

計算科学・量子計算における情報圧縮

Data Compression in Computational Science and Quantum Computing

**2022.10.13**

**#2:線形代数の復習**

**Review of linear algebra**

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# Outline

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- Review of linear algebra
  - Vector space- Abstract vectors-
    - General vector space (with inner product)
    - Basis and relation to coordinate vector space
    - Vector subspace and spanned vector subspace
  - Matrix and linear map
    - Relation between matrices and linear maps
  - Eigenvalue problem and diagonalization

# Review of linear algebra

Vector space -Abstract vectors-

A quantum state is a **vector**

$$|\Psi\rangle$$

# Geometric vector

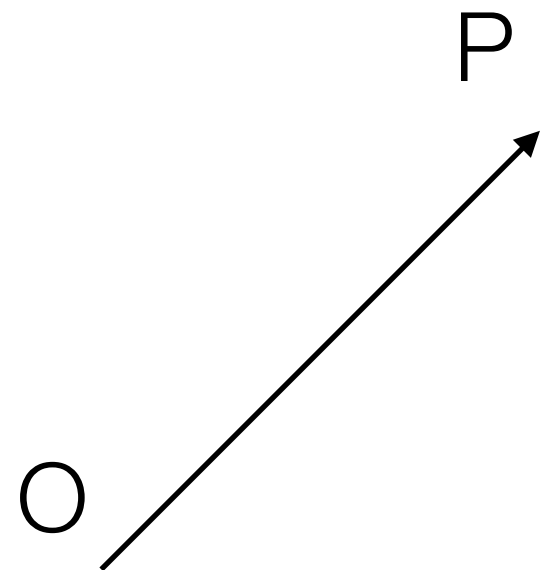
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Geometric vector: Arrow on the plane (or the space) ,  
which has "Direction" and "Length"

$$\vec{v} \equiv \overrightarrow{OP}$$

We can express a vector by its component:

$$\vec{v} = \begin{pmatrix} v_x \\ v_y \\ v_z \end{pmatrix} = \begin{pmatrix} x_p - x_o \\ y_p - y_o \\ z_p - z_o \end{pmatrix}$$



# Inner product of vector

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Inner product:

$$\begin{aligned}(\vec{a}, \vec{b}) &\equiv \vec{a} \cdot \vec{b} \\ &= a_x b_x + a_y b_y + a_z b_z\end{aligned}$$

Example:

$$\vec{a} = \begin{pmatrix} a_x \\ a_y \\ a_z \end{pmatrix}, \vec{b} = \begin{pmatrix} b_x \\ b_y \\ b_z \end{pmatrix}$$

Properties:

$$(\vec{a}, \vec{a}) \geq 0$$

$$(\vec{a}, \vec{b}) = (\vec{b}, \vec{a})$$

$$(\vec{a} + \vec{b}, \vec{c}) = (\vec{a}, \vec{c}) + (\vec{b}, \vec{c})$$

$$(c\vec{a}, \vec{b}) = c(\vec{a}, \vec{b}) \quad c \in \mathbb{R}$$

Norm (length):

$$\|\vec{a}\| \equiv \sqrt{(\vec{a}, \vec{a})}$$

# Vector space (linear space)

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Vector space  $\mathbb{V}$  : generalization of geometric vector

Set of elements (vectors) satisfying following **axioms** (公理)

## Properties of addition:

$$\vec{a} + \vec{b} = \vec{b} + \vec{a}$$

Commutative property (交換法則)

$$(\vec{a} + \vec{b}) + \vec{c} = \vec{a} + (\vec{b} + \vec{c})$$

Associative property (結合法則)

$$\vec{a} + \vec{0} = \vec{a}$$

Existence of **unique** zero vector

$$\vec{a} + (-\vec{a}) = \vec{0}$$

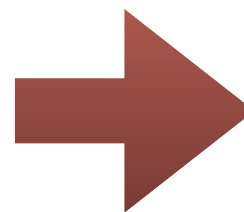
Existence of **unique** inverse vector

## Multiplication of scalar $c$ :

$$c(\vec{a} + \vec{b}) = c\vec{b} + c\vec{a}$$

$$(c + d)\vec{a} = c\vec{a} + d\vec{a}$$

$$(cd)\vec{a} = c(d\vec{a})$$



$c \in \mathbb{R}$  : Real vector space

$c \in \mathbb{C}$  : Complex vector space

# Inner product space (metric vector space)

(計量空間)

Inner product space:

Vector space + definition of **inner product**

Inner product:  $(\vec{a}, \vec{b})$

**Axiom:**

$$(\vec{a}, \vec{a}) \geq 0$$

$$(\vec{a}, \vec{b}) = (\vec{b}, \vec{a})^*$$

$$(\vec{a} + \vec{b}, \vec{c}) = (\vec{a}, \vec{c}) + (\vec{b}, \vec{c})$$

$$(c\vec{a}, \vec{b}) = c(\vec{a}, \vec{b})$$

\*If a norm defined from the inner product is "complete" (完備) ,  
that space is called **Hilbert space**.



# Examples of vector spaces

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(1) Coordinate space (数ベクトル空間)  $\mathbb{R}^n, \mathbb{C}^n$

Vector:  $\vec{v} = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix} \quad v_i \in \mathbb{R} \text{ or } \mathbb{C}$

Inner product:  $(\vec{a}, \vec{b}) \equiv \vec{a} \cdot \vec{b}^*$

(2) Wave vectors in quantum physics

Vector:  $|\Psi\rangle$

Inner product:  $(|a\rangle, |b\rangle) = \langle b|a\rangle$

Linearly independent or dependent

————— (線形独立) ————

(線形従属) —————

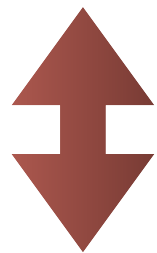
## Linear combination:

$$\vec{x} = c_1 \vec{v}_1 + c_2 \vec{v}_2 + \cdots c_k \vec{v}_k$$

$$\vec{v}_i \in \mathbb{V} \quad c_i \in \mathbb{R} \text{ or } \mathbb{C}$$

A set  $\{\vec{v}_1, \vec{v}_2, \cdots \vec{v}_k\}$  is **linearly independent** when

$\vec{x} = \vec{0}$  is satisfied **if and only if**  $c_1 = c_2 = \cdots = c_k = 0$



A set  $\{\vec{v}_1, \vec{v}_2, \cdots \vec{v}_k\}$  is **linearly dependent** when

it is not linearly independent.

# Basis of vector space

(基底)

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A set  $\{\vec{e}_1, \vec{e}_2, \dots, \vec{e}_n\}$  ( $\vec{e}_i \in \mathbb{V}$ ) is a basis (基底) of  $\mathbb{V}$  when

$\{\vec{e}_1, \vec{e}_2, \dots, \vec{e}_n\}$  is linearly independent.

and

Any vectors in  $\mathbb{V}$  are represented by its linear combination.

$\vec{e}_i$  : basis vector

# of basis vectors ( $n$ ) is called **dimension** (次元) of  $\mathbb{V}$  .

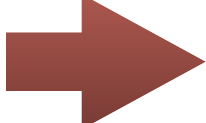
$$n = \dim \mathbb{V}$$

# Relation (map) to coordinate vector space

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By using a basis  $\{\vec{e}_1, \vec{e}_2, \dots, \vec{e}_n\}$ ,  $\vec{v} \in \mathbb{V}$  is **uniquely represented** as  
(<sup>\*</sup> From linear independency)

$$\vec{v} = v_1 \vec{e}_1 + v_2 \vec{e}_2 + \dots + v_n \vec{e}_n$$

 We can represent  $\vec{v}$  as a coordinate vector

$$\vec{v} \rightarrow \begin{pmatrix} v_1 \\ v_2 \\ \dots \\ v_n \end{pmatrix} \in \mathbb{C}^n \text{ ( or } \mathbb{R}^n \text{ )}$$

By selecting a basis, we obtain a "**concrete**" coordinate vector  
for an "**abstract**" vector

## Orthonormal basis (正規直交基底)

When a vector space has an inner product,

$\vec{a}, \vec{b}$  is **orthogonal** (直交) if  $(\vec{a}, \vec{b}) = 0$ .

### Orthonormal basis

A basis  $\{\vec{e}_1, \vec{e}_2, \dots, \vec{e}_n\}$  is an orthonormal basis when

$$\|\vec{e}_i\| = 1 \quad (i = 1, 2, \dots, n)$$

$$(\vec{e}_i, \vec{e}_j) = 0 \quad (i \neq j; i, j = 1, 2, \dots, n)$$

\*A basis can be transformed into an orthonormal basis.

**cf. Gram-Schmidt orthonormalization**

# Example: quantum states

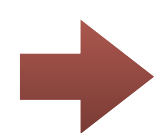
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2 qbits: We can choose the following **four vectors as the (orthonormal) basis**.



$$|0\rangle \otimes |0\rangle, |0\rangle \otimes |1\rangle, |1\rangle \otimes |0\rangle, |1\rangle \otimes |1\rangle$$

Simple notation:  $|00\rangle, |01\rangle, |10\rangle, |11\rangle$

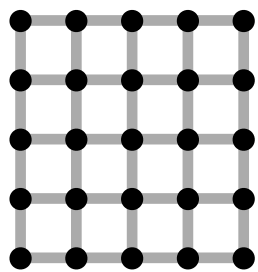


$$|\Psi\rangle = \sum_{\alpha, \beta=0,1} C_{\alpha, \beta} |\alpha\beta\rangle$$

$C_{\alpha, \beta} = \langle \alpha\beta | \Psi \rangle$  :complex number

$$C \in \mathbb{C}^4$$

Many qbits:



**basis:**  $|m_1, m_2, \dots, m_N\rangle = |00 \dots 0\rangle, |00 \dots 1\rangle, |01 \dots 0\rangle, \dots$

$$|\Psi\rangle = \sum_{\{m_i=0,1\}} T_{m_1, m_2, \dots, m_N} |m_1, m_2, \dots, m_N\rangle$$

$$T_{m_1, m_2, \dots, m_N} = \langle m_1, m_2, \dots, m_N | \Psi \rangle \Rightarrow T \in \mathbb{C}^{2^N}$$

# Bell basis

Cf. Nobel prize in physics 2022

We may consider a different orthonormal basis for 2-qubit states.

**Bell basis**

$$|\Phi^\pm\rangle = \frac{1}{\sqrt{2}} (|00\rangle \pm |11\rangle)$$
$$|\Psi^\pm\rangle = \frac{1}{\sqrt{2}} (|01\rangle \pm |10\rangle)$$

\*These are *maximally entangled states*.

**Basis 1**  
**(computational basis)**

$$\begin{aligned} \vec{e}_1 &= |00\rangle \\ \vec{e}_2 &= |01\rangle \\ \vec{e}_3 &= |10\rangle \\ \vec{e}_4 &= |11\rangle \end{aligned} \quad \Rightarrow \quad |00\rangle \rightarrow \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

**Basis 2**  
**(Bell basis)**

$$\begin{aligned} \vec{e}_1 &= |\Phi^+\rangle \\ \vec{e}_2 &= |\Phi^-\rangle \\ \vec{e}_3 &= |\Psi^+\rangle \\ \vec{e}_4 &= |\Psi^-\rangle \end{aligned} \quad \Rightarrow \quad |00\rangle \rightarrow \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

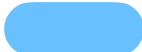
When we change the basis, the corresponding coordinate vector changes.

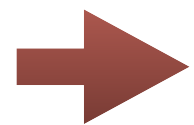
# As an aside...

We may use unusual vector spaces in quantum many-body problems.

## Quantum dimer model

(D. S. Rokhsar and S. A. Kivelson, Phys. Rev. Lett. **61**, 2376 (1988))

“Dimer”   $\frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle)$

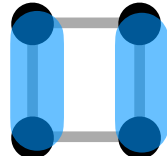


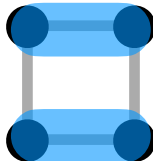
Orthonormal basis: All closed packing of dimers

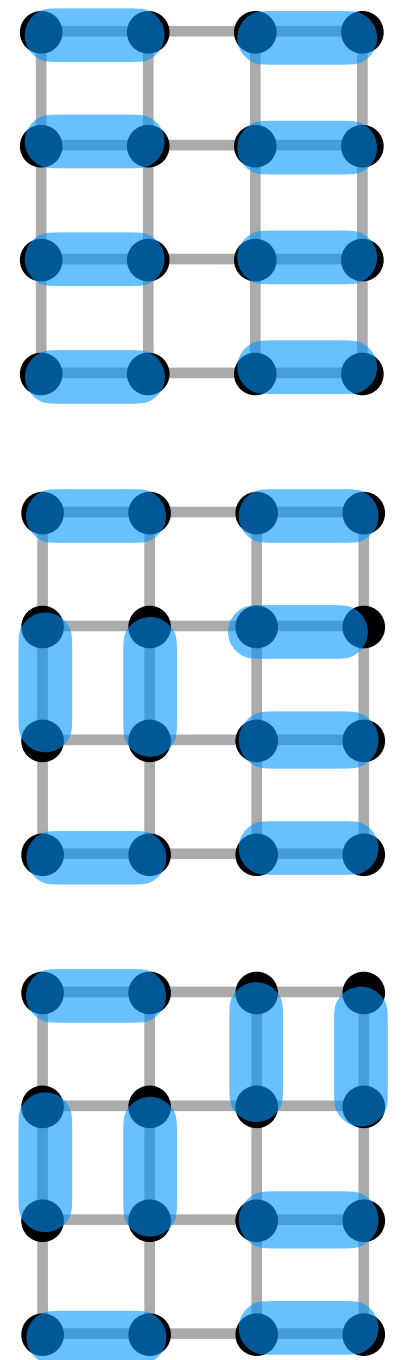
(When we use the usual  $S=1/2$  spins, **they are not orthogonal.**)

$$\mathcal{H} = \sum_{\text{plaquettes}} \left[ \underbrace{-J (|\parallel\rangle\langle=| + \text{H.C.})}_{\text{Flip of a dimer pair}} + \underbrace{V (|\parallel\rangle\langle\parallel| + |=\rangle\langle=|)}_{\text{Potential energy}} \right]$$

States of a plaquette

$|\parallel\rangle =$  

$|=\rangle =$  





# Vector subspace (linear subspace)

## **Vector subspace** (ベクトル部分空間) :

A subset  $\mathbb{W}$  of a vector space  $\mathbb{V}$  is a **vector subspace** of  $\mathbb{V}$  when  $\mathbb{W}$  satisfies the same axioms of vector space with  $\mathbb{V}$ .

The following conditions are necessary and sufficient.

$$\begin{array}{ccc} \vec{a}, \vec{b} \in \mathbb{W} & \longrightarrow & \vec{a} + \vec{b} \in \mathbb{W} \\ \vec{a} \in \mathbb{W}, c \in \mathbb{C} & \longrightarrow & c\vec{a} \in \mathbb{W} \end{array}$$

(In the case of **complex** vector space)

\*The **dimension of a vector subspace can be smaller** than that of the original vector space.

 If we construct an efficient vector subspace, it can be **a kind of data compression**.

Matrix and linear map

# Matrix (行列)

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**Matrix:** "Table" of (complex) numbers in a rectangular form

$M \times N$  matrix

$$A = \begin{pmatrix} A_{11} & A_{12} & \cdots & A_{1,N} \\ A_{21} & A_{22} & \cdots & A_{2,N} \\ \vdots & \vdots & & \vdots \\ A_{M1} & A_{M2} & \cdots & A_{M,N} \end{pmatrix}$$

Product of matrices:  $C = AB$   $A_{ij} \in \mathbb{C} \text{ ( or } \mathbb{R} \text{ )}$

$$C_{ij} = \sum_{k=1}^K A_{ik} B_{kj}$$

$$A : M \times K$$

$$B : K \times N$$

$$C : M \times N$$

In general:  $XY \neq YX$

\*We also know addition, multiplication of scalar.

# Identity matrix (単位行列)

---

## Identity matrix:

$N \times N$  matrix  
(Square matrix)

$$I = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}$$

Product:

$$IA = A$$

$$A : N \times M$$

$$BI = B$$

$$B : K \times N$$

\* Element of the identity matrix:  $I_{ij} = \delta_{ij}$  (Kronecker delta)

$$\delta_{ij} = \begin{cases} 1 & (i = j) \\ 0 & (i \neq j) \end{cases}$$

# Transpose, complex conjugate and adjoint

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Transpose:  
(転置)

$$A^t \quad (A^t)_{ij} = A_{ji}$$

Complex conjugate:  
(複素共役)

$$A^* \quad (A^*)_{ij} = A_{ij}^*$$

Adjoint:  
(随伴)

$$A^\dagger = (A^t)^* = (A^*)^t$$

or

$$(A^\dagger)_{ij} = A_{ji}^*$$

Hermitian conjugate:  
(エルミート共役)

("Dagger" is convention in physics)

# Multiplication to coordinate vector

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$$\begin{array}{ccc} A : M \times N & \vec{v} \in \mathbb{C}^N & \vec{v}' \in \mathbb{C}^M \\ \begin{pmatrix} A_{11} & A_{12} & \cdots & A_{1,N} \\ A_{21} & A_{22} & \cdots & A_{2,N} \\ \vdots & \vdots & & \vdots \\ \vdots & \vdots & & \vdots \\ A_{M1} & A_{M2} & \cdots & A_{M,N} \end{pmatrix} & \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_N \end{pmatrix} & = \begin{pmatrix} v'_1 \\ v'_2 \\ \vdots \\ v'_M \end{pmatrix} \end{array}$$

$M \times N$  matrix **transforms** an  $N$ -dimensional coordinate vector to an  $M$ -dimensional coordinate vector.

**M × N matrix**  **Linear map:**  $\mathbb{C}^N \rightarrow \mathbb{C}^M$   
**1 to 1** (線形写像)

# General linear map

---

Map:  $f : \mathbb{V} \rightarrow \mathbb{V}'$

$$f(\vec{v}) = \vec{v}' \quad (\vec{v} \in \mathbb{V}, \vec{v}' \in \mathbb{V}')$$

Linear map:

$$f(\vec{x} + \vec{y}) = f(\vec{x}) + f(\vec{y})$$

$$f(c\vec{x}) = cf(\vec{x})$$

$$(\vec{x}, \vec{y} \in \mathbb{V}, c \in \mathbb{C})$$

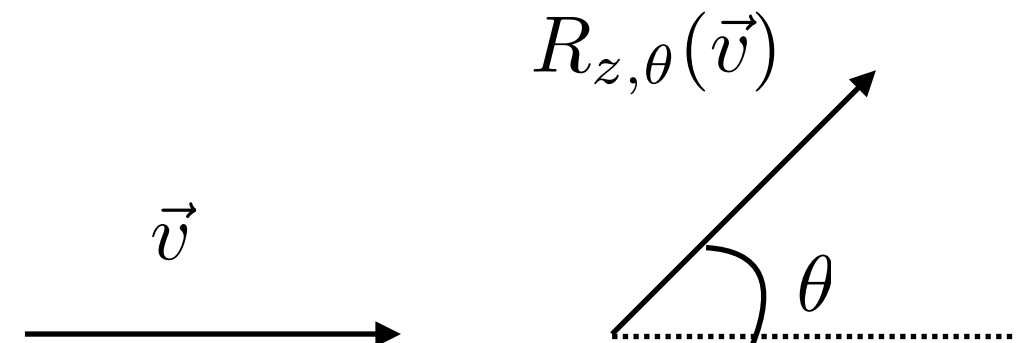
Examples:

**Rotation** (e.g.  $\theta$  rotation around z-axis)

$$R_{z,\theta} : \mathbb{C}^3 \rightarrow \mathbb{C}^3$$

**Hamiltonian operator**

$$\mathcal{H} : \mathbb{V} \rightarrow \mathbb{V}$$



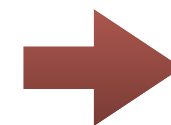
$$|\Psi\rangle \quad \rightarrow \quad \mathcal{H}|\Psi\rangle$$

# Matrix representation of linear map

By using a basis, we can represent a linear map in a matrix.

$$f : \mathbb{V} \rightarrow \mathbb{V}'$$

**Vector space**  $\mathbb{V} : \dim \mathbb{V} = N$



$\mathbb{V}' : \dim \mathbb{V}' = M$

**Basis**

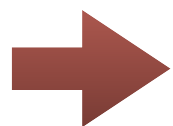
$$\{\vec{e}_1, \vec{e}_2, \dots, \vec{e}_N\}$$

$$\{\vec{e}'_1, \vec{e}'_2, \dots, \vec{e}'_M\}$$

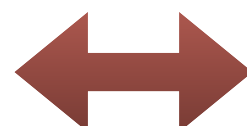
Transformation of basis vectors:

$$f(\vec{e}_j) = f_{1j}\vec{e}'_1 + f_{2j}\vec{e}'_2 + \dots + f_{Mj}\vec{e}'_M$$

\* For orthonormal basis,  $f_{ij} = (f(\vec{e}_j), \vec{e}'_i)$



$$f : \mathbb{V} \rightarrow \mathbb{V}'$$



**1 to 1**

(if we fix basis)

$$\begin{pmatrix} f_{11} & f_{12} & \dots & f_{1,N} \\ f_{21} & f_{22} & \dots & f_{2,N} \\ \vdots & \vdots & & \vdots \\ f_{M1} & f_{M2} & \dots & f_{M,N} \end{pmatrix}$$



# Examples of matrix

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**Rotation** (e.g.  $\theta$  rotation around z-axis)

$$R_{z,\theta} : \mathbb{C}^3 \rightarrow \mathbb{C}^3$$

$$R_{z,\theta} = \begin{pmatrix} \cos \theta & -\sin \theta & 0 \\ \sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$



**Hamiltonian operator**

$$\mathcal{H} : \mathbb{V} \rightarrow \mathbb{V} \quad \mathcal{H} \rightarrow \begin{pmatrix} H_{0,0;0,0} & H_{0,0;0,1} & H_{0,0;1,0} & H_{0,0;1,1} \\ H_{0,1;0,0} & H_{0,1;0,1} & H_{0,1;1,0} & H_{0,1;1,1} \\ H_{1,0;0,0} & H_{1,0;0,1} & H_{1,0;1,0} & H_{1,0;1,1} \\ H_{1,1;0,0} & H_{1,1;0,1} & H_{1,1;1,0} & H_{1,1;1,1} \end{pmatrix}$$

**Matrix element:**  $H_{\alpha,\beta;\alpha',\beta'} \equiv \langle \alpha\beta | \mathcal{H} | \alpha'\beta' \rangle$   
(行列要素)

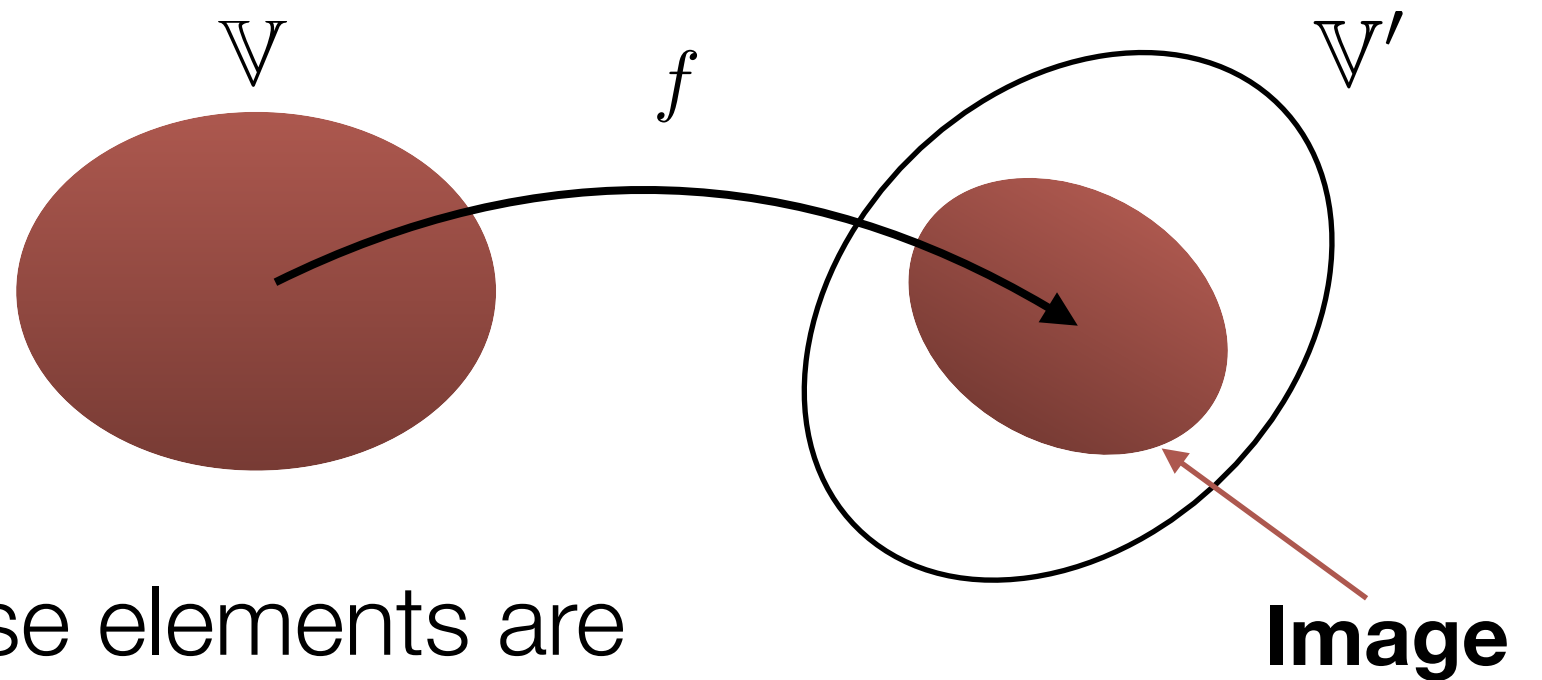
\* In this notation, **basis should be orthonormal.**

# Image of a map

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$$f : \mathbb{V} \rightarrow \mathbb{V}'$$

Image of  $f$ :  
(像)



Vector subspace whose elements are mapped from  $\mathbb{V}$  by  $f$ .

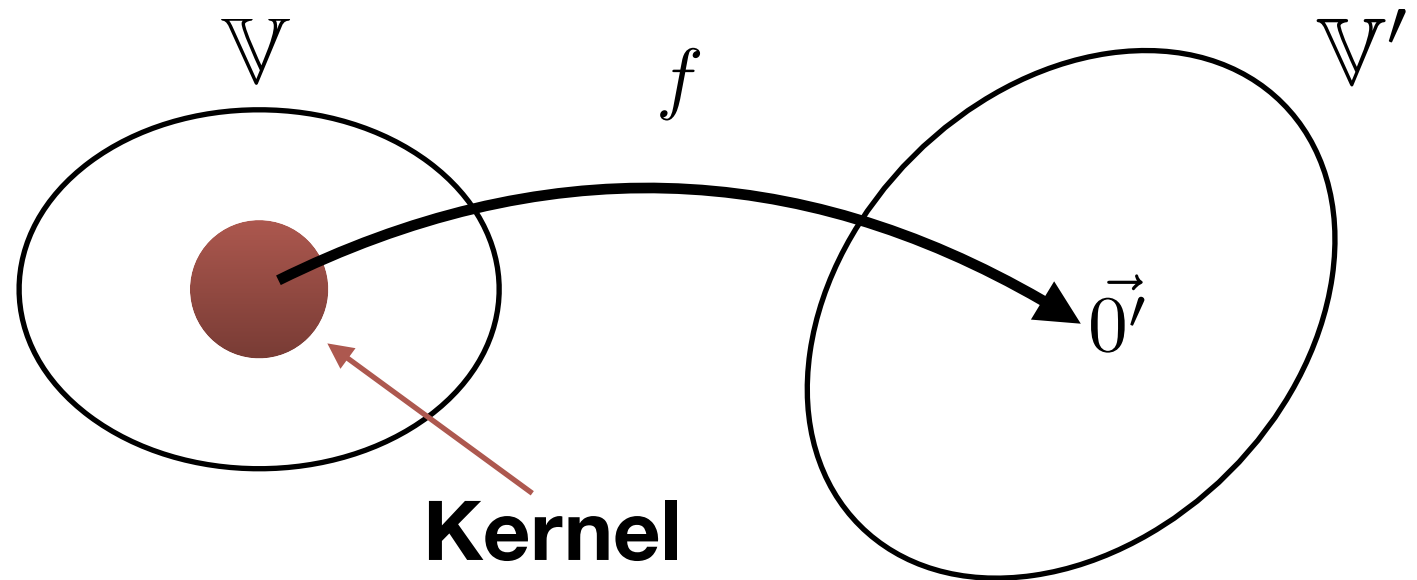
$$\text{img}(f) = \{\vec{v}' \mid \vec{v} \in \mathbb{V}, \vec{v}' = f(\vec{v})\}$$

# Kernel of a map

---

$$f : \mathbb{V} \rightarrow \mathbb{V}'$$

Kernel of  $f$ :  
(核)



Vector subspace whose elements are mapped into zero vector by  $f$ .

$$\ker(f) = \{\vec{v} | \vec{v} \in \mathbb{V}, f(\vec{v}) = \vec{0}'\}$$

## Theorem:

$$\dim(V) = \dim(\ker(f)) + \dim(\text{img}(f))$$

# Rank of matrix

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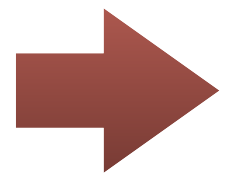
**Rank** (ランク or 階数) of a matrix  $A$ :

$$\text{rank}(A) \equiv \dim(\text{img}(A))$$

**Rank** is identical with

Maximum # of linearly independent column vectors (列ベクトル) in  $A$

Maximum # of linearly independent row vectors (行ベクトル) in  $A$



$$\text{rank}(A) \leq \min(M, N)$$

for an  $M \times N$  matrix  $A$ .

$$\begin{pmatrix} A_{11} & A_{12} & \cdots & A_{1,N} \\ A_{21} & A_{22} & \cdots & A_{2,N} \\ \vdots & \vdots & & \vdots \\ \vdots & \vdots & & \vdots \\ A_{M1} & A_{M2} & \cdots & A_{M,N} \end{pmatrix}$$

# Regular matrix and its inverse matrix

---

A square matrix  $A$  is a **regular matrix** (正則) if a matrix  $X$  satisfying

$$AX = XA = I$$

exists. The matrix  $X$  is called inverse matrix (逆行列) of  $A$  and it is written as  $X = A^{-1}$ .

**Properties:**  $A^{-1}$  is unique.

$$(A^{-1})^{-1} = A$$

$$(AB)^{-1} = B^{-1}A^{-1}$$

$A$  is a regular matrix  $\longleftrightarrow \text{rank}(A) = N$

Can we consider an "inverse matrix" of a non-regular matrix (including a rectangular matrix) ?

# Simultaneous linear equation

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## **Simultaneous linear equation** (連立一次方程式)

can be represented by a matrix and a vector as

$$A\vec{x} = \vec{b} \quad A : M \times N, \vec{x} \in \mathbb{C}^N, \vec{b} \in \mathbb{C}^M$$

If  $A$  is a **square matrix** ( $N=M$ ), and it **has an inverse matrix** ( $\text{rank}(A) = N$ ),  
we can solve the equation as

$$\vec{x} = A^{-1}\vec{b}$$

$N > M$ : Underdetermined problem (劣決定問題)

$N < M$ : Overdetermined problem (優決定問題)

How can we find a "solution" when  $A$  does not have the "inverse"?

➡ It is related to "sparse modeling".

(Although we will not treat it, it is a kind of data compression)

# Eigenvalue problems and diagonalization

# Eigenvalue and Eigenvector

---

For a square matrix  $A$

$$A\vec{v} = \lambda\vec{v}$$

$\vec{v} \neq \vec{0}$  :eigenvector (固有ベクトル)

$\lambda \in \mathbb{C}$  :eigenvalue (固有値)

Properties:

If  $\vec{v}$  is an eigenvector,  $c\vec{v}$  is also an eigenvector.

Eigenspace (固有空間) :

The set of eigenvectors corresponds an eigenvalue  $\lambda$ .

Eigenvectors corresponding to different eigenvalues are  
**linearly independent.**



# Example: quantum many-body problems

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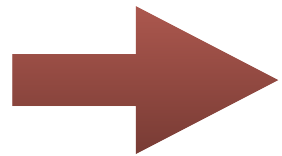
Schrödinger equation

$$\mathcal{H}|\Psi\rangle = E|\Psi\rangle$$

$\mathcal{H}$  :Hamiltonian

$|\Psi\rangle$  :Wave function (state vector)

$E$  :Energy



Eigenvalue = energy

Eigenvector = quantum state (eigenstate)

Typical questions:

- How does the lowest energy vary when we change the Hamiltonian?
  - What is the energy gap between the lowest and the 2nd lowest states?
- What is the property of the eigenstate?
  - Quantum phase transitions

# Right and left eigenvectors

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In general, **left eigenvectors** can be different from the right eigenvectors.

$$\begin{aligned} A\vec{v} &= \lambda\vec{v} \\ (\vec{u}^*)^t A &= \lambda(\vec{u}^*)^t \end{aligned}$$

Cf.  $|\psi\rangle, \langle\phi|$   
in quantum state

$\vec{v}$  : Right eigenvector  
 $(\vec{u}^*)^t$  : Left eigenvector

## Properties:

The set of **eigenvalues** is **identical** between the right and the left eigenvectors.

A left eigenvector and a right eigenvector are **orthogonal** when they correspond to different eigenvalues.

$$\vec{u}_i^* \cdot \vec{v}_j = 0 \quad (\lambda_i \neq \lambda_j)$$

# Diagonalization

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Diagonalization (対角化) :

$$A : N \times N$$
$$P^{-1}AP = \begin{pmatrix} \alpha_1 & & & \\ & \alpha_2 & & \\ & & \ddots & \\ & & & \alpha_N \end{pmatrix}$$

$A$  can be diagonalized.   $A$  has  $N$  linearly independent eigenvectors.

**necessary  
and  
sufficient**

$$\alpha_i = \lambda_i$$

$$P = (\vec{v}_1, \vec{v}_2, \dots, \vec{v}_N)$$

$$(P^{-1})^t = (\vec{u}_1^*, \vec{u}_2^*, \dots, \vec{u}_N^*)$$

$$\text{Normalization: } \vec{u}_i^* \cdot \vec{v}_i = 1$$

# Unitary matrix

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**Unitary matrix** (ユニタリ行列) :  $U^\dagger = U^{-1}$

**Real Orthogonal matrix** (実直交行列) :  $P^t = P^{-1}, (P_{ij} \in \mathbb{R})$

When we consider a unitary matrix as a set of vectors:

$$U = (\vec{v}_1, \vec{v}_2, \dots, \vec{v}_N)$$

it is an orthonormal basis:  $\vec{v}_i^* \cdot \vec{v}_j = \delta_{i,j}$

 The linear map represented by a unitary matrix  
(**unitary transformation**) does not change

- the norm of a vector

$$\|U\vec{v}\| = \|\vec{v}\|$$

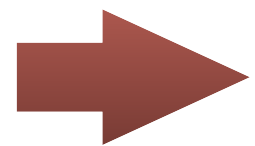
- "distance" between two vectors

$$\|U\vec{v}_1 - U\vec{v}_2\| = \|\vec{v}_1 - \vec{v}_2\|$$

# Normal matrix

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**Normal matrix** (正規行列) :  $A^\dagger A = AA^\dagger$



We can **always diagonalize it** by a unitary matrix

$$U^\dagger = U^{-1}$$

as

$$U^\dagger A U = \begin{pmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_N \end{pmatrix} \quad \lambda_i \in \mathbb{C}$$

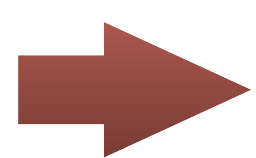
Its eigenvalues could be **complex**.  
(even if  $A$  is a real matrix)

# Hermitian matrix and its eigenvalue

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**Hermitian matrix** (エルミート行列) :  $A^\dagger = A$

**Real symmetric matrix** (実対称行列) :  $A^t = A$ , ( $A_{ij} \in \mathbb{R}$ )



It is a special **normal matrix**.  $A^\dagger A = AA^\dagger = AA$

Its eigenvalues are **real**.

We can **always diagonalize it** by a unitary matrix

$$U^\dagger A U = \begin{pmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_N \end{pmatrix} \quad \lambda_i \in \mathbb{R}$$

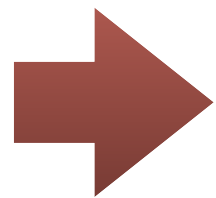
**Hermitian (or real symmetric) matrices often appear in physics.**

Ex. Hamiltonian,  $\mathcal{H}$

# Generalization of diagonalization

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- Eigenvalue problems and diagonalizations are defined for a square matrix.
- Even if  $A$  is a square matrix, it may not be diagonalized.



- Is it possible to transform all square matrixes into diagonal forms by generalizing the diagonalization?
- Is it possible to generalize it to a rectangular matrices?

Yes. **The singular value decomposition**  
(特異値分解) is an generalization of the diagonalization.

(We can also consider a decomposition of a tensor.)

## Notice

Next week (Oct. 20)

- No classes on Nov. 3, Nov. 17, and Nov. 22
- Classes will be also held on Jan. 5 and Jan. 19

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1. Computational science, quantum computing, and data compression
  2. Review of linear algebra
  3. Singular value decomposition
  4. Application of SVD and generalization to tensors
  5. Entanglement of information and matrix product states
  6. Application of MPS to eigenvalue problems
  7. Tensor network representation
  8. Data compression in tensor network
  9. Tensor network renormalization
  10. Quantum mechanics and quantum computation
  11. Simulation of quantum computers
  12. Quantum-classical hybrid algorithms and tensor network
  13. Quantum error correction and tensor network