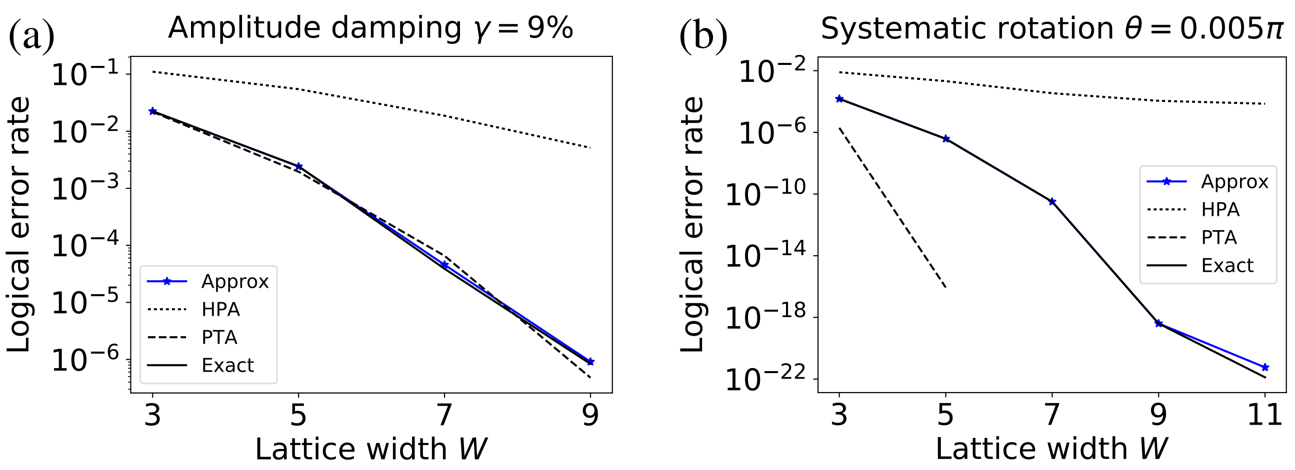
A. S. Darmawan and D. Poulin, Phys. Rev. Lett. 119, 040502 (2017).

Simulation of the surface code

Tensor network diagram



Calculated logical error rate



Exact = exact contraction
Approx = approximated contraction

Amplitude damping:

$$\mathcal{E}_{\mathrm{AD}}(\rho) = \sum_i K_i \rho K_i^\dagger$$

$$K_0 = |0\rangle\langle 0| + \sqrt{1-\gamma}|1\rangle\langle 1|, \qquad K_1 = \sqrt{\gamma}|0\rangle\langle 1|,$$

Systematic rotation:

$$\mathcal{E}_{\mathrm{SR}}(\rho) = e^{-i\theta Z}\rho e^{i\theta Z},$$

A. S. Darmawan and D. Poulin, Phys. Rev. Lett. 119, 040502 (2017).

Report No.2

- Deadline: 2023/02/02 23:59
- Submission: via ITC-LMS

- tensornetwork.ipynb
  - simulation of quantum circuit using TensorNetwork library
    - problem 1 (mandatory): probability of states 11 and 01 of Bell state
    - problem 2 (mandatory): GHZ state
    - problem 3 (optional): entangled state of n qubits
- grover.ipynb
  - simulation of Grover's algorithm using Qiskit and TensorNetwork library
    - problem 1 (mandatory): running Qiskit simulation of Grover's algorithm
    - problem 2 (mandatory): running TensorNetwork simulation
    - problem 3(mandatory): scaling of computation time