Solution for Exercises in

— Linear Algebra Done Right

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1 Vector Space

1.1 $\mathbf{R}^n$ and $\mathbf{C}^n$

1. Suppose a and b are real numbers, not both 0. Find real numbers c and d such that

$$1/(a+bi) = c+di$$

Solution:

Since $c+di$ is the multiplicative inverse of $a+bi$, we have

$$(a+bi)(c+di) = 1,$$

expand the left-hand side of the equation above, we have

$$(ac-bd)+(ad+bc)i = 1,$$

i.e.,

$$\begin{cases} ac-bd &= 1 \\ ad+bc &= 0, \end{cases}$$

since $a \neq 0$, from the below equation $ad+bc=0$ we have $d=-bc/a$, substitute it in the above equation $ac-bd=0$, we have

$$c = \frac{a}{a^2+b^2},$$

use $d=-bc/a$ we have

$$d = -\frac{b}{a^2+b^2}.$$

2. Show that

$$\frac{-1+\sqrt{3}i}{2}$$

is a cube root of 1 (meaning that its cube equals 1).

Solution:

We have

$$\begin{aligned}\left(\frac{-1+\sqrt{3}i}{2}\right)^3 &= (\frac{-1+\sqrt{3}i}{2}\times\frac{-1+\sqrt{3}i}{2})\times\frac{-1+\sqrt{3}i}{2}\\ &= \frac{(1-3)+(-\sqrt{3}-\sqrt{3})i}{4}\times\frac{-1+\sqrt{3}i}{2}\\ &= \frac{-1-\sqrt{3}i}{2}\times\frac{-1+\sqrt{3}i}{2}\\ &= \frac{(1+3)+(-\sqrt{3}+\sqrt{3})i}{4}\\ &= 1.\end{aligned}$$

3. Find two distinct square root of i.

Solution:

One can be

$$\frac{1+i}{2}$$

since

$$\frac{1+i}{2}\times\frac{1+i}{2}=\frac{(1-1)+(1+1)i}{2}=i.$$

Another can be

$$\frac{-1-i}{2}$$

since

$$\frac{-1-i}{2}\times\frac{-1-i}{2}=\frac{(1-1)+(1+1)i}{2}=i.$$

4. Show that $\alpha+\beta=\beta+\alpha$ for all $\alpha,\beta\in\mathbf{C}$.

Solution:

Suppose $\alpha=a+bi$ and $\beta=c+di$, we have

$$\begin{aligned}\alpha+\beta &= (a+bi)+(c+di)\\ &= (a+c)+(b+d)i\\ &= (c+a)+(d+b)i\\ &= (c+di)+(a+bi)\\ &= \beta+\alpha,\end{aligned}$$

where the second and forth equations above hold because of the definition of addition in C and the third equation holds because of the usual commutativity of addition in R.

5. Show that $(\alpha+\beta)+\lambda=\alpha+(\beta+\lambda)$ for all $\alpha,\beta,\lambda\in\mathbb{C}$.

Solution:

Suppose $\alpha=a+bi$, $\beta=c+di$, $\lambda=e+fi$, we have

$$\begin{aligned}
(\alpha+\beta)+\lambda &= ((a+bi)+(c+di))+(e+fi)\\
&= ((a+c)+(c+b)i)+(e+fi)\\
&= (a+c+e)+(b+d+f)i\\
&= (a+(c+e))+(b+(d+f))i\\
&= (a+bi)+((c+e)+(d+f)i)\\
&= \alpha+(\beta+\lambda),
\end{aligned}$$

where second, third and fifth equations above hold because of the definition of the addition in C and the fourth equation holds because of the normal associativity in R.

6. Show that $(\alpha\beta)\lambda=\alpha(\beta\lambda)$ for all $\alpha,\beta,\lambda\in\mathbb{C}$.

Solution:

Suppose $\alpha=a+bi$, $\beta=c+di$ and $\lambda=e+fi$, we have

$$\begin{aligned}
(\alpha\beta)\lambda &= ((a+bi)(c+di))(e+fi)\\
&= ((ac-bd)+(ad+bc)i)(e+fi)\\
&= ((ac-bd)e-(ad+bc)f)+((ac-bd)f+(ad+bc)e)i\\
&= ((ce-df)a-(de+cf)b)+((ce-df)b+(de+cf)a)i\\
&= (a+bi)((ce-df)+(de+cf)i)\\
&= (a+bi)((c+di)(e+fi))\\
&= \alpha(\beta\lambda),
\end{aligned}$$

where the second, third, fifth and sixth equations above hold because of the definition of multiplicative in C and the fourth equation holds because of the normal form changing in R.

7. Show that for every $\alpha\in\mathbb{C}$, there exists a unique $\beta\in\mathbb{C}$ such that $\alpha+\beta=0$.

Solution:

Suppose $\alpha = a + bi$, it is easy to verify that $\beta = -a - bi$ satisfy the equaiton $\alpha + \beta = 0$, now suppose there is another $\beta' \neq -a - bi$ which satisfy the equation, denote it as

$$\beta' = c + di,$$

we have

$$\alpha + \beta' = (a + c) + (b + d)i = 0,$$

implies that

$$a + c = 0, \qquad b + d = 0,$$

therefore

$$c = -a, \qquad d = -b,$$

which is a contradictt with our assumption, therefore there exists a unique β holds the equation.

8. Show that for every $\alpha \in \mathbf{C}$ with $\alpha \neq 0$, there exists a unique $\beta \in \mathbf{C}$ such $\alpha\beta = 1$.

Solution:

Suppose $\alpha = a + bi$, let $\beta = c + di$ satisfy the equation $\alpha\beta = 1$, implies

$$(ac - bd) + (ad + bc)i = 1,$$

from the solution of exercise 1, we have

$$c = \frac{a}{a^2 + b^2}, \qquad d = -\frac{b}{a^2 + b^2},$$

and its unique.

9. Show that $\lambda(\alpha + \beta) = \lambda\alpha + \lambda\beta$ for all $\alpha, \beta, \lambda \in \mathbf{C}$.

Solution:

Suppose $\alpha = a + bi$, $\beta = c + di$ and $\lambda = e + fi$, we have

$$\begin{aligned}\lambda(\alpha+\beta) &= (e+fi)((a+bi)+(c+di))\\ &= (e+fi)((a+c)+(b+d)i)\\ &= (e(a+c)-f(b+d))+(e(b+d)+f(a+c))i\\ &= ((ea-fb)+(ec-fd))+((eb+fa)+(ed+fc))i\\ &= ((ea-fb)+(eb+fa)i)+((ec-fd)+(ed+fc)i)\\ &= (e+fi)(a+bi)+(e+fi)(c+di)\\ &= \lambda\alpha+\lambda\beta,\end{aligned}$$

where the second the fifth equations above hold because of the definition of addition in C and the third and sisth equations hold because of the definition of multiplicative in C, and the fourth equation holds in R.

10. Find $x \in \mathbf{R}^4$ such that

$$(4,-3,1,7)+2x = (5,9,-6,8).$$

Solution:

Denote x as

$$x = (x_1, x_2, x_3, x_4),$$

we have

$$2x = (2x_1, 2x_2, 2x_3, 2x_4)$$

because of the definition of scalar multiplication in $\mathbf{F}^n$, therefore we have

$$(4+2x_1, -3+2x_2, 1+2x_3, 7+2x_4) = (5,9,-6,8),$$

which is same as

$$\begin{cases} 4+2x_1 &= 5\\ -3+2x_2 &= 9\\ 1+2x_3 &= -6\\ 7+2x_4 &= 8\end{cases}$$

solve the equation above we have

$$x_1 = 0.5, \qquad x_2 = 6, \qquad x_3 = -3.5, \qquad x_4 = 0.5,$$

thus

$$x = (0.5, 6, -3.5, 0.5).$$

11. Explain why there does not exist $\lambda \in \mathbf{C}$ such that

$$\lambda(2-3i, 5+4i, -6+7i) = (12-5i, 7+22i, -31-9i).$$

Solution:

This problem is same with finding $\lambda \in \mathbf{C}$ which satisfies

$$\begin{cases} \lambda(2-3i) & = 12-5i \\ \lambda(5+4i) & = 7+22i \\ \lambda(-6+7i) & = -31-9i \end{cases},$$

let $\lambda = a+bi$, where $a, b \in \mathbf{R}$, from the first equation $\lambda(2-3i) = 12-5i$ and $(a+bi)(2-3i) = (2a+3b)+(2b-3a)i$ we have

$$2a+3b = 12, \qquad 2b-3a = -5,$$

which have solution

$$a = 3, \qquad b = 2,$$

but $\lambda = 3+2i$ does not satisfies the third equation, since

$$(3+2i)(-6+7i) = -32+9i \neq -31-9i.$$

12. Show that $(x+y)+z = x+(y+z)$ for all $x, y, z \in \mathbf{F}^n$.

Solution:

$$
\begin{aligned}
(x+y)+z &= \big((x_1,\dots,x_n)+(y_1,\dots,y_n)\big)+(z_1,\dots,z_n)\\
&= (x_1+y_1,\dots,x_n+y_n)+(z_1,\dots,z_n)\\
&= (x_1+y_1+z_1,\dots,x_n+y_n+z_n)\\
&= \big(x_1+(y_1+z_1),\dots,x_n+(y_n+z_n)\big)\\
&= (x_1,\dots,x_n)+(y_1+z_1,\dots,y_n+z_n)\\
&= x+(y+z),
\end{aligned}
$$

where the second, third, fifth and sixth equations hold because of the definition of addition in $\mathbf{F}^n$, and the fourth equaiton holds because of the associativity in $\mathbf{F}$.

13. Show that $(ab)x = a(bx)$ for all $x \in \mathbf{F}^n$ and all $a, b \in \mathbf{F}$.

Solution:

Let $x = (x_1,\dots,x_n)$, we have

$$
\begin{aligned}
(ab)x &= (ab)(x_1,\dots,x_n)\\
&= (abx_1,\dots,abx_n)\\
&= \big(a(bx_1),\dots,a(bx_n)\big)\\
&= a(bx_1,\dots,bx_n) \qquad = a(bx),
\end{aligned}
$$

where the second and fourth equations above hold because of the definition of the scalar-multiplication in $\mathbf{F}^n$, and the third equation holds because of the associativity in $\mathbf{F}$.

14. Show that $1x = x$ for all $x \in \mathbf{F}^n$.

Solution:

Let $x = (x_1,\dots,x_n)$, we have

$$
\begin{aligned}
1x &= 1(x_1,\dots,x_n)\\
&= (1x_1,\dots,1x_n)\\
&= (x_1,\dots,x_n)\\
&= x,
\end{aligned}
$$

where the second and fourth equations hold because of the definition of scalar-multiplication in $\mathbf{F}^n$, and the third equation hold because of the identity in $\mathbf{F}$.

15. Show that $\lambda(x+y)=\lambda x+\lambda y$ for all $\lambda\in\mathbf{F}$ and all $x,y\in\mathbf{F}^n$.

Solution:

Let $x=(x_1,\ldots,x_n)$ and $y=(y_1,\ldots,y_n)$, we have

$$\begin{aligned}\lambda(x+y)&=\lambda\big((x_1,\ldots,x_n)+(y_1,\ldots,y_n)\big)\\&=\lambda(x_1+y_1,\ldots,x_n+y_n)\\&=\big(\lambda(x_1+y_1),\ldots,\lambda(x_n+y_n)\big)\\&=(\lambda x_1+\lambda y_1,\ldots,\lambda x_n+\lambda y_n)\\&=(\lambda x_1,\ldots,\lambda x_n)+(\lambda y_1,\ldots,\lambda y_n)\\&=\lambda(x_1,\ldots,x_n)+\lambda(y_1,\ldots,y_n)\\&=\lambda x+\lambda y,\end{aligned}$$

where the second and fifth equations hold because of definition of addition in $\mathbf{F}^n$, the third and sixth equations hold because of the definition of scalar-multiplication in $\mathbf{F}^n$, the fourth equation holds because of the distributive property in $\mathbf{F}$.

16. Show that $(a+b)x=ax+bx$ for all $a,b\in\mathbf{F}$ and all $x\in\mathbf{F}^n$.

Solution:

Let $x=(x_1,\ldots,x_n)$ we have

$$\begin{aligned}(a+b)x&=(a+b)(x_1,\ldots,x_n)\\&=\big((a+b)x_1,\ldots,(a+b)x_n\big)\\&=(ax_1+bx_1,\ldots,ax_n+bx_n)\\&=(ax_1,\ldots,ax_n)+(bx_1,\ldots,bx_n)\\&=a(x_1,\ldots,x_n)+b(x_1,\ldots,x_n)\\&=ax+bx,\end{aligned}$$

where the second and fifth equations hold because of the definition of scalar-multiplication in $\mathbf{F}^n$, and the fourth equation holds because of the definition of addition in F^n, and the third equation holds because of the associativity and distributive property in F, since

$$(a+b)x_i = x_i(a+b) = x_i a + x_i b = ax_i + bx_i, \qquad i = 1, \ldots, n.$$

1.2 Definition of Vector Space

1. Prove that $-(-v) = v$ for every $v \in \mathrm{V}$.

Solution:

Let $-(-v) = w$, since w is the additive invers of $-v$, we have

$$w + (-v) = 0,$$

add by v in the both sides of the equation above, we have

$$w + (-v) + v = v,$$

we can imply

$$w = v,$$

since

$$(-v) + v = 0.$$

2. Suppose $a \in \mathrm{F}, v \in \mathrm{V}$, and $av = 0$. Prove that $a = 0$ or $v = 0$.

Solution:

If $a = 0$, then we solve the problem, therefore suppose $a \neq 0$, since $a \in \mathrm{F}$, it have multiplicative inverse, denote it to be a^{-1}, *i.e.*

$$aa^{-1} = a^{-1}a = 1,$$

from $av = 0$, we have

$$a^{-1}av = a^{-1}0,$$

from the left-hand side of the equation above we have

$$a^{-1}av = (a^{-1}a)v = 1v = v,$$

from the right-hand side of the equation above we have

$$a^{-1}0 = 0,$$

therefore we imply that

$$v = 0,$$

which is as desired.

3. Suppose $v, w \in \mathrm{V}$. Explain why there exists a unique $x \in \mathrm{V}$ such that $v + 3x = w$.

Solution:

From $v + 3x = w$ we can imply that

$$\frac{1}{3}(v - w) + x = 0,$$

which means x is the additive inverse of $\frac{1}{3}(v - w)$, thus x is unique, since **Every element in a vector space has a unique additive inverse.**

4. The empty set is not a vector space. The empty set fails to satisfy only one of the requirements listed in 1.19. Which one?

Solution:

The **Additive identity,** since the additive identity requires at least one element 0 exists in the vector space.

5. Show that in the definition of a vector space (1.19), the additive inverse condition can be replaced with the condition that

$$0v = 0 \text{ for all } v \in \mathrm{V}.$$

Here the 0 on the left side is the number 0, and the 0 on the right side is the additive identity of V. (The phrase "a condition can be replaced" in a definition means that the collection of objects satisfying the definition is unchanged if the original condition is replaced with the new condition.)

Solution:

Suppose v is a element in such set, We have

$$\begin{aligned} v + 0 &= 1v + 0v \\ &= (1+0)v \\ &= 1v \\ &= v, \end{aligned}$$

where the second equaiton holds because of the distributive properties of this set, and the fourth equation holds because of the multiplicative identity of this set.

Thus, the condition $0v = 0$ is same with $v + 0 = v$ if a set satisfies other conditions listed in 1.19.

6. Let ∞ and $-\infty$ denote two distinct objects, neither of which is in $\mathbb{R}$. Define an addition and scalar multiplication on $\mathbb{R} \cup \{\infty\} \cup \{-\infty\}$ as you could guess from the notation. Specifically, the sum and product of two real number is as usual, and for $t \in \mathbb{R}$ define

$$t\infty = \begin{cases} -\infty & \text{if } t < 0, \\ 0 & \text{if } t = 0, \\ \infty & \text{if } t > 0, \end{cases} \qquad t(-\infty) = \begin{cases} \infty & \text{if } t < 0, \\ 0 & \text{if } t = 0, \\ -\infty & \text{if } t > 0, \end{cases}$$

$$t + \infty = \infty + t = \infty, \qquad t + (-\infty) = (-\infty) + t = -\infty,$$

$$\infty + \infty = \infty, \qquad (-\infty) + (-\infty) = -\infty, \qquad \infty + (-\infty) = 0.$$

Is $\mathbb{R} \cup \{\infty\} \cup \{-\infty\}$ a vector space over $\mathbb{R}$? Explain.

Solution:

Yes. Because it satisfies all condition that the vector space should satisfy. For example let's check the commutative, *i.e.*,

$$u + v = v + u, \qquad \text{for all } u, v \in \mathbb{R} \cup \{\infty\} \cup \{-\infty\},$$

- If $u, v \in \mathbb{R}$, the equation holds;
- otherwise, one of u and v should be ∞ or $-\infty$, since we have

$$t + \infty = \infty + t = \infty, \qquad t + (-\infty) = (-\infty) + t = -\infty,$$

which means the equation holds.

1.3 Subspaces

1. For each of the following subsets of $\mathbb{F}^3$, determine whether it is a subspace of $\mathbb{F}^3$;
   (a) $\{(x_1, x_2, x_3) \in \mathbb{F}^3 : x_1 + 2x_2 + 3x_3 = 0\}$;
   (b) $\{(x_1, x_2, x_3) \in \mathbb{F}^3 : x_1 + 2x_2 + 3x_3 = 4\}$;
   (c) $\{(x_1, x_2, x_3) \in \mathbb{F}^3 : x_1x_2x_3 = 0\}$;
   (d) $\{(x_1, x_2.x_3) \in \mathbb{F}^3 : x_1 = 5x_3\}$.

Solution:

For all solution below, we suppose u and v are the element of corresponded subsets. and a to be a scalar, *i.e.*, $a \in \mathbb{F}$, and denote V to be the corresponded subsets.

(a) $u + v = (u_1 + v_1, u_2 + v_2, u_3 + v_3)$ and we have

$$(u_1 + v_1) + 2(u_2 + v_2) + 3(u_3 + v_3) = (u_1 + 2u_2 + 3u_3) + (v_1 + 2v_2 + 3v_3) = 0 + 0 = 0,$$

therefore

$$u + v \in V.$$

Also we have $au = (au_1, au_2, au_3)$ which holds the equation

$$au_1 + 2au_2 + 3au_3 = a(u_1 + 2u_2 + eu_3) = 0,$$

therefore

$$au \in V,$$

which menas V is closed under addition and scalar multiplication, *i.e.*, **it is a subspace.**

(b) Since $u + v = (u_1 + v_1, u_2 + v_2, u_3 + v_3)$ and

$$(u_1 + v_1) + 2(u_2 + v_2) + 3(u_3 + v_3) = (u_1 + 2u_2 + 3u_3) + (v_1 + 2v_2 + 3v_3) = 8 \neq 0,$$

which means that V is not closed under addition, *i.e.*, it is **not** a subspace.

(c) It's **not** a subspace, we can give a counterexample, it is easy to verify that

$$u = (1,1,0), \qquad v = (0,1,1)$$

are in V, but $u+v=(1,2,1)$ is not in V.

(d) For $u+v=(u_1+v_1, u_2+v_2, u_3+v_3)$ we have

$$(u_1+v_1) = 5(u_3+v_3),$$

therefore

$$u+v \in V,$$

and for $au=(au_1, au_2, au_3)$ we have

$$au_1 = 5au_3,$$

implies that

$$au \in V,$$

means that V is closed under addition and scalar multiplication, *i.e.*, it is a subspace.

2. Verify all the assertions in Example 1.35.

Solution:

(a) If the set

$$V = \{(x_1, x_2, x_3, x_4) \in \mathbf{F}^4 : x_3 = 5x_4 + b\}$$

is a subspace, means it is closed under addition and scalar multiplication, *i.e.*, for $u,v \in V$, we have $u+v \in V$, implies that

$$u_3 + v_3 = 5(u_4+v_4) + b,$$

since we have $u,v \in V$, means

$$u_3 = 5u_4 + b, \qquad v_3 = 5v_4 + b,$$

we can imply that $b=0$.

In another direction, if $b=0$, if $u,v \in V$, we have

$$u_3 + v_3 = 5(u_4+v_4),$$

which means for $u+vinV$, and

$$au_3 = 5au_4,$$

which means that

$$au \in V,$$

therefore V is a subspace.

(b) The set of continuous real-valued functions on the interval $[0,1]$ is a subspace of $\mathbb{R}^{[0,1]}$.

Denote the set as $C^{[0,1]}$, since the function $0:[0,1] \to R$ defined as

$$0(x) = 0$$

is continuous, *i.e.*,

$$0 \in C^{[0,1]}.$$

Suppose $f, g \in C^{[0,1]}$, since f and g are continuous real-value functions, the $f+g$ is also a continuous real-valued function, *i.e.*,

$$f+g \in C^{[0,1]},$$

which means $C^{[0,1]}$ is closed under addtion.

let $\lambda \in \mathbb{F}$, the function λf is also a continuous function, means it exists in $C^{[0,1]}$, *i.e.*,

$$\lambda f \in C^{[0,1]},$$

means that $C^{[0,1]}$ is closed under the scalar multiplication.

Finally, we conclude that $C^{[0,1]}$ is a subspace of $\mathbb{R}^{[0,1]}$.

(c) The set of differentiable real-valued functions on R is a subspace of $\mathbb{R}^{\mathbb{R}}$.

The function $0:\mathbb{R} \to \mathbb{R}$ defined as

$$0(x) = 0$$

is differentiable, which means 0 is in the set.

And for any functions f, g in this set, since f and g are differentiable, implies that $f+g$ is differentiable, too, *i.e.*, $f+g$ is in this set.

For any scalar λ and function f in this set, we have λf is differentiable, *i.e.*, λf is in this set.

Summary, we show that the set is not empty, and is closed under additon and scalar multiplication, *i.e.*, it is a subspace.

(d) The set of differentiable real-valued functions f on the interval $(0,3)$ such that $f'(2)=b$ is a subspace of $\mathbf{R}^{(0,3)}$ if and only if $b=0$.

If this set is a subspace, means that function 0 defined as $0(x)=0$ is in this set, and we have $0'(x)=0$, which implies that $b=0$.

Now suppose $b=0$, it is easy to verify that function 0 is in this set, and for functions f and g which are in this set, we have $f+g$ is differentiable, and

$$(f+g)'(2)=f'(2)+g'(2)=0$$

which means $f+g$ is in this set, too.

and for any scalar λ, the function λf is also differentiable, and we have

$$(\lambda f)'(2)=\lambda f'(2)=0$$

which means that λf is in this set.

Summary, we show that this set is not empty, and is closed under addition and scalar multiplication, we can conclude that this set is a subspace.

(e) The set of all sequences of complex numbers with limit 0 is a subspace of $\mathbf{C}^{\infty}$.

Since the sequences

$$0,0,0,\ldots$$

has limit 0, which means it is in this set.

Suppose sequences u and v is in this set, we have

$$\lim_{k\to\infty} u_k = 0, \qquad \lim_{k\to\infty} v_k = 0,$$

therefore the limit of $u+v$ is

$$\lim_{k\to\infty}(u_k+v_k)=\lim_{k\to\infty}u_k+\lim_{k\to\infty}v_k=0,$$

which means $u+v$ is in this set, *i.e.*, this set is closed under addition.

For any scalar $a\in\mathbf{C}$, we have

$$\lim_{k\to\infty}au_k=a\lim_{k\to\infty}u_k=0,$$

which means au is in this set, *i.e.*, this set is closed under scalar multiplication.

Thus, the set is a subspace.

3. Show that the set of differentiable real-valued functions f on the interval $(-4,4)$ such that $f'(-1)=3f(2)$ is a subspace of $R^{(-4,4)}$.

Solution:

The function $0:(-4,4)\to\mathbf{R}$ define as

$$0(x)=0,$$

satisfies condition, means it's in this set.

Suppose f and g are any functions in this set, which means they satisfies that

$$f'(-1)=3f(2),\qquad g'(-1)=3g(2),$$

then we have

$$(f+g)'(-1)=f'(-1)+g'(-1)=3(f(2)+g(2))=3(f+g)(2),$$

means $f+g$ is in this set.

And with any $\lambda\in\mathbf{F}$, we have

$$(\lambda f)'(-1)=\lambda f'(-1)=3\lambda f(2)=3(\lambda f)(2),$$

means that λf is in this set.

Thus, this set is a subspace.

4. Suppose $b\in\mathbf{R}$. Show that the set of continuous real-valued functions f on the interval $[0,1]$ such that $\int_0^1 f=b$ is a subspace of $\mathbf{R}^{[0,1]}$ if and

only if $b = 0$.

Solution:

Suppose the set is a subspace, and f and g are any functions in this set, therefore $f+g$ is in this set, too, means

$$\int_0^1 f = b, \qquad \int_0^1 g = b, \qquad \int_0^1 (f+g) = b,$$

since

$$\int_0^1 (f+g) = \int_0^1 f + \int_0^1 g = 2b,$$

implies $2b = b$, *i.e.*

$$b = 0.$$

In another direction, suppose $b = 0$, *i.e.*, the function in the set satisfies $\int_0^1 f = 0$.

Obviously, the function $0 : (-4, 4) \to \mathbb{R}$ defined as

$$0(x) = 0$$

satisfies the condition, therefore it is in this set.

And for any functions f and g in this set, we have

$$\int_0^1 (f+g) = \int_0^1 f + \int_0^1 g = 0,$$

means $f+g$ is in this set, too.

For any scalar λ, we have

$$\int_0^1 (\lambda f) = \lambda \int_0^1 f = 0,$$

means λf is in this set.

Thus, we conclude that if $b = 0$, the set described above is a subspace.

5. Is $\mathbb{R}^2$ a subspace of the complex vector space $\mathbb{C}^2$?

Solution:

$\mathbb{R}^2$ is **not** a subspace of $\mathbb{C}^2$, as we can give a counterexample: Suppose it is, then we know $\mathbb{R}^2$ is closed under scalar multiplication, as $(1,0) \in \mathbb{R}^2$, let $\alpha = i$, we have $i(1,0) = (i,0) \in \mathbb{R}^2$, which is contradictt.

6. (a) Is $\{(a,b,c) \in \mathbb{R}^3 : a^3 = b^3\}$ a subspace of $\mathbb{R}^3$?
(b) Is $\{(a,b,c) \in \mathbb{R}^3 : a^3 = b^3\}$ a subspace of $\mathbb{C}^3$?

Solution:

(a) Yes, since the additive identity is

$$(0,0,0),$$

and it close under addition and scalar multiplication, since

$$(a+x, b+y, c+z) \in \mathbb{R}^3 \text{ and } (a+x)^3 = (b+y)^3$$

if $(a,b,c),(x,y,z) \in \mathbb{R}^3$ and $a^3 = b^3, x^3 = y^3$, with same method, we can prove that it's closed satisfies the conditions.

(b) No, we can use the method of the previous problem.

7. Give an example of a nonempty subset U of $\mathbb{R}^2$ such that U is closed under addition and under taking additive inverse (meaning $-u \in U$ whenever $u \in U$), but U is not a subspace of $\mathbb{R}^2$.

Solution:

Let $\mathbb{N}$ denote set of all the integer number, therefore

$$\mathbb{N}^2 \subset \mathbb{R}^2,$$

and

$$(0,0) \in \mathbb{N}^2$$

is the additive identity, and it is closed under addition, since for $(a,b) \in \mathbb{N}^2$ and $(c,d) \in \mathbb{N}^2$, we have

$$(a+c, b+d) \in \mathbb{N}^2,$$

meanwhile it is under taking additive inverse, for a element $u = (a,b) \in \mathbb{N}^2$, we have

$$-u = (-a,-b) \in \mathbb{N}^2,$$

but it is not closed under scalar multiplication, if we choos $a = 0.5$ and $(1,1) \in \mathbb{N}^2$ we have

$$(0.5, 0.5) \notin \mathbb{N}^2.$$

8. Give an example of a nonempty subset U of $\mathbf{R}^2$ such that U is closed under scalar multiplication, but U is not a subspace of $\mathbf{R}^2$.

Solution:

Let the set of U to be

$$U = \{(x, y) \in \mathbf{R}^2 : x^2 = y^2\},$$

for any $u = (x, y) \in U$ and $a \in \mathbf{R}$, we have $au = (ax, ay)$ which satisfies $(ax)^2 = (ay)^2$, *i.e.*,

$$au \in U,$$

but for $u = (1, -1), v = (1, 1) \in U$, we have

$$u + v = (2, 0) \notin U.$$

9. A function $f : \mathbf{R} \to \mathbf{R}$ is called **periodic** if there exists a positive number p such that $f(x) = f(x + p)$ for all $x \in \mathbf{R}$. Is the set of periodic functions from $\mathbf{R}$ to $\mathbf{R}$ a subspace of $\mathbf{R}^{\mathbf{R}}$? Explain.

Solution:

Yes.

First, we give a basic fact: if a function has periodic p, *i.e.*, $f(x) = f(x + p)$, then we have

$$f(x) = f(x + pk), \qquad k \in \mathbf{N}.$$

Now, we can figure out that the additive identity is the periodic function $0 : \mathbf{R} \to \mathbf{R}$ which is defined as

$$0(x) = 0.$$

Suppose functions f, g are element of this set, and we have

$$f(x) = f(x + p_1), \qquad g(x) = g(x + p_2),$$

let p to be the lowest common multiple of p_1 and p_2, which means there exist $k_1, k_2 \in \mathbf{N}$, that

$$k_1 p_1 = k_2 p_2 = p.$$

according the fact we gived above, we have

$$f(x) = f(x + p_1) = f(x + p_1 k_1) = f(x + p), \qquad g(x) = g(x + p_2) = g(x + p_2 k_2) = g(x + p),$$

therefore

$$(f + g)(x) = f(x) + g(x) = f(x + p) + g(x + p) = (f + g)(x + p),$$

which means that $f + g$ is a perodic function, *i.e.*, this set is closed under addition.
Suppose f is in this set and it's periodic is p, and $\lambda \in \mathrm{R}$, we have

$$(\lambda f)(x) = \lambda f(x) = \lambda f(x + p) = (\lambda f)(x + p),$$

which means this set is closed under scalar multiplication.

10. Suppose U_1 and U_2 are subspaces of V. Prove that the intersection $U_1 \cap U_2$ is a subspace of V.

Solution:

Since U_1 and U_2 are subspaces, the additive identity 0 is in both set, which means

$$0 \in U_1 \cap U_2.$$

Now suppose $u, v \in U_1 \cap U_2$, *i.e.*, $u, v \in U_1$ and $u, v \in U_2$, we have

$$u + v \in U_1, \qquad u + v \in U_2$$

since U_1 and U_2 are subspaces, therefore we have

$$u + v \in U_1 \cap U_2,$$

which means it is closed under addition.
Suppose $u \in U_1 \cap U_2$, *i.e.*, $u \in U_1$ and $u \in U_2$, and let $a \in \mathrm{F}$, we have

$$au \in U_1, \qquad au \in U_2$$

since U_1 and U_2 are subspaces, therefore

$$au \in U_1 \cap U_2,$$

which means it is closed under scalar multiplication.
Thus, $U_1 \cap U_2$ is a subspace of V.

11. Prove that the intersection of every collection of subspaces of V is a subspace of V.

Solution:

For convenient, Let $S_1, S_2, \ldots$ denote the subspaces of V, and $I_1, I_2, \ldots$ denote the intersection of the collections of subspaces. which means

$$I_i = S_{k_1} \cap S_{k_2} \cap \ldots.$$

It is easy to verify that the additive identity 0 is an element of I_i since 0 is an element of every subspaces of V.
Suppose $u, v \in I_i$, which means they are in every $S_{k_1}, S_{k_2}, \ldots$, since $S_{k_1}, S_{k_2}, \ldots$ are subsapces, which means $u + v$ are in every $S_{k_1}, S_{k_2}, \ldots$, *i.e.*,

$$u + v \in I_i,$$

which means I_i is closed under addition.
Suppose $a \in \mathrm{F}$, $u \in I_i$, implies that u is in every $S_{k_1}, S_{k_2}, \ldots$, since $S_{k_1}, S_{k_2}, \ldots$ are subspaces, we have au in every $S_{k_1}, S_{k_2}, \ldots$, *i.e.*

$$au \in I_i,$$

which means I_i is closed under scalar multiplication.

12. Prove that the union of two subspaces of V is a subspace of V if and only if one of the subspaces is contained in the other.

Solution:

Denote the two subspaces of V as U_1 and U_2.
First we suppose $U_1 \cup U_2$ is a subspace but neither U_1 nor U_2 is contained in the other, which means there are elements u and v with

$$u \in U_1 \text{ and } u \notin U_2, \qquad v \in U_2 \text{ and } v \notin U_1,$$

therefore

$$u, v \in U_1 \cup U_2,$$

with assumption $U_1 \cup U_2$ is a subpace, which means

$$u + v \in U_1 \cup U_2,$$

i.e.,

$$u + v \in U_1 \text{ or } u + v \in U_2.$$

If $u + v \in U_1$ with $-u \in U_1$ we have $u + v + (-u) = v \in U_1$ which is contradictt. If $u + v \in U_2$ with $-v \in U_2$ we have $u + v + (-v) = u \in U_2$ which is contradictt, too. Therefore if $U_1 \cup U_2$ is a subspace, one of the U_1 and U_2 is contained in the other.
In conversely direction, we can assume $U_1 \subseteq U_2$, implies that

$$U_1 \cup U_2 = U_2$$

which is a subspace.

13. Prove that the union of three subspaces of V is a subspace of V if and only if one of the subspaces contains the other two.

Solution:

Denote the three subspaces of V as U_1, U_2, U_3.

14. Verify the assertion in Example 1.38.

Solution:

For any $u \in U$ and $v \in W$, we have

$$u + v = (x_1 + x_2, x_1 + x_2, y_1 + x_2, y_1 + y_2) \in \{(x, x, y, z) \in \mathbf{F}^4 : x, y, z \in \mathbf{F}\},$$

and for any (x, x, y, z), we can find

$$u = (x - y + z/2, x - y + z/2, z/2, z/2) \in U, \qquad v = (y - z/2, y - z/2, y - z/2, z/2) \in W,$$

which satisfies

$$u + v = (x, x, y, z),$$

as we desired.

15. Suppose U is a subspace of V. What is $U + U$?

Solution:

As U is a subspace, it is closed under addtion, *i.e.*, for any $u, v \in U$, we have $u+v \in U$, therefore

$$U + U = U.$$

16. Is the operation of addition on the subspaces of V commutative? In other words, if U and W are subspaces of V, is $U+W = W+U$?

Solution:

Suppose $x \in U+W$, according the definition, we have

$$u \in U, \qquad v \in W,$$

with

$$x = u + v = v + u,$$

which means $x \in W+U$. With same method, we can prove that if $x \in W+U$, we have $x \in U+W$. Thus,

$$U + W = W + U.$$

17. Is the operation of addition on the subspaces of V associative? In other words, if $U_1, U_2.U_3$ are subspaces of V, is

$$(U_1 + U_2) + U_3 = U_1 + (U_2 + U_3)?$$

Solution:

For any $u_1 \in U_1$, $u_2 \in U_2$, $u_3 \in U_3$, we have

$$u_1 + u_2 + u_3 = (u_1 + u_2) + u_3 \in (U_1 + U_2) + U_3,$$

and

$$u_1 + u_2 + u_3 = u_1 + (u_2 + u_3) \in U_1 |(U_2 + U_3),$$

therefore

$$(U_1 + U_2) + U_3 = U_1 + (U_2 + U_3).$$

18. Does the operation of addition on the subspaces of V have an additive identity? Which subspaces have additive inverses?

Solution:

The subspace $0 = \{0\}$ is an addition identity, since for any subspace U, we have

$$U + 0 = U.$$

Suppose U has additive inverse V, *i.e.*,

$$U + V = 0,$$

therefore for any $u \in U$, and $v \in V$, we have

$$u + v = 0,$$

since U is a subspace, means for $-u \in U$, and

$$-u + v = 0,$$

we have

$$v = 0,$$

with same method, we have $-v \in V$ and $u + (-v) = 0$, which implies

$$u = 0,$$

therefore 0 is the only one subspace who has an additive inverse.

19. Prove or give a counterexample: if U_1, U_2, W are subspaces of V such that

$$U_1 + W = U_2 + W,$$

then $U_1 = U_2$.

Solution:

We give a counterexample, let $V = \mathbf{F}^2$, and

$$W = \mathbf{F}^2, \qquad U_1 = \{(x, 0) : x \in \mathbf{F}\}, \qquad U_2 = \{(0, x) : x \in \mathbf{F}\},$$

therefore W, U_1, U_2 are subspace of V, and

$$U_1 + W = W = U_2 + W,$$

but

$$U_1 \neq U_2.$$

20. Suppose

$$U = \{(x, x, y, y) \in \mathbf{F}^4 : x, y \in \mathbf{F}\}.$$

Find a subspace W of $\mathbf{F}^4$ such that $\mathbf{F}^4 = U \oplus W$.

Solution:

Let

$$W = \{(x, y, y, 0) \in \mathbf{F}^4 : x, y, z \in F\},$$

it is easy to verify that W is a subspace, and

$$U + W = \mathbf{F}^4,$$

now we will show every element in $\mathbf{F}^4$ can be written in only one way as a sum of $u + w$: For $(x, y, z, w) \in \mathbf{F}^4$, we suppose $u = (p, p, q, q) \in U$ and $w = (m, n, n, 0) \in W$ with

$$u + w = (x, y, z, w),$$

which is same as

$$\begin{cases} p + m & = x \\ p + n & = y \\ q + n & = z \\ q & = w \end{cases},$$

we have

$$p = y - z + w, \qquad q = w, \qquad m = x - y + z - w, \qquad n = z - w,$$

which are unique as desired.

21. Suppose

$$U = \{(x, y, x + y, x - y, 2x) \in \mathbf{F}^5 : x, y \in F\}.$$

Find a subspace W of $\mathbf{F}^5$ such that $\mathbf{F}^5 = U \oplus W$.

Solution:

Let

$$W = \{(0,0,x,y,z) \in \mathbf{F}^5 : x,y,z \in \mathbf{F}\},$$

W is a subspace of $\mathbf{F}^5$, and

$$U + W = \mathbf{F}^5,$$

For any element $(x_1,x_2,x_3,x_4,x_5) \in \mathbf{F}^5$, the only way to format as sum of elements of U and W is

$$(x_1,x_2,x_1+x_2,x_1-x_2,2x_2) \in U, \qquad (0,0,x_3-x_1-x_2,x_4-x_1+x_2,x_5-2x_1) \in W.$$

22. Suppose

$$U = \{(x,y,x+y,x-y,2x) \in \mathbf{F}^5 : x,y \in \mathbf{F}\}.$$

Find three subspaces W_1,W_2,W_3 of $\mathbf{F}^5$, none of which equals $\{0\}$, such that $\mathbf{F}^5 = U \oplus W_1 \oplus W_2 \oplus W_3$.

Solution:

We give W_1,W_2,W_3 as following:

$$W_1 = \{(0,0,x,x,x) \in \mathbf{F}^5 : x \in \mathbf{F}\}$$
$$W_2 = \{(0,0,0,x,x) \in \mathbf{F}^5 : x \in \mathbf{F}\}$$
$$W_3 = \{(0,0,0,0,x) \in \mathbf{F}^5 : x \in \mathbf{F}\}$$

it is not difficult to verify that

$$U \oplus W_1 \oplus W_2 \oplus W_3 = \mathbf{F}^5,$$

since for any element $(x_1,x_2,x_3,x_4,x_5) \in \mathbf{F}^5$, we have

$$u = (x_1,x_2,x_1+x_2,x_1-x_2,2x_1) \in U$$
$$w_1 = (0,0,x_3-x_1-x_2,x_3-x_1-x_2,x_3-x_1-x_2) \in W_1$$
$$w_2 = (0,0,0,x_4-x_3+2x_2,x_4-x_3+2x_2) \in W_2$$
$$w_3 = (0,0,0,0,x_5-x_4-x_1-x_2) \in W_3.$$

23. Prove or give a counterexample: if U_1,U_2,W are subspaces of V such that

$$V = U_1 \oplus W \quad \text{and} \quad V = U_2 \oplus W,$$

then $U_1 = U_2$.

Solution:

I will give a counterexample, let $V = \mathbb{R}^2$, and

$$W = \{(x,x) \in \mathbb{R}^2 : x \in \mathbb{R}\},$$

we have

$$U_1 = \{(0,x) \in R^2 : x \in \mathbb{R}\}, \qquad U_2 = \{(x,0) \in \mathbb{R}^2 : x \in \mathbb{R}\},$$

which satisfies

$$U_1 \oplus W = V = U_2 \oplus W,$$

but

$$U_1 \neq U_2.$$

24. A function $f : \mathbb{R} \to \mathbb{R}$ is called ***even*** if

$$f(-x) = f(x)$$

for all $x \in \mathbb{R}$. A function $f : \mathbb{R} \to \mathbb{R}$ is called ***odd*** if

$$f(-x) = -f(x)$$

for all $x \in \mathbb{R}$. Let U_e denote the set of real-valued even functions on $\mathbb{R}$ and let U_o denote the set of real-valued odd functions on $\mathbb{R}$. Show that $\mathbb{R}^{\mathbb{R}} = U_e \oplus U_o$.

Solution:

It is easy to show that any real-valued function f can be expressed as a sum of an even function and an odd function, we let

$$g(x) = \frac{f(x) + f(-x)}{2}, \qquad \phi(x) = \frac{f(x) - f(-x)}{2},$$

we have $g(-x) = g(x)$ which means g is even and $\phi(-x) = -\phi(x)$ means ϕ is odd, and

$$f = g + \phi.$$

The problem is that the form of g and ϕ unique? Let suppose g is an even function and ϕ is an odd function with

$$f(x) = g(x) + \phi(x),$$

let $x = -x$, we have

$$f(-x) = g(-x) + \phi(-x) = g(x) - \phi(x),$$

from above two equations we have

$$g(x) = \frac{f(x) + f(-x)}{2}, \qquad \phi(x) = \frac{f(x) - f(-x)}{2},$$

as we desired.

2 Finit-Dimensional Vector Spaces

2.1 Span and Linear Independence

1. Suppose v_1, v_2, v_3, v_4 spans V. Prove that the list

$$v_1 - v_2, v_2 - v_3, v_3 - v_4, v_4$$

also spans V.

Solution:

If $u \in \text{span}(v_1, v_2, v_3, v_4)$, *i.e.*

$$\begin{aligned} u &= a_1 v_1 + a_2 v_2 + a_3 v_3 + a_4 v_4 \\ &= a_1(v_1 - v_2) + (a_2 + a_1)(v_2 - v_3) + (a_3 + a_2 + a_1)(v_3 - v_4) + (a_4 + a_3 + a_2 + a_1)v_4, \end{aligned}$$

therefore $u \in \text{span}(v_1 - v_2, v_2 - v_3, v_3 - v_4, v_4)$.

Now suppose $u \in \text{span}(v_1 - v_2, v_2 - v_3, v_3 - v_4, v_4)$, *i.e.*

$$\begin{aligned} u &= a_1(v_1 - v_2) + a_2(v_2 - v_3) + a_3(v_3 - v_4) + a_4 v_4 \\ &= a_1 v_1 + (a_2 - a_1)v_2 + (a_3 - a_2)v_3 + (a_4 - a_3)v_4 \end{aligned}$$

therefore $u \in \text{span}(v_1, v_2, v_3, v_4)$.

Thus we have

$$\text{span}(v_1, v_2, v_3, v_4) = \text{span}(v_1 - v_2, v_2 - v_3, v_3 - v_4, v_4).$$

2. Verify the assertions in Example 2.18.

Solution:

(a) A list v of one vector $v \in V$ is linearly independent if and only if $v \neq 0$. First suppose v is linearly independent and $v = 0$, for any $a \in \mathbf{F}$, we have $av = 0$, therefore v is linearly dependent, which is a contradict.
If $v \neq 0$, the solution of equation $av = 0$, for $a \in \mathbf{F}$ is $a = 0$, therefore v is linearly independent.

(b) A list of two vectors in V is linearly independent if and only if neither vector is a scalar multiple of the other. Suppose these two vectors are u and v. If one is a scalar multiple of another, assume that $u = \lambda v$ for $\lambda \neq 0$, we have

$$-u + \lambda v = 0$$

which means that $\{u, v\}$ is not linearly independent.

Now we assume that neither of u and v is a scalar multiple of the other and $\{u, v\}$ is linearly dependent, which menns there are nonzero a_1, a_2 with

$$a_1 u + a_2 v = 0,$$

we can easy suppose that $a_1 \neq 0$, we have

$$u = -(a_2/a_1)v$$

which is a contradict to our assumption.

(c) $(1,0,0,0),(0,1,0,0),(0,0,1,0)$ is linearly independent in F^4. Let a_1, a_2, a_3 to be the coeffecients which make the sum of $(1,0,0,0),(0,1,0,0),(0,0,1,0)$ be zero, *i.e.*,

$$a_1(1,0,0,0) + a_2(0,1,0,0) + a_3(0,0,1,0) = 0,$$

which is

$$(a_1, a_2, a_3, 0) = 0,$$

therefore

$$a_1 = 0, \qquad a_2 = 0, \qquad a_3 = 0.$$

(d) The list $1, z, \ldots, z^m$ is linearly independent in $\mathcal{P}(\mathrm{F})$ for each nonnegative integer m. When $m = 0$, this list is 1 which is linearly independent, now suppose the list is linearly independent for $m = 0, \ldots k$, let

$$a_0 + a_1 z + \cdots + a_{k+1} z^{k+1} = 0,$$

let $z = 0$ we have $a_0 = 0$, the equation above is same with

$$a_1 z + \cdots + a_{k+1} z^{k+1} = z(a_1 + \cdots + a_{k+1} z^k) = 0,$$

which is same that

$$a_1 + \cdots + a_{k+1} z^k = 0,$$

since $1, \ldots, z^k$ is linearly independent as we assumed, we have

$$a_1 = \cdots = a_{k+1} = 0,$$

with

$$a_0 = 0,$$

we conclude that $1, \ldots, z^{k+1}$ is linearly independent.

3. Find a number t such that

$$(3, 1, 4), (2, -3, 5), (5, 9, t)$$

is not linearly independent in $\mathbf{R}^3$.

Solution:

Let $t = 2$, we have

$$3(3, 1, 4) + (-2)(2, -3, 5) + (-1)(5, 9, 2) = (0, 0, 0),$$

which means the list is not linearly independent.

4. Verify the assertion in the second bullet point in Example 2.20.

Solution:

The list $(2, 3, 1), (1, -1, 2), (7, 3, c)$ is linearly dependent in $\mathbf{F}^3$ if and only if $c = 8$, as you should verify.

If $c = 8$, with the first bullet point we know it is linearly dependent.

Now suppose $c \neq 8$ and it is linearly dependent, let a_1, a_2, a_3 be the coeffecients with

$$a_1(2, 3, 1) + a_2(1, -1, 2) + a_3(7, 3, c) = 0,$$

which is same as

$$\begin{cases} 2a_1 + a_2 + 7a_3 & = 0 \\ 3a_1 - a_2 + 3a_3 & = 0 \\ a_1 + 2a_2 + ca_3 & = 0 \end{cases},$$

the first equation multiple by 7 add the second equation multiple by -3 add the third equation multiple by -1, we have

$$(40-5c)a_3 = 0,$$

since $c \neq 8$, we have $a_3 = 0$, substitue it to the equations above, we have $a_1 = a_2 = 0$, therefore it's linearly independent, which is contradict.

5. (a) Show that if we think of $\mathbb{C}$ as a vector space over $\mathbb{R}$, then the list $(1+i, 1-i)$ is linearly independent.

   (b) Show that if we think of $\mathbb{C}$ as a vector space over $\mathbb{C}$, then the list $(1+i, 1-i)$ is linearly dependent.

Solution:

(a) Suppose $a_1, a_2 \in \mathbb{R}$ with

$$a_1(1+i) + a_2(1-i) = 0,$$

it is same with

$$\begin{cases} a_1 + a_2 & = 0 \\ a_1 - a_2 & = 0 \end{cases},$$

which has only solution

$$a_1 = 0, \qquad a_2 = 0,$$

thus it is linearly independent.

(b) We can find $a_1 = 1+i, a_2 = 1-i \in \mathbb{C}$, that

$$a_1(1+i) + a_2(1-i) = 2i - 2i = 0,$$

therefore it is linearly dependent.

6. Suppose v_1, v_2, v_3, v_4 is linearly independent in V. Prove that the list

$$v_1 - v_2, v_2 - v_3, v_3 - v_4, v_4$$

is also linearly independent.

Solution:

Assume that $v_1 - v_2, v_2 - v_3, v_3 - v_4, v_4$ is linearly dependent, there are nonzero a_1, a_2, a_3, a_4 with

$$a_1(v_1 - v_2) + a_2(v_2 - v_3) + a_3(v_3 - v_4) + a_4 v_4 = 0,$$

which is same with

$$a_1 v_1 + (a_2 - a_1)v_2 + (a_3 - a_2)v_3 + (a_4 - a_3)v_4 = 0,$$

since v_1, v_2, v_3, v_4 is linearly independent, therefore

$$\begin{cases} a_1 & = 0 \\ a_2 - a_1 & = 0 \\ a_3 - a_2 & = 0 \\ a_4 - a_3 & = 0 \end{cases},$$

which means

$$a_1 = a_2 = a_3 = a_4 = 0,$$

which is a contradict to our assumption (that a_1, a_2, a_3, a_4 are nonzero).

7. Prove or give a counterexample: If $v_1, v_2, \ldots, v_m$ is a linearly independent list of vectors in V, then

$$5v_1 - 4v_2, v_2, v_3, \ldots, v_m$$

is linearly independent.

Solution:

Assume that it is linearly dependent, therefore there are nonzero $a_1, \ldots, a_m$ that

$$a_1(5v_1 - 4v_2) + a_2 v_2 + \cdots + a_m v_m = 0,$$

which can be formed as

$$5a_1 v_1 + (a_2 - 4a_1)v_2 + a_3 v_3 + \cdots + a_m v_m = 0,$$

since $v_1, \ldots, v_m$ is linearly independent, we have

$$\begin{cases} 5a_1 & = 0 \\ a_2 - 4a_1 & = 0 \\ a_3 & = 0 \\ \vdots \\ a_m & = 0 \end{cases},$$

from the first equaiton $5a_1 = 0$ we have $a_1 = 0$, substitute it to the second equation $a_2 - 4a_1 = 0$, we have $a_2 = 0$, therefore

$$a_1 = a_2 = \cdots = a_m = 0,$$

which is a contradict to our assumption.
Thus, this list is linearly independent.

8. Prove or give a counterexample: If $v_1, v_2, \ldots, v_m$ is a linearly independent list of vectors in v and $\lambda \in \mathrm{F}$ with $\lambda \neq 0$, then $\lambda v_1, \lambda v_2, \ldots, \lambda v_m$ is linearly independent.

Solution:

Suppose $a_1, \ldots, a_m \in \mathrm{F}$ that

$$a_1 \lambda v_1 + \cdots + a_m \lambda v_m = 0,$$

since $v_1, \ldots, v_m$ is linearly independent, means

$$a_i \lambda = 0, \qquad i = 1, \ldots, m,$$

as $\lambda \neq 0$, we have

$$a_i = 0, \qquad i = 1, \ldots, m,$$

thus, the list is linearly independent.

9. Prove or give a counterexample: If $v_1, v_2, \ldots, v_m$ and $w_1, \ldots, w_m$ are linearly independent lists of vectors in V, then $v_1 + w_1, \ldots, v_m + w_m$ is linearly independent.

Solution:

Let

$$v_1 = (1, 0), \qquad v_2 = (0, 1), \qquad w_1 = (0, 1), \qquad w_2 = (1, 0),$$

it is easy to verify that v_1, v_2 is linearly independent and w_1, w_2 is linearly independent, but $v_1 + w_1, v_2 + w_2$ which is

$$(1, 1), (1, 1)$$

is linearly dependent.

10. Suppose $v_1, \ldots, v_m$ is linearly independent in V and $W \in V$. Prove that if $v_1 + w, \ldots, v_m + w$ is linearly dependent, then $w \in \operatorname{span}(v_1, \ldots, v_m)$.

Solution:

Since $v_1 + w, \ldots, v_m + w$ are linearly dependent, which means there are nonzero $a_1, \ldots, a_m$ with

$$a_1(v_1 + w) + \cdots + a_m(v_m + w) = 0,$$

let $S = a_1 + \cdots + a_m$, the equation above is same with

$$a_1 v_1 + \cdots + a_m v_m + Sw = 0,$$

we have $S \neq 0$, or the equation above will be same with

$$a_1 v_1 + \cdots + a_m v_m = 0,$$

and $v_1, \ldots v_m$ is linearly independent, implies

$$a_1 = \cdots = a_m = 0,$$

which is a contradict with our assumption. Since $S \neq 0$, we can formed w as

$$w = \frac{-a_1}{S} v_1 + \cdots + \frac{-a_m}{S} v_m,$$

i.e.,

$$w \in \operatorname{span}(v_1, \ldots, v_m).$$

11. Suppose $v_1, \ldots, v_m$ is linearly independent in V and $w \in V$. Show that

$v_1, \ldots, v_m, w$ is linearly independent if and only if

$$w \notin \operatorname{span}(v_1, \ldots, v_m).$$

Solution:

First we suppose that $v_1, \ldots, v_m, w$ is linearly independent and $w \in \operatorname{span}(v_1, \ldots, v_m)$, therefore

$$w = a_1 v_1 + \cdots + a_m v_m,$$

i.e.,

$$a_1 v_1 + dots + a_m v_m - w = 0,$$

which means $v_1, \ldots, v_m, w$ is linearly dependent, which is contradict. Now we suppose $w \notin \operatorname{span}(v_1, \ldots, v_m)$ and $v_1, \ldots, v_m, w$ is linearly dependent, therefore there are nonzero $a_1, \ldots, a_m, b$ that

$$a_1 v_1 + \cdots + a_m v_m + bw = 0,$$

we have $b \neq 0$, or the equation above will be

$$a_1 v_1 + \cdots + a_m v_m = 0,$$

as $v_1, \ldots, v_m$ is linearly independent, implies that

$$a_1 = \cdots = a_m = 0,$$

which is a contradict, since $b \neq 0$, we have

$$w = \frac{-a_1}{b} v_1 + \cdots + \frac{-a_m}{b} v_m,$$

which means $w \in \operatorname{span}(v_1, \ldots, v_m)$, which is a contradict.

12. Explain why no list of four polynomials spans $\mathcal{P}_4(\mathbf{F})$.

Solution:

We know that the list $1, z, z^2, z^3, z^4$ is linearly independent and they are in $\mathcal{P}_4(\mathbf{F})$, with the fact that

Length of linearly independent list $\leq$ *Length of span list*.

We know the length of span list of $\mathcal{P}_4(\mathbf{F})$ is at least 5.

13. Prove that V is infinite-dimensional if and only if there is a sequence $v_1, v_2, \ldots$ of vectors in V such that $v_1, \ldots, v_m$ is linearly independent for every positive integer m.

Solution:

Firts we will show that if V is infinite-dimensional, then such a list exists: we can use induction method in m,

- If $m = 1$, there is non-zero elements in V, we choose $v_1 \neq 0 \in V$, therefore v_1 is linearly independent;
- Now suppose for $m = 1, \ldots, k$, we can find $v_1, \ldots, v_k$ is linearly independent;
- For $m = k+1$, we can find $v_{k+1} \in V$ which can not be formed as linear combination of $v_1, \ldots, v_k$, or we have $\text{span}(v_1, \ldots, v_k) = V$, which means V is finit-dimensional, is contradict. Since v_{k+1} can not be formed as linear combination of $v_1, \ldots, v_k$, means $v_1, \ldots, v_k, v_{k+1}$ is linearly independent.

Now we will show the inversely direction, that if such a list exist, that V is infinite-dimensional: As fact it is easy, suppose that V is finite-dimensional and $v_1, \ldots, v_k$ spans V, therefore there is not linearly independent list with length greater than k, which is contradict to our assumption.

14. Prove that $\mathbf{F}^\infty$ is infinite-dimensional.

Solution:

15. Prove that the real vector space of all continuous real-valued functions on the interval $[0,1]$ is infinite-dimensional.

16. Suppose $p_0, p_1, \ldots, p_m$ are polynomials in $\mathcal{P}_m(\mathbf{F})$ such that $p_j(2) = 0$ for each j. Prove that $p_0, p_1, \ldots, p_m$ is not linearly independent in $\mathcal{P}_m(\mathbf{F})$.

2.2 Bases

1. Find all vector spaces that have exactly one basis.

Solution:

The only vector space that has exactly one basis is $\{0\}$ and it is easy to verify.

Now suppose there is a vector space $U \neq \{0\}$ that has only one basis, assume the basis is

$$v_1, \ldots, v_m.$$

it is easy to verify that $v_i \neq 0$, and we will show that

$$v_1 + v_2, v_2, \ldots, v_m$$

is also a basis of U: let

$$a_1(v_1 + v_2) + a_2 v_2 + \cdots + a_m v_m = 0,$$

which is same with

$$a_1 v_1 + (a_1 + a_2) v_2 + \cdots + a_m v_m = 0,$$

since $v_1, \ldots, v_m$ is basis therefore they are linearly independent, *i.e.*,

$$a_1 = \cdots = a_m = 0,$$

which means $v_1 + v_2, v_1, \ldots, v_m$ is linearly independent.

since $v_1, \ldots, v_m$ spans U, means for any $u \in U$, we have

$$u = a_1 v_1 + \cdots + a_m v_m = a_1(v_1 + v_2) + (a_2 - a_1) v_2 + \cdots + a_m v_m,$$

means $v_1 + v_2, v_2, \ldots, v_m$ spans U.

Thus $v_1 + v_2, v_2, \ldots, v_m$ is basis of U, but it is different with $v_1, \ldots, v_m$.

2. Verify all the assertions in Example 2.28.

Solution:

(a) The list $(1,0,\ldots,0), (0,1,0,\ldots,0), (0,\ldots,0,1)$ is a basis of $\mathbf{F}^n$, called the **standard basis** of $\mathbf{F}^n$.

For any $(x_1, \ldots, x_n) \in \mathbf{F}^n$, we can find

$$(x_1, \ldots, x_n) = x_1(1,0,\ldots) + \cdots + x_n(0,\ldots,1),$$

therefore these list spans $\mathbb{F}^n$.

Suppose the we have

$$a_1(1,0,\ldots,0)+\cdots+a_n(0,\ldots,1)=(a_1,\ldots,a_n)=0,$$

which means

$$a_1=\ldots=a_n=0,$$

therefore this list is linearly independent.

(b) The list $(1,2),(3,5)$ is a basis of $\mathbb{F}^2$.

For any $(x,y)\in\mathbb{F}$, we can find a_1,a_2 with

$$a_1(1,2)+a_2(3,5)=(x,y),$$

since the equations

$$\begin{cases} a_1+3a_2 & =x \\ 2a_1+5a_2 & =y \end{cases}$$

has solution

$$a_1=3y-5x, \qquad a_2=2x-y.$$

Let $x=0,y=0$, we have

$$a_1=a_2=0,$$

means this list is linearly independent.

Thus this list is a basis of $\mathbb{F}^2$.

(c) The list $(1,2,-4),(7,-5,6)$ is linearly independent in $\mathbb{F}^3$ but is not a basis of $\mathbb{F}^3$ because it does not span $\mathbb{F}^3$.

We can prove that this list is linearly independent since the equations

$$\begin{cases} a_1+7a_2 & =0 \\ 2a_1-5a_2 & =0 \\ -4a_1+6a_2 & =0 \end{cases}$$

has only solution

$$a_1=a_2=a_3=0.$$

(d) The list $(1,1,0),(0,0,1)$ is a basis of $\{(x,x,y)\in\mathbb{F}^3 : x,y\in\mathbb{F}\}$.

It is easy to verigy that the list spans the vector space and it is linearly independent.s

(e) The list $(1,-1,0),(1,0,-1)$ is a basis of

$$\{(x,y,z) \in \mathbf{F}^3 : x+y+z=0\}.$$

It is easy to verify this list is linearly independent, now we will show that it spans the vector space: since for any (x,y,z) in this vector space, we can find

$$a_1 = -y, \qquad a_2 = -z$$

that

$$a_1(1,-1,0) + a_2(1,0,-1) = (x,y,z).$$

(f) The list $1, z, \ldots, z^m$ is a basis of $\mathcal{P}_m(\mathbf{F})$.
We have prove that $1, z, \ldots, z^m$ is linearly independent, since it also spans $\mathcal{P}_m(\mathbf{F})$, therefore it is a basis.

3. (a) Let U be the subspace of $\mathbf{R}^5$ defined by

$$U = \{(x_1, x_2, x_3, x_4, x_5) \in \mathbf{R}^5 : x_1 = 3x_2 \text{ and } x_3 = 7x_4\}.$$

Find a basis of U.

(b) Extend the basis in part (a) to a basis of $\mathbf{R}^5$.

(c) Find a subspace W of $\mathbf{R}^5$ such that $\mathbf{R}^5 = U \oplus W$.

Solution:

(a) The list $(3,1,0,0,0),(0,0,7,1,0),(0,0,0,0,1)$ is a basis of this vector space.

(b) Since $(1,0,0,0,0)$ is not in the span of previous list, we can add $(1,0,0,0,0)$; then we find $(0,0,1,0,0)$ is not in the span of the previous list, therefore

$$(3,1,0,0,0),(0,0,7,1,0),(0,0,0,0,1),(1,0,0,0,0),(0,0,1,0,0)$$

is a basis of $\mathbf{R}^5$.

(c) let $W = \operatorname{span}((1,0,0,0,0),(0,0,1,0,0))$, we have $U \oplus W = \mathbf{R}^5$.

4. Prove or disprove: there exists a basis p_0, p_1, p_2, p_3 of $\mathcal{P}_3(\mathrm{F})$ such that none of the polynomials p_0, p_1, p_2, p_3 has degree 2.

Solution:

We will show that the polynomial list

$$1, x, x^3 - x^2, x^3, x^4$$

satisfies the requirement, since none of the these polynomials has degree 2, now we will show that it is a basis: since the list has length 5, we only need to show that it is linearly independent: suppose that

$$a_1 + a_2 x + a_3(x^3 - x^2) + a_4 x^3 + a_5 x^4 = 0,$$

observed that only $a_5 x^4$ has term x^4 implies $a_5 = 0$ also we have

$$a_1 = a_2 = 0,$$

and only $a_3(x^3 - x^2)$ has term x^2 implies that $a_2 = 0$, means $a_3 = 0$, too, thus it is linearly independent as desired.

5. Suppose v_1, v_2, v_3, v_4 is a basis of V. Prove that

$$v_1 + v_2, v_2 + v_3, v_3 + v_4, v_4$$

is also a basis of V.

Solution:

We have

$$\dim V = 4,$$

means we only need to show that $v_1 + v_2, v_2 + v_3, v_3 + v_4, v_4$ is linearly independent, then it is a basis of V since it has the right length. suppose that

$$a_1(v_1 + v_2) + a_2(v_2 + v_3) + a_3(v_3 + v_4) + a_4 v_4 = 0,$$

which can be formed as

$$a_1 v_1 + (a_1 + a_2)v_2 + (a_2 + a_3)v_3 + (a_3 + a_4)v_4 = 0,$$

since v_1, v_2, v_3, v_4 is linearly independent, then we have

$$a_1 = 0, \quad a_1 + a_2 = 0, \quad a_2 + a_3 = 0, \quad a_3 + a_4 = 0,$$

which implies

$$a_1 = a_2 = a_3 = a_4 = 0,$$

as we desired.

6. Prove or give a counterexample: If v_1, v_2, v_3, v_4 is a basis of V and U is a subspace of V such that $v_1, v_2 \in U$ and $v_3 \notin U$ and $v_4 \notin U$, then v_1, v_2 is a basis of U.

Solution:

Consider $U = \text{span}(v_1, v_2, v_3 + v_4)$, we will show that U satisfies the conditions above, but v_1, v_2 is not the basis of U.
It is obviousely that $v_1, v_2 \in U$, soppuse that $v_3 \in U$, we have $v_4 \in U$, means there is a linearly independent list with length 4, *i.e.*, v_1, v_2, v_3, v_4 but the span list has length 3, which is imposible, therefore

$$v_3 \notin U,$$

with same method, we have

$$v_4 \notin U,$$

thus U satisfies the condition, and we know that the element $v_3 + v_4$ can not be formed as linear combination of v_1 and v_2, therefore v_1, v_2 is not a basis of U.

7. Suppose U and W are subspaces of V such that $V = U \oplus W$. Suppose also that $u_1, \ldots, u_m$ is a basis of U and $w_1, \ldots, w_n$ is a basis of W. Prove that

$$u_1, \ldots, u_m, w_1, \ldots, w_n$$

is a basis of V.

Solution:

First we will show that

$$V = \text{span}(u_1, \ldots, u_m, w_1, \ldots, w_n),$$

since $V = U \oplus W$, we suppose that $p \in V$, therefore p can be formed as

$$p = u + w, \qquad u \in U, w \in W,$$

and with $u_1, \ldots u_m$ is basis of U and u is an element of U, we have

$$u = a_1 u_1 + \cdots + a_m u_m,$$

and simily we have

$$w = b_1 w_1 + \cdots + b_n w_n,$$

therefore

$$p = a_1 u_1 + \cdots + a_m u_m + b_1 w_1 + \cdots + b_n w_n,$$

means

$$p \in \text{span}\,(u_1, \ldots, u_m, w_1, \ldots, w_n)\,.$$

Then we consider p as an element of the span of list $u_1, \ldots, u_m, w_1, \ldots, w_n$, *i.e.*,

$$p = a_1 u_1 + \cdots + a_m u_m + b_1 w_1 + \cdots + b_n w_n,$$

since $u_1, \ldots, u_m$ is basis of U, we have

$$a_1 u_1 + \cdots + a_m u_m \in U,$$

let it to be u, *i.e.*,

$$u = a_1 u_1 + \cdots + a_m u_m \in U,$$

similarly we have

$$w = b_1 w_1 + \cdots + b_n w_n \in W,$$

therefore

$$p \in U + W = V,$$

thus we have

$$V = \text{span}\,(u_1, \ldots, u_m, w_1, \ldots, w_n)\,.$$

Since $U + W$ is a direct sum and therefore $u_1, \ldots, u_m, w_1, \ldots, w_n$ is linearly independent, implies that

$$u_1, \ldots, u_m, w_1, \ldots, w_n$$

is a basis of V.

2.3 Dimension

1. Suppose V is finite-dimensional and U is a subspace of V such that $\dim U = \dim V$. Prove that $U = V$.
2. Show that the subspaces of $\mathbb{R}^2$ are precisely $\{0\}$, $\mathbb{R}^2$, and all lines in $\mathbb{R}^2$ through the origin.
3. Show that the subspaces of $\mathbb{R}^3$ are precisely $\{0\}$, $\mathbb{R}^3$, and all lines in $\mathbb{R}^3$ through the origin, and all planes in $\mathbb{R}^3$ through the origin.
4. (a) Let $U = \{p \in \mathcal{P}_4(\mathbb{F}) : p(6) = 0.\}$. Find a basis of U.
   (b) Extend the basis in part (a) to a basis of $\mathcal{P}_4(\mathbb{F})$.
   (c) Find a subspace W of $\mathcal{P}_4(\mathbb{F})$ such that $\mathcal{P}_4(\mathbb{F}) = U \oplus W$.
5. (a) Let $U = \{p \in \mathcal{P}_4(\mathbb{R}) : p''(6) = 0\}$.Find a basis of U.
   (b) Extend the basis in part (a) to a basis of $\mathcal{P}_4(\mathbb{R})$.
   (c) Find a subspace W of $\mathcal{P}_4(\mathbb{R})$ such that $\mathcal{P}_4(\mathbb{R}) = U \oplus W$.
6. (a) Let $U = \{p \in \mathcal{P}_4(\mathbb{R}) : p(2) = p(5) = p(6)\}$.Find a basis of U.
   (b) Extend the basis in part (a) to a basis of $\mathcal{P}_4(\mathbb{R})$.
   (c) Find a subspace W of $\mathcal{P}_4(\mathbb{R})$ such that $\mathcal{P}_4(\mathbb{R}) = U \oplus W$.
7. (a) Let $U = \{p \in \mathcal{P}_4(\mathbb{R}) : \int_{-1}^{1} p = 0\}$.Find a basis of U.
   (b) Extend the basis in part (a) to a basis of $\mathcal{P}_4(\mathbb{R})$.
   (c) Find a subspace W of $\mathcal{P}_4(\mathbb{R})$ such that $\mathcal{P}_4(\mathbb{R}) = U \oplus W$.
8. Suppose $v_1, \ldots, v_m$ is linearly independent in V and $w \in V$. Prove that
   $$\dim \operatorname{span}(v_1 + w, \ldots, v_m + w) \geq m - 1.$$
9. Suppose $p_0, p_1, \ldots, p_m \in \mathcal{P}(\mathbb{F})$ are such that each p_j has degree j. Prove that $p_0, p_1, \ldots, p_m$ is a basis of $\mathcal{P}_m(\mathbb{F})$.
10. Suppose that U and W are subspaces of $\mathbb{R}^8$ such that $\dim U = 3$, $\dim W = 5$, and $U + W = \mathbb{R}^8$. Prove that $\mathbb{R}^8 = U \oplus W$.

11. Suppose U and W are both five-dimensional subspaces of $\mathbb{R}^9$. Prove that $U \cap W \neq \{0\}$.

12. Suppose U and W are both 4-dimensional subspaces of $\mathbb{C}^6$. Prove that there exist two vectors in $U \cap W$ such that neigher of these vectors is a scalar multiple of the other.

13. Suppose $U_1, \ldots, U_m$ are finite-dimensional subspaces of V. Prove that $U_1 + \cdots + U_m$ is finite-dimensional and

$$\dim (U_1 + \cdots + U_m) \leq \dim U_1 + \cdots + \dim U_m.$$

Solution:

First We will show that **the sum of two finite-dimensional vector space is a finite-dimensional.**
Suppose U and V are these two finite-dimensional vector space, therefore they have basis, since there are span list of them and span list include basis, suppose the that $u_1, \ldots, u_n$ and $v_1, \ldots, v_m$ are basis of U and V, respectively. Now We will show that

$$u_1, \ldots, u_n, v_1, \ldots, v_m$$

spans $U + V$: for any element $p \in U + V$, we have $p = u + v, u \in U, v \in V$, and we have

$$u = a_1 u_1 + \cdots + a_n u_n, \qquad v = b_1 v_1 + \cdots + b_m v_m,$$

therefore

$$p = a_1 u_1 + \cdots + a_n u_n + b_1 v_1 + \cdots + b_m v_m$$

which is an element of span$(u_1, \ldots, u_n, v_1, \ldots, v_m)$, and we can prove that for any element in the span$(u_1, \ldots, u_n, v_1, \ldots, v_m)$, it is also a element of $U + v$, therefore

$$U + V = \text{span}(u_1, \ldots, u_n, v_1, \ldots, v_m),$$

i.e., $U + V$ is finite-dimensional.
Use the inductive method, we can prove that $U_1 + \cdots + U_m$ is finite-dimensional.

In order to prove

$$\dim(U_1 + \cdots + U_m) \leq \dim U_1 + \cdots + \dim U_m,$$

we show that for any finite-dimensional vector space U and V, we have

$$\dim(U+V) = \dim U + \dim V - \dim(U \cap V) \leq \dim U + \dim V$$

since $\dim U \cap V \geq 0$, therefore

$$\begin{aligned}\dim(U_1 + \cdots + U_m) &\leq \dim U_1 + \dim(U_2 + \cdots + U_m)\\ &\leq \ldots\\ &\leq \dim U_1 + \cdots + \dim U_m.\end{aligned}$$

14. Suppose V is finite-dimensional, with $\dim V = n \geq 1$. Prove that there exist 1-dimensional subspaces $U_1, \ldots, U_n$ of V such that

$$V = U_1 \oplus \cdots \oplus U_n.$$

Solution:

In order to show this, we prove another satetement: **For give V, we can find 1-dimensional subsapce U and $(n-1)$-dimensional W, that**

$$V = U \oplus W.$$

We suppose that list

$$v_1, \ldots, v_n$$

is basis of V, let $U = \operatorname{span}(v_1)$ and $W = \operatorname{span}(v_2, \ldots, v_m)$, From 2.34 we know that

$$V = U \oplus W,$$

continue this method, we can prove that

$$V = \operatorname{span}(v_1) + \cdots + \operatorname{span}(v_n).$$

15. Suppose $U_1, \ldots, U_m$ are finite-dimensional subspaces of V such that $U_1 + \cdots + U_m$ is direct sum. Prove that $U_1 \oplus \cdots \oplus U_m$ is finite-dimensional

and

$$\dim U_1 \oplus \cdots \oplus U_m = \dim U_1 + \cdots + \dim U_m.$$

Solution:

If we can prove that when $m = 2$, the statement hold, then then statement hold for any positive interge m.

Suppose that $u_1, \ldots, u_n$ and $v_1, \ldots, v_m$ are the basis of U_1 and U_2, respectively, then

$$u_1, \ldots, u_n, v_1, \ldots, v_m$$

spans $U_1 + U_2$, which means $U + V$ is finite-dimensional.

Since $U_1 + U_2$ is direct sum, we have $U_1 \cap U_2 = \{0\}$, *i.e.*, $\dim (U_1 \cap U_2) = 0$ therefore

$$\dim (U_1 \oplus U_2) = \dim U_1 + \dim U_2.$$

16. You might guess, by analogy with the formula for the number of elements in the union of three subsets of a finite set, that if U_1, U_2, U_3 are subspaces of a finite-dimensional vector space, then

$$\begin{aligned}\dim (U_1 + U_2 + U_3) = \dim U_1 + \dim U_2 + \dim U_3 \\ - \dim(U_1 \cap U_2) - \dim (U_1 \cap U_3) - \dim (U_2 \cap U_3) \\ + \dim (U_1 \cap U_2 \cap U3).\end{aligned}$$

Prove this or give a counterexample.

Solution:

We will give a counterexample: Consider that

$$U_1 = \{(x,x) \in \mathrm{R}^2 : x \in \mathrm{R}\}, \quad U_2 = \{(x,0) \in \mathrm{R}^2 : x \in \mathrm{R}\}, \quad U_3 = \{(0,x) \in \mathrm{R}^2 : x \in \mathrm{R}\},$$

we have

$$\dim U_1 = \dim U_2 = \dim U_3 = 1,$$

and since $U_1 + U_2 + U_3 = \mathrm{R}^2$, we have

$$\dim (U_1 + U_2 + U_3) = 2,$$

and

$$U_1 \cap U_2 = U_1 \cap U_3 = U_2 \cap U_3 = U_1 \cap U_2 \cap U_3 = \{0\},$$

we have

$$\dim(U_1 \cap U_2) = \dim(U_1 \cap U_3) = \dim(U_2 \cap U_3) = \dim(U_1 \cap U_2 \cap U_3) = 0,$$

therefore the left-hand side of the equation is 2, the right-hand side is 2, since

$$3 \neq 2,$$

we conclude that the equation does not hold all the time.

3 Linear Maps

3.1 The Vector Space of Linear Map

1. Suppose $b, c \in \mathbf{R}$. Define $T : \mathbf{R}^3 \to \mathbf{R}^3$ by

$$T(x, y, z) = (2x - 4y + 3z + b, 6x + cxyz).$$

   Show that T is linear if and only if $b = c = 0$.

2. Suppose $b, c \in \mathbf{R}$. Define $T : \mathcal{P}(\mathbf{R}) \to \mathbf{R}^2$ by

$$Tp = \left(3p(4) + 5p'(6) + bp(1)p(2), \int_{-1}^{2} x^3 p(x) dx + c \sin p(0)\right).$$

   Show that T is linear if and only if $b = c = 0$.

3. Suppose $T \in \mathcal{L}(\mathbf{F}^n, \mathbf{F}^m)$. Show that there exist scalars $A_{j,k} \in \mathbf{F}$ for $j = 1, \ldots, m$ and $k = 1, \ldots, n$ such that

$$T(x_1, \ldots, x_n) = (A_{1,1}x_1 + \cdots + A_{1,n}x_n, \ldots, A_{m,1}x_1 + \cdots + A_{m,n}x_n)$$

   for every $(x_1, \ldots, x_n) \in \mathbf{F}^n$.

4. Suppose $T \in \mathcal{L}(V, W)$ and $v_1, \ldots, v_m$ is a list of vectors in V such that $Tv_1, \ldots, Tv_m$ is a linearly independent list in W. Prove that $v_1, \ldots, v_m$ is linearly independent.

5. Prove the assertion in 3.7.

6. Prove the assertion in 3.9.

7. Show that every linear map from a 1-dimensional vector space to itself is multiplication by some scalar. More precisely, prove that if $\dim V = 1$ and $T \in \mathcal{L}(V, V)$, then there exists $\lambda \in \mathbf{F}$ such that $Tv = \lambda v$ for all $v \in V$.

8. Give an example of a function $\phi : \mathbf{R}^2 \to \mathbf{R}$ such that

$$\phi(av) = a\phi(v)$$

   for all $a \in \mathbf{R}$ and all $v \in \mathbf{R}^2$ but ϕ is not linear.

9. Give an example of a function $\phi : \mathbf{C} \to \mathbf{C}$ such that

$$\phi(w + z) = \phi(w) + \phi(z)$$

for all $w, z \in \mathbf{C}$ but ϕ is not linear.

10. Suppose U is a subspace of V with $U \neq V$. Suppose $S \in \mathcal{L}(U, W)$ and $S \neq 0$(which means that $Su \neq 0$ for some $u \in U$). Define $T : V \to W$ by

$$Tv = \begin{cases} Sv & \text{if } v \in U, \\ 0 & \text{if } v \in V \text{ and } v \notin U. \end{cases}$$

Prove that T is not a linear map on V.

11. Suppose V is finite-dimensional. Prove that every linear map on a subspace of V can be extended to a linear map on V. In other words, show that if U is a subspace of V and $S \in \mathcal{L}(U, W)$, then there exists $T \in \mathcal{L}(V, W)$ such that $Tu = Su$ for all $u \in U$.

Solution:

Since U is a subspace of U, let $u_1, \ldots, u_n$ to be the basis of U, and it can extended to be basis of V, suppose it is

$$u_1, \ldots, u_n, w_1, \ldots, w_m,$$

We define $T : V \to W$:

$$T(a_1u_1 + \cdots + a_nv_n + b_1w_1 + \cdots + b_mw_m) = S(a_1u_1 + \cdots + a_nu_n) + b_1w_1 + \cdots + b_mw_m,$$

where $a_1, \ldots, a_n, b_1, \ldots, b_m$ are arbitrary elements of $\mathbf{F}$, since $u_1, \ldots, u_n, w_1, \ldots, w_m$ is a basis of V, therefore the equation define a function T from V to V.
Now for $u \in U$, we have

$$u = c_1u_1 + \cdots + c_nu_n,$$

therefore

$$Tu = Su.$$

Now we will show that T is linear map, for $u, v \in V$, suppose that

$$u = a_1u_1 + \cdots + a_nu_n + b_1w_1 + \cdots + b_mw_m,$$

and

$$v = c_1u_1 + \cdots + c_nu_n + d_1w_1 + \cdots + d_mw_m,$$

we have

$$
\begin{aligned}
T(u+v) &= T\left((a_1+c_1)u_1+\cdots+(a_n+c_n)u_n+(b_1+d_1)w_1+\cdots+(b_m+d_m)w_m\right)\\
&= S\left((a_1+c_1)u_1+\cdots+(a_n+c_n)u_n\right)+(b_1+d_1)w_1+\cdots+(b_m+d_m)w_m\\
&= S(a_1u_1+\cdots+a_nu_n)+b_1w_1+\cdots+b_mw_m\\
&\quad+S(c_1u_1+\cdots+c_nu_n)+d_1w_1+\cdots+d_mw_m\\
&= Tu+Tv,
\end{aligned}
$$

Similarly, if $\lambda \in \mathrm{F}$, and $v = a_1u_1+\ldots+a_nu_n+b_1w_1+\cdots+b_mw_m$, then

$$
\begin{aligned}
T(\lambda v) &= T\left(\lambda(a_1u_1+\ldots+a_nu_n+b_1w_1+\cdots+b_mw_m)\right)\\
&= S(\lambda a_1u_1+dots+\lambda a_nu_n)+\lambda b_1w_1+\cdots+\lambda b_mw_m\\
&= S\left(\lambda(a_1u_1+\cdots+a_nu_n)\right)+\lambda(b_1w_1+\cdots+b_mw_m)\\
&= \lambda S(a_1u_1+\cdots+a_nu_n)+\lambda(b_1w_1+\cdots+b_mw_m)\\
&= \lambda\left(S(a_1u_1+\cdots+a_nu_n)+b_1w_1+\cdots+b_mw_m\right)\\
&= \lambda Tv,
\end{aligned}
$$

therefore T is a linear map from V to V.

12. Suppose V is finite-dimensional with $\dim V > 0$, and suppose W is infinite-dimensional. Prove that $\mathcal{L}(V,W)$ is infinite-dimensional.

Solution:

Since W is infinite-dimensional, from exercise 2A-13, we know that there is a linearly independent list

$$w_1,\ldots,w_m \in W$$

for any positive m. Now consider the fucntions $T_1,\ldots,T_k : V \to W$ define as:

$$T_i(a_1v_1+\ldots+a_nv_n) = a_1w_1+\cdots+a_{n-1}w_{n-1}+a_nw_{n-1+i}, \quad i=1,\ldots,k$$

where $a_1,\ldots,a_n$ are arbitrary elements in F and $v_1,\ldots,v_n$ is basis of V, it is easy to verify that T_i is linear map from V to W, Now we will prove that $T_1,\ldots,T_k$ is linearly independent:

Suppose that

$$b_1T_1 + \cdots + b_kT_k = 0,$$

consider a special $v = v_n \in V$, we have

$$b_1T_1(v_n) + \ldots b_kT_k(v_n) = b_1w_n + \cdots + b_kw_{n+k-1},$$

since $w_n, \ldots, w_{n+k-1}$ is linearly independent, we have

$$b_1 = \cdots = b_k = 0.$$

Obeserved that we didn't the only assumption we made on k is that k is a postive integer, means that $\mathcal{L}(V, W)$ is infinite-dimensional.

13. Suppose $v_1, \ldots, v_m$ is a linearly **dependent** list of vectors in V. Suppose also $W \neq \{0\}$. Prove that there exist $w_1, \ldots, w_m \in W$ such that no $T \in \mathcal{L}(V, W)$ satisfies $Tv_k = w_k$ for each $k = 1, \ldots, m$.

Solution:

Since $W \neq \{0\}$, means there exists $w \in W$, which $w \neq 0$.

- If one of $v_1, \ldots, v_m$ is 0, assume that $v_k = 0$, consider the list $w, \ldots, w$, if such T exists, we have

  $$Tv_k = w,$$

  since $Tv_k = T0 = 0$, implies that

  $$w = 0,$$

  which is contradict.

- If none of $v_1, \ldots, v_m$ is 0, there is non-zero list $a_1, \ldots, a_m$ with

  $$a_1v_1 + \ldots + a_mv_m = 0,$$

  since $v_1, \ldots, v_m$ is linearly dependent, consider the list

  $$w_i = \begin{cases} \dfrac{1}{a_i}w & \text{if } a_i \neq 0 \\ w & \text{if } a_i = 0 \end{cases},$$

if such T exists, we have

$$T(a_1v_1+\cdots+a_mv_m)=kw,$$

where k is the number of non-zero values in $a_1,\ldots,a_m$, we have $0<k\le m$ the left-hand side term of equation above is

$$T(a_1v_1+\cdots+a_mv_m)=T0=0,$$

implies

$$kw=0,$$

means

$$w=0,$$

which is contradict, too.

Thus, we can find such $w_1,\ldots,w_m$ that no $T\in\mathcal{L}(V,W)$ satisfies the condition, notice that in the prove above should provided that W is a vector space.

14. Suppose V is finite-dimensional with $\dim V\ge 2$. Prove that there exist $S,T\in\mathcal{L}(V,V)$ such that $ST\ne TS$.

Solution:

Since $\dim V\ge 2$, we suppose that $u_1,u_2,\ldots u_m$ is basis of V, we have

$$u_2,u_1+u_2,\ldots,u_m\in V,$$

therefore there is a linear map S with

$$Su_1=u_2,Su_2=u_1+u_2,\ldots Su_m=u_m.$$

Similarly, we have

$$u_2,u_1-u_2,\ldots,u_m\in V,$$

therefore there is a linearly map T with

$$Tu_1=u_2,Tu_2=u_1-u_2,\ldots,Tu_m=u_m,$$

now we have

$$TSu_1=Tu_2=u_1-u_2,\qquad STu_1=Su_2=u_1+u_2,$$

i.e.,

$$TS \neq ST.$$

3.2 Null Spaces and Ranges

1. Give an example of a linear map T such that $\dim \text{null } T = 3$ and $\dim \text{range } T = 2$.

Solution:

Consider such a linear map $T : \mathbb{R}^5 \to \mathbb{R}^2$ defined by

$$T(x_1, x_2, x_3, x_4, x_5) = (x_1 + x_2 + x_3, x_4 + x_5),$$

we have

$$\text{null } T = \{(x_1, x_2, x_3, x_4, x_5) \in \mathbb{R}^5 : x_1 + x_2 + x_3 = 0 \text{ and } x_4 + x_5 = 0\},$$

a basis of null T is

$$(1, -1, 0, 0, 0), \qquad (1, 0, -1, 0, 0), \qquad (0, 0, 0, 1, -1),$$

therefore

$$\dim \text{null } T = 3.$$

And we can easily verify that

$$\text{range } T = \mathbb{R}^2,$$

therefore

$$\dim \text{range } T = 2.$$

Thus, the linear map T we defined above satisfies the conditions.

Remark. We can get the example from the idear of **Fundamental Theorem of Linear Maps:**

$$\dim V = \dim \text{null } T + \dim \text{range } T.$$

2. Suppose V is a vector space and $S, T \in \mathcal{L}(V, W)$ are such that

$$\text{range } S \subset T.$$

Prove that $(ST)^2 = 0$.

Solution:

Suppose $v \in V$, since S,T are linear maps from V to V, we have

$$Tv \in V,$$

and

$$STv \in \texttt{range}\ (S),$$

since we have range $S \subset \texttt{null}\ T$, we have

$$STv \in \texttt{null}\ T,$$

i.e.,

$$TSTv = 0,$$

with S is a linear map means $S0 = 0$, therefore

$$STSTv = 0,$$

for any $v \in V$, thus

$$(ST)^2 = 0.$$

3. Suppose $v_1, \ldots, v_m$ is a list of vectors in V. Define $T \in \mathcal{L}(\mathbf{F}^m, V)$ by

$$T(z_1, \ldots, z_m) = z_1 v_1 + \cdots + z_m v_m.$$

(a) What property of T correspnods to $v_1, \ldots, v_m$ spanning V?

(b) what property of T corresponds to $v_1, \ldots, v_m$ being linearly independent?

Solution:

(a) Since $v_1, \ldots, v_m$ spans V, means

$$\texttt{range}\ T = V.$$

(b) If $v_1, \ldots, v_m$ is linearly independent, means

$$\texttt{null}\ T = \{0\}.$$

4. Show taht

$$\{T \in \mathcal{L}(\mathbf{R}^5, \mathbf{R}^4) : \dim \texttt{null}\ T > 2\}$$

is not subspace of $\mathcal{L}(\mathbf{R}^5, \mathbf{R}^4)$.

Solution:

Consider the linear maps $T_1, T_2 \in \mathcal{L}(\mathbb{R}^5, \mathbb{R}^4)$ defined by

$$T_1(x_1, x_2, x_3, x_4, x_5) = (0, 0, x_3, x_4), \qquad T_2(x_1, x_2, x_3, x_4, x_5) = (0, 0, -x_3, x_4),$$

they are both linear maps, and

$$\dim \texttt{null}\ T_1 = \dim \texttt{null}\ T_2 = 3 > 2,$$

but we have

$$(T_1 + T_2)(x_1, x_2, x_3, x_4, x_5) = (0, 0, 0, x_4)$$

means

$$\dim \texttt{null}\ (T_1 + T_2) = 2$$

which is not in this set, *i.e.*, it is not a subspace.

5. Give an example of a linear map $T : \mathbb{R}^4 \to \mathbb{R}^4$ such that

$$\texttt{range}\ T = \texttt{null}\ T.$$

Solution:

Consider the function $T : \mathbb{R}^4 \to \mathbb{R}^4$ defined by

$$T(x_1, x_2, x_3, x_4) = (0, 0, x_1, x_2).$$

We will show that T satisfies the conditions:
It is easy to verify that T is a linear map. And we have

$$\texttt{range}\ T = \{(0, 0, x, y) \in \mathbb{R}^4 : x, y \in \mathbb{R}\},$$

also we have

$$\texttt{null}\ T = \{(0, 0, x, y) \in \mathbb{R}^4 : x, y \in \mathbb{R}\},$$

i.e.,

$$\texttt{range}\ T = \texttt{null}\ T.$$

6. Prove that there does not exist a linear map $T : \mathbb{R}^5 \to \mathbb{R}^5$ such that

$$\texttt{range}\ T = \texttt{null}\ T.$$

Solution:

Assume that T is such a linear map, according to Fundamental Theorem we have

$$5 = \dim \texttt{range}\ T + \dim \texttt{null}\ T,$$

and since range $T = \texttt{null}\ T$, implies

$$\dim \texttt{range}\ T = \dim \texttt{null}\ T,$$

i.e.,

$$\dim \texttt{range}\ T = \dim \texttt{null}\ T = 2.5,$$

which is impossible since the dimension of vector space should be integer.

7. Suppose V and W are finite-dimensional with $2 \le \dim V \le \dim W$. Show that $\{T \in \mathcal{L}(V,W) : T \text{ is not injective}\}$ is not a subspace of $\mathcal{L}(V,W)$.

Solution:

Suppose $\dim V = n$, and $v_1, \ldots, v_m$ is a basis of V, since $\dim W \ge \dim V$, there exists linearly independence list $w_1, \ldots, w_n \in W$, we define the function $T_1 : V \to W$ by

$$T_1 v_1 = w_1, \quad \ldots, \quad T_1 v_{n-1} = w_{n-1}, \quad T_1 v_n = 0,$$

and $T_2 : V \to W$ by

$$T_2 v_1 = w_1, \quad \ldots, \quad T_2 v_{n-1} = 0, \quad T_2 v_n = w_n.$$

For T_1 we have

$$T_1(0) = 0, \qquad T_1(v_n) = 0,$$

means T_1 is not injective, with same method we have T_2 is not injective, now we consider $T_1 + T_2$, for any element in V, denoted by $v = a_1 v_1 + \cdots + v_n$, we have

$$\begin{aligned}(T_1 + T_2)(v) &= T_1 v + T_2 v \\ &= T_1(a_1 v_1 + \cdots + a_n v_n) + T_2(a_1 v_1 + \cdots + a_n v_n) \\ &= 2a_1 w_1 + \ldots + 2a_{n-2} w_{n-2} + a_{n-1} w_{n-1} + a_n w_n.\end{aligned}$$

Since $w_1, \ldots, w_n$ is linearly independent, means the solution of

$$(T_1 + T_2)(v) = 0$$

has only one solution

$$v = 0,$$

i.e., linear map $T_1 + T_2$ is injective, which means the set is not closed under addition, thus it is not subspace.

8. Suppose V and W are finite-dimensional with $\dim V \geq \dim W \geq 2$. Show that $\{T \in \mathcal{L}(V, W) : T$ is not surjective$\}$ is not a subspace of $\mathcal{L}(V, W)$.

Solution:

This question is quite similar with the one above, suppose that $\dim V = n$ and $v_1, \ldots, v_n$ is a basis of V, and $\dim W = m$, $w_1, \ldots, w_m$ is a basis of W. $T_1, T_2 \in \mathcal{L}(V, W)$ which defined by

$$T_1(a_1 v_1 + \cdots + a_n v_n) = a_1 w_1 + a_3 w_3 + \cdots + a_m w_m,$$

and

$$T_2(a_1 v_1 + \cdots + a_n v_n) = a_2 w_2 + a_3 w_3 + \cdots + a_m w_m,$$

where $a_1, \ldots, a_n \in \mathbf{F}$ are arbitrary.

It is easy to verify that

$$w_2 \notin \text{range } T_1, \qquad w_1 \notin \text{range } T_2,$$

i.e., T_1, T_2 are not surjective.

But for $T_1 + T_2$, we have

$$(T_1 + T_2)(a_1 v_1 + \cdots + a_n v_n) = a_1 w_1 + a_2 w_2 + 2a_3 w_3 + \cdots + 2a_m w_m,$$

since $w_1, \ldots, w_m$ is a basis of W, implies that $T_1 + T_2$ is surjective, which means the vector space defined in question is not closed under addition, *i.e.*, it is not a subspace.

9. Suppose $T \in \mathcal{L}(V, W)$ is injective and $v_1, \ldots, v_n$ is linearly independent in V. Prove that $Tv_1, \ldots, Tv_n$ is linearly independent in W.

Solution:

Assume that $Tv_1,\ldots,Tv_n$ is linearly dependent, implies there are nonzero $a_1,\ldots,a_n \in \mathbf{F}$ with

$$a_1Tv_1 + \cdots + a_nTv_n = 0,$$

as we have

$$\begin{aligned} a_1Tv_1 + \cdots + a_nTv_n &= T(a_1v_1) + \cdots + T(a_nv_n) \\ &= T(a_1v_1 + \cdots + a_nv_n), \end{aligned}$$

i.e.,

$$T(a_1v_1 + \cdots + a_nv_n) = 0,$$

since T is injective, means

$$a_1v_1 + \cdots + a_nv_n = 0,$$

with $v_1,\ldots,v_n$ is linearly independent, implies

$$a_1 = \cdots = a_n = 0,$$

which is contradict with our assumption.

10. Suppose $v_1,\ldots,v_n$ spans V and $T \in \mathcal{L}(V,W)$. Prove that the list $Tv_1,\ldots,Tv_n$ spans range T.

Solution:

This question is equivalent to prove that

$$\text{span}\,(Tv_1,\ldots,Tv_n) = \text{range } T.$$

First we show that

$$\text{span}\,(Tv_1,\ldots,Tv_n) \subseteq \text{range } T,$$

suppose $w \in \text{span}\,(Tv_1,\ldots,Tv_n)$, implies w can be written in

$$w = a_1Tv_1 + \cdots + a_nTv_n,$$

here $a_1, \ldots, a_n \in \mathbb{F}$, which means

$$\begin{aligned} w &= a_1 T v_1 + \cdots + a_n T v_n \\ &= T(a_1 v_1) + \cdots + T(a_n v_n) \\ &= T(a_1 v_1 + \cdots + a_n v_n), \end{aligned}$$

since $v_1, \ldots, v_n$ spans V, means there exists $v \in V$ with

$$v = a_1 v_1 + \cdots + a_n v_n,$$

i.e., we have

$$w = T(v),$$

means

$$w \in \texttt{range } T.$$

Second we show that

$$\texttt{range } T \subseteq \texttt{span}(T v_1, \ldots, T v_n),$$

suppose $w \in$ range T with $w = T(v), v \in V$, since $v_1, \ldots, v_n$ spans V, we can form v as

$$v = a_1 v_1 + \cdots + a_n v_n,$$

where $a_1, \ldots, a_n \in \mathbb{F}$, therefore

$$\begin{aligned} w &= T v \\ &= T(a_1 v_1 + \cdots + a_n v_n) \\ &= T(a_1 v_1) + \cdots + T(a_n v_n) \\ &= a_1 T v_1 + \cdots + a_n T v_n, \end{aligned}$$

i.e.,

$$w \in \texttt{span}(T v_1, \ldots, T v_n).$$

Thus we conclude that

$$\texttt{span}(T v_1, \ldots, T v_n) = \texttt{range } T.$$

11. Suppose $S_1, \ldots, S_n$ are injective linear maps such that $S_1 S_2 \ldots S_n$ makes sense. Prove that $S_1 S_2 \ldots S_n$ is injective.

Solution:

Assume that $S_1S_2 \ldots S_n$ is not injective, means there exists $u \neq 0$ with

$$S_1S_2 \ldots S_n u = 0,$$

since

$$S_1S_2 \ldots S_n u = S_1(S_2 \ldots S_n u)$$

and S_1 is injective, implies

$$S_2 \ldots S_n u = 0,$$

continue this iteration, we conclude that

$$S_n u = 0$$

which is contradict since S_n is injective.

12. Suppose that V is finite-dimensional and that $T \in \mathcal{L}(V, W)$. Prove that there exists a subspace U of V such that $U \cap \text{null } T = \{0\}$ and range $T = \{Tu : u \in U\}$.

Solution:

Since V is finite-dimensional, suppose $\dim \text{null } T = m$, and $v_1, \ldots, v_m$ is a basis of null T, since $v_1, \ldots, v_m$ is linearly independent list of V, it can be extended to be a basis of V, suppose that

$$v_1, \ldots, v_m, u_1, \ldots, u_n$$

to be the extended basis of V, *i.e.*, $\dim V = n + m$.
Now consider subspace U obtained by

$$U = \text{span}(u_1, \ldots, u_n).$$

It is easy to verify that $U \cap \text{null } T = \{0\}$ since $v_1, \ldots, v_m, u_1, \ldots, u_n$ is linearly independent. Now we will show that $Tu_1, \ldots, Tu_n$ is linearly independent, or we have nonzero $a_1, \ldots, a_n \in \mathbf{F}$ with

$$a_1Tu_1 + \cdots + a_nTu_n = 0,$$

implies that

$$T(a_1u_1 + \cdots + a_nu_n) = 0,$$

means

$$a_1u_1 + \cdots + a_nu_n \in \texttt{null}\ T,$$

and with $v_1, \ldots, v_m$ is basis of null T, we have

$$a_1u_1 + \cdots + a_nu_n = b_1v_1 + \cdots + b_mv_m,$$

since $v_1, \ldots, v_m, u_1, \ldots, u_n$ is linearly independent, we have

$$a_1 = \cdots = a_n = 0,$$

which is contradict.

From above we have $Tu_1, \ldots, Tu_n \in \texttt{range}\ T$ is linearly independent, and with Fundamental Theorem we have

$$\dim \texttt{range}\ T = \dim V - \dim \texttt{null}\ T = n,$$

which means $Tu_1, \ldots, Tu_n$ is a basis of range T, *i.e.*,

$$\texttt{range}\ T = \texttt{span}(Tu_1, \ldots, Tu_n),$$

since $\texttt{span}(Tu_1, \ldots, Tu_n) = \{Tu : u \in U\}$ we have

$$\texttt{range}\ T = \{Tu : u \in U\}.$$

13. Suppose T is a linear map from $\mathbf{F}^4$ to $\mathbf{F}^2$ such that

$$\texttt{null}\ T = \{(x_1, x_2, x_3, x_4) \in \mathbf{F}^4 : x_1 = 5x_2 \text{ and } x_3 = 7x_4\}.$$

Prove that T is surjective.

Solution:

We know that

$$(5, 1, 0, 0), \qquad (0, 0, 7, 1)$$

is a basis of null T, means

$$\dim \texttt{null}\ T = 2,$$

with $\dim \mathbf{F}^4 = 4$ and Fundamental Theorem we have

$$\dim \texttt{range}\ T = 4 - 2 = 2,$$

since $\dim \mathbf{F}^2 = 2$, and range $T \subseteq \mathbf{F}^2$, we have

$$\texttt{range}\ T = \mathbf{F}^2,$$

i.e., it is surjective.

14. Suppose U is a 3-dimensional subspace of $\mathbf{R}^8$ and that T is a linear map from $\mathbf{R}^8$ to $\mathbf{R}^5$ such that null $T = U$. Prove that T is surjective.

Solution:

This is similar to the question above, we can use the same method to prove it, with Fundamental Theorem we have

$$\dim \texttt{range}\ T = \dim \mathbf{R}^8 - \dim \texttt{null}\ T = 8 - 3 = 5,$$

and since

$$\texttt{range}\ T \subseteq \mathbf{R}^5, \qquad \dim \texttt{range}\ T = \dim \mathbf{R}^5$$

we have

$$\texttt{range}\ T = \mathbf{R}^5.$$

Or there is an element with $u \in \mathbf{R}^5$ and $u \notin$ range T, mean u and the basis of range T is linearly independent, implies that the dimension of $\mathbf{R}^5$ is greater than 5, which is contradict.

15. Prove that there does not exist a linear map from $\mathbf{F}^5$ to $\mathbf{F}^2$ whose null space equals

$$\{(x_1, x_2, x_3, x_4, x_5) \in \mathbf{F}^5 : x_1 = 3x_2 \text{ and } x_3 = x_4 = x_5\}.$$

Solution:

We can figure out that

$$(3, 1, 0, 0, 0), \qquad (0, 0, 1, 1, 1)$$

is a basis of the vector space above.

Assume that such linear map T exists, we have

$$\dim \texttt{null}\ T = 2,$$

with Fundamental Theorem we have

$$\dim \texttt{range}\ T = \dim \mathbf{F}^5 - \dim \texttt{null}\ T = 5 - 2 = 3,$$

since range $T \subseteq \mathbf{F}^2$, implies

$$\dim \texttt{range}\ T \leq \dim \mathbf{F}^2 = 2,$$

thus, there is no such linear map.

16. Suppose there exist a linear map on V whose null space and range are both finite-dimensional. Prove that V is finite-dimensional.

Solution:

With Fundamental Theorem we have

$$\dim V = \dim \texttt{null}\ T + \dim \texttt{range}\ T,$$

i.e., V is finite-dimensional.

17. Suppose V and W are both finite-dimensional. Prove that there exists an injective linear map from V to W if and only if $\dim V \leq \dim W$.

Solution:

First we show that if $\dim V \leq \dim W$, then such linear map exists:
Since both V and W are finite-dimensional, suppose that $v_1, \ldots, v_n$ is a basis of V, and with $\dim V \leq \dim W$, we know there exist a linearly independent list $w_1, \ldots, w_n \in W$, consider the function $T \in \mathcal{L}(V, W)$ defined by

$$T(a_1 v_1 + \cdots + a_n v_n) = a_1 w_1 + \cdots + a_n w_n,$$

where $a_1, \ldots, a_n \in \mathbf{F}$ are arbitrary.
Since $w_1, \ldots, w_n$ is linearly independent, means

$$a_1 w_1 + \cdots + a_n w_n = 0$$

holds only when $a_1 = \cdots = a_n = 0$, *i.e.*,

$$\text{null } T = \{0\},$$

means T is injective.

The reverse is obviously since from the conclusion of the text-book, we know that when $\dim V > \dim W$, there is no injective linear map from V to W.

18. Suppose V and W are both finite-dimensional. Prove that there exists a surjective linear map from V onto W if and only if $\dim V \geq \dim W$.

Solution:

This is quite similar to the previous one, suppose that $w_1, \ldots, w_m$ is the basis of W, and $v_1, \ldots, v_n$ is the basis of V, since we have $m \leq n$, consider the linear map $T \in \mathcal{L}(V, W)$ defined by

$$T(a_1 v_1 + \cdots + a_n v_n) = a_1 w_1 + \cdots + a_m w_m,$$

where $a_1, \ldots, a_n \in \mathbf{F}$ are arbitrary.

Since

$$\text{span}\,(w_1, \ldots, w_m) = W,$$

we have

$$\text{range } T = W,$$

i.e., T is surjective.

19. Suppose V and W are finite-dimensional and that U is subspace of V. Prove that there exists $T \in \mathcal{L}(V, W)$ such that null $T = U$ if and only if $\dim U \geq \dim V - \dim W$.

Solution:

First we show that if $\dim U \geq \dim V - \dim W$, then such linearly map T exist, suppose that $u_1, \ldots, u_m$ is a basis of U, since $U \subseteq V$, means that $u_1, \ldots, u_m$ can be extended to be a basis of V, suppose the extended basis of V is

$$u_1, \ldots, u_m, v_1, \ldots, v_n,$$

therefore we have

$$\dim U = m, \qquad \dim V = m + n,$$

means

$$\dim W \geq \dim V - \dim U = m + n - m = n.$$

Therefore there are linearly independent list $w_1, \ldots, w_n \in W$, consider the linear map $T \in \mathcal{L}(V, W)$ defined by

$$T(a_1 u_1 + \cdots + a_m u_m + b_1 v_1 + \cdots + b_n v_n) = b_1 w_1 + \cdots + b_n w_n,$$

where $a_1, \ldots, a_m, b_1, \ldots, b_n \in \mathbf{F}$ are arbitrary.

Let

$$b_1 w_1 + \cdots + b_n w_n = 0,$$

implies that

$$b_1 = \cdots = b_n = 0,$$

means

$$\texttt{null}\ T = \texttt{span}\,(u_1, \ldots, u_m) = U.$$

Seconde we show the reverse: if $\dim U < \dim V - \dim W$, then there is no such linear map: according to Fundamental Theorem we have

$$\begin{aligned}\dim \texttt{null}\ T &= \dim V - \dim \texttt{range}\ T \\ &> \dim V - \dim W \\ &> \dim U,\end{aligned}$$

means there is no $T \in \mathcal{L}(V, W)$ that null $T = U$.

20. Suppose W is finite-dimensional and $T \in \mathcal{L}(V, W)$. Prove that T is injective if and only if there exists $S \in \mathcal{L}(W, V)$ such that ST is the identity map on V.

Solution:

First, we will show that if T is injective, then such S exists: since T is injective, then $\dim V \leq \dim W$, suppose that $v_1, \ldots, v_n$ is a basis of V, according to exercise 3B-9, we know that

$$Tv_1, \ldots, Tv_n$$

is linearly independent in W, which means it can extend to be a basis of W, suppose the extended basis is

$$Tv_1, \ldots, Tv_n, w_1, \ldots, w_m$$

consider the function $S \in \mathcal{L}(W, V)$ defined by

$$S(a_1 Tv_1 + \cdots + a_n Tv_n + b_1 w_1 + \cdots + b_m w_m) = a_1 v_1 + \cdots + a_n v_n,$$

where $a_1, \ldots, a_n, b_1, \ldots, b_m \in \mathbf{F}$ are arbitrary. For any $v \in V$, we can form v as

$$v = a_1 v_1 + \cdots + a_n v_n,$$

therefore

$$\begin{aligned} STv &= ST(a_1 v_1 + \cdots + a_n v_n) \\ &= S(Ta_1 v_1 + \cdots + Ta_n v_n) \\ &= S(a_1 Tv_1 + \cdots + a_n Tv_n) \\ &= a_1 v_1 + \cdots + a_n v_n \\ &= v, \end{aligned}$$

i.e., ST is identity map on V.
Second, we will show the reverse direction: If ST is identity then T is injective, or, assume that $u \neq 0$ with

$$Tu = 0,$$

we have

$$STu = S0 = 0,$$

but since ST is identity map, we have

$$STu = u,$$

implies that

$$u = 0,$$

which is contradict to our assumption.

21. Suppose W is finite-dimensional and $T \in \mathcal{L}(V, W)$. Prove that T is

surjective if and only if there exists $S \in \mathcal{L}(W,V)$ such that TS is the identity map on W.

Solution:

First, we will show that if T is surjective, then such S exists: Suppose $w_1,\ldots,w_n$ is a basis of W, since T is surjective, means we can find $v_1,\ldots,v_n \in V$ that

$$Tv_i = w_i, \qquad i = 1,\ldots,n,$$

Consider the function $S \in \mathcal{L}(W,V,)$ defined by

$$S(a_1w_1 + \cdots + a_nw_n) = a_1v_1 + \cdots + a_nv_n,$$

where $a_1,\ldots,a_n \in \mathbf{F}$ are arbitrary. Now for any $w \in W$, we can form w as

$$w = a_1w_1 + \cdots + a_nw_n,$$

then

$$\begin{aligned}
TSw &= TS(a_1w_1 + \cdots + a_nw_n) \\
&= T(a_1v_1 + \cdots + a_nv_n) \\
&= Ta_1v_1 + \cdots + Ta_nv_n \\
&= a_1Tv_1 + \cdots + a_nTv_n \\
&= a_1w_1 + \cdots + a_nw_n \\
&= w,
\end{aligned}$$

i.e., TS is identity map on W.
Second, we will show the reverse direction: Assume that T is not surjective, means that there is $w \in W$, that $w \notin \operatorname{range} T$, but we have

$$T(Sw) = w, \qquad Sw \in V,$$

means

$$w \in \operatorname{range} T,$$

which is contradict.

22. Suppose U and V are finite-dimensional vector spaces and $S \in \mathcal{L}(V,W)$

and $T \in \mathcal{L}(U,V)$. Prove that

$$\dim \texttt{null}\ ST \leq \dim \texttt{null}\ S + \dim \texttt{null}\ T.$$

Solution:

Suppose that $u_1,\ldots,u_n$ is a basis of null T, it is easy to verify that $u_i \in \texttt{null}\ ST$ for $i = 1,\ldots,n$ since

$$STu_i = S0 = 0, \qquad i = 1,\ldots,n,$$

therefore it can extended to be a basis of null ST, suppose to be

$$u_1,\ldots,u_n,v_1,\ldots,v_m.$$

Now we will show that $Tv_i \in \texttt{null}\ S$ for $i = 1,\ldots,m$ and $Tv_1,\ldots,Tv_n$ is linearly independent: Obviously that $Tv_i \in \texttt{null}\ S$ since

$$STv_i = S(Tv_i) = 0,$$

assume the list is not linearly independent, means there exists nonzero $a_1,\ldots,a_m \in \mathrm{F}$ with

$$a_1Tv_1 + \cdots + a_mTv_m = 0,$$

since

$$a_1Tv_1 + \cdots + a_mTv_m = T(a_1v_1 + \cdots + a_mv_m),$$

means

$$a_1v_1 + \cdots + a_mv_m \in \texttt{null}\ T,$$

and with $u_1,\ldots,u_n$ is a basis of null T, means there exists $b_1,\ldots,b_n \in \mathrm{F}$ that

$$a_1v_1 + \cdots + a_mv_m = b_1u_1 + \cdots + b_nu_n,$$

and with $u_1,\ldots,u_n,v_1,\ldots,v_m$ is linearly independent, we have

$$a_1 = \cdots = a_m = b_1 = \cdots = b_n = 0,$$

which is a contradict, therefore $Tv_1,\ldots,Tv_m$ is linearly independent in null S,

$$m \leq \dim \texttt{null}\ S,$$

with $\dim \texttt{null}\ ST = n+m$ and $\dim \texttt{null}\ T = n$ we have

$$\dim \texttt{null}\ ST \leq \dim \texttt{null}\ T + \dim \texttt{null}\ S.$$

23. Suppose U and V are finite-dimensional vector space and $S \in \mathcal{L}(V,W)$ and $T \in \mathcal{L}(U,V)$. Prove that

$$\dim \texttt{range}\ ST \leq \min\{\dim \texttt{range}\ S, \dim \texttt{range}\ T\}.$$

Solution:

First we will show that

$$\dim \texttt{range}\ ST \leq \dim \texttt{range}\ S,$$

this is easy, since the basis of range ST must be linearly independent in range S, therefore the inequality holds.
Second we will show that

$$\dim \texttt{range}\ ST \leq \dim \texttt{range}\ T,$$

suppose that $u_1, \ldots, u_n$ is basis of range T, we will show that $Su_1, \ldots, Su_n$ spans range S, for any $s \in \texttt{range}\ ST$, suppose that

$$s = STu, u \in U,$$

since $Tu \in \texttt{range}\ T$, we can form it as

$$Tu = a_1 u_1 + \cdots + a_n u_n,$$

here $a_1, \ldots, a_n \in \mathrm{F}$, therefore above equation can be written as

$$\begin{aligned} s &= STu \\ &= S(a_1 u_1 + \cdots + a_n u_n) \\ &= a_1 Su_1 + \cdots + a_n Su_n, \end{aligned}$$

which means any $s \in \texttt{range}\ ST$ can be formed as a linear combination of $Su_1, \ldots, Su_n$, *i.e.*, $Su_1, \ldots, Su_n$ spans range ST, therefore it can be reduced to be a basis of range TS, means

$$\dim \texttt{range}\ ST \leq n = \texttt{range}\ T.$$

24. Suppose W is finite-dimensional and $T_1, T_2 \in \mathcal{L}(V,W)$. Prove that null $T_1 \subset$ null T_2 if and only if there exists $S \in \mathcal{L}(W,W)$ such that $T_2 = ST_1$.

Solution:

First we will show that if null $T_1 \subset$ null T_2, then such linear map S exists: suppose $u_1, \ldots, u_n$ is basis of null T_1, since null $T_1 \subset$ null T_2, means $u_1, \ldots, u_n$ is linearly independent in null T_2, it can extended to be a basis of null T_2, suppose to be

$$u_1, \ldots, u_n, v_1, \ldots, v_m,$$

this list can be extended to be a basis of U, suppose to be

$$u_1, \ldots, u_n, v_1, \ldots, v_m, s_1, \ldots, s_p$$

we will show that

$$T_1 v_1, \ldots, T_1 v_m, T_1 s_1, \ldots, T_1 s_p$$

is linearly independent in W, obviously that $T_1 v_i \in W$ for $i = 1, \ldots, m$ and $T_1 s_i \in W$ for $i = 1, \ldots, p$, assume that it is not linearly independent, we have nonzero $a_1, \ldots, a_m, b_1, \ldots, b_p \in \mathrm{F}$ with

$$a_1 T_1 v_1 + \cdots + a_m T_1 v_m + b_1 T_1 s_1 + \cdots + b_p T_1 s_p = 0,$$

since

$$a_1 T_1 v_1 + \cdots + a_m T_1 v_m + b_1 T_1 s_1 + \cdots + b_p T_1 s_p = T_1(a_1 v_1 + \cdots + a_m v_m + b_1 s_1 + \cdots + b_p s_p) = 0,$$

means

$$a_1 v_1 + \cdots + a_m v_m + b_1 s_1 + \cdots + b_p s_p \in \text{null } T_1,$$

with $u_1, \ldots, u_n$ is a basis of null T_1, we have

$$a_1 T_1 v_1 + \cdots + a_m T_1 v_m + b_1 s_1 + \cdots + b_p s_p = c_1 u_1 + \cdots + c_n u_n,$$

with $u_1, \ldots, u_n, v_1, \ldots, v_m, s_1, \ldots, s_p$ is linearly independent, we have

$$a_1 = \cdots = a_m = b_1 = \cdots = b_p = 0,$$

which is contradict, therefore $T_1 v_1, \ldots, T_1 v_m, T_1 s_1, \ldots, T_1 s_p$ is linearly independent in W, implies it can extended to be a basis of W, suppose the basis to be

$$T_1v_1,\ldots,T_1v_m,T_1s_1,\ldots,T_1s_p,w_1,\ldots,w_k,$$

consider the function $S:W\to W$ defined by

$$\begin{aligned}&S(a_1T_1v_1+\cdots+a_mT_1v_m+b_1T_1s_1+\cdots+b_pT_1s_p+c_1w_1+\cdots+c_kw_k)\\&\quad=b_1T_2s_1+\cdots+b_pT_2s_p+c_1w_1+\cdots+c_kw_k\end{aligned}$$

it is easy to verify that $S\in\mathcal{L}(W,W)$, now we will show that

$$T_2=ST_1,$$

for $u\in U$. we can form u as

$$u=a_1u_1+\cdots+a_nu_n+b_1v_1+\cdots+b_mv_m+\cdots+c_1s_1+\cdots+c_ps_p,$$

therefore

$$\begin{aligned}T_2u&=T_2(a_1u_1+\cdots+a_nu_n+b_1v_1+\cdots+b_mv_m+\cdots+c_1s_1+\cdots+c_ps_p)\\&=T_2(c_1s_1+\cdots+c_ps_p),\end{aligned}$$

and

$$\begin{aligned}ST_1u&=ST_1(a_1u_1+\cdots+a_nu_n+b_1v_1+\cdots+b_mv_m+\cdots+c_1s_1+\cdots+c_ps_p)\\&=S(b_1T_1v_1+\cdots+b_mT_1v_m+c_1T_1s_1+\cdots+c_pT_1s_p)\\&=c_1T_2s_1+\cdots+c_pT_2s_p\\&=T_2(c_1s_1+\cdots+c_ps_p),\end{aligned}$$

which means

$$T_2=ST_1.$$

Now we will show it in the reverse direction: If $T_2=ST_1$, then for any $u\in\text{null }T_1$, we have

$$T_2u=ST_1u=S0=0,$$

means

$$u\in\text{null }T_2,$$

i.e., null $T_1 \subset$ null T_2.

25. Suppose V is finite-dimensional and $T_1, T_2 \in \mathcal{L}(V, W)$. Prove that range $T_1 \subset$ range T_2 if and only if there exists $S \in \mathcal{L}(V, V)$ such that $T_1 = T_2 S$.

26. Suppose $D \in \mathcal{L}(\mathcal{P}(\mathbf{R}), \mathcal{P}(\mathbf{R}))$ is such that $\deg Dp = (\deg p) - 1$ for every nonconstant polynomial $p \in \mathcal{P}(\mathbf{R})$. Prove that D is surjective.

27. Suppose $p \in \mathcal{P}(\mathbf{R})$. Prove that there exists a polynomial $q \in \mathcal{P}(\mathbf{R})$ such that $5q'' + 3q' = p$.

28. Suppose $T \in \mathcal{L}(V, W)$, and $w_1, \ldots, w_m$ is a basis of range T. Prove that there exist $\varphi_1, \ldots, \varphi_m \in \mathcal{L}(V, \mathbf{F})$ such that

$$Tv = \varphi_1(v)w_1 + \cdots + \varphi_m(v)w_m$$

for every $v \in V$.

29. Suppose $\varphi \in \mathcal{L}(V, \mathbf{F})$. Suppose $u \in V$ is not in null φ. Prove that

$$V = \text{null } \varphi \oplus \{au : a \in \mathbf{F}\}.$$

30. Suppose φ_1 and φ_2 are linear map from V to $\mathbf{F}$ that have the same null space. Show that there exists a constant $c \in \mathbf{F}$ such that $\varphi_1 = c\varphi_2$.

31. Give an example of two linear maps T_1 and T_2 from $\mathbf{R}^5$ to $\mathbf{R}^2$ that have the same null space but are such that T_1 is not a scalar multiple of T_2.