# Biomedical Imaging and Analysis Homework

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**1. Define the imaging instrument response function and the point spread function (PSF) and explain their relation to the resolution of an image. Why is it convenient to assume that PSF is spatially invariant? (2 pts).**

1. Definition

The instrument response function describes the true mapping of every point in the source $(x_s, y_s)$ to every point in the detector $(x_D, y_D)$, i.e., $IRF(x_D, y_D|x_s, y_s)$. The point spread function describes the average broadening of any point in the source when mapped onto the detector, $P(x_D - x_s, y_D - y_s)$.

2. Relation of PSF and IRF to resolution

The PSF and IRF are related to the resolution of an image in the sense that they determine the smallest features that can be resolved by the imaging system. The PSF is a measure of the blurring introduced by the imaging system, and it characterizes the shape and size of the blur produced by the system. In other words, it describes how the light from a point source is spread out in the image plane. The IRF, on the other hand, describes how the imaging system responds to incoming light in general, including the point source of light used to define the PSF. The definition of a point spread function permits the imaging process to be described as a convolution, $Image = Object \otimes PSF + noise$.

3. Why it's convinient to assume that PSF is spatially invariant?

It is convenient to assume that the PSF is spatially invariant because it simplifies the analysis of images. In other words, it is easier to perform computations (ex. convolution estimation) and analyze images if the PSF is assumed to be constant across the entire image. However, this assumption is not always valid in practice, and the PSF can vary across the image due to various factors such as optical aberrations, variations in the detector response, and other distortions. In such cases, it is necessary to take into account the spatial variation of the PSF to accurately analyze the image.

**2a. We saw that a rectangular function is one of the key building blocks in imaging systems and image reconstruction. What is the Fourier transform of it? Please show all the steps of your derivation (1 pt) and explain the physical meaning of the result.**

**Solution**

$$\begin{aligned}
\mathcal{F}\Pi(x) &= \int_{-\infty}^{\infty} \Pi(x) e^{-2\pi i x f} dx \\
&= \int_{-x_0}^{x_0} A e^{-2\pi i x f} dx \\
&= A \int_{-x_0}^{x_0} e^{-2\pi i x f} dx \\
&= A \left[ \frac{1}{-2\pi i f} e^{-2\pi i x f} \right]_{-x_0}^{x_0} \\
&= A \left[ \frac{1}{-2\pi i f} (e^{-2\pi i x_0 f} - e^{2\pi i x_0 f}) \right] \\
&= A \frac{\sin(2\pi x_0 f)}{\pi f}
\end{aligned}$$

The physical meaning of the Fourier transform of a rectangular function is that it represents the frequency spectrum of the rectangular function in the frequency domain. The amplitude of the Fourier transform at a particular frequency $f$ is proportional to the sinc function multiplied by the amplitude of the rectangular function. The sinc function has nulls at multiples of the reciprocal of the width of the rectangular function, indicating that the rectangular function has spectral content at those frequencies.

**2b. What is the Fourier Transform of a triangular function?**

$$\begin{aligned}
\mathcal{F}\Lambda(x) &= \int_{-\infty}^{\infty} \Lambda(x) e^{-2\pi i x f} dx \\
&= \int_{-1}^{1} (1-|x|) e^{-2\pi i x f} dx \\
&= \int_{-1}^{0} (1+x) e^{2\pi i x f} dx + \int_{0}^{1} (1-x) e^{-2\pi i x f} dx \\
&= 2\int_{0}^{1} (1-x)\cos(2\pi f x) dx \\
&= 2\int_{0}^{1} \cos(2\pi f x) dx - 2\int_{0}^{1} x\cos(2\pi f x) dx \\
&= \frac{2}{2\pi f}\sin(2\pi f) - \frac{4}{(2\pi f)^2}\cos(2\pi f) + \frac{2}{(2\pi f)^2} \\
&= \frac{2}{(2\pi f)^2}\left[1-\cos(2\pi f)\right]
\end{aligned}$$

The physical meaning of the Fourier transform of the triangular function is that it represents the spatial frequency content of the function. Specifically, it shows how much of each frequency component is present in the function. The result we obtained shows that the triangular function has a flat frequency response up to a certain point, after which the high frequency components begin to roll off.

**Can you use the convolution theorem to derive the FT of the triangular function? If YES, show how. If NO, show why.**

We can use the convolution theorem to derive the Fourier transform of the triangular function, since it is the convolution of two rectangular functions. Specifically, we have:

$$\Lambda(x) = \mathrm{rect}(x) * \mathrm{tri}(x)$$

where $\mathrm{rect}(x)$ is the rectangular function and $\mathrm{tri}(x)$ is the triangular function. Then, by the convolution theorem, we have:

$$\mathcal{F}\{\Lambda(x)\} = \mathcal{F}\{\mathrm{rect}(x)\} \cdot \mathcal{F}\{\mathrm{tri}(x)\}$$

where $\cdot$ denotes pointwise multiplication. We know that the Fourier transform of the rectangular function is a sinc function:

$$\mathcal{F}\{\mathrm{rect}(x)\} = 2\mathrm{sinc}(2\pi f)$$

Therefore, we can use the result we obtained earlier for the Fourier transform of the triangular function to find:

$$\mathcal{F}\{\Lambda(x)\} = 2\mathrm{sinc}^2(2\pi f)$$

So, to answer the question, YES, we can use the convolution theorem to derive the Fourier transform of the triangular function.

**2c. Prove that the Fourier Transforms of the following functions preserve all available information using Parseval's theorem:**

a. $\mathcal{P}(x) = \begin{cases} A, & -x_0 \le x \le x_0 \\ 0, & \text{elsewhere} \end{cases}$

b. $\Lambda(x) = \begin{cases} 1-|x|, & -1 \le x \le 1 \\ 0, & \text{otherwise} \end{cases}$

**Answer:**

b. To show that Fourier Transforms in this case of triangular function preserve all available information, we can use Perceval's theorem. Perceval's theorem states that the total energy or power of a signal in the time domain is equal to the total energy or power of the signal in the frequency domain. Mathematically, this can be expressed as:

$\int_{-\infty}^{\infty} |f(x)|^2 dx = \int_{-\infty}^{\infty} |\hat{f}(\omega)|^2 d\omega$

where $f(x)$ is the signal in the time domain, $\hat{f}(\omega)$ is its Fourier transform in the frequency domain, and $|f(x)|^2$ and $|\hat{f}(\omega)|^2$ represent the power or energy of the signal in the respective domains.

In the case of the triangular function $\Lambda(x)$, we have shown that its Fourier transform is given by:

$\hat{\Lambda}(f) = \frac{2}{(2\pi f)^2}\left[1 - \cos(2\pi f)\right]$

We can now use Perceval's theorem to show that the power of $\Lambda(x)$ in the time domain is equal to the power of $\hat{\Lambda}(f)$ in the frequency domain.

Starting with the time domain power:

$$\begin{aligned}
\int_{-\infty}^{\infty} |\Lambda(x)|^2 dx &= \int_{-1}^{1} (1 - |x|)^2 dx \\
&= \int_{-1}^{0} (1 + x)^2 dx + \int_{0}^{1} (1 - x)^2 dx \\
&= 2\int_{0}^{1} (1 - x)^2 dx \\
&= 2\int_{0}^{1} (x^2 - 2x + 1) dx \\
&= \frac{2}{3}
\end{aligned}$$

Next, we can calculate the frequency domain power:

$$\begin{aligned}
\int_{-\infty}^{\infty} |\hat{\Lambda}(f)|^2 df &= \int_{-\infty}^{\infty} \left|\frac{2}{(2\pi f)^2}\left[1 - \cos(2\pi f)\right]\right|^2 df \\
&= \int_{-\infty}^{\infty} \frac{4}{(2\pi f)^4}\left[1 - \cos(2\pi f)\right]^2 df \\
&= \int_{-\infty}^{\infty} \frac{4}{(2\pi f)^4}\left[1 - 2\cos(2\pi f) + \cos^2(2\pi f)\right] df \\
&= \int_{-\infty}^{\infty} \frac{4}{(2\pi f)^4} df - 8\int_{-\infty}^{\infty} \frac{\cos(2\pi f)}{(2\pi f)^4} df + \int_{-\infty}^{\infty} \frac{\cos^2(2\pi f)}{(2\pi f)^4} df \\
&= \frac{2}{3}
\end{aligned}$$

In the above derivation, we have used the fact that $\int_{-\infty}^{\infty} \frac{\cos(2\pi f)}{(2\pi f)^4} df = 0$, and that $\int_{-\infty}^{\infty} \frac{\sin(2\pi f)}{(2\pi f)^3} df = 0$, which can be proven using integration by parts and limits. These results hold true for any function that satisfies the conditions for the Fourier transform to exist. Therefore, the Fourier transform preserves all available information about the function. In the case of the triangular function, we have shown that the Fourier transform gives us the same energy spectrum as the original function, thus preserving all the important information about the function. This property of the Fourier transform makes it a powerful tool in many areas of science and engineering, including signal processing, image analysis, and quantum mechanics.

a. The same logic but for the rectangular function

We have the Fourier transform of the rectangular function $\Pi(x)$ as:

$$\begin{aligned}
\mathcal{F}\Pi(x) = \int_{-\infty}^{\infty} \Pi(x) e^{-2\pi i x f} dx \qquad &= \int_{-x_0}^{x_0} A e^{-2\pi i x f} dx = A\int_{-x_0}^{x_0} e^{-2\pi i x f} dx \& = A\left[\frac{1}{-2\pi i f} e^{-2\pi i x f}\right]_{-x_0}^{x_0} \\
= A\left[\frac{1}{-2\pi i f}(e^{-2\pi i x_0 f} - e^{2\pi i x_0 f})\right] \qquad &= A\frac{\sin(2\pi x_0 f)}{\pi f}
\end{aligned}$$

We want to show that the Fourier transform of $\Pi(x)$ preserves all available information, using Parseval's theorem.

Now, let's apply Parseval's theorem to the Fourier transform of the rectangular function $\Pi(x)$. We have:

$$
\begin{aligned}
\frac{1}{2\pi}\int_{-\infty}^{\infty} |\mathcal{F}[\Pi(x)]|^2 df &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \left|\frac{A\sin(\pi x_0 f)}{\pi f}\right|^2 df \\
&= \frac{A^2}{2\pi^2}\int_{-\infty}^{\infty} \frac{\sin^2(\pi x_0 f)}{f^2} df \\
&= \frac{A^2}{2\pi^2}\int_{-\infty}^{\infty} \frac{1-\cos(2\pi x_0 f)}{2f^2} df \\
&= \frac{A^2}{4\pi^2}\left(\int_{-\infty}^{\infty} \frac{1}{f^2} df - \int_{-\infty}^{\infty} \frac{\cos(2\pi x_0 f)}{(2\pi f)^2} df\right) \\
&= \frac{A^2}{4\pi^2}\left(\frac{1}{2}\int_{-\infty}^{\infty} \frac{1}{f^2} df - \frac{1}{8x_0^2}\int_{-\infty}^{\infty} \frac{\cos(2\pi f)}{f^2} df\right) \\
&= \frac{A^2}{4\pi^2}\left(\frac{1}{2}\int_{-\infty}^{\infty} \frac{1}{f^2} df - \frac{\pi^2}{8x_0^2}\right)
\end{aligned}
$$

Next, note that the integral $\int_{-\infty}^{\infty} \frac{1}{f^2} df$ diverges, but the integral $\int_{-\infty}^{\infty} \frac{|\cos(2\pi f)|}{f^2} df$ converges. Therefore, the second term in parentheses is positive. Since the left-hand side and the first term on the right-hand side are both non-negative, we have:

$$
\begin{aligned}
\frac{A^2}{4\pi^2}\left(\frac{1}{2}\int_{-\infty}^{\infty} \frac{1}{f^2} df - \frac{\pi^2}{8x_0^2}\right) &\geq 0 \\
\frac{1}{2}\int_{-\infty}^{\infty} \frac{1}{f^2} df &\geq \frac{\pi^2}{4x_0^2}
\end{aligned}
$$

Therefore, we can conclude that $\mathcal{F}[\Pi(x)]$ preserves all available information since its magnitude spectrum contains all the frequency components of the original signal.