計算科学・量子計算における情報圧縮：パイロット講義

Data Compression in Computational Science and Quantum Computing

2022.1.11, 17:00-18:30

#3: Quantum error correction and tensor network

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Lecture materials are available at
https://github.com/utokyo-qsw/data-compression

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https://qsw.phys.s.u-tokyo.ac.jp

Schedule of the pilot lecture

- 12/14: Tensor network and tensor renormalization group **[Okubo]**
 （テンソルネットワークとテンソル繰り込み群）
- 12/21: Quantum computers and simulations **[Todo]**
 （量子コンピュータ・シミュレーション）
- 1/11: Quantum error corrections and tensor network **[Okubo]**
 （量子誤り訂正とテンソルネットワーク）
- 1/25: Quantum-classical hybrid algorithms and tensor network **[Todo]**
 （量子古典ハイブリッドアルゴリズムとテンソルネットワーク）

Outline

- Introduction: purpose of error correction
- Classical error correction
- Simple quantum error correction
- Stabilizer and the surface (Toric) code
- Tensor network simulation of the surface code

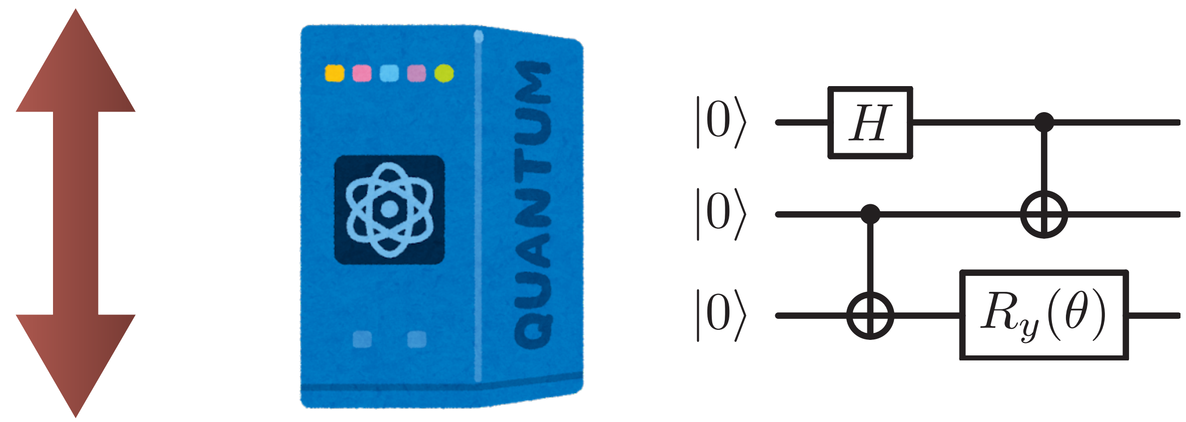
Introduction: purpose of error correction

(for quantum computers)

How can we check reliability of the computer output?

We usually use computers to calculate things we cannot calculate.

Thus, to ensure the result, we need to understand its operation principle



Actual devices do not work as the ideal system, expected in algorithms.

- Deviation from an ideal qubit
    - Effect of excitation states
- Errors in operations
- Environments
    - Thermal noise
    - Dissipation
    - Radiation

Error corrections in classical computer

During daily use of classical computers, we need not to take care the deviation from the ideal (digital) computer.

In classical computers, a lot of error correction mechanisms are employed.

e.g.,
- Reptation codes
- Parity check bit
- ECC memory
- ...
-

→ Users can deal with classical computer as if it is an ideal device.

Similarly, by correcting quantum errors, we can run various quantum algorithms as in text books.

Typical errors in quantum computer

Bit flip:

Flip the states of a qubit from $|0\rangle$ to $|1\rangle$ (and vice versa).

It can be represented by $X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

$$X|0\rangle = |1\rangle$$
$$X|1\rangle = |0\rangle$$

$$\Rightarrow \quad X(\alpha|0\rangle + \beta|1\rangle) = \alpha|1\rangle + \beta|0\rangle$$

Phase flip:

Change the phase of the states of a qubit.

It can be represented by $Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.

$$Z|0\rangle = |0\rangle$$
$$Z|1\rangle = -|1\rangle$$

$$\Rightarrow \quad Z(\alpha|0\rangle + \beta|1\rangle) = \alpha|0\rangle - \beta|1\rangle$$

Depolarization:

The states of a qubit is replaced by the completely mixed state.

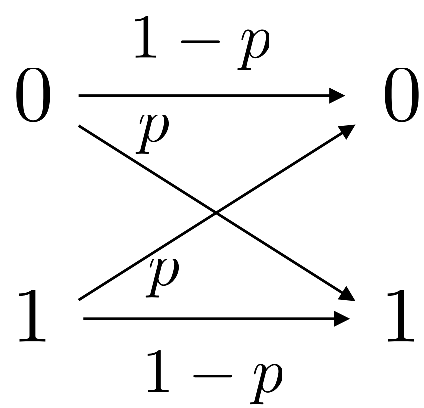
Amplitude dampling:

Energy dissipation to the environment.

Classical error correction

Classical bit flip error

Suppose a bit flip error occurs with a probability p in a classical bit.

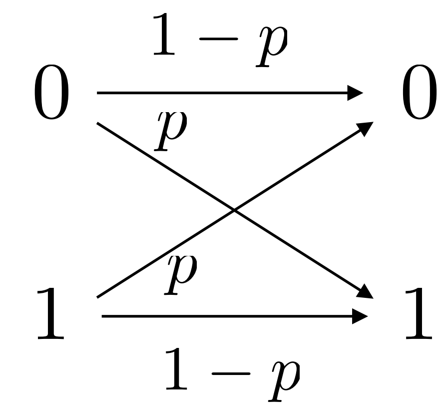


If we transfer a classical information (bit) through this channel,
the receiver will get incorrect bit with a probability p.

How do we protect our information?

Typical way is encoding of the information using more than one bit, and recover (decoding) after we receive the information.

Three-bit encoding based on reputation



Here we encode 0 and 1 into three bits.

$$0 \to 000$$
$$1 \to 111$$

000 and 111 are considered as logical 0 and logical 1, respectively.

When we send information represented by three bits, the receiver may get:

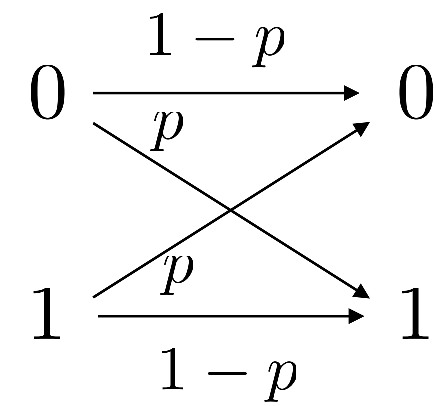
$$000, 001, 010, 011, 100, 101, 110, 111$$

We decode them based on the majority rule.

$$\begin{matrix} 000 \\ 001 \\ 010 \\ 100 \end{matrix} \to 0, \qquad \begin{matrix} 111 \\ 110 \\ 101 \\ 011 \end{matrix} \to 1$$

(Note: to decode the state, we need to measure the state.)

Error in the three-bit encoding



$$\begin{array}{c} 000 \\ 001 \\ 010 \\ 100 \end{array} \to 0, \qquad \begin{array}{c} 111 \\ 110 \\ 101 \\ 011 \end{array} \to 1$$

If only a single bit flip happens, it successfully recovers the information.

The probability of the error in this error correction is

$$p_3 = \underbrace{p^3}_{\text{3 bits flipped}} + \underbrace{3p^2(1-p)}_{\text{2 bits flipped}}$$

It is smaller than p, if $p < \dfrac{1}{2}$.

By using more than one bit, we can reduces the error probability!

Simple quantum error correction

Quantum error correction?

Can we consider similar approach for quantum error corrections?

At a glance, there are difficulties to apply the classical reputation idea.

- We cannot copy a quantum state.
    - It is impossible to make a state like $|\psi\rangle \to |\psi\rangle \otimes |\psi\rangle \otimes |\psi\rangle$.
    - It is know as the no-cloning theorem.
- Measurements at the decoding procedure may destroy the state.
- A state (and error) of the qubit is continuous.
    - $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle \to \alpha'|0\rangle + \beta'|1\rangle$

However, we can apply reputation idea to the basis, $|0\rangle$ and $|1\rangle$, instead of a quantum state.

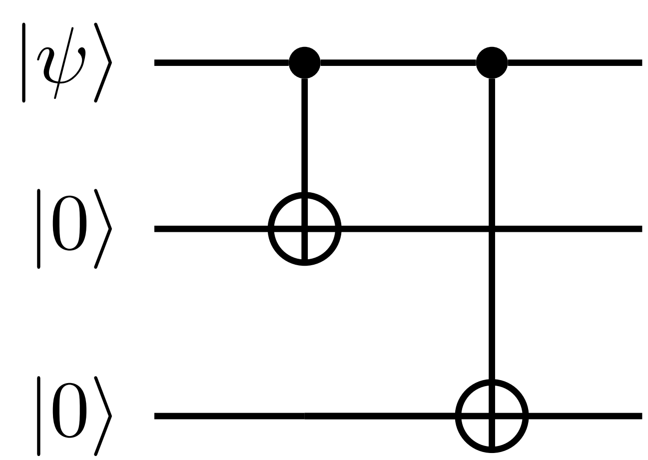
$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle \to \alpha|000\rangle + \beta|111\rangle$$

Quantum circuit for the encoding

The encoding

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle \rightarrow \alpha|000\rangle + \beta|111\rangle$$

can be done by the quantum circuit



It is easy to check

$$|0\rangle \rightarrow |000\rangle$$
$$|1\rangle \rightarrow |111\rangle$$

and it is sufficient.

Three qubit code for bit flip

Let's consider the error correction for the bit flip:

> The bit flip X is operated with probability p, independently for each three qubits.

What happens after the single bit flip, *e.g.*, for the first qubit?

$$\alpha|000\rangle + \beta|111\rangle \rightarrow \alpha|100\rangle + \beta|011\rangle$$

➡ The state goes outside of the vector space spanned by the logical qubits, $|000\rangle$ and $|111\rangle$.

Code space

Different vector spaces appears depending on the bit flip qubit:

No bit flips:	$\mathrm{Span}\{|000\rangle, |111\rangle\}$
Bit flip on qubit 1:	$\mathrm{Span}\{|100\rangle, |011\rangle\}$
Bit flip on qubit 2:	$\mathrm{Span}\{|010\rangle, |101\rangle\}$
Bit flip on qubit 3:	$\mathrm{Span}\{|001\rangle, |110\rangle\}$

***Thus, error spaces are orthogonal each other.**

Detecting the error: syndrome measurement

No bit flips:	$\mathrm{Span}\{\|000\rangle, \|111\rangle\}$
Bit flip on qubit 1:	$\mathrm{Span}\{\|100\rangle, \|011\rangle\}$
Bit flip on qubit 2:	$\mathrm{Span}\{\|010\rangle, \|101\rangle\}$
Bit flip on qubit 3:	$\mathrm{Span}\{\|001\rangle, \|110\rangle\}$

Projective measurement

$$M = \sum_{i=0}^{3} m P_m$$

can distinguish errors!

It is called syndrome measurement.

$$P_0 = |000\rangle\langle 000| + |111\rangle\langle 111|$$
$$P_1 = |100\rangle\langle 100| + |011\rangle\langle 011|$$
$$P_2 = |010\rangle\langle 010| + |101\rangle\langle 101|$$
$$P_3 = |001\rangle\langle 001| + |110\rangle\langle 110|$$

Note:

This does not destroy our quantum state. However if we consider projectors on, e.g., $|000\rangle\langle 000|$, it may destroy a quantum superposition.

Recovering errors:

$$M = \sum_{i=0}^{3} m P_m$$

$$P_0 = |000\rangle\langle 000| + |111\rangle\langle 111|$$
$$P_1 = |100\rangle\langle 100| + |011\rangle\langle 011|$$
$$P_2 = |010\rangle\langle 010| + |101\rangle\langle 101|$$
$$P_3 = |001\rangle\langle 001| + |110\rangle\langle 110|$$

Depending on the result of the syndrome measurement $P_M = \langle\psi|M|\psi\rangle$.

- $P_M = 0$: no error
  - Do nothing
- $P_M = 1$: bit flip error on the qubit 1
  - Apply X to the qubit 1.
- $P_M = 2$: bit flip error on the qubit 2
  - Apply X to the qubit 2.
- $P_M = 3$: bit flip error on the qubit 3
  - Apply X to the qubit 3.

When the bit flips occur one (or zero) qubit, this procedure perfectly recovers the original state $|\psi\rangle = |000\rangle + \beta|111\rangle$

* Thus, the error correction probability is $p_3 = p^3 + 3p^2(1-p)$

Phase flip errors

The bit flip errors are almost similar to the classical errors.

Can we correct other errors?

Let's consider the error correction for the phase flip:

> The bit flip Z is operated with probability p,
> independently for each three qubits.

$$Z(\alpha|0\rangle + \beta|1\rangle) = \alpha|0\rangle - \beta|1\rangle$$

We can correct this error by recognizing

Z acts as "bit flip" for $|+\rangle$ and $|-\rangle$.

$$Z|+\rangle = |-\rangle$$
$$Z|-\rangle = |+\rangle$$

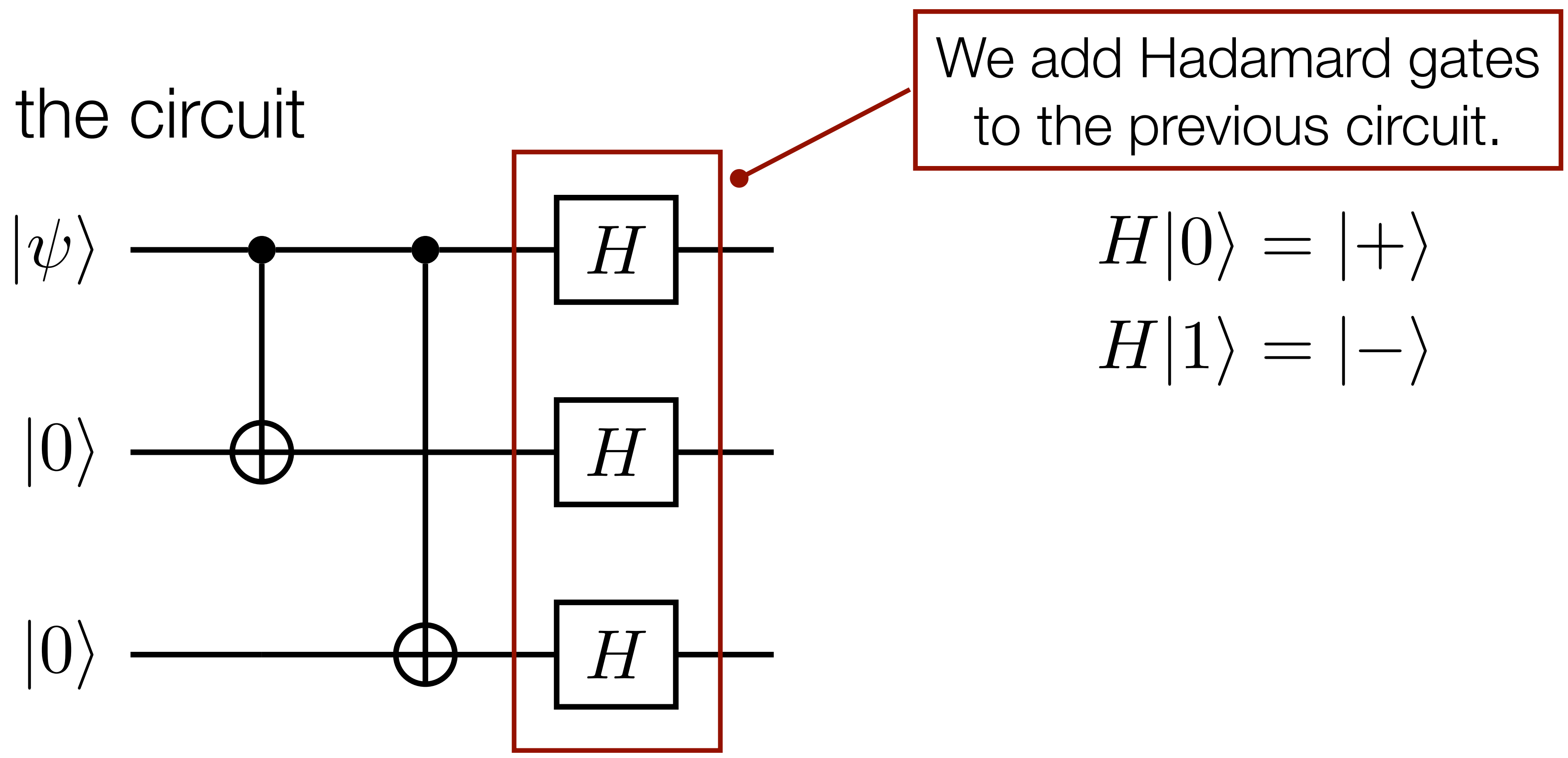
$$|+\rangle \equiv \frac{|0\rangle + |1\rangle}{\sqrt{2}}, |-\rangle \equiv \frac{|0\rangle - |1\rangle}{\sqrt{2}},$$

Three qubit code for phase flip errors

We encode our state as

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle \to \alpha|+++\rangle + \beta|---\rangle$$

It can be done by the circuit



$$H|0\rangle = |+\rangle$$
$$H|1\rangle = |-\rangle$$

Syndrome measurements and error correction:

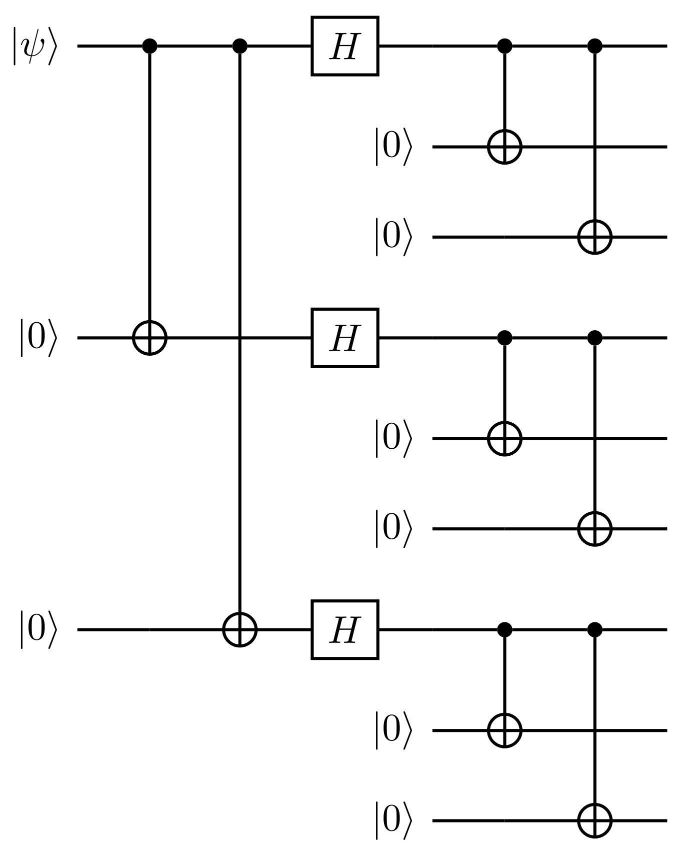
Replace previous $(|0\rangle, |1\rangle, X)$ to $(|+\rangle, |-\rangle, Z)$

Arbitrary single qubit errors: The Shor code

By combining the three qubit codes for bit flip and phase flip, we can construct an error correction code for an arbitrary *single* qubit error.

$$|0\rangle \to |+++\rangle \quad \to \frac{1}{2\sqrt{2}}(|000\rangle + |111\rangle) \otimes (|000\rangle + |111\rangle) \otimes (|000\rangle + |111\rangle)$$

$$|1\rangle \to |---\rangle \quad \to \frac{1}{2\sqrt{2}}(|000\rangle - |111\rangle) \otimes (|000\rangle - |111\rangle) \otimes (|000\rangle - |111\rangle)$$



Stabilizer and the surface (toric) code

Another view of the three qubit codes

Three qubit bit flip code

No bit flips:	$\mathrm{Span}\{\|000\rangle, \|111\rangle\}$
Bit flip on qubit 1:	$\mathrm{Span}\{\|100\rangle, \|011\rangle\}$
Bit flip on qubit 2:	$\mathrm{Span}\{\|010\rangle, \|101\rangle\}$
Bit flip on qubit 3:	$\mathrm{Span}\{\|001\rangle, \|110\rangle\}$

We can define the code states as a set of three quibit states satisfying

$$Z_1 Z_2 |\phi\rangle = |\phi\rangle$$
$$Z_2 Z_3 |\phi\rangle = |\phi\rangle$$

Simultaneous eigenstate of Z_1Z_2 and Z_2Z_3 with the eigenvalue 1.

(We call that a state $|\phi\rangle$ is *stabilized* by Z_1Z_2 and Z_2Z_3.)

Also, we can distinguish errors by the two measurements Z_1Z_2 and Z_2Z_3.

$$(Z_1 Z_2, Z_2 Z_3) = \underbrace{(1,1)}_{\text{No bit flips}}, \underbrace{(1,-1)}_{\text{Bit flip on 3}}, \underbrace{(-1,1)}_{\text{Bit flip on 1}}, \underbrace{(-1,-1)}_{\text{Bit flip on2}}$$

***Note**: Z_1Z_2 and Z_2Z_3 also do not destroy the state.

Pauli group and stabilizer

n-qubit Pauli group G_n: $\{\pm 1, \pm i\} \times \{I, X, Y, Z\}^{\otimes n}$

1-qubit G_1: $\{\pm X, \pm iI, \pm X, \pm iX, \pm Y, \pm iY, \pm Z, \pm iZ\}$

2-qubit G_2:

$$\{\pm 1, \pm i\} \times \{II, XI, IX, YI, IY, ZI, IZ, XY, YX, YZ, ZY, ZX, XZ, XX, YY, ZZ\}$$

Stabilized vector subspace:

Suppose S is a subgroup of G_n. We define a vector subspace V_s as a set of n-qubit states *stabilized* by all elements of S. And S is called the stabilizer.

Three qubit bit flip code: $S = \{I, Z_1Z_2, Z_2Z_3, Z_3Z_1\}$

*Note : because $I = (Z_1Z_2)^2, Z_3Z_1 = (Z_1Z_2)(Z_2Z_3)$, Z_1Z_2 and Z_2Z_3 are sufficient to define S. They are called generator of S.

Stabilizer codes for quantum error correction

Stabilizer code:

We encode quantum states into the vector space stabilized by stabilizer S with $-I \notin S$.

*More precisely, we define a $[n, k]$ stabilizer code with S which has $n-k$ *independent* and *commuting* generators.
The dimension of the $[n, k]$ stabilizer code is 2^{n-k}.

In the stabilizer codes, syndrome measurements correspond to the measurements of generators.

Examples:

*$\langle g_1, g_2, \ldots, \rangle$ represents a group generated by $g_1, g_2, \ldots$.

Three qubit bit flip code = a $[3,2]$ stabilizer code with $S = \langle Z_1Z_2, Z_2Z_3 \rangle$

Three qubit phase flip code = a $[3,2]$ stabilizer code with $S = \langle X_1X_2, X_2X_3 \rangle$

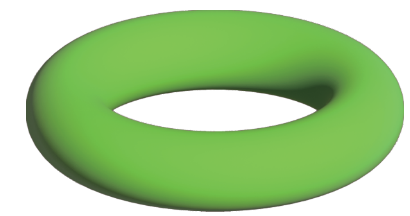
Shor code = a $[9,8]$ stabilizer code with

$$S = \langle Z_1Z_2, Z_2Z_3, Z_4Z_5, Z_5Z_6, Z_7Z_8, Z_8Z_9, X_1X_2X_3X_4X_5X_6, X_4X_5X_6X_7X_8X_9 \rangle$$

Surface code: a topological error correcting code

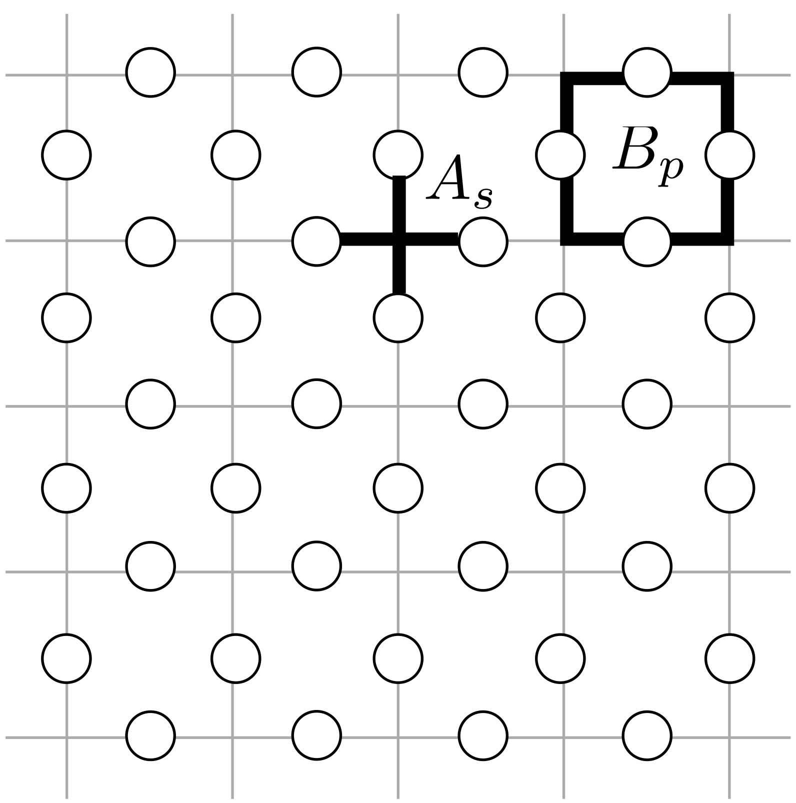
Surface (toric) code:

A. Kitaev, arXiv:quant-ph/9707021; Ann. Phys. **303**, 2 (2003).



Stabilizer $A_s = \prod_{j \in \mathrm{star}(s)} X_j \qquad B_p = \prod_{j \in \partial p} Z_j$

For all vertices s and plaquettes p on *the torus*.



Note: All A_s and B_p are commutable each other.

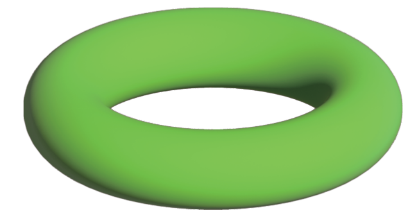
Quantum state $|\psi\rangle$ within the surface code:

$$A_s|\psi\rangle = |\psi\rangle$$
$$B_p|\psi\rangle = |\psi\rangle$$

for all, s and p.

Logical qubits represented by the surface code

of qubits in $L \times L$ torus: $2L^2$



# of vertices (stars): L^2

# of plaquettes: L^2

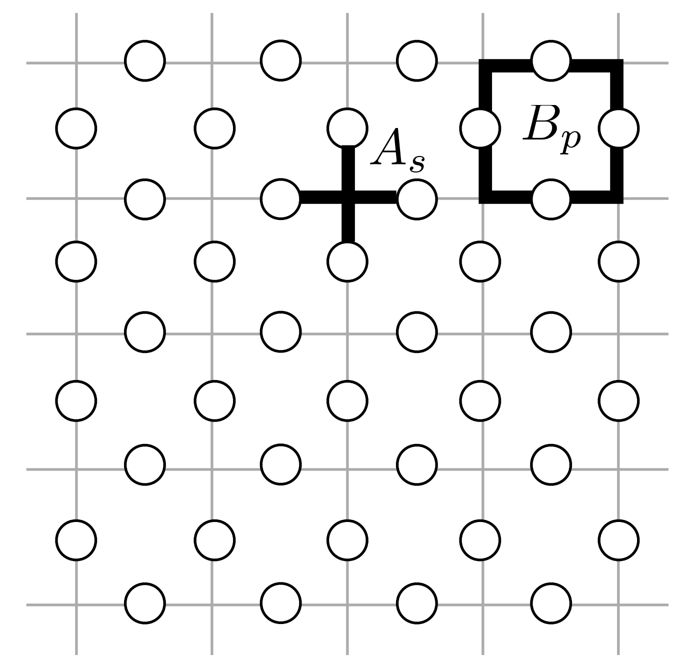
# of independent stabilizers: $2L^2 - 2$

Due to the relations

$$\prod_s A_s = I$$

$$\prod_p B_p = I$$

two of the $2L^2$ are not independent.



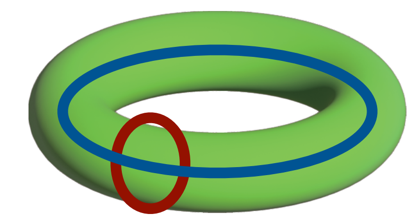
Dimension of the surface code space is

$$2^{2L^2-(2L^2-2)} = 2^2$$

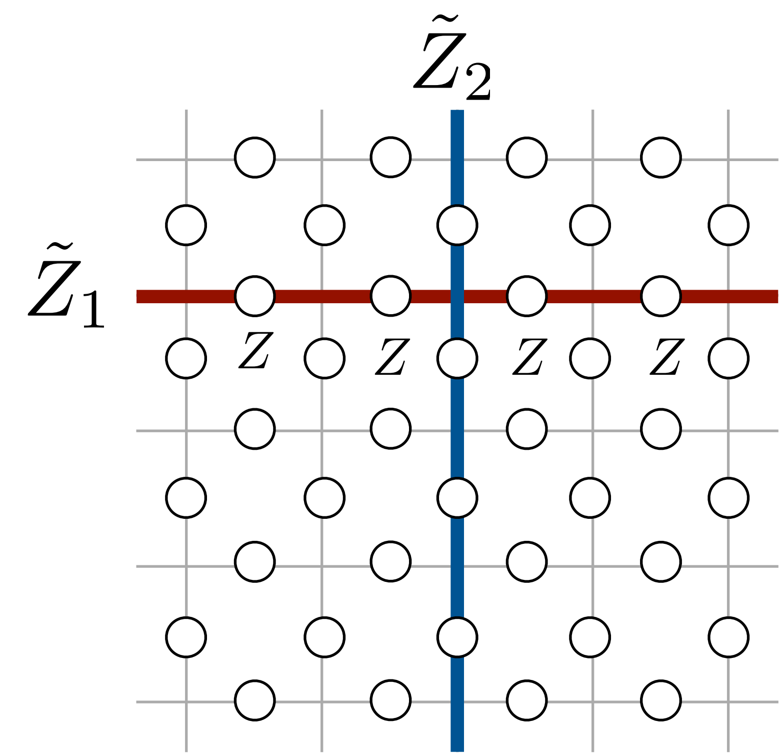
meaning that the surface code encodes a two qubits state.

Logical qubit states in the surface code

The logical qubit states are characterized by non-trivial loop operators.



$$\tilde{Z}_i = \prod_{j \in \text{non-trivial loop}} Z_j$$

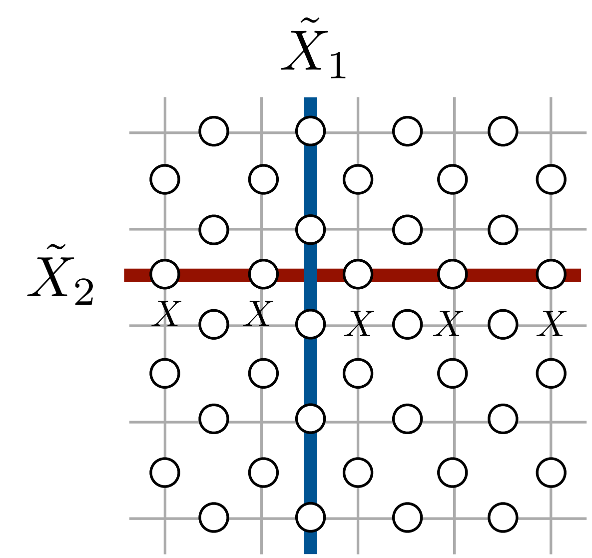


$i = 1, 2$

We can define logical states, *e.g.*,

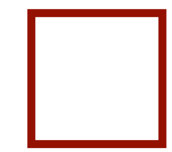
$$\tilde{Z}_1 |00\rangle_L = |00\rangle_L$$
$$\tilde{Z}_2 |00\rangle_L = |00\rangle_L$$
$$\tilde{Z}_1 |10\rangle_L = -|10\rangle_L$$
$$\tilde{Z}_2 |10\rangle_L = |10\rangle_L$$
$$\vdots$$

We can also define corresponding X operators →



Error detection (syndrome measurements)

Errors in qubits can be detected through the measurements of A_s and B_p.

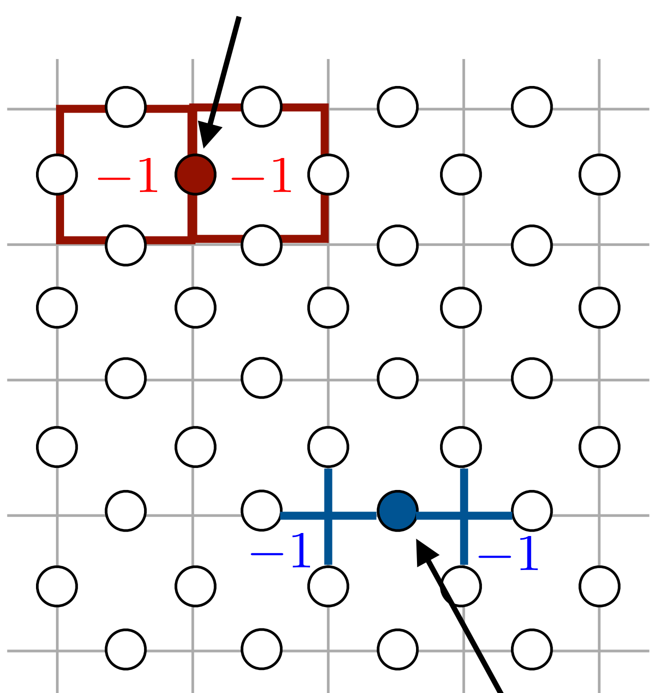


$= B_p$

$= A_s$

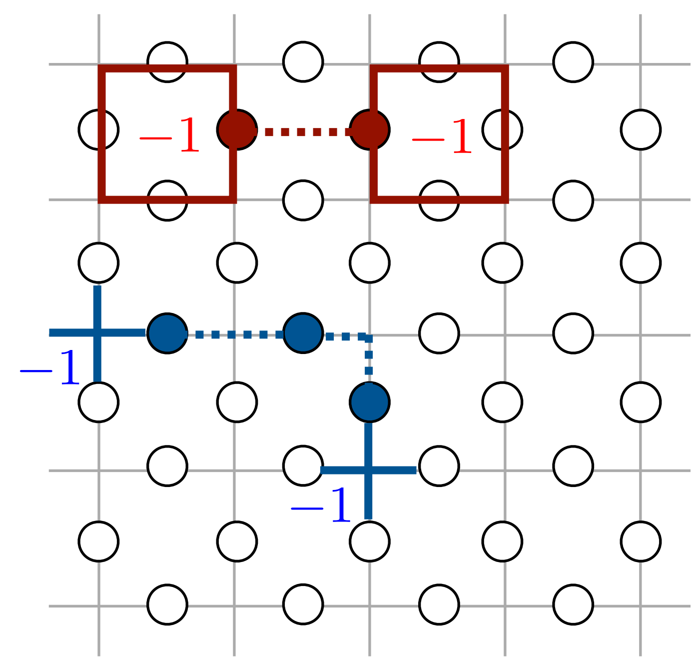
Single errors

Bit flip = X



Phase flip = Z

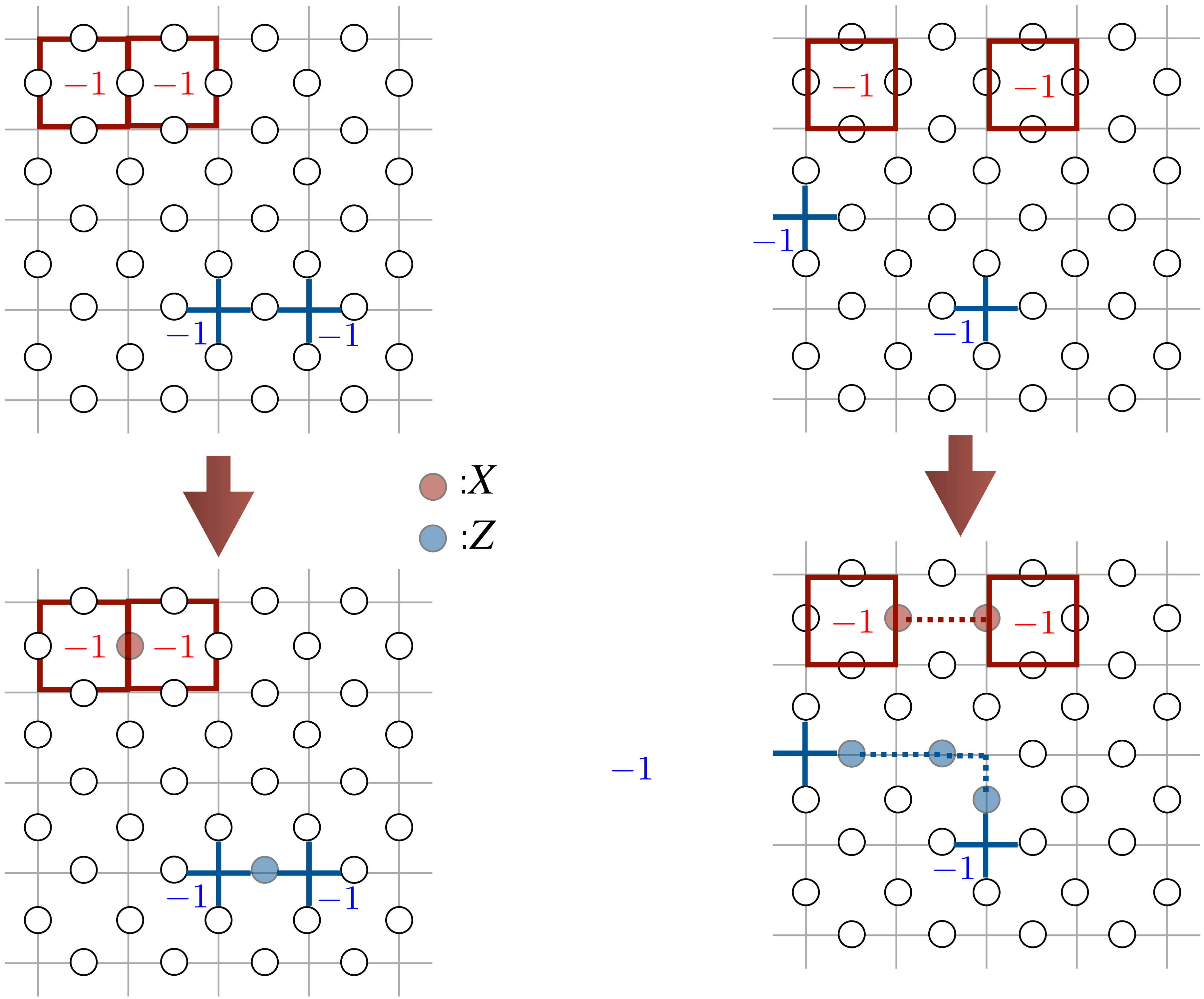
Multiple errors



Generally, the error syndromes appear at the edges (boundaries) of the error chain.

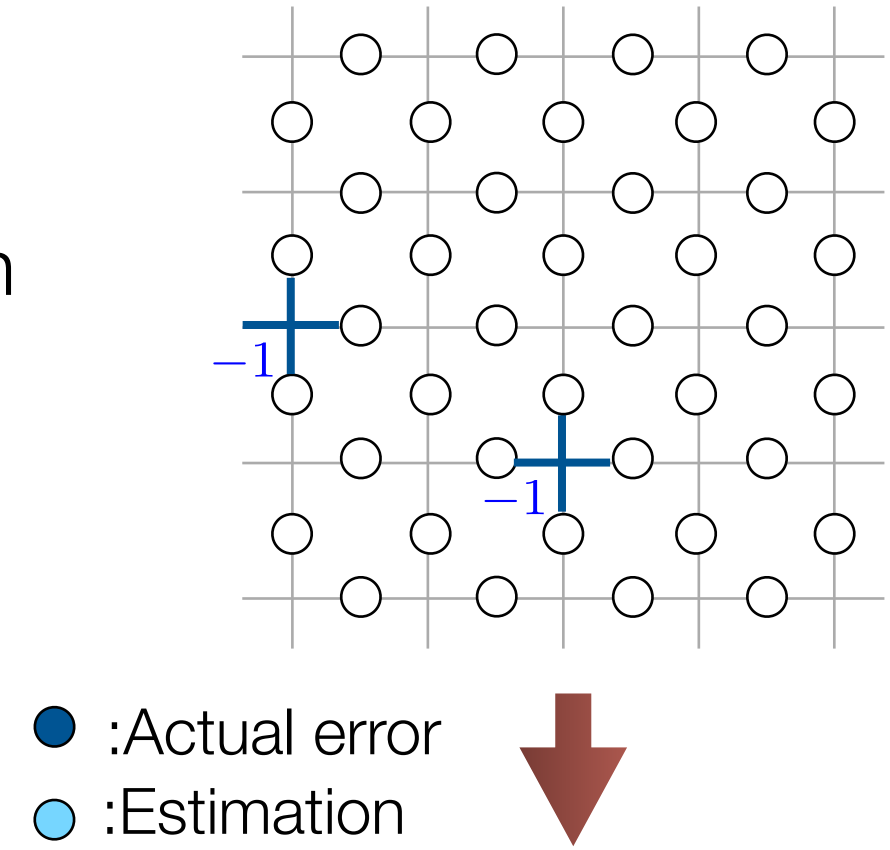
Error correction

From error syndromes, we may correct it by estimating error position.



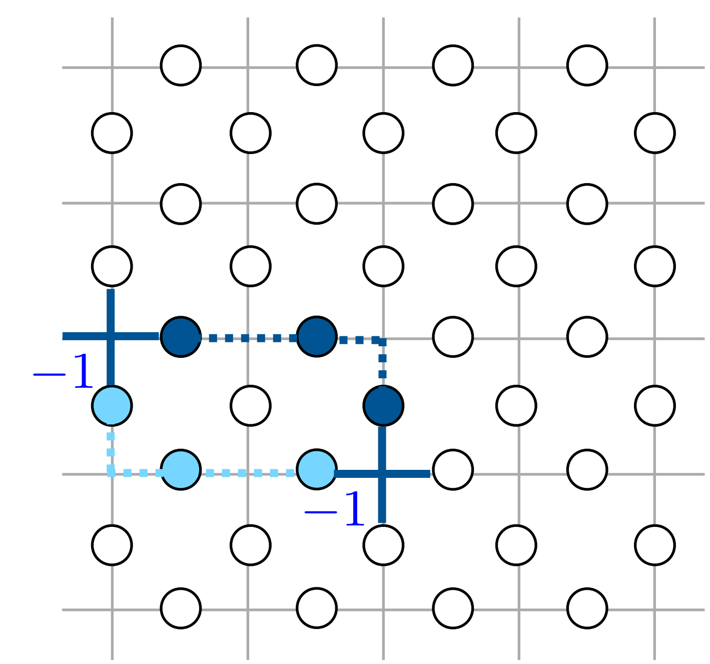
Error correction (cont.)

In some cases, we may estimate different positions from the actual errors.



→ Even in such cases, the error correction works well because the actual and estimated errors form a (trivial) loop corresponding to a stabilizer.

However, when errors are *too* dense, the loop may become non-trivial.
In such situation, error correction procedure changes the logical state.



→ There is a threshold in the error density for the successful error correction.

Simulation of the surface code

Simulation of the quantum error correction

When we investigate properties of the quantum error correction codes, we may need to simulate the error correction numerically.

c.f error threshold

Such simulations are, basically, similar to the simulations of quantum circuits.

We can use techniques discussed in lecture #2 by Prof. Todo.

Schrödinger simulation; Feynman simulation; Feynman sampling

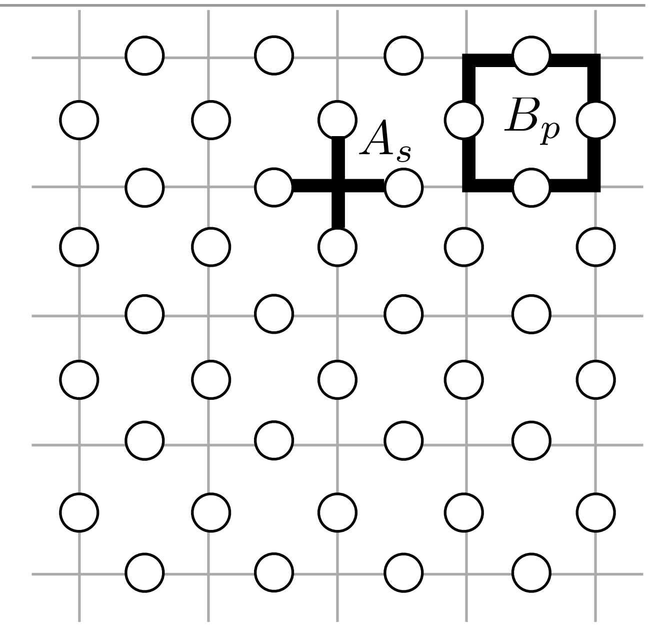
In the case of the surface code, we can also use the fact that the code state is represented by a simple tensor network to simulate the quantum error correction.

Tensor network representation of surface code

Surface (toric) code:

A. Kitaev, arXiv:quant-ph/9707021; Ann. Phys. **303**, 2 (2003).

$$A_s = \prod_{j \in \mathrm{star}(s)} X_j \qquad B_p = \prod_{j \in \partial p} Z_j$$



Surface code is a simultaneous eigenstate of A_s and B_p.

Can we represent it by a simple way?

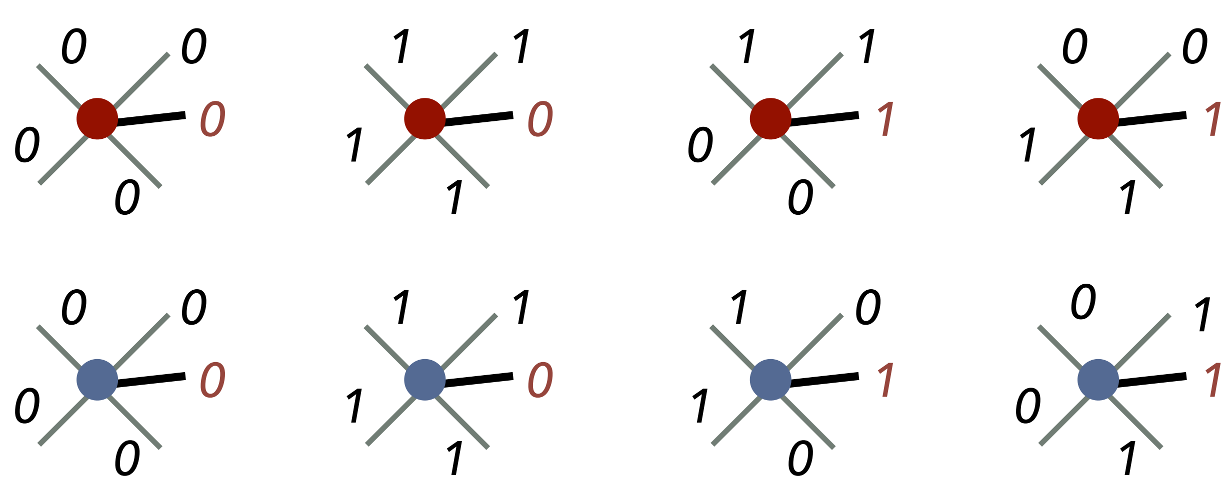
Yes! We can represent it by a D=2 tensor product state (TPS).

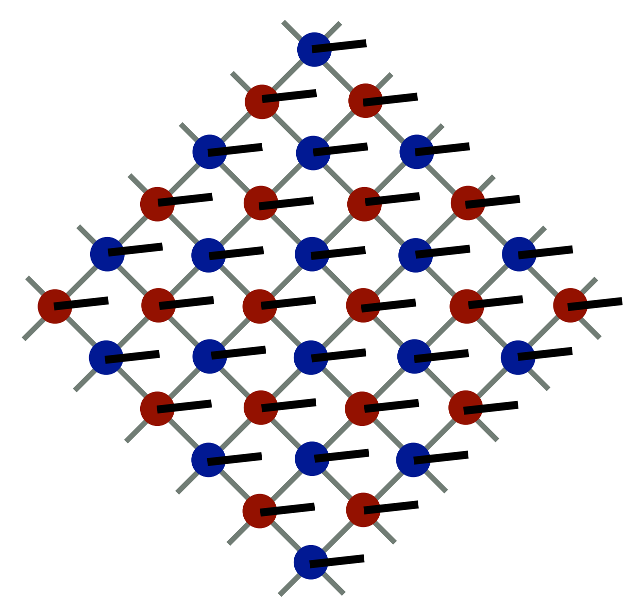
$$|\Psi\rangle = \lim_{\beta\to\infty} e^{\beta \sum_p B_p} |++\cdots+\rangle$$

(F. Verstraete, et al, Phys. Rev. Lett. **96**, 220601 (2006).

0,1: eigenstate of σ_z

(Non-zero elements of tensor)





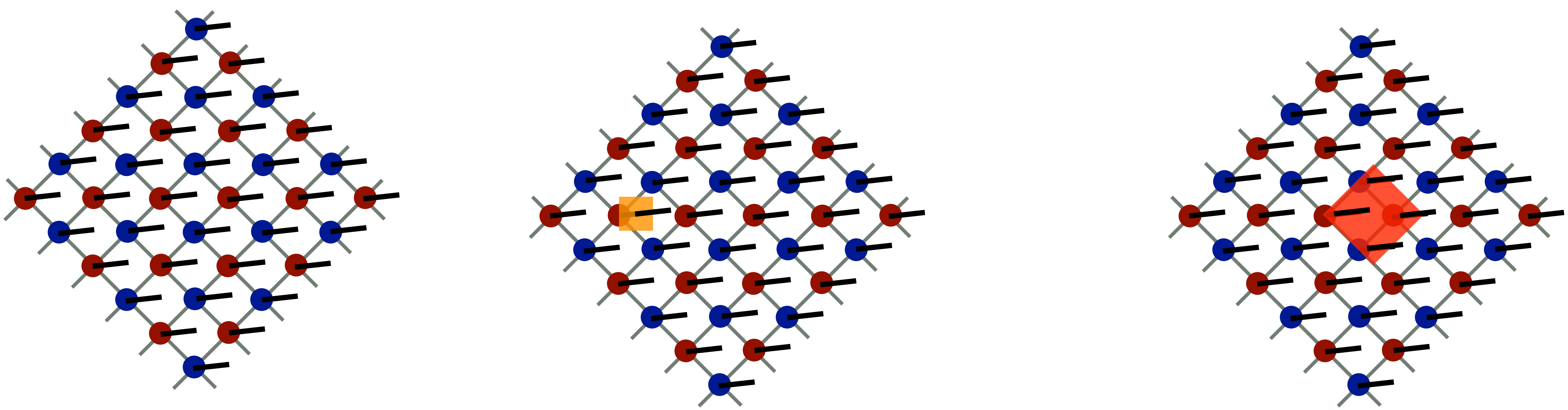
Simulation of the surface code

By using this tensor network representation, we can represent error operations, syndrome measurements, ..., by tensor networks.

:syndrome projector

Surface code state

:local error operator



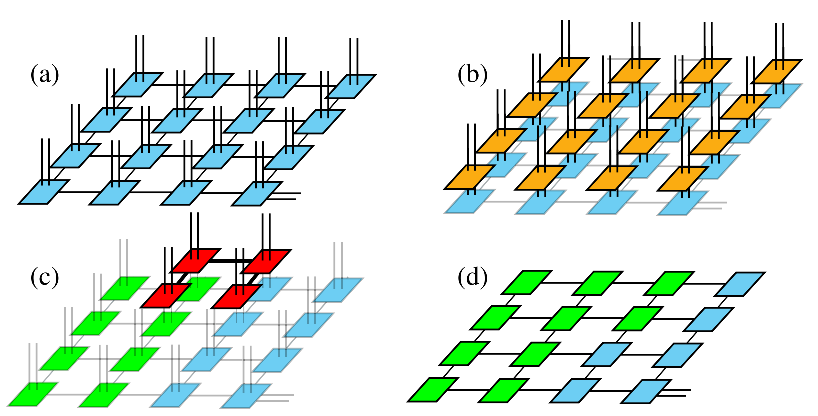
*To simulate various realistic noises, we need to consider mixed states instead of pure states. It also be represented by a tensor network.

c.f. A. S. Darmawan and D. Poulin, Phys. Rev. Lett. **119**, 040502 (2017).

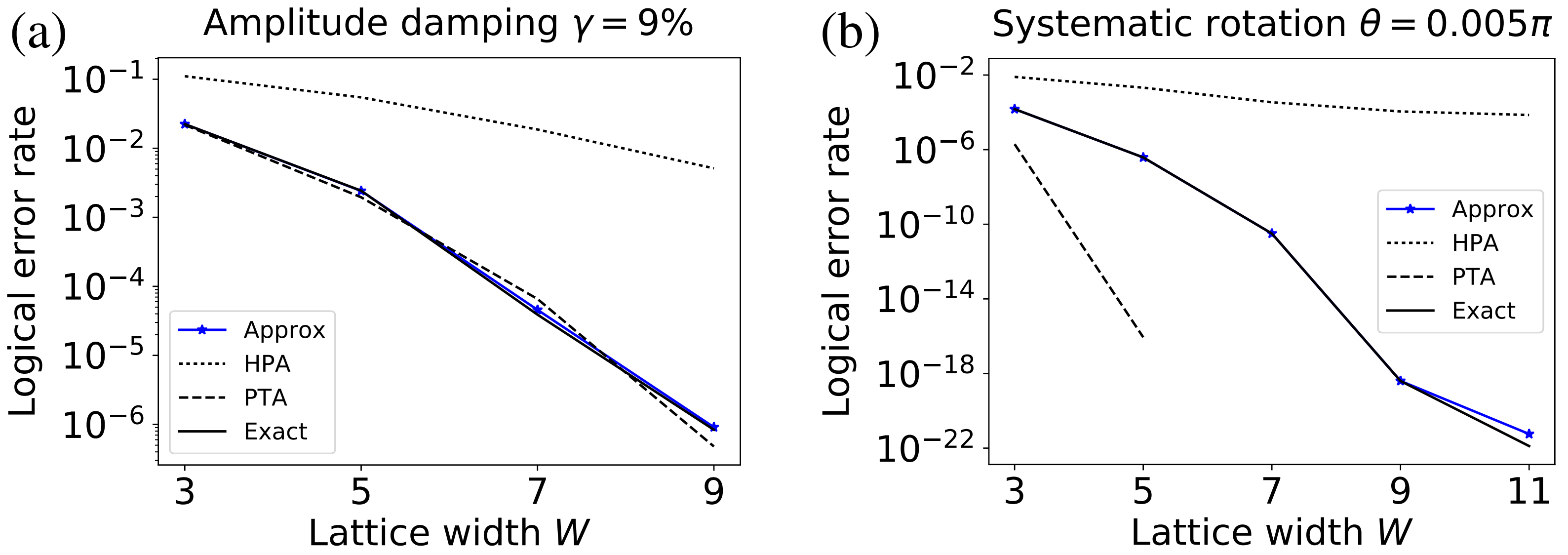
Simulation of the surface code

A. S. Darmawan and D. Poulin, Phys. Rev. Lett. **119**, 040502 (2017).

Tensor network diagram



Calculated logical error rate



Exact = exact contraction
Approx = approximated contraction

Amplitude damping:

$$\mathcal{E}_{\mathrm{AD}}(\rho) = \sum_i K_i \rho K_i^\dagger$$

$$K_0 = |0\rangle\langle 0| + \sqrt{1-\gamma}|1\rangle\langle 1|, \qquad K_1 = \sqrt{\gamma}|0\rangle\langle 1|,$$

Systematic rotation:

$$\mathcal{E}_{\mathrm{SR}}(\rho) = e^{-i\theta Z}\rho e^{i\theta Z},$$

References

今回の講義準備に（主に）参考にした資料

- "Quantum Computation and Quantum Information", M. A. Nielsen and I. L. Chung, Cambridge university press.
- 「量子アルゴリズム」中山茂著、技報堂出版
- 阪大・藤井啓祐先生の講義資料
    - https://quantphys.org/wp/qinfp/授業/

Next week (1/25) will be given by Prof. Todo

- 12/14: Tensor network and tensor renormalization group **[Okubo]**
  （テンソルネットワークとテンソル繰り込み群）

- 12/21: Quantum computers and simulations [Todo]
  （量子コンピュータ・シミュレーション）

- 1/11: Quantum error corrections and tensor network **[Okubo]**
  （量子誤り訂正とテンソルネットワーク）

- **1/25: Quantum-classical hybrid algorithms and tensor network [Todo]**
  （量子古典ハイブリッドアルゴリズムとテンソルネットワーク）