

Three-dimensional straight-line drawings of planar graphs

Tobias Feigenwinter

ETH Zürich

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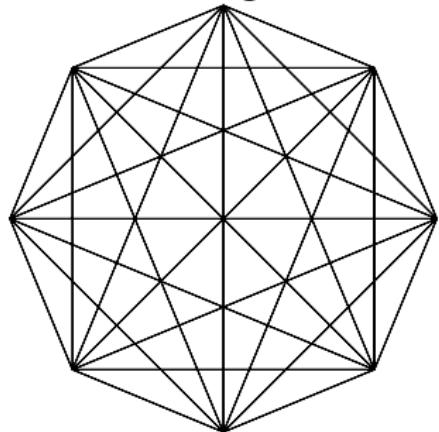
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- ▶ “Nice” drawings

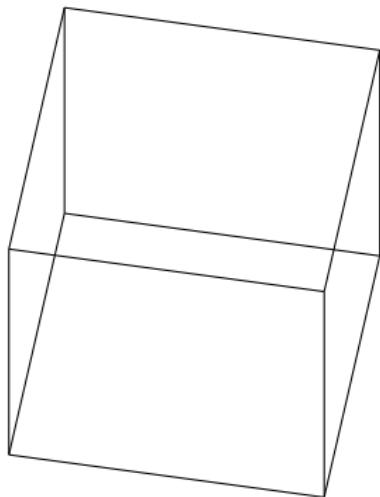
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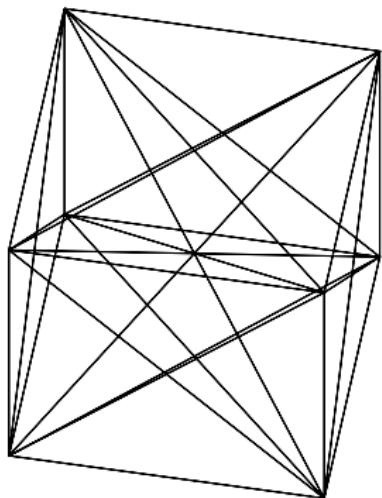
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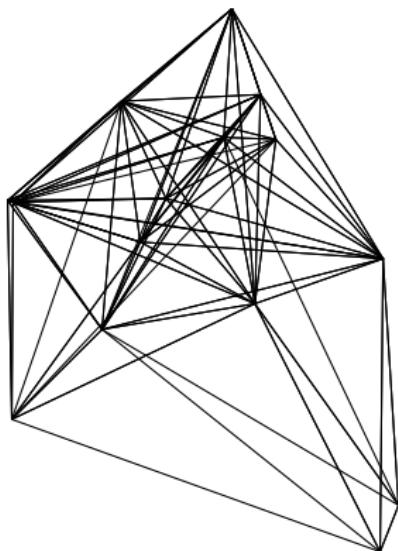
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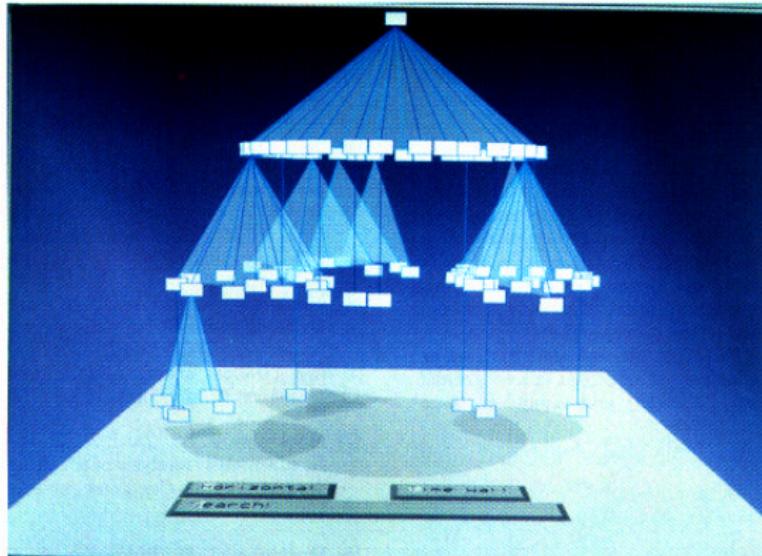


Figure from: George, Jock and Stuart (1991): *Cone Trees: Animated 3D Visualizations of Hierarchical Information*
Tobias Feigenwinter (ETH Zürich) 3D straight-line drawings of planar graphs

Why 3D drawings?

- ▶ “Nice” drawings
- ▶ Visualization of data with spacial information
- ▶ User Interfaces
 - ▶ Virtual Reality?

Main result

Theorem (Dujmović et al. 2020)

Every planar graph with n vertices has a 3-dimensional crossing-free straight-line drawing on the integer grid

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Theorem (Dujmović et al. 2020)

Every planar graph with n vertices has a 3-dimensional crossing-free straight-line drawing on the integer grid with $\mathcal{O}(n)$ Volume.

Why is this interesting?

The Moment Curve

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Upper bound for general graphs

Theorem (Cohen et al. 1996)

Every graph on n vertices has a $\mathcal{O}(n^3)$ 3D crossing-free straight-line drawing.

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We try drawing a complete graph

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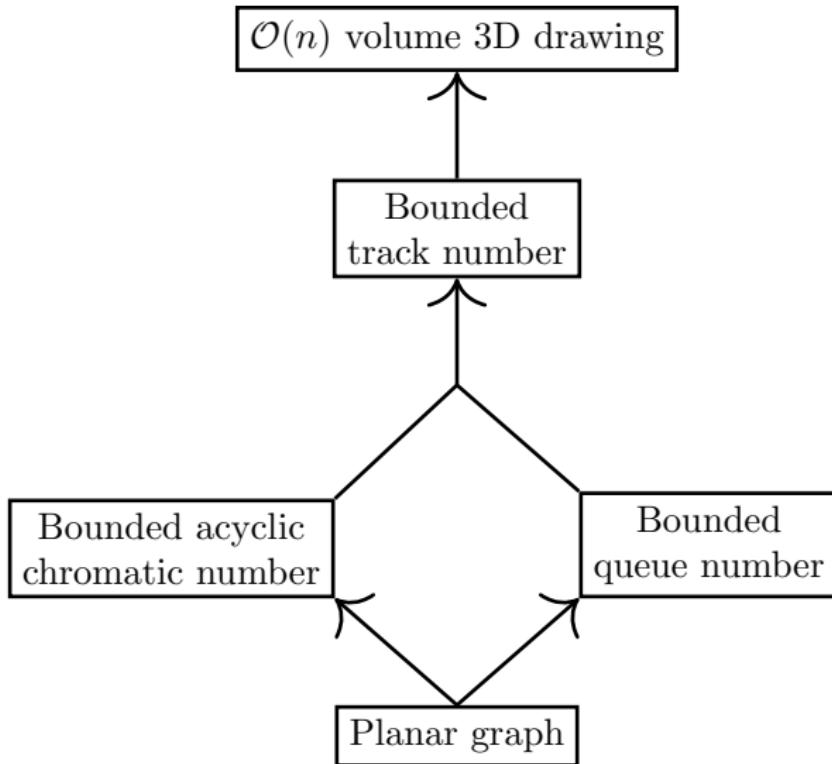
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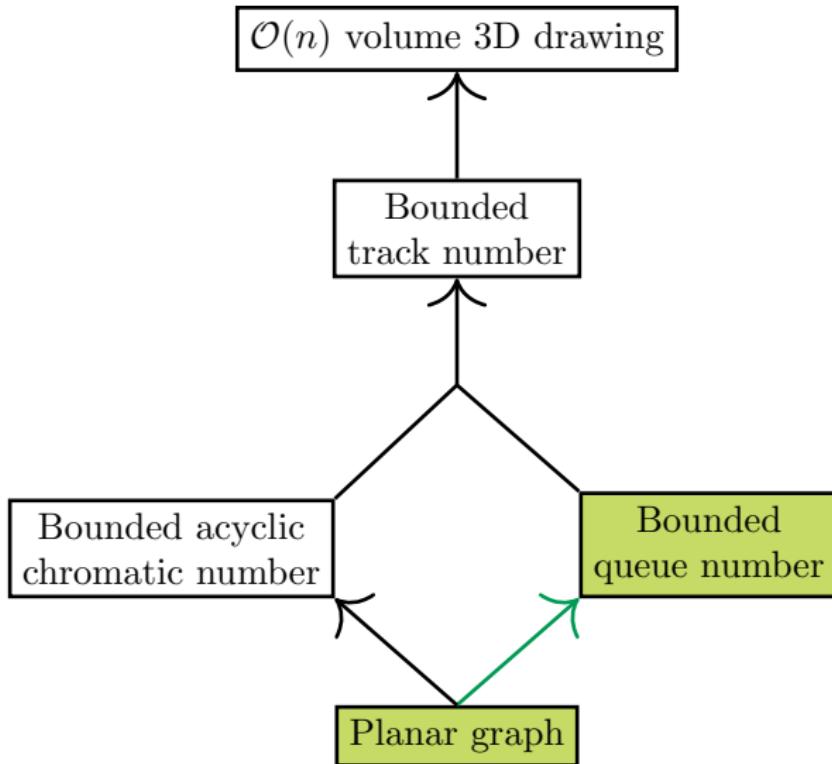
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- ▶ At least $\lceil \frac{n}{4} \rceil^3 \in \Omega(n^3)$ Volume

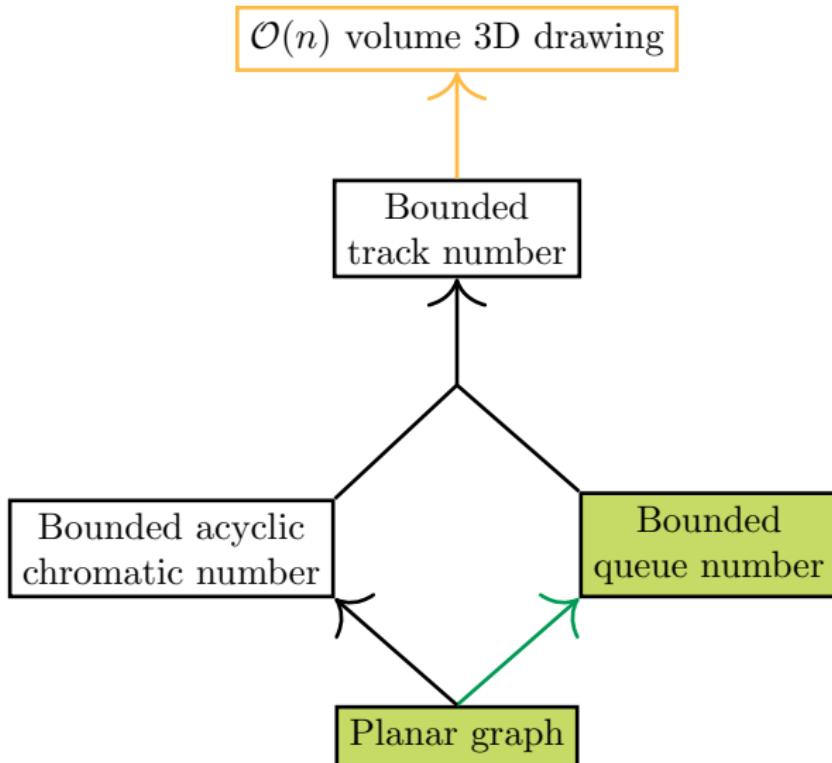
Outline of the proof



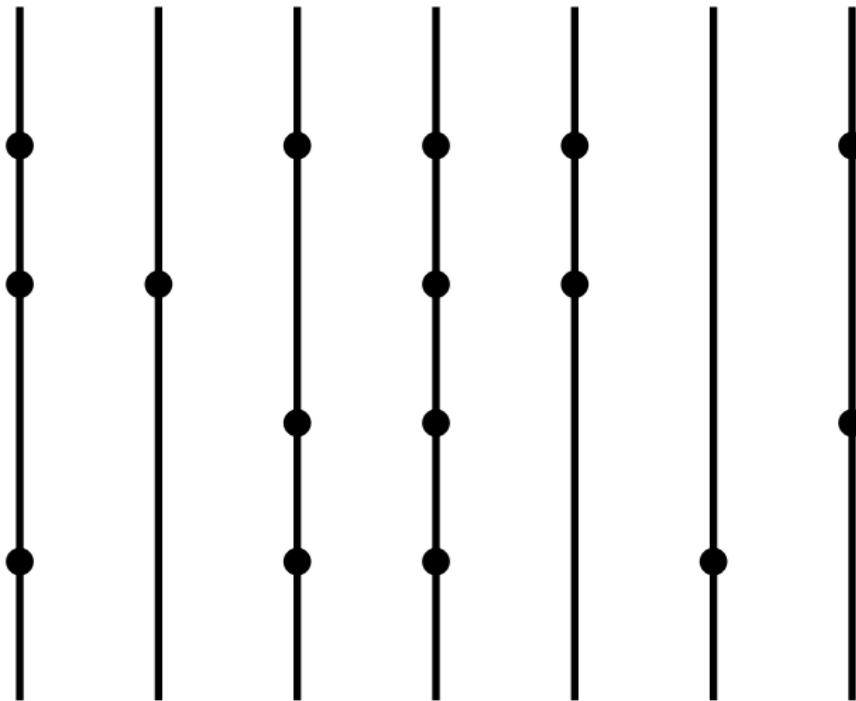
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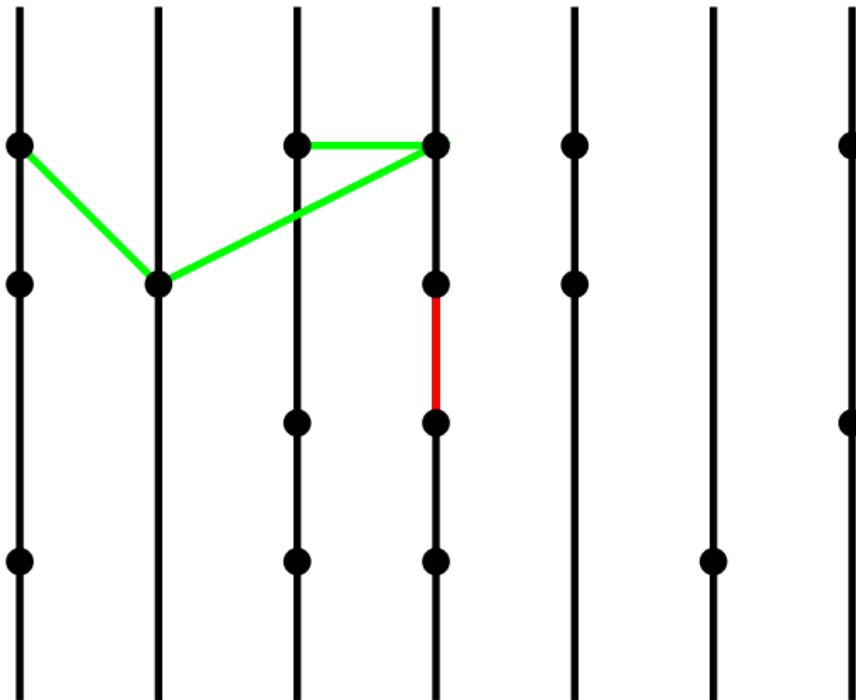
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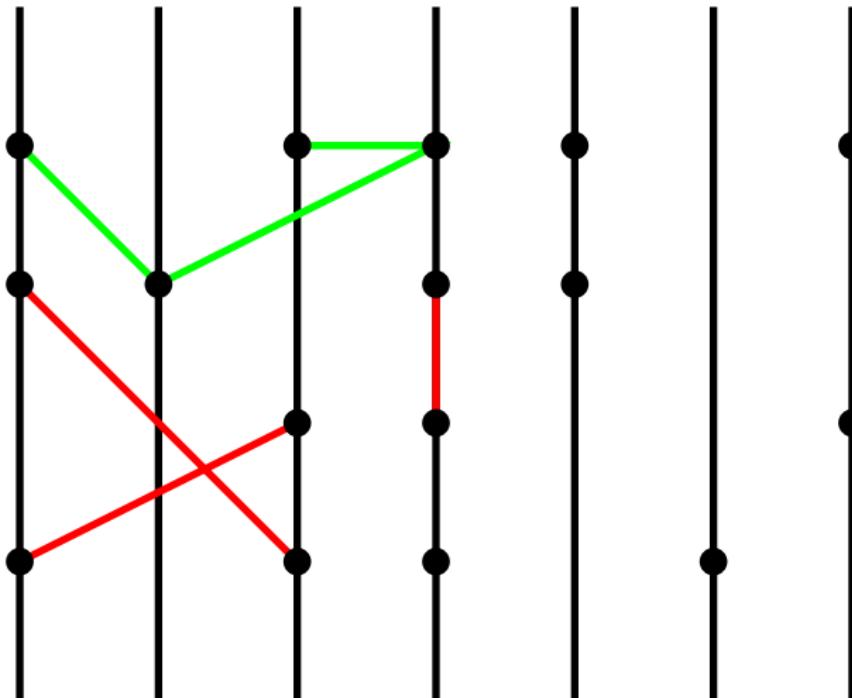
Track Layouts



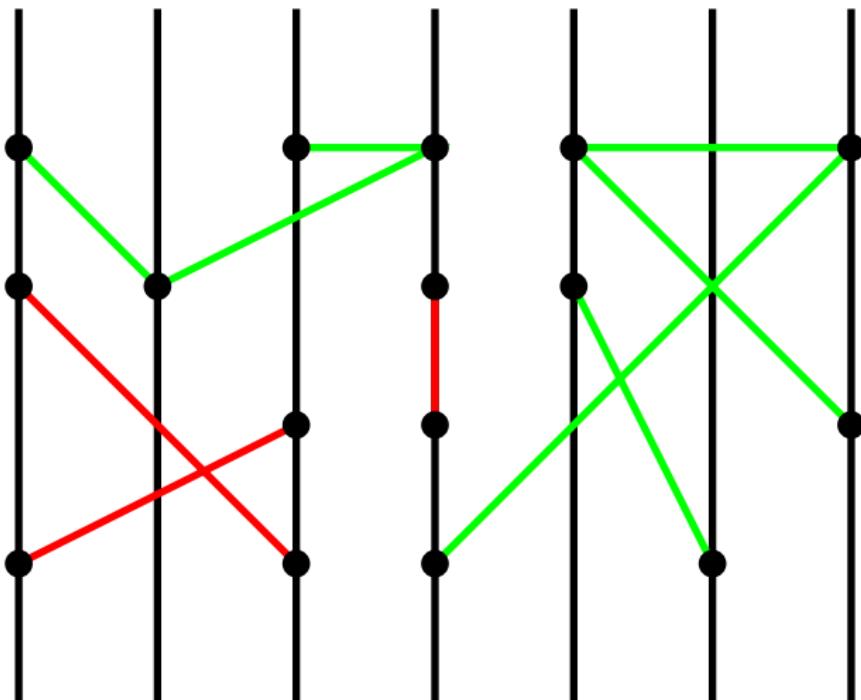
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The Modular Moment Curve with tracks

- ▶ Let p prime, $t \leq p < 2t$
- ▶ Place j -th vertex of i -th track at $(i, i^2 \pmod{p}, i^3 \pmod{p} + jp)$

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Case 1: All i 's different

- ▶ Four vertices are coplanar iff this matrix has determinant zero:

$$M = \begin{pmatrix} 1 & i_0 & i_0^2 & \pmod{p} & i_0^3 & \pmod{p} + j_0 p \\ 1 & i_1 & i_1^2 & \pmod{p} & i_1^3 & \pmod{p} + j_1 p \\ 1 & i_2 & i_2^2 & \pmod{p} & i_2^3 & \pmod{p} + j_2 p \\ 1 & i_3 & i_3^2 & \pmod{p} & i_3^3 & \pmod{p} + j_3 p \end{pmatrix}$$

- ▶ $\det(M) \equiv \prod_{0 \leq k < l < 4} \underbrace{(i_l - i_k)}_{\not\equiv 0} \pmod{p} \not\equiv 0 \pmod{p}$

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- ▶ Let p prime, $t \leq p < 2t$
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Case 2: Two i 's the same, WLOG $i_0 = i_1$

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- ▶ $\det(M) =$
- $$\underbrace{(j_1 p - j_0 p)}_{\neq 0} \cdot \det \underbrace{\begin{pmatrix} 1 & i_1 & i_1^2 & \pmod{p} \\ 1 & i_2 & i_2^2 & \pmod{p} \\ 1 & i_3 & i_3^2 & \pmod{p} \end{pmatrix}}_{\neq 0} \neq 0$$

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- ▶ Edges are between the same track
- ▶ Crossing in the 3D drawing would also be crossing in track layout

The Modular Moment Curve with tracks

- ▶ Let p prime, $t \leq p < 2t$
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Case 4: Three or more i s are the same

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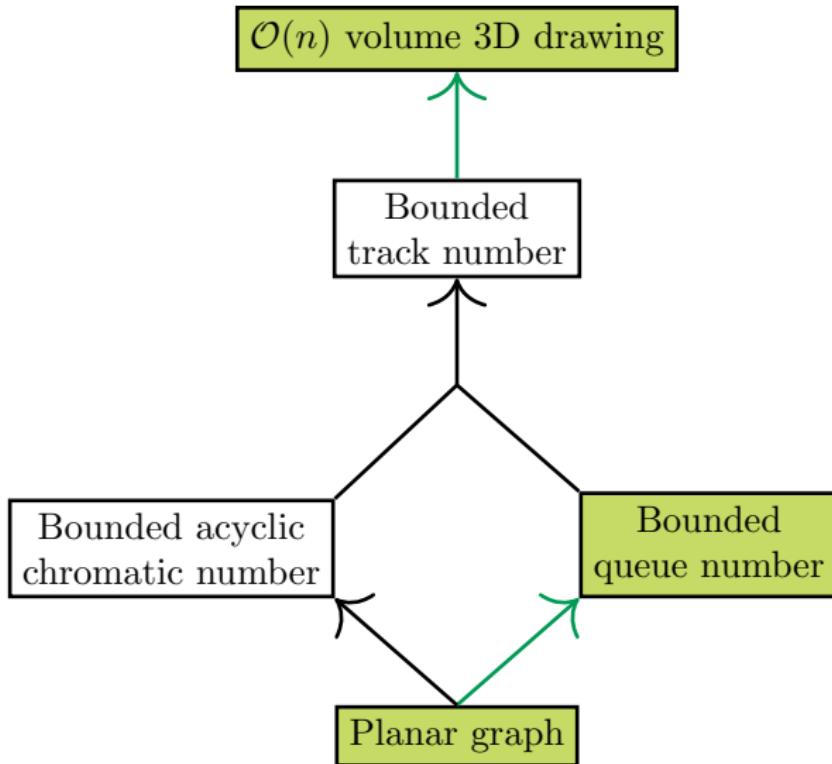
- ▶ Impossible: requires intertrack edges

The Modular Moment Curve with tracks

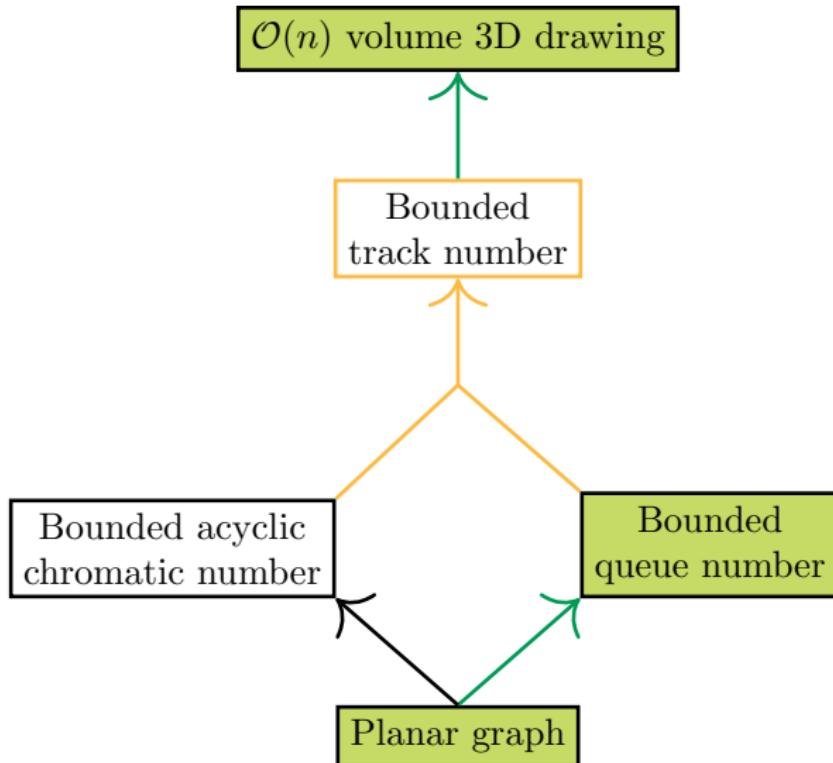
Theorem (Dujmović et al. 2005)

Every graph on n vertices with track number t has a 3-dimensional crossing-free straight-line drawing on the integer grid with $\mathcal{O}(t^3 n)$ volume.

Outline of the proof

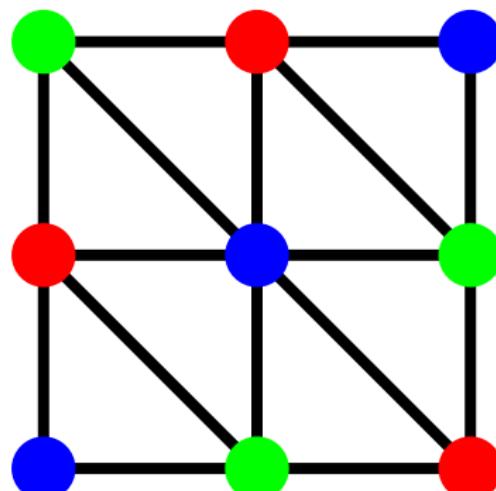


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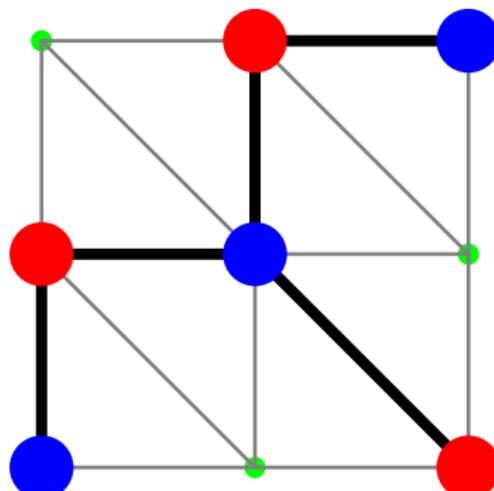
Definition (Acyclic coloring)

An acyclic coloring is a proper coloring such that every bichromatic subgraph is a forest



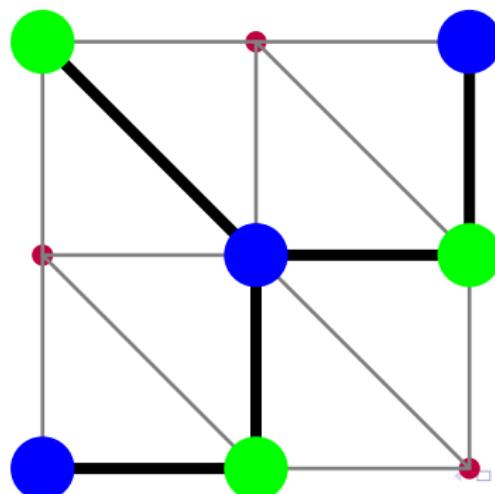
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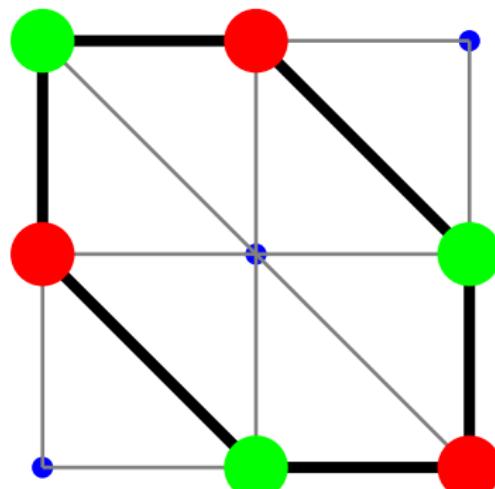
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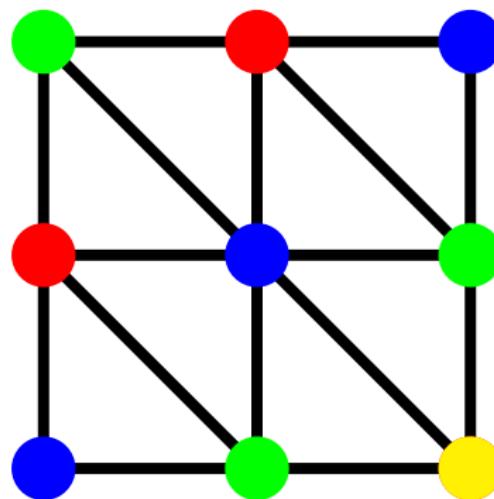
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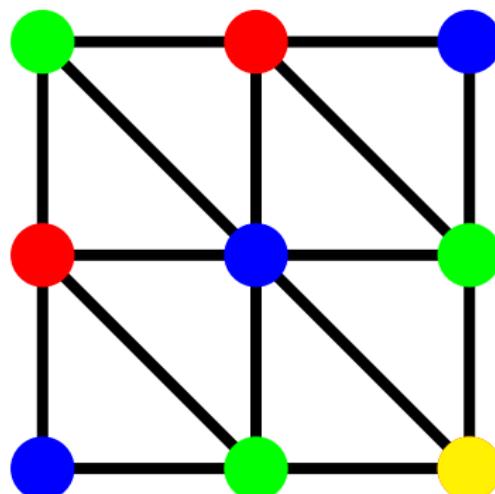
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An acyclic coloring is a proper coloring such that every bichromatic subgraph is a forest, or equivalently, such that every cycle receives at least three colors.

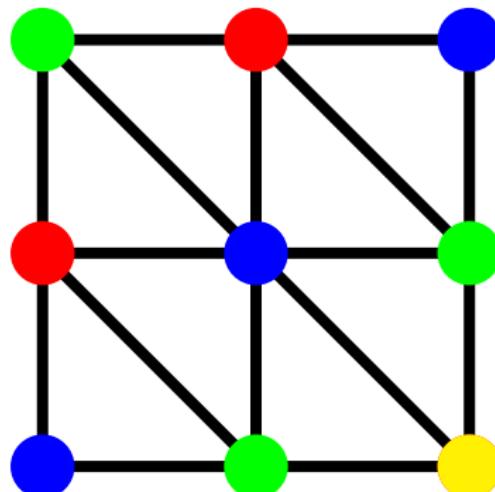


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Definition (Acyclic Chromatic Number)

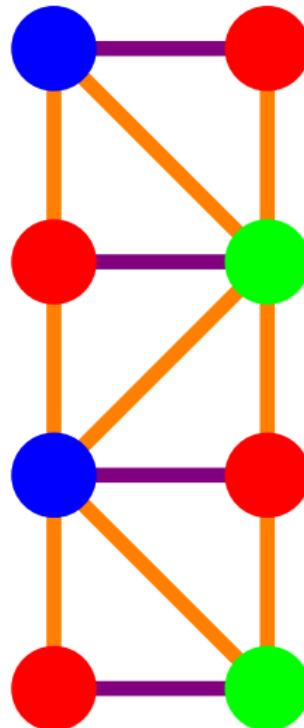
The acyclic chromatic number $\chi_a(G)$ is the smallest number of colors across all acyclic colorings of G .



The Refinement Lemma

Lemma (Alon and Marshall 1998)

Given a graph with

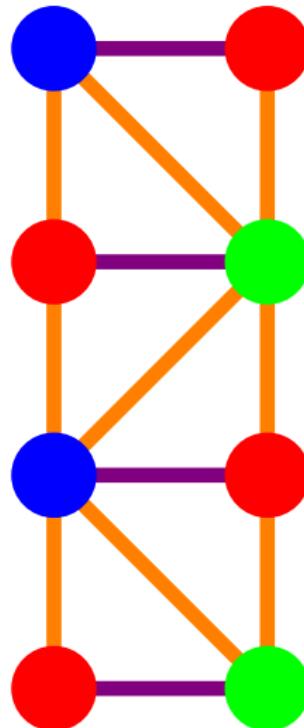


The Refinement Lemma

Lemma (Alon and Marshall 1998)

Given a graph with

- ▶ *an acyclic c -coloring*

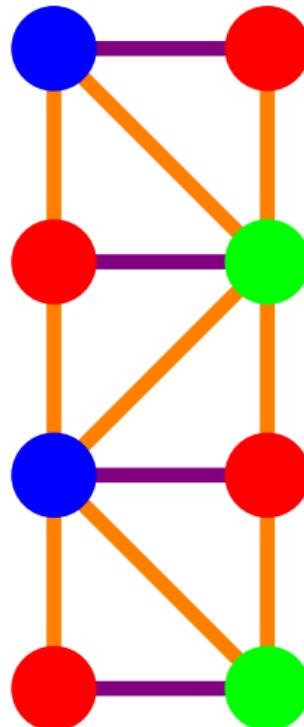


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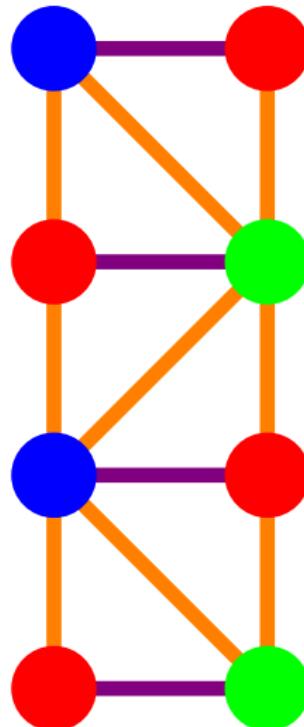
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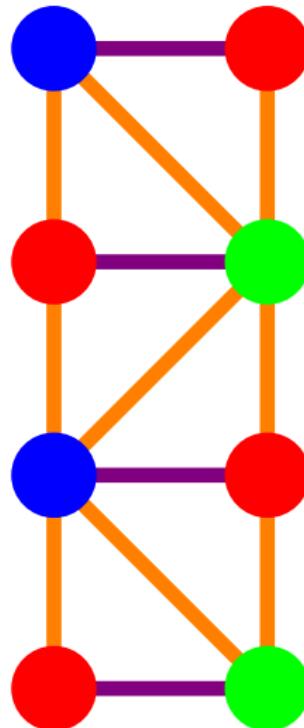
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- ▶ every new vertex color class is contained in one old vertex color class, and



The Refinement Lemma

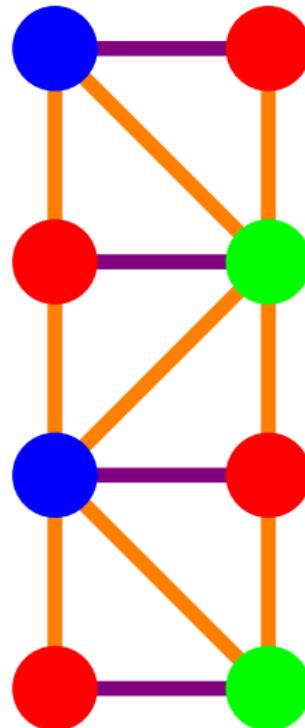
Lemma (Alon and Marshall 1998)

Given a graph with

- ▶ an acyclic c -coloring
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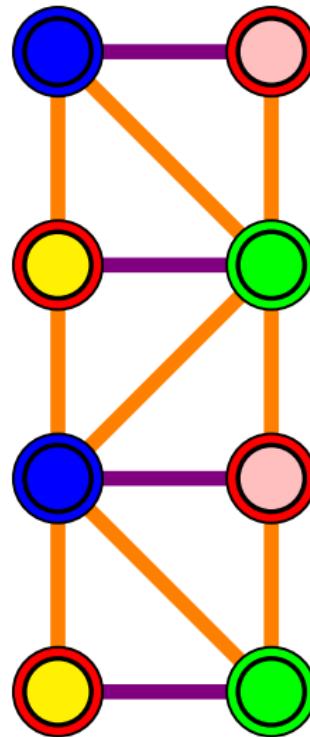
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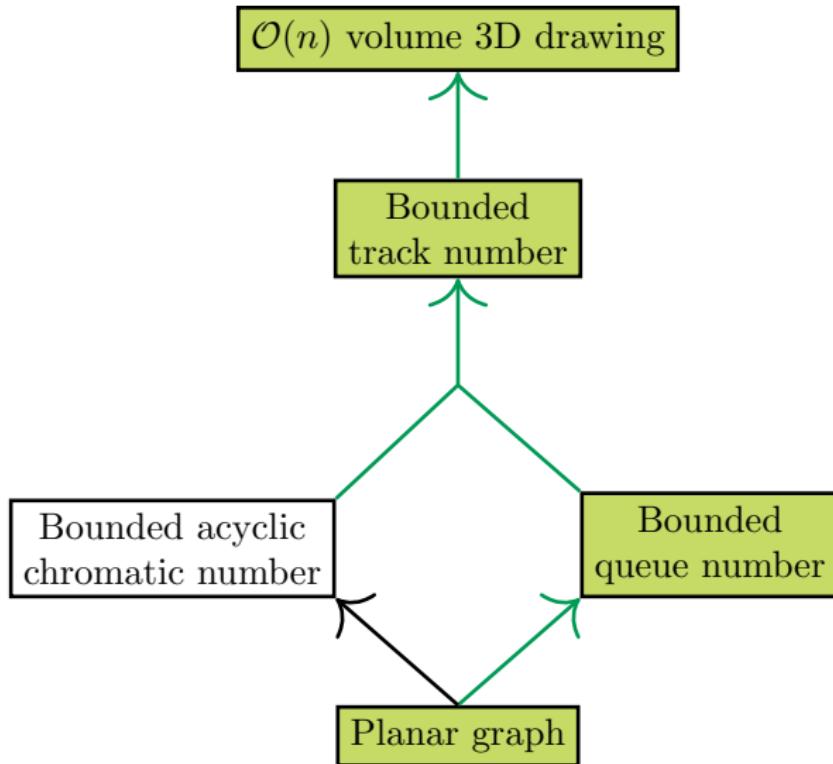
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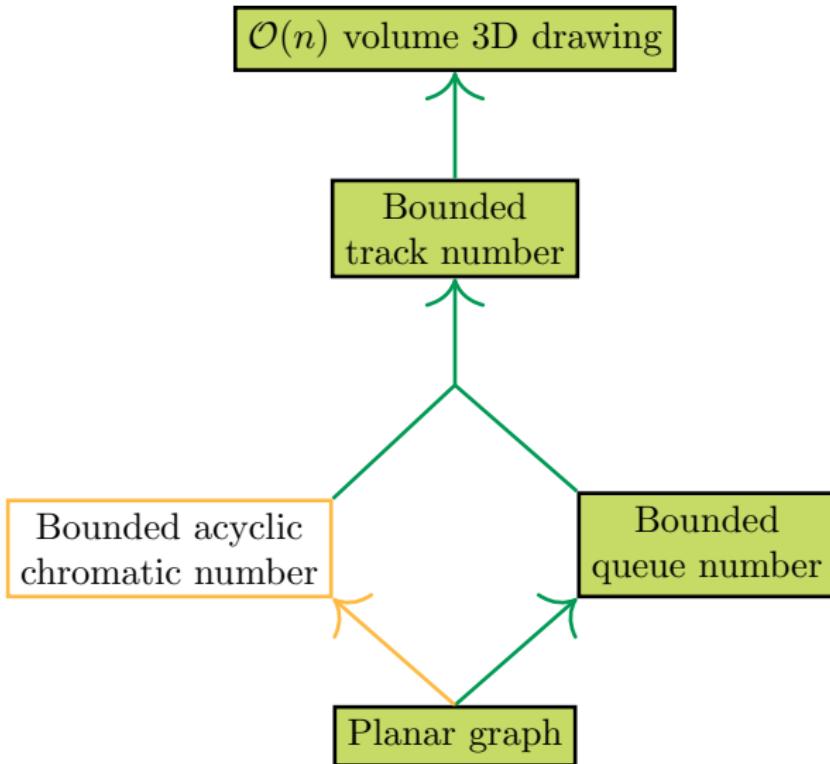
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- ▶ Observe: Edges of G are within a single shell or between two adjacent shells.
- ▶ We will now color each shell S_i such that it cannot be the innermost shell of a bichromatic cycle.

Coloring the shells

Task

Color the shell S_i such that it cannot be the innermost shell of a bichromatic cycle.

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We have shown that, using our coloring, no bichromatic cycle exists. It is easy to see that the coloring is proper. Therefore, we have found an acyclic 9-coloring. □

Better bounds

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Planar graphs have acyclic chromatic number at most five.

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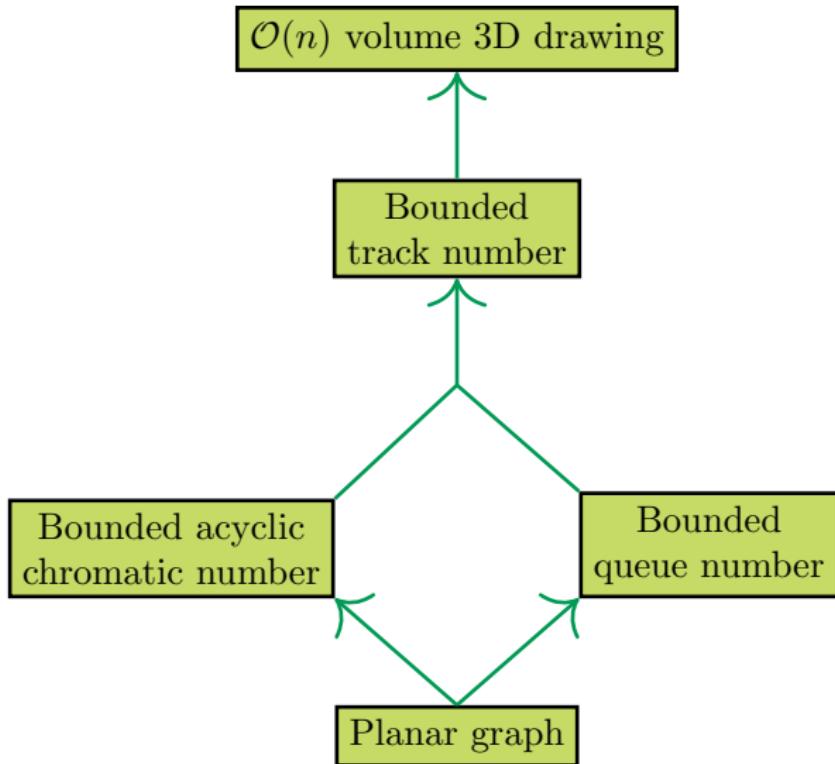
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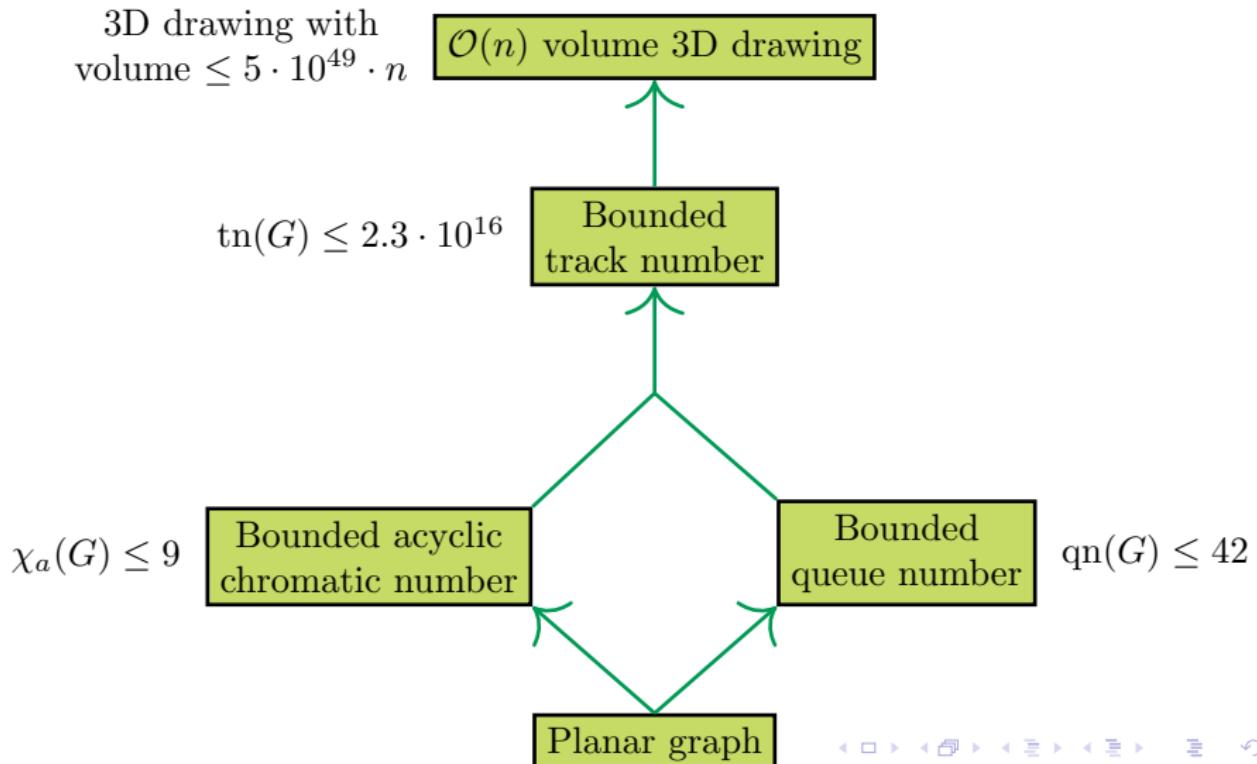
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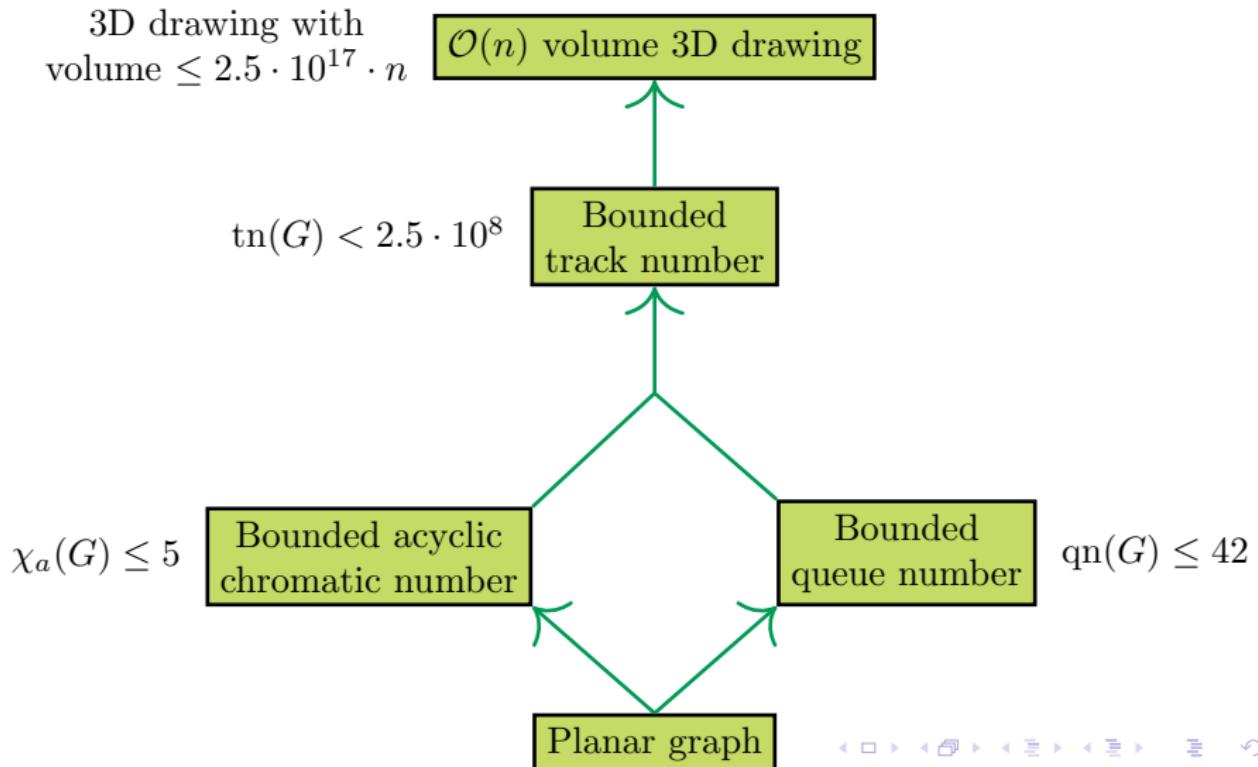
Summary



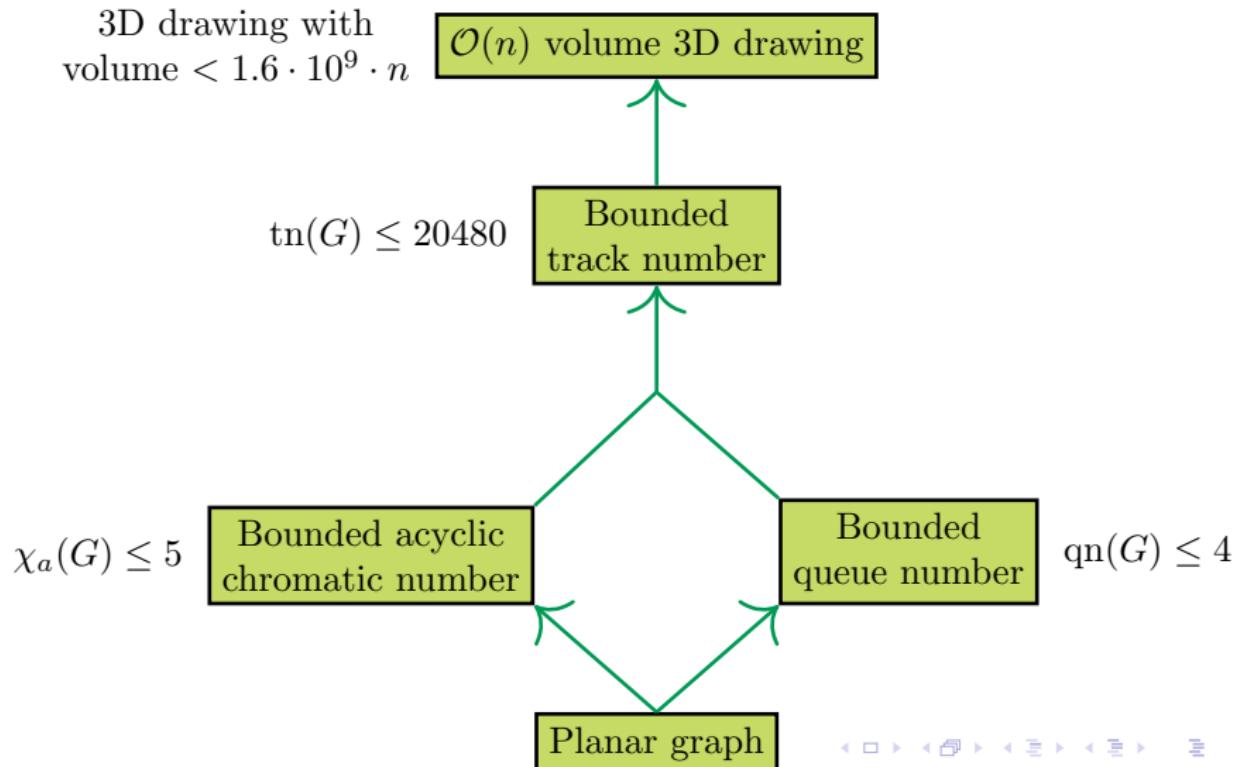
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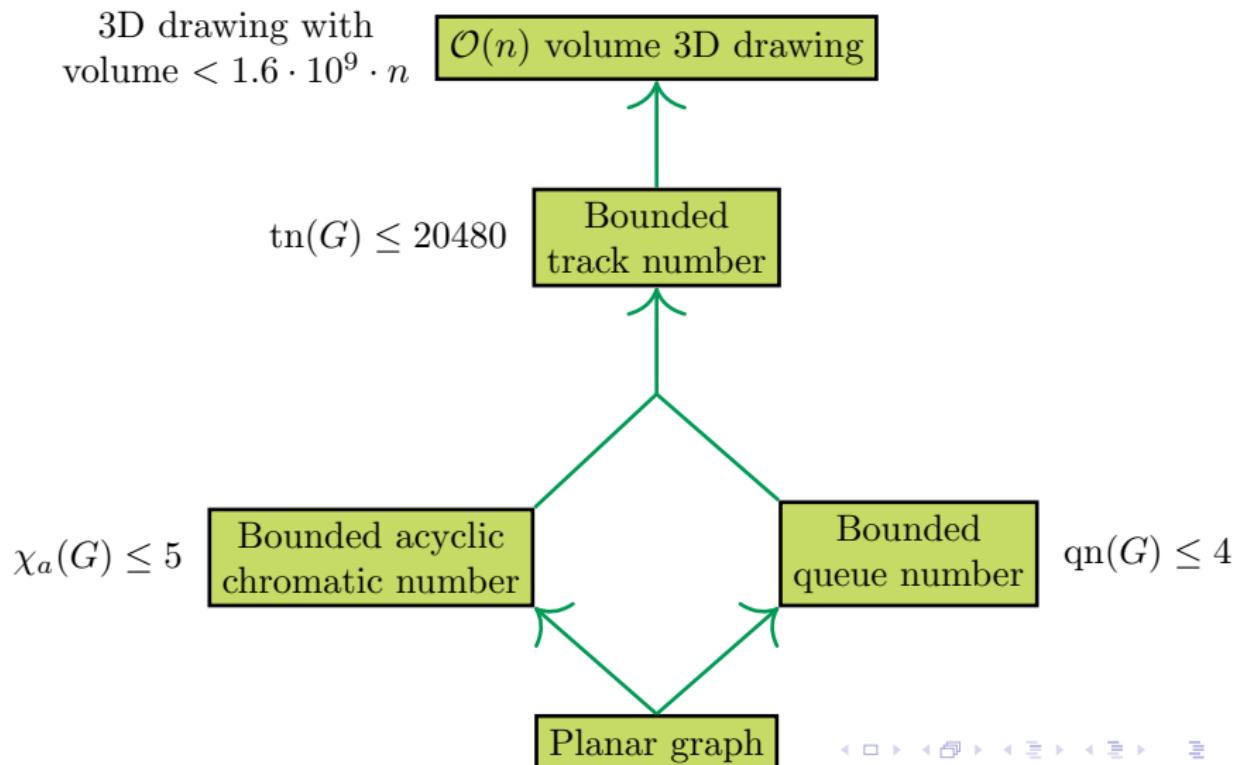
Best known upper bounds



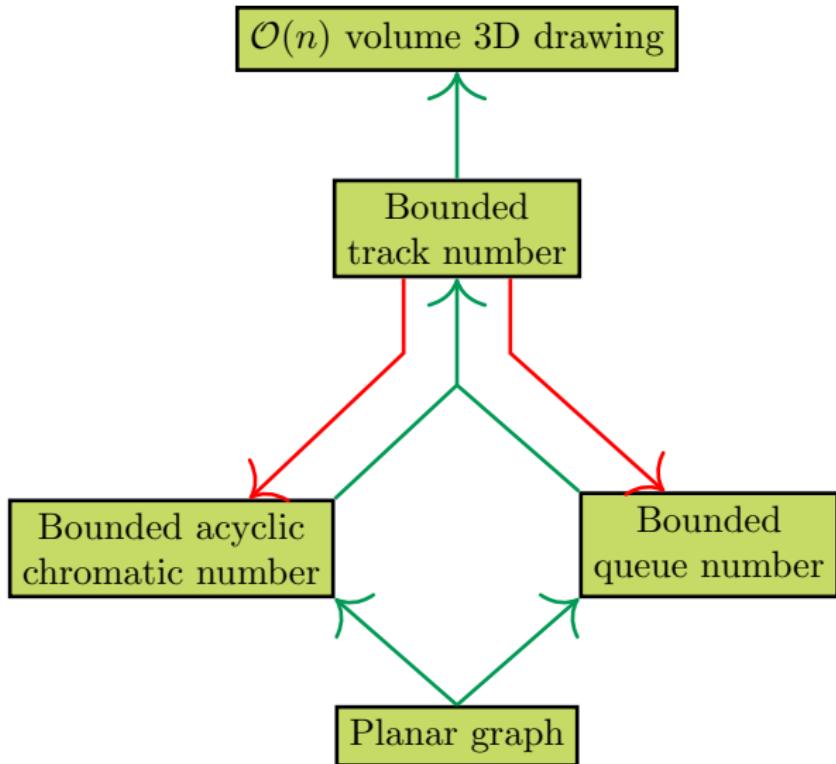
Best known lower bound of queue number



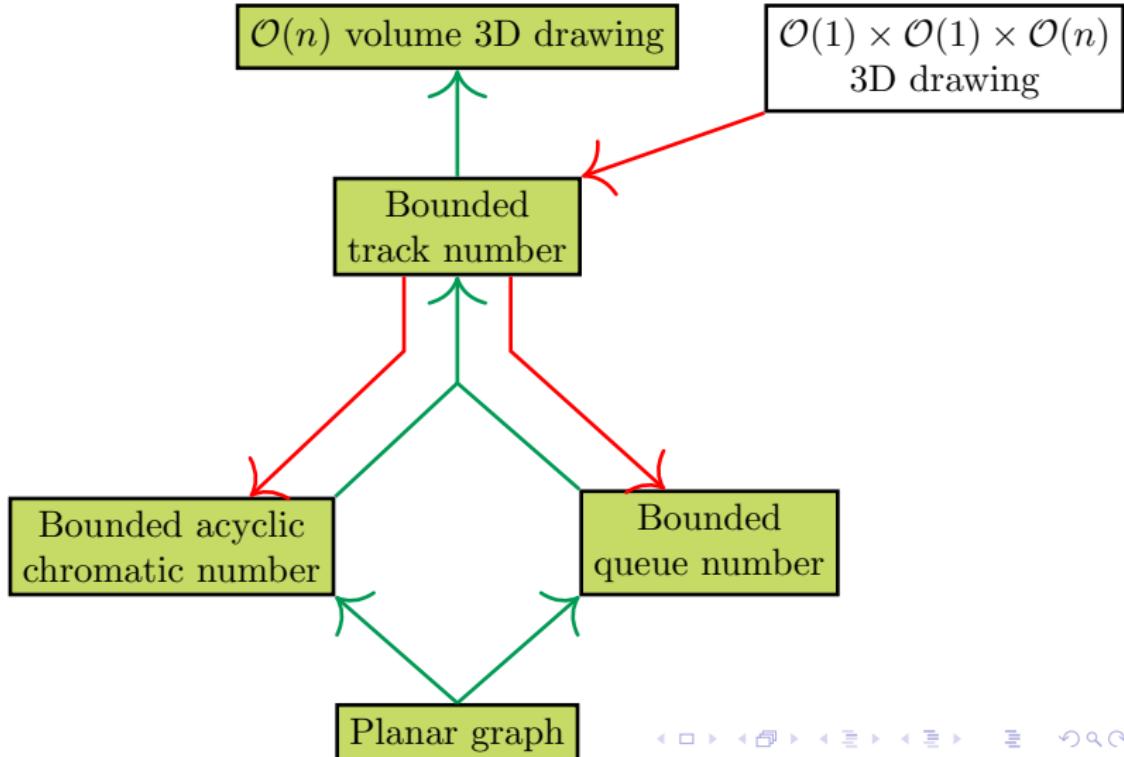
Further results



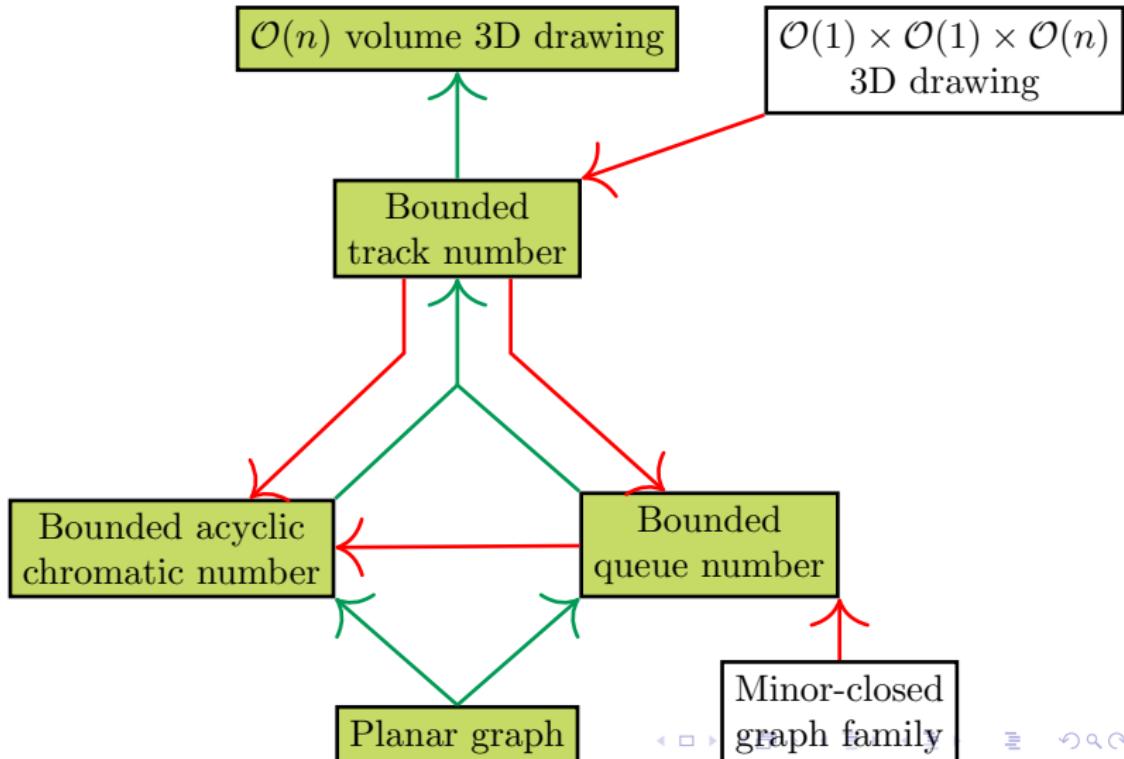
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