Three-dimensional straight-line drawings of planar graphs

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ETH Zürich

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Outline

Motivation for 3D drawings

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Results for general graphs

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Proof of the main result

- Track Layouts to 3D drawings
- Queues and chromatic number to Tracks
- Bounds on acyclic chromatic number

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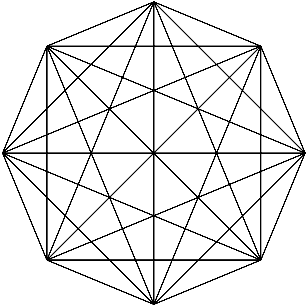

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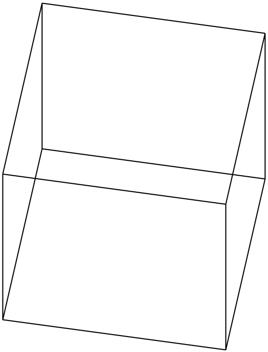

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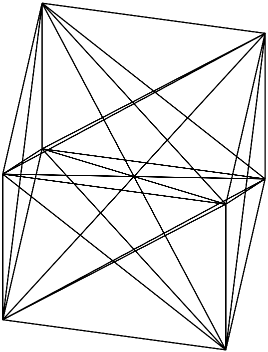

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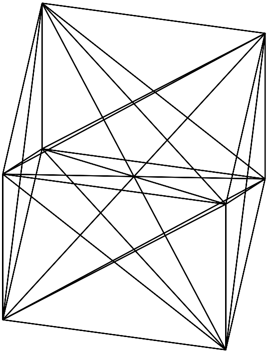

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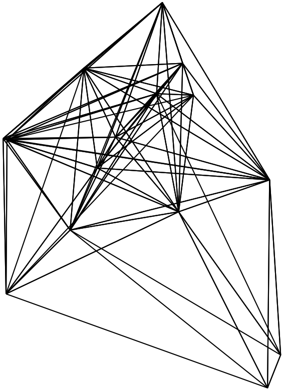

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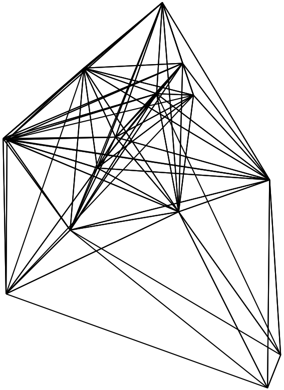

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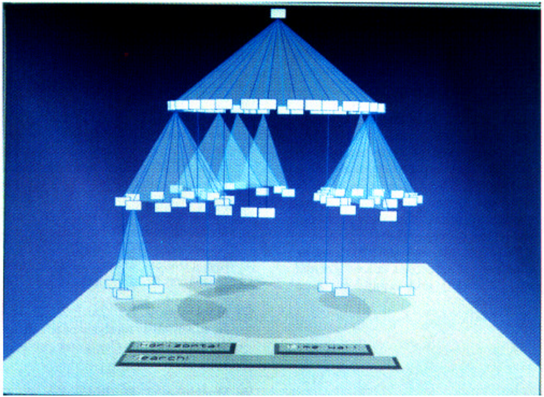

Figure from: George, Jock and Stuart (1991): *Cone Trees: Animated 3D Visualizations of Hierarchical Information*

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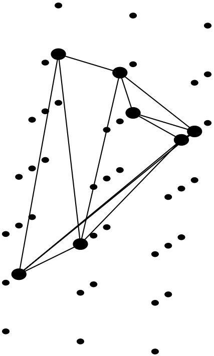
  - Virtual Reality?

Main result

Theorem

Every planar graph with n vertices has a 3-dimensional straight-line drawing on the integer grid



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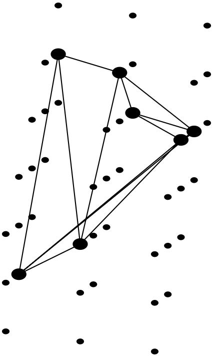

Why is this interesting?

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The Moment Curve

Theorem

Every graph with n vertices has a 3-dimensional straight-line drawing on the integer grid with $\mathcal{O}(n^6)$ Volume.

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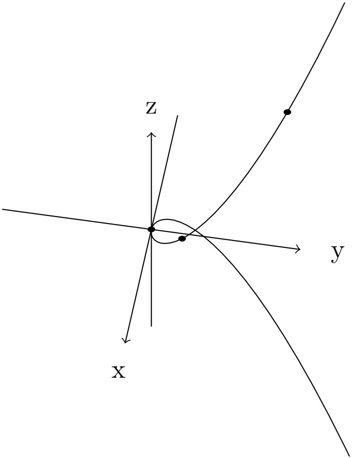

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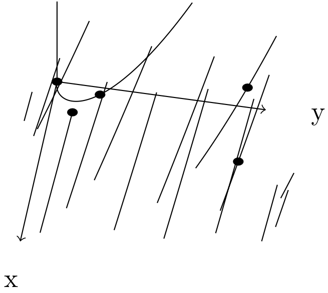

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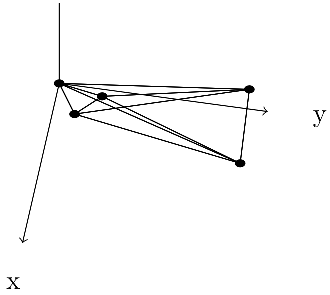

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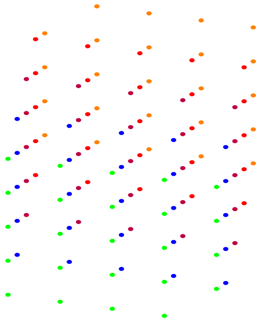
$\mathcal{O}(n^3)$ is tight for complete graphs

- Cut grid into planes



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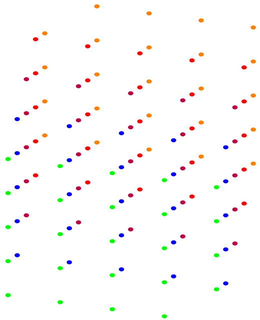

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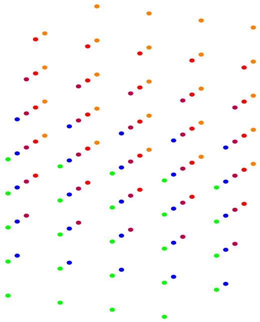

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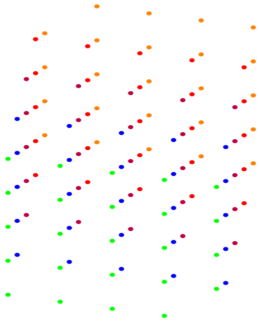

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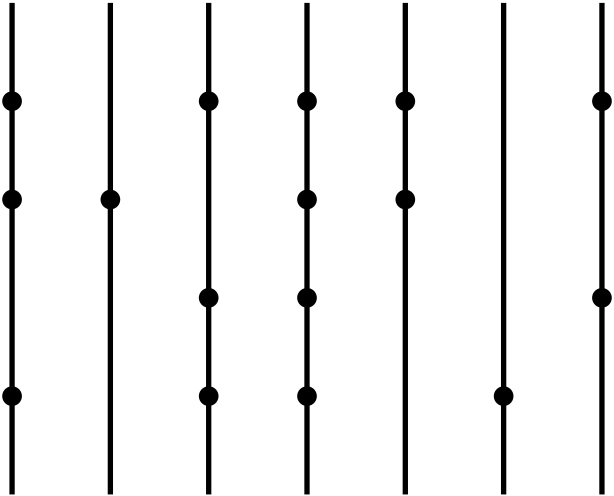

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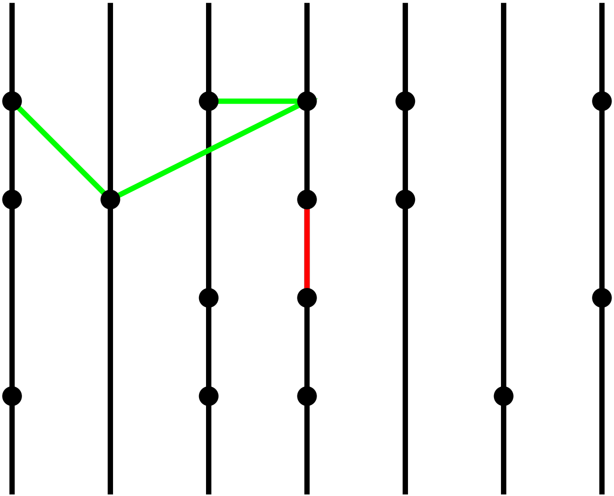

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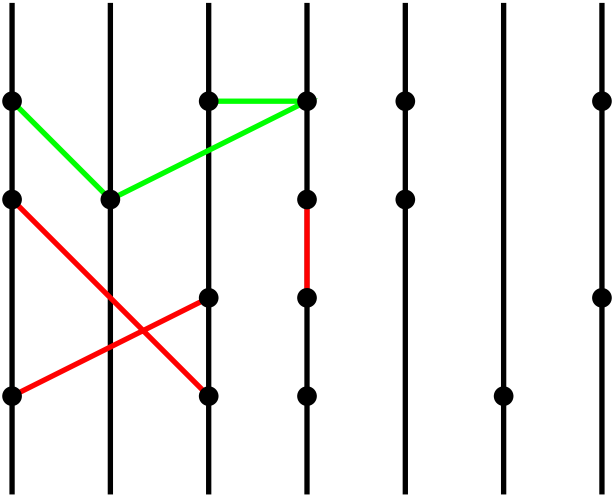

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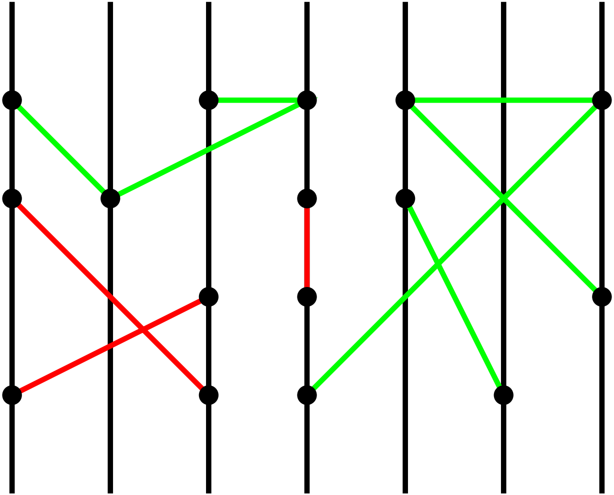

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Definition (Track Number)

The *Track Number* $\mathrm{tn}(G)$ is the smallest t such that G has a t-track layout

The Modular Moment Curve with tracks

Theorem

Every graph on n vertices with track number t has a 3-dimensional straight-line drawing on the integer grid with $\mathcal{O}(t^3 n')$ volume, where n' is the maximal number of vertices on any single track.

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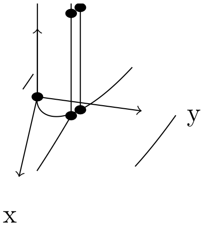

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□

Theorem

Every c-colorable graph on n vertices with track number t has a 3-dimensional straight-line drawing on the integer grid with $\mathcal{O}(c^7 tn)$ volume.

Proof.

proof sketch Similar to how we put vertices of the same track into the same xy-axis, put vertices of the same color to the same x-coordinate. This works if we put the k-th vertex of the j-th track and the i-th color to

$$\begin{pmatrix} i \\ p(2it+j)+(i^2 \mod p) \\ p(20cin(p(2it+j)+(i^2 \mod p))+k)+(i^3 \mod p) \end{pmatrix}$$

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Outline

Theorem

Every graph on n vertices with queue number q and acyclic chromatic number c has track number $c(2q)^{c-1}$

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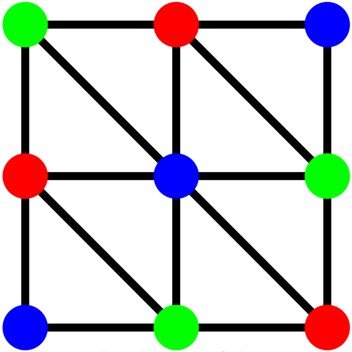

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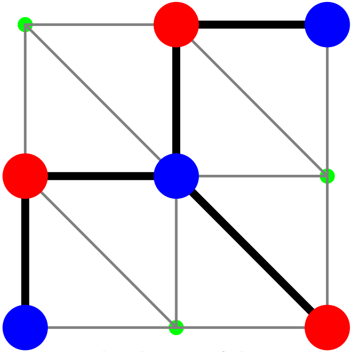

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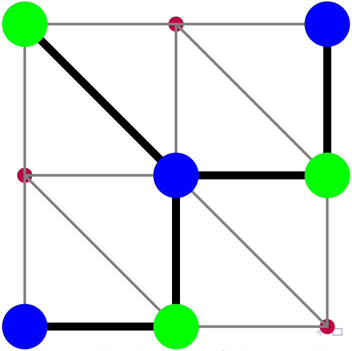

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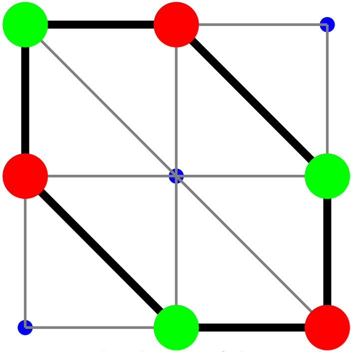

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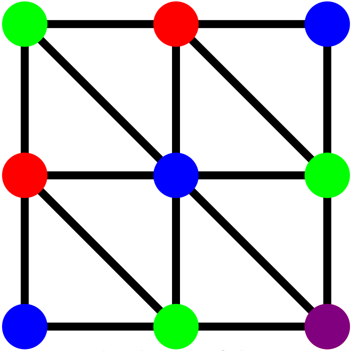

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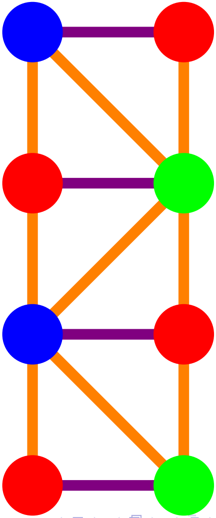

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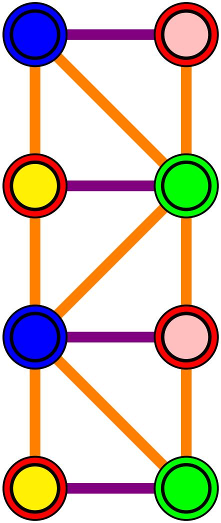

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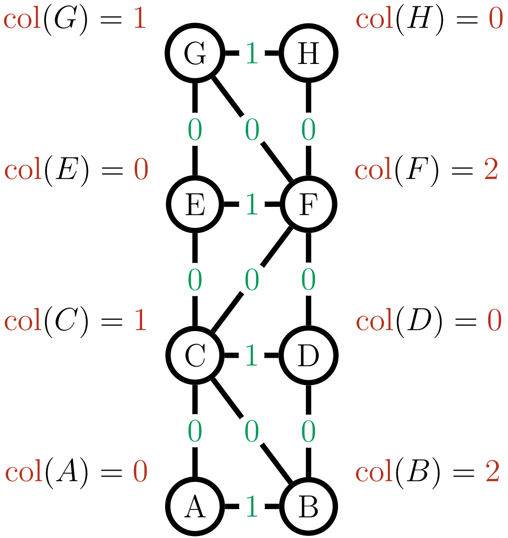

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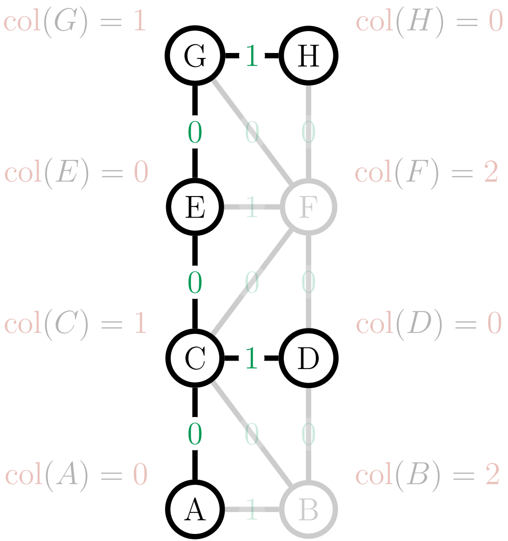

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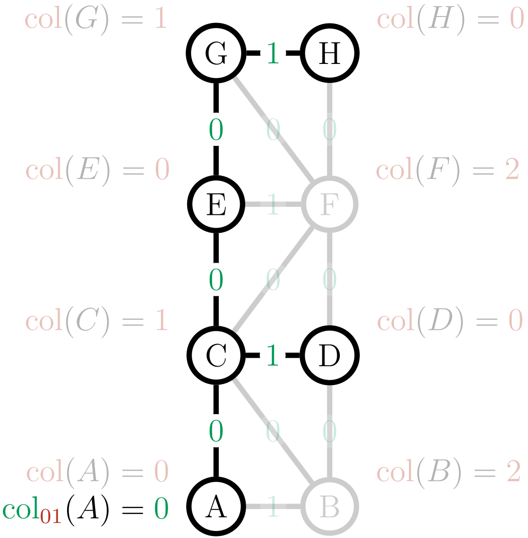

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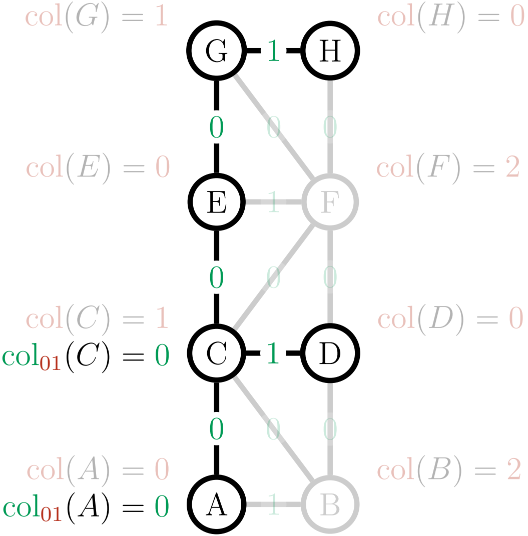

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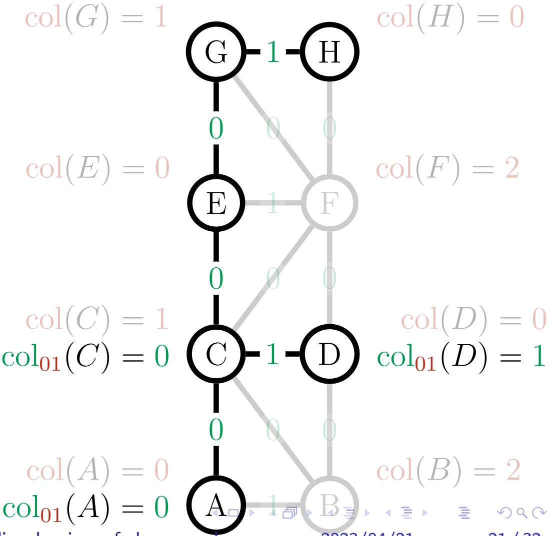

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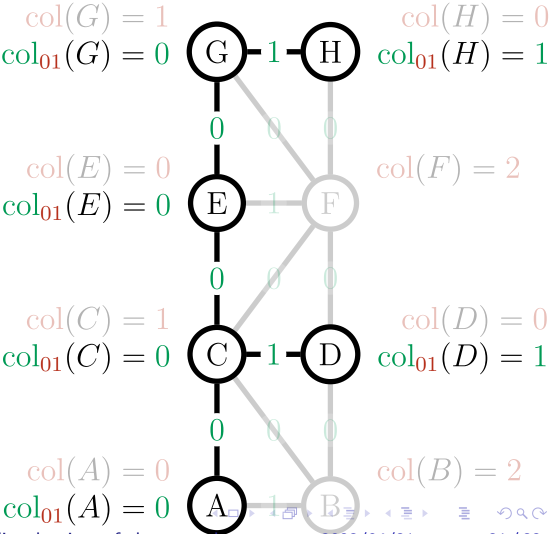

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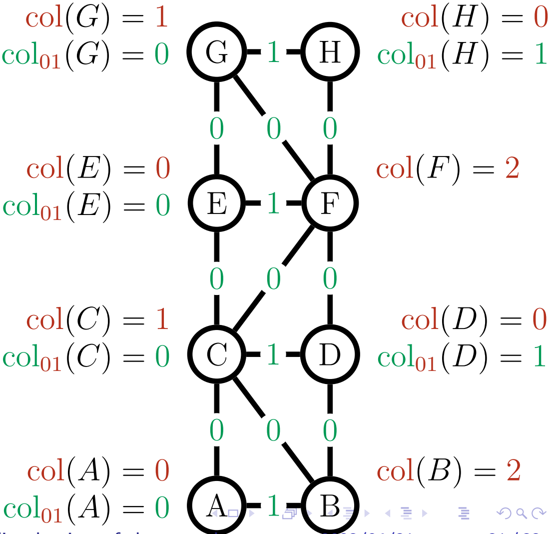

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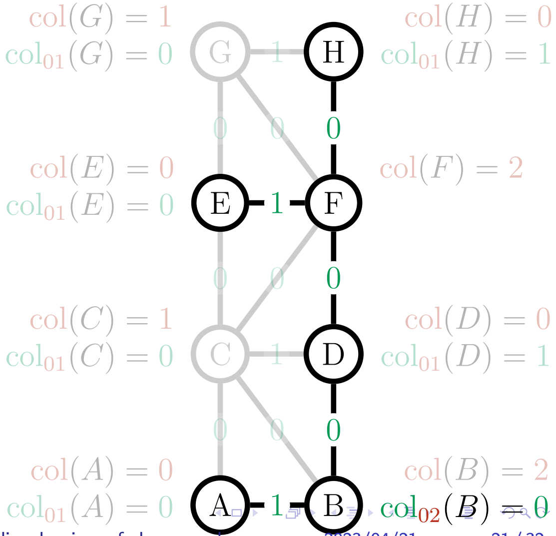

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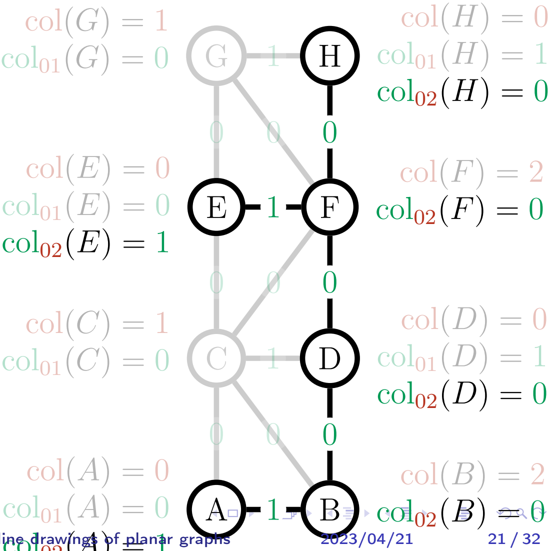

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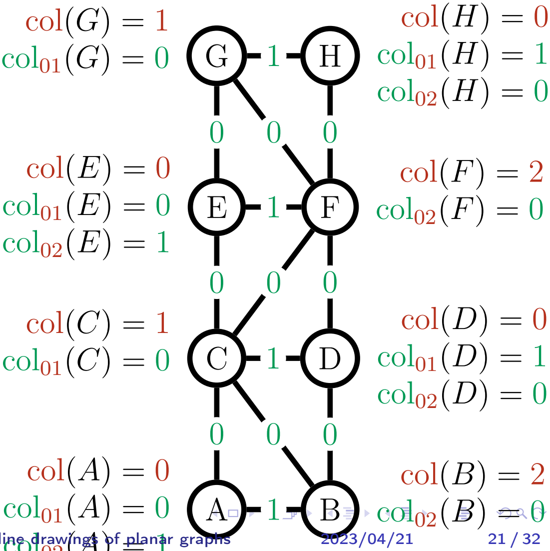

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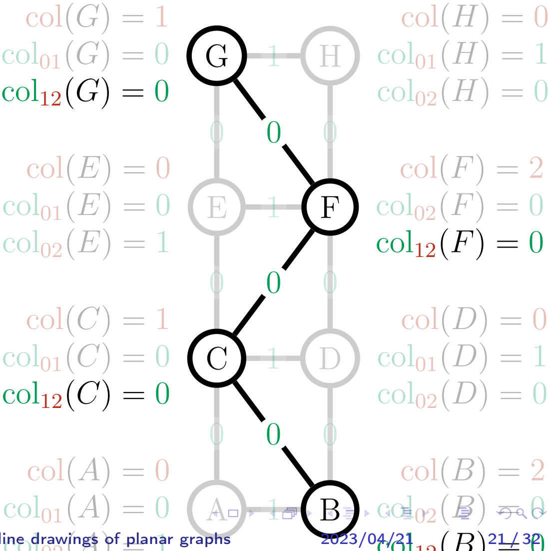

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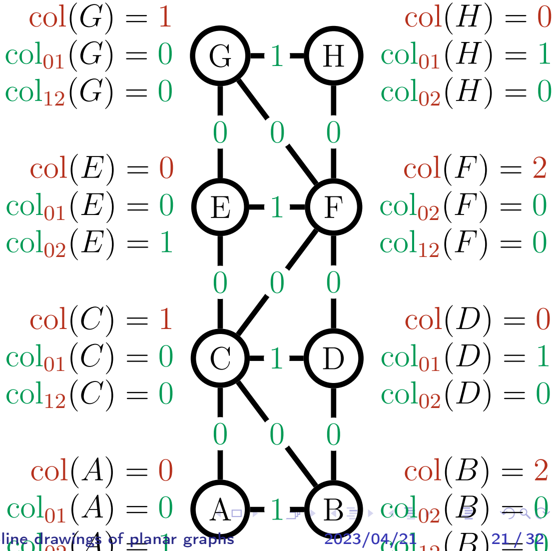

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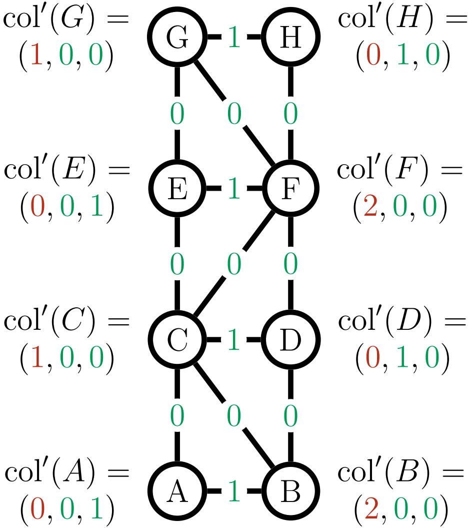

Theorem

Every graph on n vertices with queue number q and acyclic chromatic number c has track number at most $c(2q)^{c-1}$

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- But we can fix them

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Outline

Theorem

Outerplanar graphs have acyclic chromatic number at most three

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- Choose v with degree 2 and neighbors u and w.
- Color $G \setminus v$.
- Color v using the color u and w don't use.
- The addition of v can't create a new bichromatic cycle: Otherwise, u and w would have the same color, but they are neighbors.

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- Observe: Edges of G are within a single circle or between two adjacent circles.
- We will now color each circle S_i such that it cannot be the innermost circle of a bichromatic cycle.

Coloring the circles

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Color the circle C_i such that it cannot be the innermost shell of a bichromatic cycle.

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- For each vertex $v \in C_{j+1}$, let $v_1, v_2, \ldots, v_j$ be the cyclic order of the vertices in C_i adjacent to v. Add edges $v_1v_2, v_2v_3, \ldots, v_jv_1$ if they don't exist.

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So, we can color C_i^* acyclically with three colors.

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- Degree 1: Impossible, because v_0, v_2 are both neighbors of v_1
- Degree 2 or 3: Impossible, because v_0, v_2 are neighbors in C_i^*
- Degree 4 or more: Take the vertices to the left and right of v_0 in cyclic ordering around v_1. They, together with v_1 and $C_{<i}$, cut off v_0 (or v_1) from the other end of the cycle.

□

Concluding the proof

Theorem

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Proof (cont.)

We have shown that, using our coloring, no bichromatic cycle exists. It is easy to see that the coloring is proper. Therefore, we have found an acyclic 9-coloring. □

Better bounds

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Planar graphs have acyclic chromatic number at most five.

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Same as our proof for outerplanar graphs (i.e. reducing any graph to another graph with less vertices), except with 450 cases. □

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Theorem

Some planar graphs have acyclic chromatic number five.

Summary

	Queue Number	Acyclic Chromatic Number	Track Number	3D Drawing Volume
Proven now	42			

Summary

	Queue Number	Acyclic Chromatic Number	Track Number	3D Drawing Volume
Proven now	42	9		

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	Queue Number	Acyclic Chromatic Number	Track Number	3D Drawing Volume
Proven now	42	9	$9 \cdot 10^{13}$	

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Proven now	42	9	$9 \cdot 10^{13}$	$8 \cdot 10^{27} \cdot n$

Summary

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Proven now	42	9	$9 \cdot 10^{13}$	$8 \cdot 10^{27} \cdot n$
Proven anytime	42			

Summary

	Queue Number	Acyclic Chromatic Number	Track Number	3D Drawing Volume
Proven now	42	9	$9 \cdot 10^{13}$	$8 \cdot 10^{27} \cdot n$
Proven anytime	42	5		

Summary

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Proven now	42	9	$9 \cdot 10^{13}$	$8 \cdot 10^{27} \cdot n$
Proven anytime	42	5	$3 \cdot 10^{7}$	

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Proven now	42	9	$9 \cdot 10^{13}$	$8 \cdot 10^{27} \cdot n$
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Best lower bound of qn	4			

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Best lower bound of qn	4	5	1280	$1.5 \cdot 10^{6} \cdot n$

Further results

- Some of the steps we've taken also work in reverse:
  - Queue number is bounded by track number.
  - If a graph has a $\mathcal{O}(1) \times \mathcal{O}(1) \times \mathcal{O}(n)$ drawing, then it has track number $\mathcal{O}(1)$
- Acyclic chromatic number is also bounded by queue number
  - In particular, any proper minor-closed family has 3D drawings with linear volume

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  - Adjacent vertices being close