Scientific Report on Modeling Tumor Growth in the Patient

Cedric Gerdes, John Kunkee, John Williams

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Problem and Data

On February 1st, the patient was found by MRI to have a malignant tumor of volume 650 cubic millimeters in the suprasellar cavity of the brain. Another MRI was performed a week later on February 8th which measured the tumor's volume to have grown to 810 cubic millimeters. The goal of this report is to give and explain a mathematical model for the tumor's growth which is both simple and accurate enough to advise the specialists at AOS, Inc. on the treatment of the patient. We are asked to

- find a mathematical model for the growth of the tumor,
- find a closed form solution to this model and graph it,
- extrapolate when the tumor will reach 90% the volume of the suprasellar cavity,
- find the growth rate of the tumor on February 1st,
- identify when the tumor is growing at its fastest rate and the size of the tumor at that time.

Model

A Gompertz curve is the generally excepted mathematical model for the growth of tumors large enough to detect and typically subject to some constraint.[1] The constraint in this scenario is the volume of the suprasellar cavity, which is 3,200 cubic millimeters. We therefore apply it here as

$$\frac{dV(t)}{dt} = rV(t)\ln(\frac{k}{V(t)}) \tag{1}$$

where $V(t)$ is the volume of the tumor in cubic millimeters as a function of time, r and k are constants, and t is time, which is measured in days with $t = 0$ referring to February 1st. By analysis shown later, we find

$$V(t) = e^{-\frac{A}{e^{rt}}} \tag{2}$$

where A is some constant. Then, from the measurements on February 1st and 8th respectively,

$$V(0) = 650\text{mm}^3 \tag{3}$$

$$V(7) = 810\text{mm}^3 \tag{4}$$

Analyses

Returning to equation (1), seperating variables, and using properties of logs, we have

$$\frac{dV(t)}{dt} = rV(t)\ln(\frac{k}{V(t)})$$

$$\frac{dV(t)}{V(t)(\ln(k) - \ln(V(t))} = r\,dt \tag{5}$$

We integrate both sides, using a u-substitution on the left with $u = \ln(V(t))$, $du = \frac{dV(t)}{V(t)}$, to find

$$-\ln(\ln(\frac{k}{V(t)})) = rt + C \tag{6}$$

Simplifying this we find equation (2)

$$V(t) = \frac{k}{e^{\frac{A}{e^{rt}}}}$$

where $A = e^C$ which is some constant. Now to find the constant k we return to the original equation and observe that since the volume of the tumor is constrained to 3200 cubic millimeters, then the change in volume when $V(t) = 3200$ is 0. We also know that r must be nonzero, otherwise the tumor wouldn't have grown by this model, so k must equal 3200 so that $\ln(\frac{k}{V(t)}) = 0$ when $V(t) = 3200$. Then, to find A, notice that $V(0) = 650$ so

$$V(0) = 650 = \frac{3200}{e^{\frac{A}{e^0}}}$$

$$A = \ln(\frac{3200}{650}) \approx 1.5939$$

Then, to find r, notice that $V(7) = 810$ so

$$V(7) = 810 = \frac{3200}{e^{\frac{1.5939}{e^{7r}}}}$$

$$r = \frac{\ln(\frac{1.5939}{\ln(\frac{3200}{810})})}{7} \approx 0.021222$$

Therefore,

$$V(t) = \frac{3200}{e^{\frac{1.5939}{e^{0.021222t}}}} \tag{7}$$

and

$$\frac{dV(t)}{dt} = 0.021222V(t)\ln(\frac{3200}{V(t)}) \tag{8}$$

Now with the complete equation (graphed in fig. 0.1) we can find when the tumor will reach 90% of the total volume of the suprasellar cavity as

$$\frac{90}{100}3200 = \frac{3200}{e^{\frac{1.5939}{e^{0.021222t}}}}$$

$$\frac{\ln\frac{1.5939}{\ln(\frac{10}{9})}}{0.021222} = t \approx 128\text{days}$$

To find the growth rate of the tumor on Febraury 1st, we can use equations (8) and (3) for

$$\frac{dV(0)}{dt} = 0.021222(650)\ln(\frac{3200}{650}) \approx 29.985$$

Substituting our solved values for the constants and functions in (1) and simplifying, we find

$$\frac{dV(t)}{dt} = 108.242e^{-1.5939e^{-0.021222t}-0.021222t} \tag{9}$$

which is graphed in fig. 0.2. To find when the tumor is growing at its fastest rate, we find the zeros of the derivative of (9)
which is

$$\frac{d^2V(t)}{dt^2} = (3.66137 - 2.29711e^{0.021222t})e^{-0.042444t-1.5939e^{-0.021222t}} \tag{10}$$

Setting this equal to zero we find the value of t for which $\frac{dV(t)}{dt}$ is maximized, because we know $\frac{dV(t)}{dt}$is always positive

$$\frac{\ln(\frac{3.66137}{2.29711})}{0.021222} = t \approx 21.967\text{days}$$

Entering this value of t into (9) we find that the maximum rate of change is

$$\frac{dV(21.967)}{dt} \approx 24.983\frac{\text{mm}^3}{\text{day}}$$

and its volume will be

$$V(21.967) \approx 1177.2\text{mm}^3$$

Summary

The requested results are hereafter provided:

- the model for the growth of the tumor is $V(t) = \frac{3200}{e^{\frac{1.5939}{e^{0.02122t}}}}$, which is graphed in fig. 0.1,
- the tumor will grow to 90% the volume of the suprasellar cavity in 128 days if left untreated,
- the growth rate of the tumor on February 1st is 29.985mm^{3}per day
- the tumor will be growing at its fastest rate on Febraury 26th when it will have a volume of 1177.2mm^3.

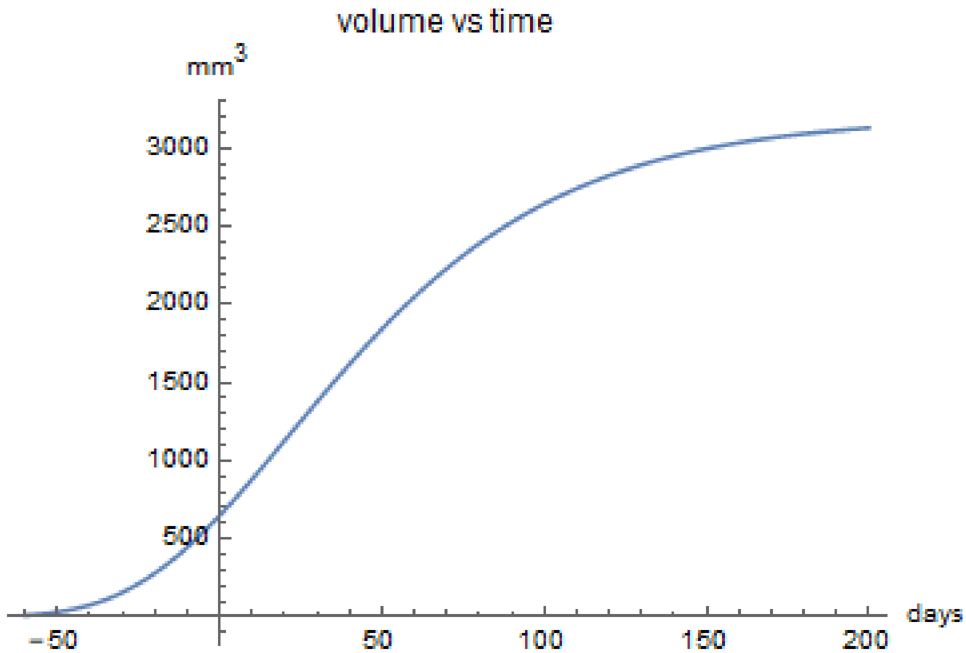
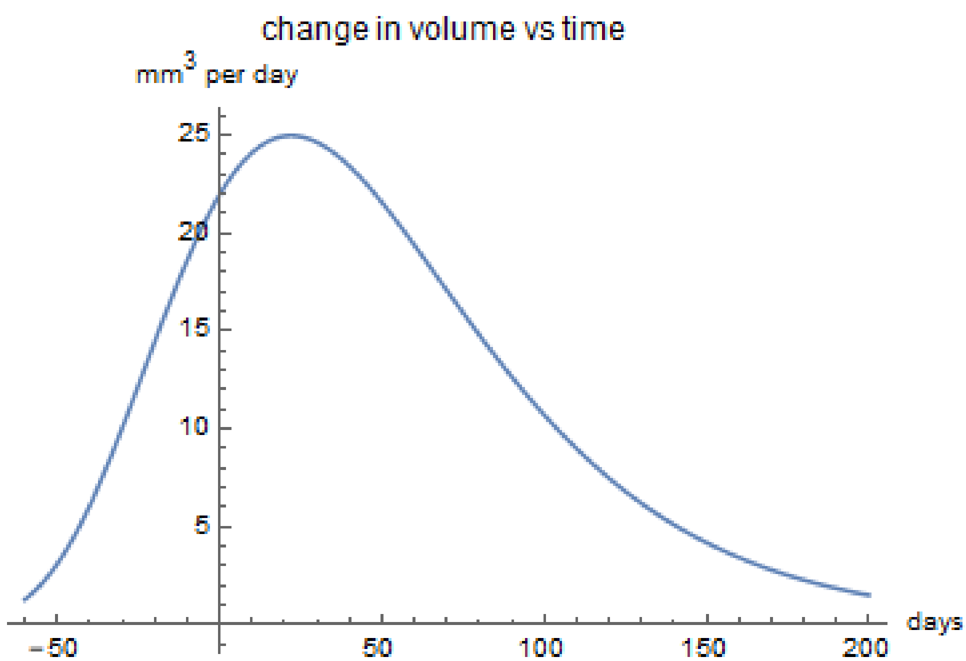
Figure 1: 

Figure 2: 

Bibliography

[1] Fornalski, K. W., Reszczyńska, J., Dobrzyński, L., Wysocki, P., & Janiak, M. K. (2020). Possible Source of the Gompertz Law of Proliferating Cancer Cells: Mechanistic Modeling of Tumor Growth. Acta Physica Polonica A, 138(6), 854–862. https://doi.org/10.12693/aphyspola.138.854