Three-dimensional straight-line drawings of planar graphs

Tobias Feigenwinter

ETH Zürich

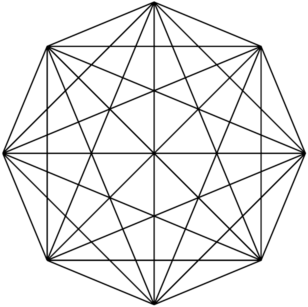
2023/04/21

Why 3D drawings?

- "Nice" drawings

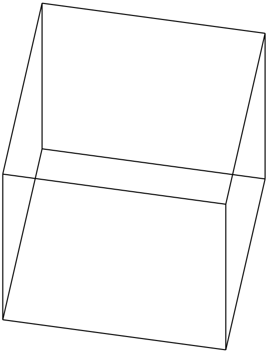
Why 3D drawings?

- "Nice" drawings



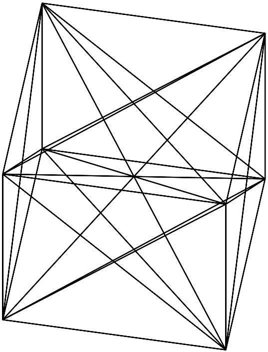
Why 3D drawings?

- "Nice" drawings



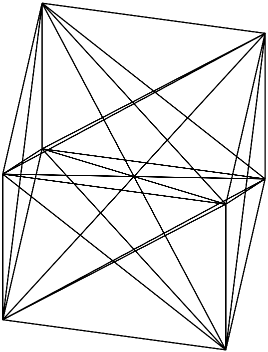
Why 3D drawings?

- "Nice" drawings



Why 3D drawings?

- "Nice" drawings

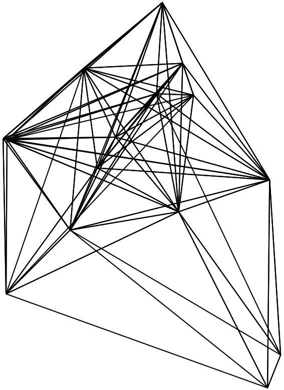


Why 3D drawings?

- "Nice" drawings
- Visualization of data with spacial information

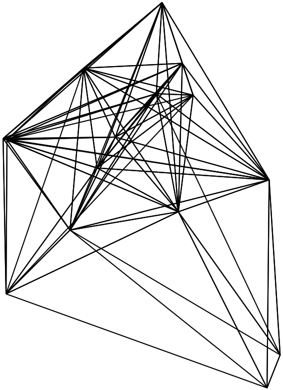
Why 3D drawings?

- "Nice" drawings
- Visualization of data with spacial information



Why 3D drawings?

- "Nice" drawings
- Visualization of data with spacial information



Why 3D drawings?

- "Nice" drawings
- Visualization of data with spacial information
- User Interfaces

Why 3D drawings?

- "Nice" drawings
- Visualization of data with spacial information
- User Interfaces

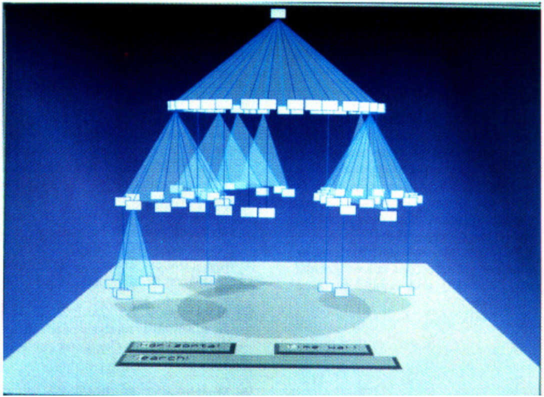


Figure from: George, Jock and Stuart (1991): *Cone Trees: Animated 3D Visualizations of Hierarchical Information*

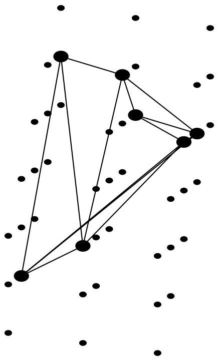
Why 3D drawings?

- "Nice" drawings
- Visualization of data with spacial information
- User Interfaces
  - Virtual Reality?

Main result

Theorem

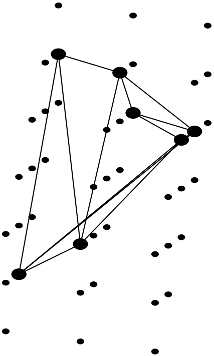
Every planar graph with n vertices has a 3-dimensional crossing-free straight-line drawing on the integer grid



Main result

Theorem

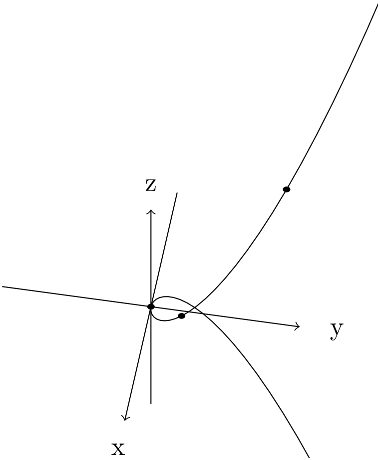
Every planar graph with n vertices has a 3-dimensional crossing-free straight-line drawing on the integer grid with $\mathcal{O}(n)$ Volume.



Why is this interesting?

The Moment Curve

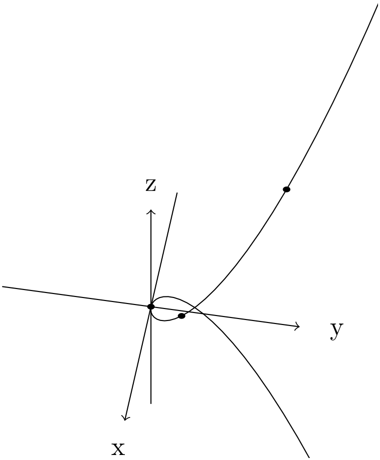
- Place i-th vertex at (i, i^2, i^3)



The Moment Curve

- Place i-th vertex at (i, i^2, i^3)
- Four vertices are coplanar iff this matrix has determinant zero:

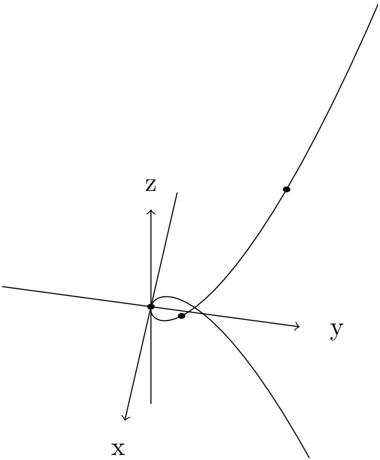
$$M = \begin{pmatrix} & i_0 & i_0^2 & i_0^3 \\ & i_1 & i_1^2 & i_1^3 \\ & i_2 & i_2^2 & i_2^3 \\ & i_3 & i_3^2 & i_3^3 \end{pmatrix}$$



The Moment Curve

- Place i-th vertex at (i, i^2, i^3)
- Four vertices are coplanar iff this matrix has determinant zero:

$$M = \begin{pmatrix} 1 & i_0 & i_0^2 & i_0^3 \\ 1 & i_1 & i_1^2 & i_1^3 \\ 1 & i_2 & i_2^2 & i_2^3 \\ 1 & i_3 & i_3^2 & i_3^3 \end{pmatrix}$$



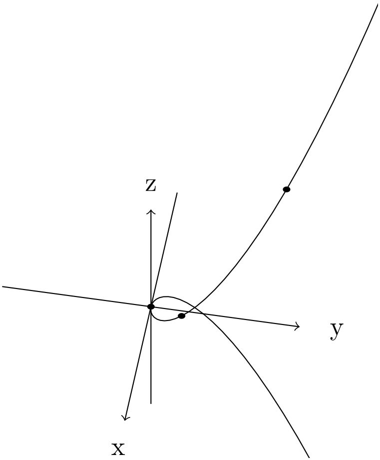
The Moment Curve

- Place i-th vertex at (i, i^2, i^3)
- Four vertices are coplanar iff this matrix has determinant zero:

$$M = \begin{pmatrix} 1 & i_0 & i_0^2 & i_0^3 \\ 1 & i_1 & i_1^2 & i_1^3 \\ 1 & i_2 & i_2^2 & i_2^3 \\ 1 & i_3 & i_3^2 & i_3^3 \end{pmatrix}$$

- This is a Vandermonde Matrix with

$$\det(M) = \prod_{0 \le k < l < 4} (v_l - v_k)$$



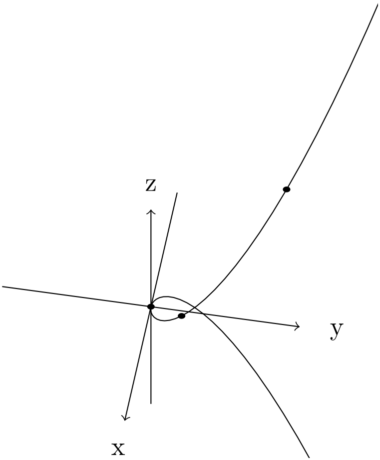
The Moment Curve

- Place i-th vertex at (i, i^2, i^3)
- Four vertices are coplanar iff this matrix has determinant zero:

$$M = \begin{pmatrix} 1 & i_0 & i_0^2 & i_0^3 \\ 1 & i_1 & i_1^2 & i_1^3 \\ 1 & i_2 & i_2^2 & i_2^3 \\ 1 & i_3 & i_3^2 & i_3^3 \end{pmatrix}$$

- This is a Vandermonde Matrix with

$$\det(M) = \prod_{0 \le k < l < 4} \underbrace{(v_l - v_k)}_{\neq 0}$$



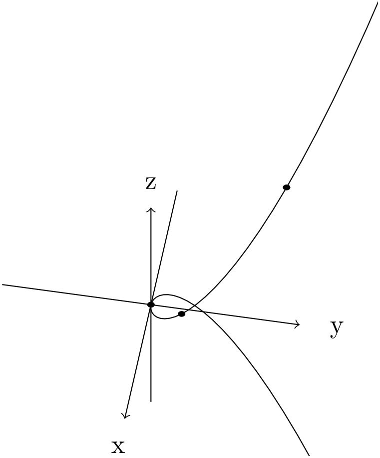
The Moment Curve

- Place i-th vertex at (i, i^2, i^3)
- Four vertices are coplanar iff this matrix has determinant zero:

$$M = \begin{pmatrix} 1 & i_0 & i_0^2 & i_0^3 \\ 1 & i_1 & i_1^2 & i_1^3 \\ 1 & i_2 & i_2^2 & i_2^3 \\ 1 & i_3 & i_3^2 & i_3^3 \end{pmatrix}$$

- This is a Vandermonde Matrix with

$$\det(M) = \prod_{0 \le k < l < 4} \underbrace{(v_l - v_k)}_{\neq 0} \neq 0$$



The Moment Curve

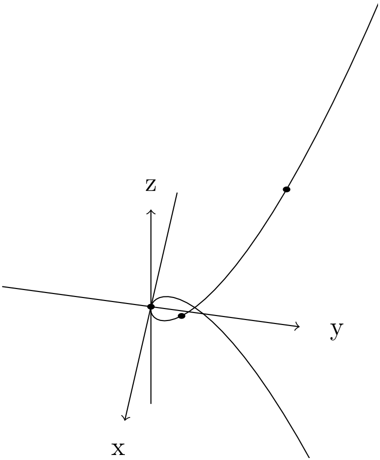
- Place i-th vertex at (i, i^2, i^3)
- Four vertices are coplanar iff this matrix has determinant zero:

$$M = \begin{pmatrix} 1 & i_0 & i_0^2 & i_0^3 \\ 1 & i_1 & i_1^2 & i_1^3 \\ 1 & i_2 & i_2^2 & i_2^3 \\ 1 & i_3 & i_3^2 & i_3^3 \end{pmatrix}$$

- This is a Vandermonde Matrix with

$$\det(M) = \prod_{0 \leq k < l < 4} \underbrace{(v_l - v_k)}_{\neq 0} \neq 0$$

- No four vertices coplanar
$\Longrightarrow$ no two edges cross

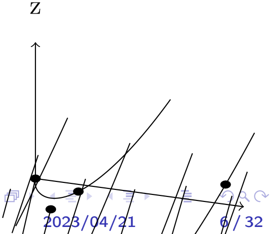


The Modular Moment Curve

- Let p prime, $n \leq p < 2n$
- Place i-th vertex at (i, i^2, i^3)
- Four vertices are coplanar iff this matrix has determinant zero:

$$M = \begin{pmatrix} 1 & i_0 & i_0^2 & i_0^3 \\ 1 & i_1 & i_1^2 & i_1^3 \\ 1 & i_2 & i_2^2 & i_2^3 \\ 1 & i_3 & i_3^2 & i_3^3 \end{pmatrix}$$

- $\det(M) = \prod_{0\leq k<l<4} \underbrace{(v_l - v_k)}_{\neq 0} \neq 0$
- No four vertices coplanar $\implies$ no two edges cross

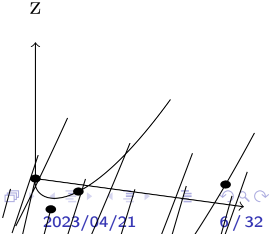


The Modular Moment Curve

- Let p prime, $n \leq p < 2n$
- Place i-th vertex at $(i, i^2 \mod p, i^3 \mod p)$
- Four vertices are coplanar iff this matrix has determinant zero:

$$M = \begin{pmatrix} 1 & i_0 & i_0^2 & i_0^3 \\ 1 & i_1 & i_1^2 & i_1^3 \\ 1 & i_2 & i_2^2 & i_2^3 \\ 1 & i_3 & i_3^2 & i_3^3 \end{pmatrix}$$

- $\det(M) = \prod_{0 \leq k < l < 4} \underbrace{(v_l - v_k)}_{\neq 0} \neq 0$
- No four vertices coplanar $\implies$ no two edges cross

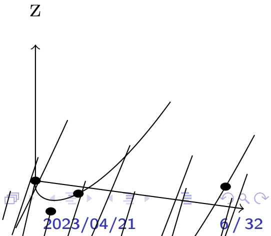


The Modular Moment Curve

- Let p prime, $n \leq p < 2n$
- Place i-th vertex at $(i, i^2 \mod p, i^3 \mod p)$
- Four vertices are coplanar iff this matrix has determinant zero:

$$M = \begin{pmatrix} 1 & i_0 & i_0^2 \mod p & i_0^3 \mod p \\ 1 & i_1 & i_1^2 \mod p & i_1^3 \mod p \\ 1 & i_2 & i_2^2 \mod p & i_2^3 \mod p \\ 1 & i_3 & i_3^2 \mod p & i_3^3 \mod p \end{pmatrix}$$

- $\det(M) = \prod_{0 \leq k < l < 4} \underbrace{(v_l - v_k)}_{\neq 0} \neq 0$
- No four vertices coplanar $\Longrightarrow$ no two edges cross

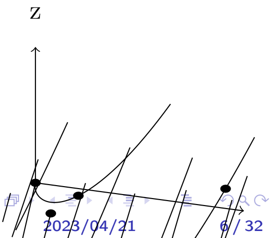


The Modular Moment Curve

- Let p prime, $n \le p < 2n$
- Place i-th vertex at $(i, i^2 \mod p, i^3 \mod p)$
- Four vertices are coplanar iff this matrix has determinant zero:

$$M = \begin{pmatrix} 1 & i_0 & i_0^2 \mod p & i_0^3 \mod p \\ 1 & i_1 & i_1^2 \mod p & i_1^3 \mod p \\ 1 & i_2 & i_2^2 \mod p & i_2^3 \mod p \\ 1 & i_3 & i_3^2 \mod p & i_3^3 \mod p \end{pmatrix}$$

- $\det(M) \equiv \prod_{0 \le k < l < 4} \underbrace{(v_l - v_k)}_{\neq 0} \neq 0 \pmod p$
- No four vertices coplanar $\implies$ no two edges cross

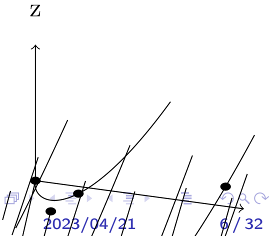


The Modular Moment Curve

- Let p prime, $n \leq p < 2n$
- Place i-th vertex at $(i, i^2 \mod p, i^3 \mod p)$
- Four vertices are coplanar iff this matrix has determinant zero:

$$M = \begin{pmatrix} 1 & i_0 & i_0^2 \mod p & i_0^3 \mod p \\ 1 & i_1 & i_1^2 \mod p & i_1^3 \mod p \\ 1 & i_2 & i_2^2 \mod p & i_2^3 \mod p \\ 1 & i_3 & i_3^2 \mod p & i_3^3 \mod p \end{pmatrix}$$

- $\det(M) \equiv \prod_{0 \leq k < l < 4} \underbrace{(v_l - v_k)}_{\not\equiv 0} \not\equiv 0 \pmod p$
- No four vertices coplanar $\implies$ no two edges cross

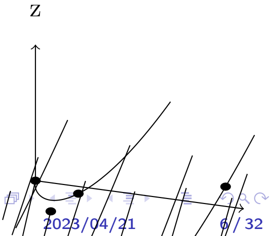


The Modular Moment Curve

- Let p prime, $n \leq p < 2n$
- Place i-th vertex at $(i, i^2 \mod p, i^3 \mod p)$
- Four vertices are coplanar iff this matrix has determinant zero:

$$M = \begin{pmatrix} 1 & i_0 & i_0^2 \mod p & i_0^3 \mod p \\ 1 & i_1 & i_1^2 \mod p & i_1^3 \mod p \\ 1 & i_2 & i_2^2 \mod p & i_2^3 \mod p \\ 1 & i_3 & i_3^2 \mod p & i_3^3 \mod p \end{pmatrix}$$

- $\det(M) \equiv \prod_{0 \leq k < l < 4} \underbrace{(v_l - v_k)}_{\not\equiv 0} \not\equiv 0 \pmod p$
- No four vertices coplanar $\implies$ no two edges cross

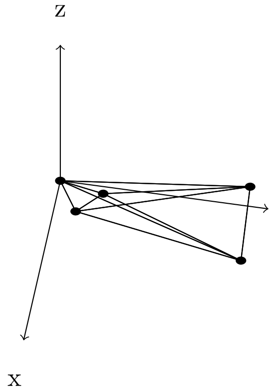


The Modular Moment Curve

- Let p prime, $n \leq p < 2n$
- Place i-th vertex at $(i, i^2 \mod p, i^3 \mod p)$
- Four vertices are coplanar iff this matrix has determinant zero:

$$M = \begin{pmatrix} 1 & i_0 & i_0^2 \mod p & i_0^3 \mod p \\ 1 & i_1 & i_1^2 \mod p & i_1^3 \mod p \\ 1 & i_2 & i_2^2 \mod p & i_2^3 \mod p \\ 1 & i_3 & i_3^2 \mod p & i_3^3 \mod p \end{pmatrix}$$

- $\det(M) \equiv \prod_{0 \leq k < l < 4} \underbrace{(v_l - v_k)}_{\not\equiv 0} \not\equiv 0 \pmod p$
- No four vertices coplanar $\implies$ no two edges cross

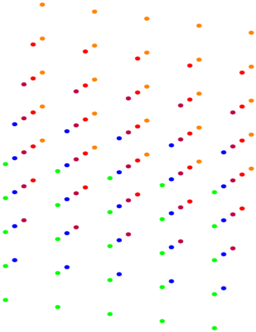


Upper bound for general graphs

Theorem

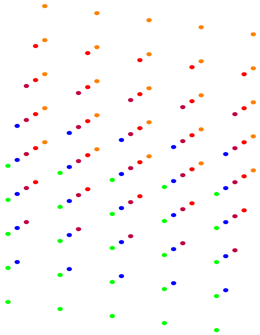
Every graph on n vertices has a $\mathcal{O}(n^3)$ 3D crossing-free straight-line drawing.

Can we do better?



Can we do better?

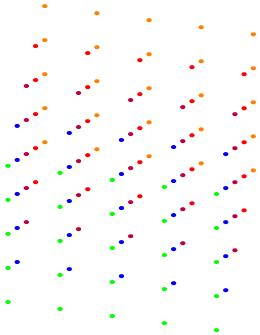
We try drawing a complete graph



Can we do better?

We try drawing a complete graph

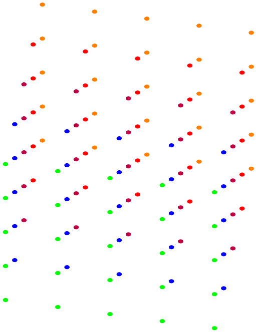
- Cut grid into planes



Can we do better?

We try drawing a complete graph

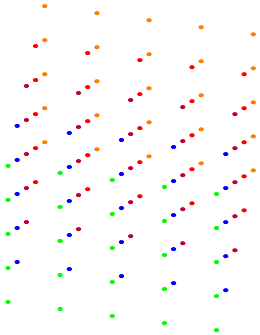
- Cut grid into planes
- At most four vertices per plane: K_5 doesn't fit



Can we do better?

We try drawing a complete graph

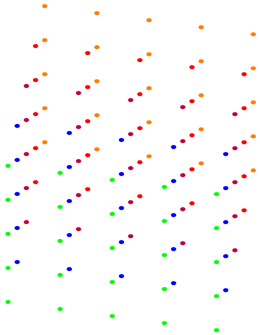
- Cut grid into planes
- At most four vertices per plane: K_5 doesn't fit
- At least $\lceil \frac{n}{4} \rceil$ planes in each dimension



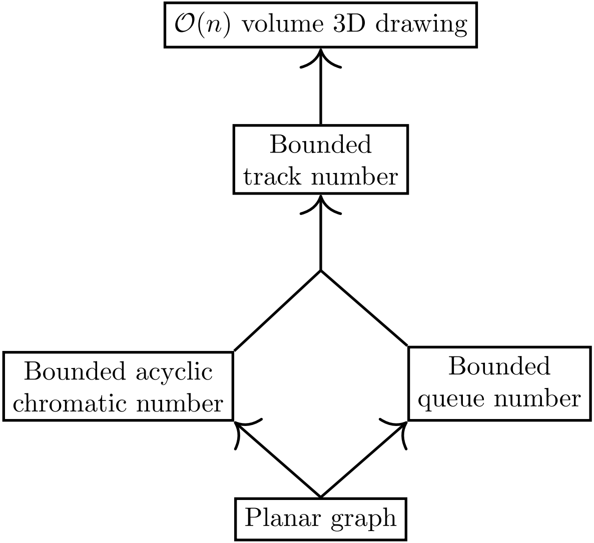
Can we do better?

We try drawing a complete graph

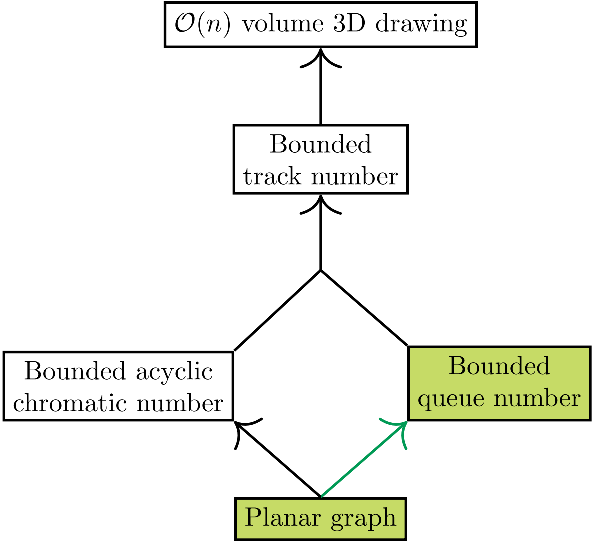
- Cut grid into planes
- At most four vertices per plane: K_5 doesn't fit
- At least $\lceil \frac{n}{4} \rceil$ planes in each dimension
- At least $\lceil \frac{n}{4} \rceil^3 \in \Omega(n^3)$ Volume



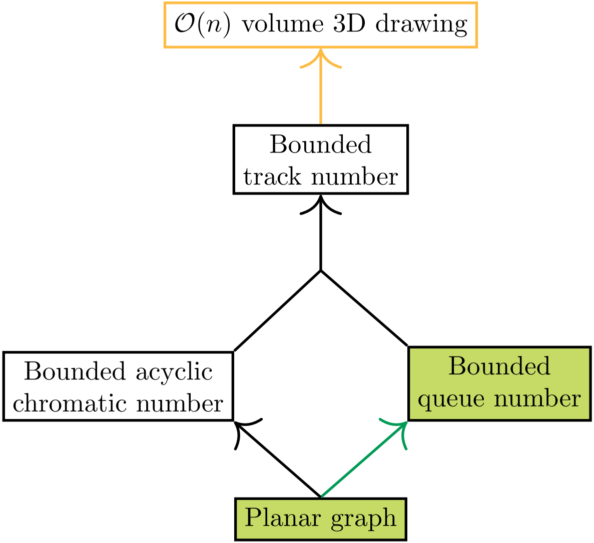
Outline of the proof



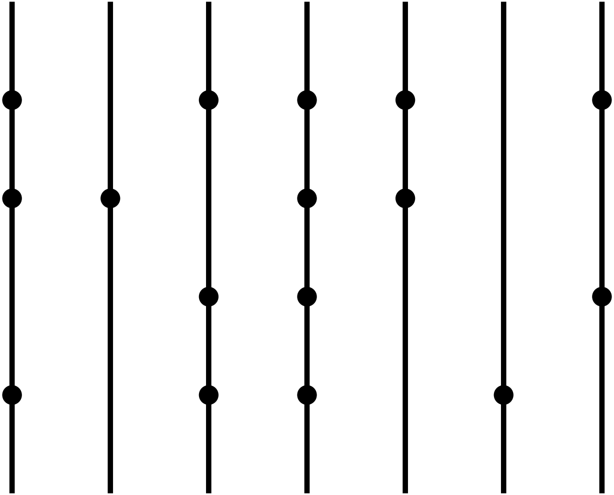
Outline of the proof



Outline of the proof

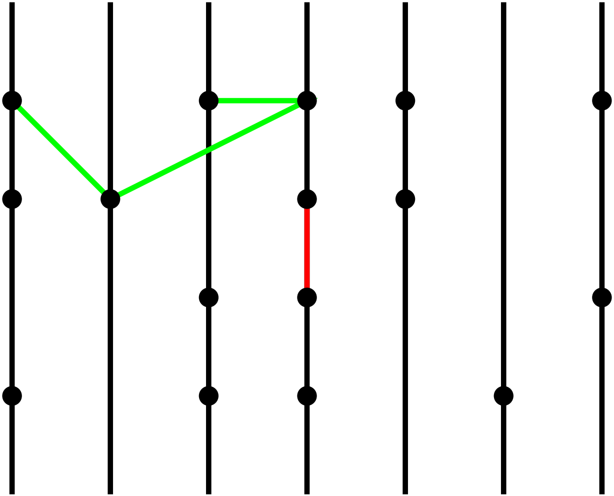


Track Layouts



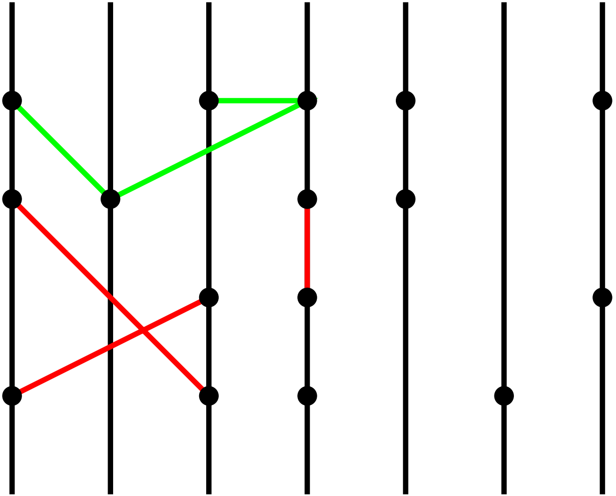
Theorem from: Dujmović, Joret, Micek, Morin, Ueckerdt and Wood (2020): *Planar Graphs Have Bounded Queue Number*

Track Layouts



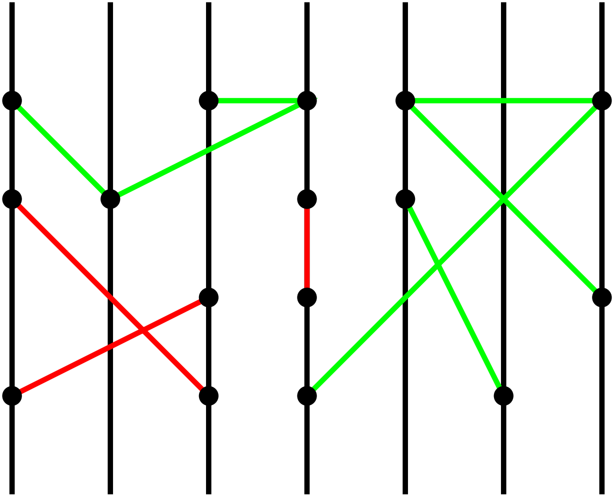
Theorem from: Dujmović, Joret, Micek, Morin, Ueckerdt and Wood (2020): *Planar Graphs Have Bounded Queue Number*

Track Layouts



Theorem from: Dujmović, Joret, Micek, Morin, Ueckerdt and Wood (2020): *Planar Graphs Have Bounded Queue Number*

Track Layouts

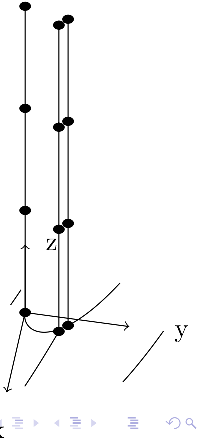


Theorem from: Dujmović, Joret, Micek, Morin, Ueckerdt and Wood (2020): *Planar Graphs Have Bounded Queue Number*

The Modular Moment Curve with tracks

- Let p prime, $n \leq p < 2n$
- Place j-th vertex of i-th track at $(i, i^2 \mod p, i^3 \mod p + jp)$
- Four vertices are coplanar iff this matrix has determinant zero:

$$M = \begin{pmatrix} 1 & i_0 & i_0^2 \mod p & i_0^3 \mod p + jp \\ 1 & i_1 & i_1^2 \mod p & i_1^3 \mod p + jp \\ 1 & i_2 & i_2^2 \mod p & i_2^3 \mod p + jp \\ 1 & i_3 & i_3^2 \mod p & i_3^3 \mod p + jp \end{pmatrix}$$



- $\det(M) \equiv \prod_{0 \leq k < l < 4} \underbrace{(v_l - v_k)}_{\not\equiv 0} \not\equiv 0 \pmod p$
- No four vertices coplanar $\implies$ no two edges cross

The Modular Moment Curve with tracks

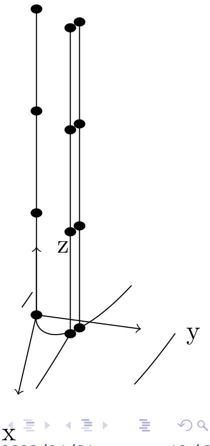
- Let p prime, $n \leq p < 2n$
- Place j-th vertex of i-th track at $(i, i^2 \mod p, i^3 \mod p + jp)$

Case 1: All i's different

- Four vertices are coplanar iff this matrix has determinant zero:

$$M = \begin{pmatrix} 1 & i_0 & i_0^2 \mod p & i_0^3 \mod p + j_0 p \\ 1 & i_1 & i_1^2 \mod p & i_1^3 \mod p + j_1 p \\ 1 & i_2 & i_2^2 \mod p & i_2^3 \mod p + j_2 p \\ 1 & i_3 & i_3^2 \mod p & i_3^3 \mod p + j_3 p \end{pmatrix}$$

- $\det(M) \equiv \prod_{0 \leq k < l < 4} \underbrace{(v_l - v_k)}_{\neq 0} \not\equiv 0 \pmod{p}$
- No four vertices coplanar $\implies$ no two edges cross



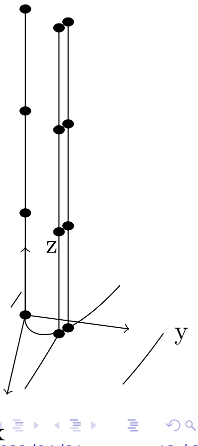
The Modular Moment Curve with tracks

- Let p prime, $n \leq p < 2n$
- Place j-th vertex of i-th track at $(i, i^2 \mod p, i^3 \mod p + jp)$

Case 2: Two i's the same, WLOG $i_0 = i_1$

- Four vertices are coplanar iff this matrix has determinant zero:

$$M = \begin{pmatrix} 1 & i_1 & i_1^2 \mod p & i_1^3 \mod p + j_0 p \\ 1 & i_1 & i_1^2 \mod p & i_1^3 \mod p + j_1 p \\ 1 & i_2 & i_2^2 \mod p & i_2^3 \mod p + j_2 p \\ 1 & i_3 & i_3^2 \mod p & i_3^3 \mod p + j_3 p \end{pmatrix}$$



The Modular Moment Curve with tracks

- Let p prime, $n \le p < 2n$
- Place j-th vertex of i-th track at $(i, i^2 \mod p, i^3 \mod p + jp)$

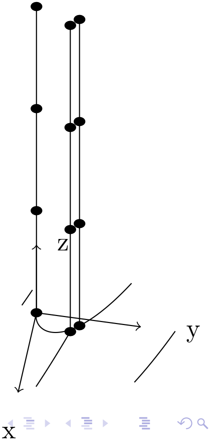
Case 2: Two i's the same, WLOG $i_0 = i_1$

- Four vertices are coplanar iff this matrix has determinant zero:

$$M = \begin{pmatrix} 1 & i_1 & i_1^2 \mod p & i_1^3 \mod p + j_0 p \\ 1 & i_1 & i_1^2 \mod p & i_1^3 \mod p + j_1 p \\ 1 & i_2 & i_2^2 \mod p & i_2^3 \mod p + j_2 p \\ 1 & i_3 & i_3^2 \mod p & i_3^3 \mod p + j_3 p \end{pmatrix}$$

- $\det(M) =$

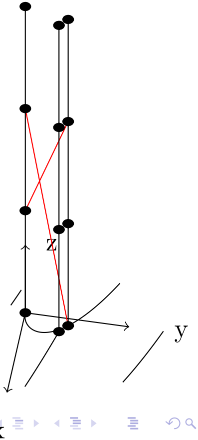
$$\underbrace{(j_1 p - j_0 p)}_{\neq 0} \cdot \det \underbrace{\begin{pmatrix} 1 & i_1 & i_1^2 \mod p \\ 1 & i_2 & i_2^2 \mod p \\ 1 & i_3 & i_3^2 \mod p \end{pmatrix}}_{\neq 0} \neq 0$$



The Modular Moment Curve with tracks

- Let p prime, $n \leq p < 2n$
- Place j-th vertex of i-th track at $(i, i^2 \mod p, i^3 \mod p + jp)$

Case 3: Two pairs of is are the same,
e.g. $i_0 = i_2$ and $i_1 = i_3$

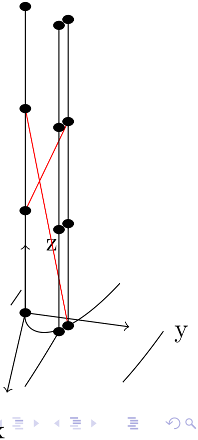


The Modular Moment Curve with tracks

- Let p prime, $n \leq p < 2n$
- Place j-th vertex of i-th track at $(i, i^2 \mod p, i^3 \mod p + jp)$

Case 3: Two pairs of is are the same,
e.g. $i_0 = i_2$ and $i_1 = i_3$

- Edges $i_0 i_2$ and $i_1 i_3$ are between the same track

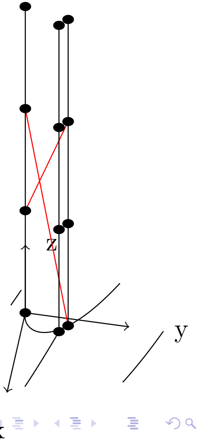


The Modular Moment Curve with tracks

- Let p prime, $n \leq p < 2n$
- Place j-th vertex of i-th track at $(i, i^2 \mod p, i^3 \mod p + jp)$

Case 3: Two pairs of is are the same, e.g. $i_0 = i_2$ and $i_1 = i_3$

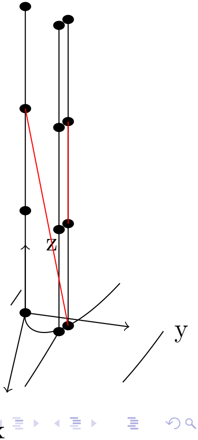
- Edges $i_0 i_2$ and $i_1 i_3$ are between the same track
- Crossing in the 3D drawing would also be crossing in track layout



The Modular Moment Curve with tracks

- Let p prime, $n \leq p < 2n$
- Place j-th vertex of i-th track at $(i, i^2 \mod p, i^3 \mod p + jp)$

Case 4: Three or more is are the same

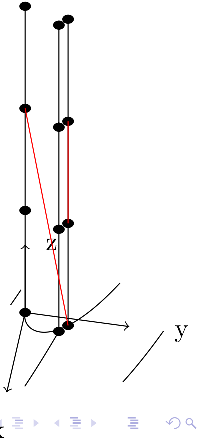


The Modular Moment Curve with tracks

- Let p prime, $n \le p < 2n$
- Place j-th vertex of i-th track at $(i, i^2 \mod p, i^3 \mod p + jp)$

Case 4: Three or more is are the same

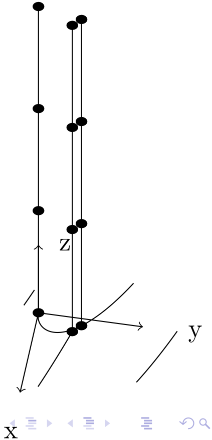
- Impossible: requires intertrack edges



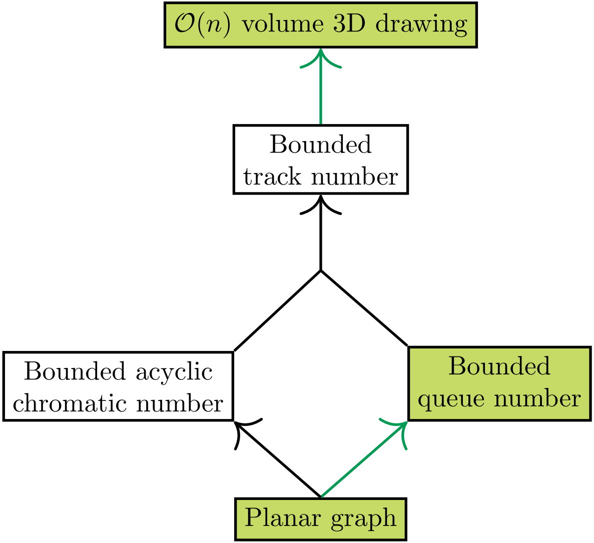
The Modular Moment Curve with tracks

Theorem

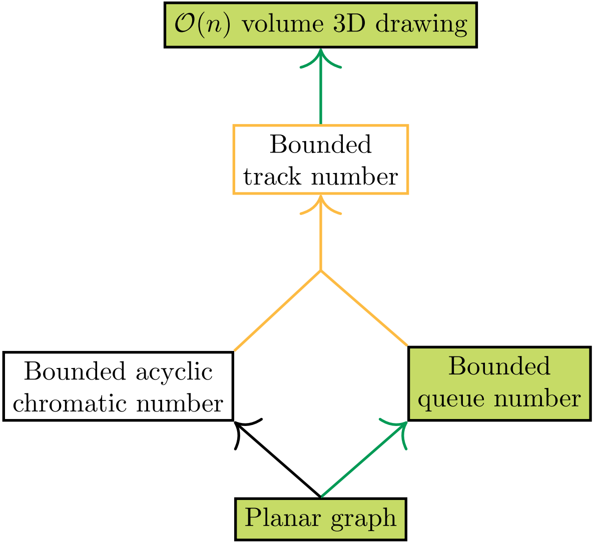
Every graph on n vertices with track number t has a 3-dimensional crossing-free straight-line drawing on the integer grid with $\mathcal{O}(t^3 n)$ volume.



Outline of the proof



Outline of the proof



Definition (Acyclic coloring)

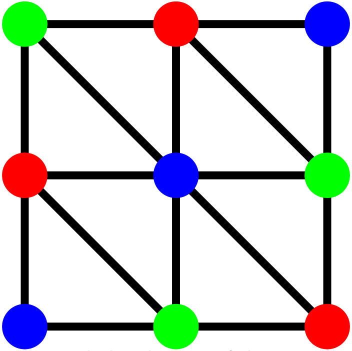
An acyclic coloring is a proper coloring such that every bichromatic subgraph is a forest, or equivalently, such that every cycle receives at least three colors.

Definition (Acyclic coloring)

An acyclic coloring is a proper coloring such that every bichromatic subgraph is a forest, or equivalently, such that every cycle receives at least three colors.

Definition (Acyclic Chromatic Number)

The acyclic chromatic number $\chi_a(G)$ is the smallest number of colors across all acyclic colorings of G.

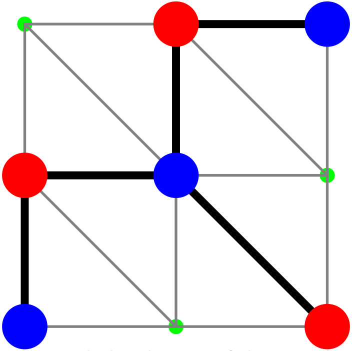


Definition (Acyclic coloring)

An acyclic coloring is a proper coloring such that every bichromatic subgraph is a forest, or equivalently, such that every cycle receives at least three colors.

Definition (Acyclic Chromatic Number)

The acyclic chromatic number $\chi_a(G)$ is the smallest number of colors across all acyclic colorings of G.

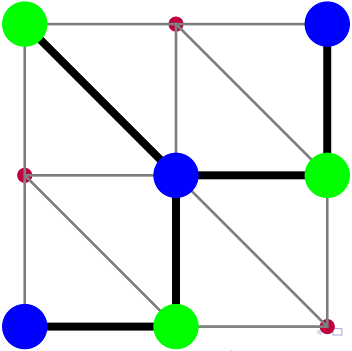


Definition (Acyclic coloring)

An acyclic coloring is a proper coloring such that every bichromatic subgraph is a forest, or equivalently, such that every cycle receives at least three colors.

Definition (Acyclic Chromatic Number)

The acyclic chromatic number $\chi_a(G)$ is the smallest number of colors across all acyclic colorings of G.

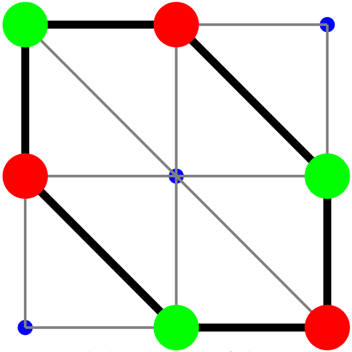


Definition (Acyclic coloring)

An acyclic coloring is a proper coloring such that every bichromatic subgraph is a forest, or equivalently, such that every cycle receives at least three colors.

Definition (Acyclic Chromatic Number)

The acyclic chromatic number $\chi_a(G)$ is the smallest number of colors across all acyclic colorings of G.

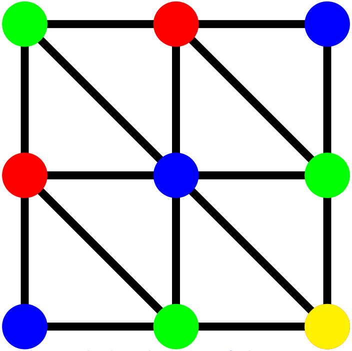


Definition (Acyclic coloring)

An acyclic coloring is a proper coloring such that every bichromatic subgraph is a forest, or equivalently, such that every cycle receives at least three colors.

Definition (Acyclic Chromatic Number)

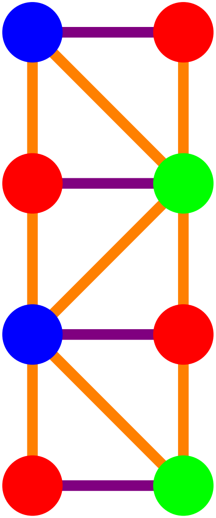
The acyclic chromatic number $\chi_a(G)$ is the smallest number of colors across all acyclic colorings of G.



The Refinement Lemma

Lemma

Given a graph with

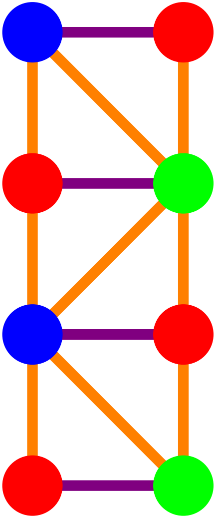


The Refinement Lemma

Lemma

Given a graph with

- *an acyclic c-coloring*

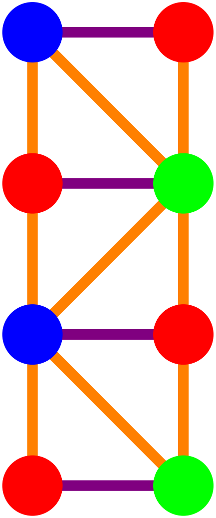


The Refinement Lemma

Lemma

Given a graph with

- *an acyclic c-coloring*
- *a (not necessarily proper) k-edge-coloring,*



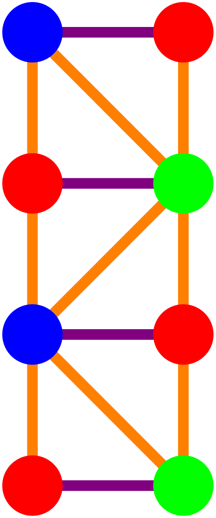
The Refinement Lemma

Lemma

Given a graph with

- *an acyclic c-coloring*
- *a (not necessarily proper) k-edge-coloring,*

there is a proper ck^{c-1}-vertex-coloring of G such that



The Refinement Lemma

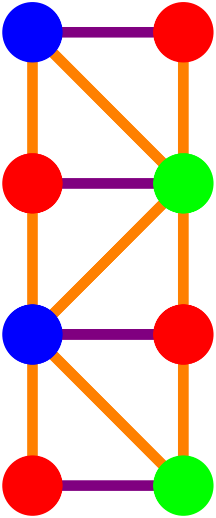
Lemma

Given a graph with

- *an acyclic c-coloring*
- *a (not necessarily proper) k-edge-coloring,*

there is a proper ck^{c-1}-vertex-coloring of G such that

- *every new vertex color class is contained in one old vertex color class, and*



The Refinement Lemma

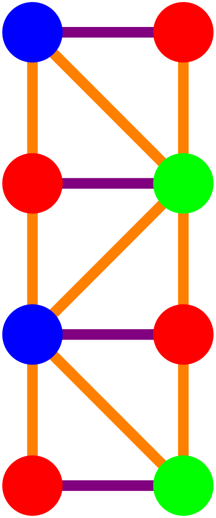
Lemma

Given a graph with

- *an acyclic c-coloring*
- *a (not necessarily proper) k-edge-coloring,*

there is a proper ck^{c-1}-vertex-coloring of G such that

- *every new vertex color class is contained in one old vertex color class, and*
- *the edges between any two vertex color classes are monochromatic.*



The Refinement Lemma

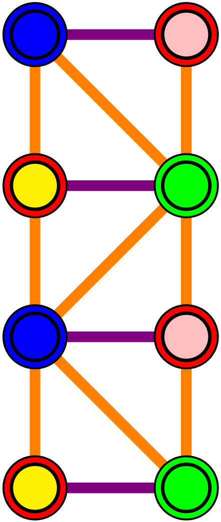
Lemma

Given a graph with

- *an acyclic c-coloring*
- *a (not necessarily proper) k-edge-coloring,*

there is a proper ck^{c-1}-vertex-coloring of G such that

- *every new vertex color class is contained in one old vertex color class, and*
- *the edges between any two vertex color classes are monochromatic.*



Theorem

Every graph on n vertices with queue number q and acyclic chromatic number c has track number at most $c(2q)^{c-1}$

Proof.

Theorem

Every graph on n vertices with queue number q and acyclic chromatic number c has track number at most $c(2q)^{c-1}$

Proof.

- Let $\leq_Q$ be the linear ordering from a q-queue-layout.

Theorem

Every graph on n vertices with queue number q and acyclic chromatic number c has track number at most $c(2q)^{c-1}$

Proof.

- Let $\leq_Q$ be the linear ordering from a q-queue-layout.
- Use the q queues for a q-edge-coloring.

Theorem

Every graph on n vertices with queue number q and acyclic chromatic number c has track number at most $c(2q)^{c-1}$

Proof.

- Let $\leq_Q$ be the linear ordering from a q-queue-layout.
- Use the q queues for a q-edge-coloring.
- Apply the refinement lemma to an acyclic c-coloring and the above edge $2q$-coloring.

Theorem

Every graph on n vertices with queue number q and acyclic chromatic number c has track number at most $c(2q)^{c-1}$

Proof.

- Let $\leq_Q$ be the linear ordering from a q-queue-layout.
- Use the q queues for a q-edge-coloring.
- Apply the refinement lemma to an acyclic c-coloring and the above edge $2q$-coloring.
- The tracks are the refined colors, ordered by $\leq_Q$

□

Theorem

Every graph on n vertices with queue number q and acyclic chromatic number c has track number at most $c(2q)^{c-1}$

Proof.

- Let $\leq_Q$ be the linear ordering from a q-queue-layout.
- Use the q queues for a q-edge-coloring.
- Apply the refinement lemma to an acyclic c-coloring and the above edge $2q$-coloring.
- The tracks are the refined colors, ordered by $\leq_Q$

□

Do the "tracks" actually form a track layout?

Theorem

Every graph on n vertices with queue number q and acyclic chromatic number c has track number at most $c(2q)^{c-1}$

Proof.

- Let $\leq_Q$ be the linear ordering from a q-queue-layout.
- Use the q queues for a q-edge-coloring.
- Apply the refinement lemma to an acyclic c-coloring and the above edge $2q$-coloring.
- The tracks are the refined colors, ordered by $\leq_Q$

$\square$

Do the "tracks" actually form a track layout?

- NO!

Theorem

Every graph on n vertices with queue number q and acyclic chromatic number c has track number at most $c(2q)^{c-1}$

Proof.

- Let $\leq_Q$ be the linear ordering from a q-queue-layout.
- Use the q queues for a q-edge-coloring.
- Apply the refinement lemma to an acyclic c-coloring and the above edge $2q$-coloring.
- The tracks are the refined colors, ordered by $\leq_Q$

□

Do the "tracks" actually form a track layout?

- NO!
- But we can fix them

Theorem

Every graph on n vertices with queue number q and acyclic chromatic number c has track number at most $c(2q)^{c-1}$

Proof.

- Let $\leq_Q$ be the linear ordering from a q-queue-layout.
- Partition the edges of each queue into upwards and downwards edges, then use the resulting $2q$ sets as a $2q$-edge-coloring.
- Apply the refinement lemma to an acyclic c-coloring and the above edge $2q$-coloring.
- The tracks are the refined colors, ordered by $\leq_Q$

Do the "tracks" actually form a track layout?

$\square$

Theorem

Every graph on n vertices with queue number q and acyclic chromatic number c has track number at most $c(2q)^{c-1}$

Proof.

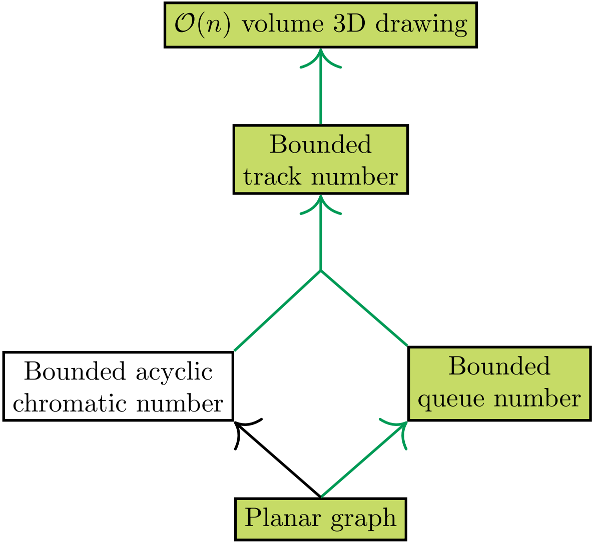
- Let $\leq_Q$ be the linear ordering from a q-queue-layout.
- Partition the edges of each queue into upwards and downwards edges, then use the resulting $2q$ sets as a $2q$-edge-coloring.
- Apply the refinement lemma to an acyclic c-coloring and the above edge $2q$-coloring.
- The tracks are the refined colors, ordered by $\leq_Q$

Do the "tracks" actually form a track layout?

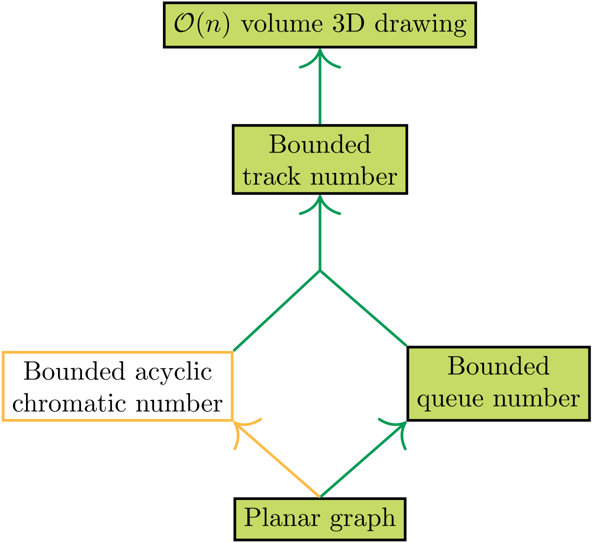
- Yes (we fixed them!)

□

Outline of the proof



Outline of the proof



Theorem

Outerplanar graphs have acyclic chromatic number at most three

Theorem

Outerplanar graphs have acyclic chromatic number at most three

Proof.

Let G be an outerplanar graph.

Theorem

Outerplanar graphs have acyclic chromatic number at most three

Proof.

Let G be an outerplanar graph. WLoG assume G is a triangulated polygon. Recursively color the graph as follows:

Theorem

Outerplanar graphs have acyclic chromatic number at most three

Proof.

Let G be an outerplanar graph. WLoG assume G is a triangulated polygon. Recursively color the graph as follows:

- Choose v with degree 2 and neighbors u and w.

Theorem

Outerplanar graphs have acyclic chromatic number at most three

Proof.

Let G be an outerplanar graph. WLoG assume G is a triangulated polygon. Recursively color the graph as follows:

- Choose v with degree 2 and neighbors u and w.
- Color $G \setminus v$.

Theorem

Outerplanar graphs have acyclic chromatic number at most three

Proof.

Let G be an outerplanar graph. WLoG assume G is a triangulated polygon. Recursively color the graph as follows:

- Choose v with degree 2 and neighbors u and w.
- Color $G \setminus v$.
- Color v using the color u and w don't use.

Theorem

Outerplanar graphs have acyclic chromatic number at most three

Proof.

Let G be an outerplanar graph. WLoG assume G is a triangulated polygon. Recursively color the graph as follows:

- Choose v with degree 2 and neighbors u and w.
- Color $G \setminus v$.
- Color v using the color u and w don't use.
- The addition of v can't create a new bichromatic cycle: Otherwise, u and w would have the same color, but they are neighbors.

□

Theorem

Planar graphs have acyclic chromatic number at most nine.

Theorem

Planar graphs have acyclic chromatic number at most nine.

Proof.

Let G be a planar graph.

Theorem

Planar graphs have acyclic chromatic number at most nine.

Proof.

Let G be a planar graph. WLOG assume G is connected.

Theorem

Planar graphs have acyclic chromatic number at most nine.

Proof.

Let G be a planar graph. WLOG assume G is connected. We will construct a 9-coloring and show that it does not contain a bichromatic cycle

Theorem

Planar graphs have acyclic chromatic number at most nine.

Proof.

Let G be a planar graph. WLOG assume G is connected. We will construct a 9-coloring and show that it does not contain a bichromatic cycle (and that it is proper).

Theorem

Planar graphs have acyclic chromatic number at most nine.

Proof.

Let G be a planar graph. WLOG assume G is connected. We will construct a 9-coloring and show that it does not contain a bichromatic cycle (and that it is proper).

- Choose some *center* v_0.

Theorem

Planar graphs have acyclic chromatic number at most nine.

Proof.

Let G be a planar graph. WLOG assume G is connected. We will construct a 9-coloring and show that it does not contain a bichromatic cycle (and that it is proper).

- Choose some *center* v_0.
- We define the *circle* $C_i = \{v \in G | d(v, v_0) = i\}$.

Theorem

Planar graphs have acyclic chromatic number at most nine.

Proof.

Let G be a planar graph. WLOG assume G is connected. We will construct a 9-coloring and show that it does not contain a bichromatic cycle (and that it is proper).

- Choose some *center* v_0.
- We define the *circle* $C_i = \{v \in G | d(v, v_0) = i\}$.
- Use a different set of three colors for the circles C_{3k}, C_{3k+1}, C_{3k+2}

Theorem

Planar graphs have acyclic chromatic number at most nine.

Proof.

Let G be a planar graph. WLOG assume G is connected. We will construct a 9-coloring and show that it does not contain a bichromatic cycle (and that it is proper).

- Choose some *center* v_0.
- We define the *circle* $C_i = \{v \in G | d(v, v_0) = i\}$.
- Use a different set of three colors for the circles C_{3k}, C_{3k+1}, C_{3k+2}
- Observe: Edges of G are within a single circle or between two adjacent circles.

Theorem

Planar graphs have acyclic chromatic number at most nine.

Proof.

Let G be a planar graph. WLOG assume G is connected. We will construct a 9-coloring and show that it does not contain a bichromatic cycle (and that it is proper).

- Choose some *center* v_0.
- We define the *circle* $C_i = \{v \in G | d(v, v_0) = i\}$.
- Use a different set of three colors for the circles C_{3k}, C_{3k+1}, C_{3k+2}
- Observe: Edges of G are within a single circle or between two adjacent circles.
- We will now color each circle S_i such that it cannot be the innermost circle of a bichromatic cycle.

Coloring the circles

Task

Color the circle C_i such that it cannot be the innermost circle of a bichromatic cycle.

Concluding the proof

Theorem

Planar graphs have acyclic chromatic number at most nine.

Concluding the proof

Theorem

Planar graphs have acyclic chromatic number at most nine.

Proof (cont.)

We have shown that, using our coloring, no bichromatic cycle exists.

Concluding the proof

Theorem

Planar graphs have acyclic chromatic number at most nine.

Proof (cont.)

We have shown that, using our coloring, no bichromatic cycle exists. It is easy to see that the coloring is proper.

Concluding the proof

Theorem

Planar graphs have acyclic chromatic number at most nine.

Proof (cont.)

We have shown that, using our coloring, no bichromatic cycle exists. It is easy to see that the coloring is proper. Therefore, we have found an acyclic 9-coloring. □

Better bounds

Theorem

Planar graphs have acyclic chromatic number at most five.

Better bounds

Theorem

Planar graphs have acyclic chromatic number at most five.

Proof.

Same as our proof for outerplanar graphs (i.e. reducing any graph to another graph with less vertices)

Better bounds

Theorem

Planar graphs have acyclic chromatic number at most five.

Proof.

Same as our proof for outerplanar graphs (i.e. reducing any graph to another graph with less vertices), except with 450 cases. □

Better bounds

Theorem

Planar graphs have acyclic chromatic number at most five.

Proof.

Same as our proof for outerplanar graphs (i.e. reducing any graph to another graph with less vertices), except with 450 cases. Details left as exercise. □

Better bounds

Theorem

Planar graphs have acyclic chromatic number at most five.

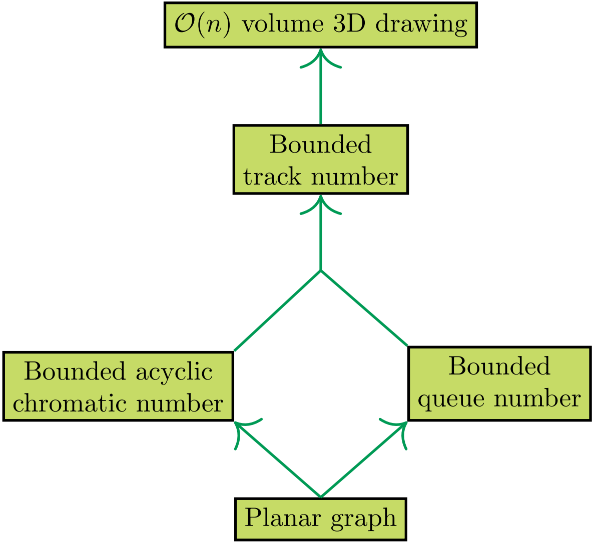
Proof.

Same as our proof for outerplanar graphs (i.e. reducing any graph to another graph with less vertices), except with 450 cases. □

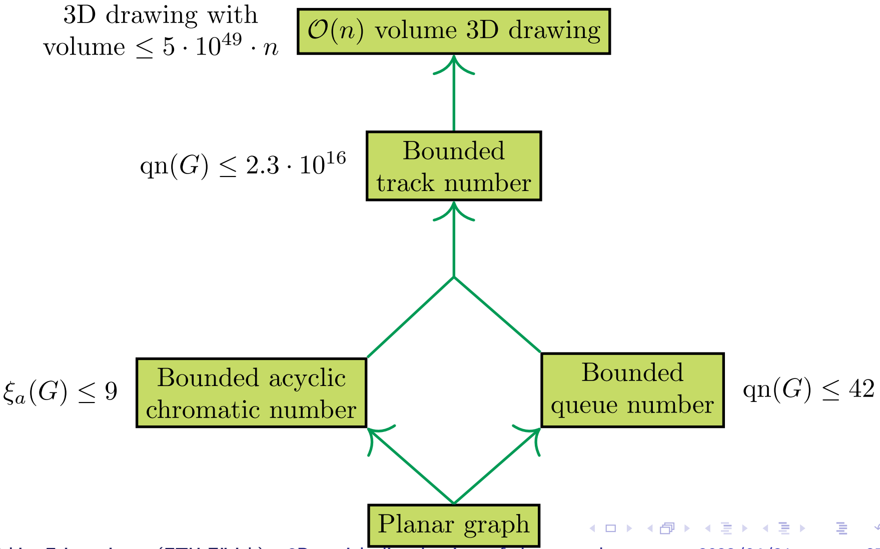
Theorem

Some planar graphs have acyclic chromatic number five.

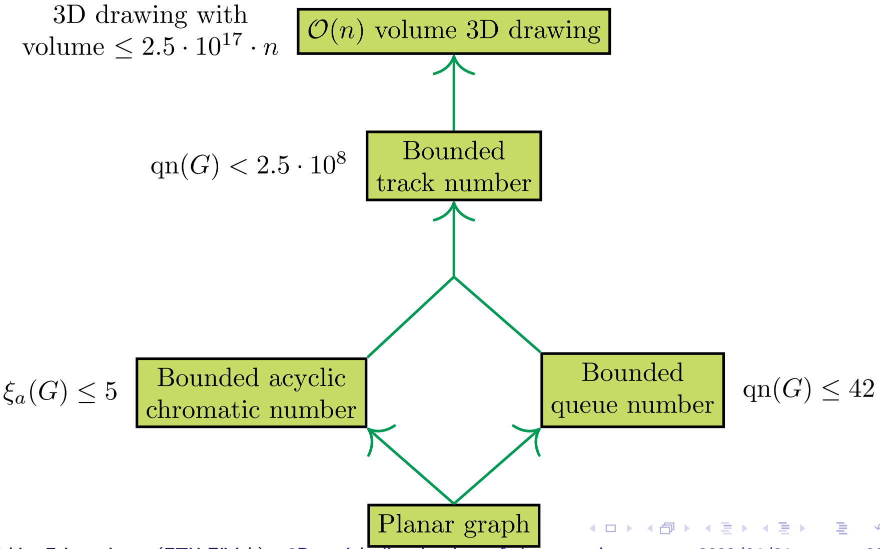
Summary


$\mathcal{O}(n)$ volume 3D drawing
Bounded track number
Bounded acyclic chromatic number
Bounded queue number
Planar graph

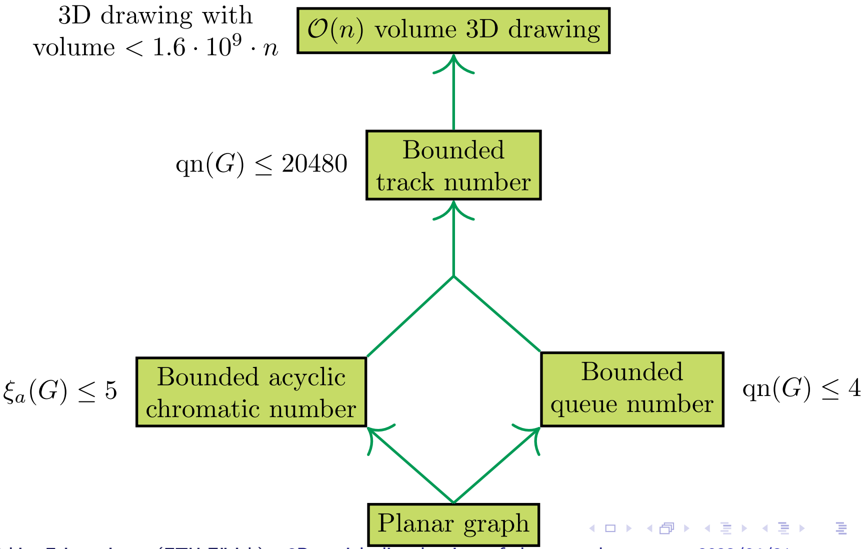
Summary


3D drawing with volume $\leq 5 \cdot 10^{49} \cdot n$
$\mathcal{O}(n)$ volume 3D drawing
$\mathrm{qn}(G) \leq 2.3 \cdot 10^{16}$
Bounded track number
$\xi_a(G) \leq 9$
Bounded acyclic chromatic number
Bounded queue number
$\mathrm{qn}(G) \leq 42$
Planar graph

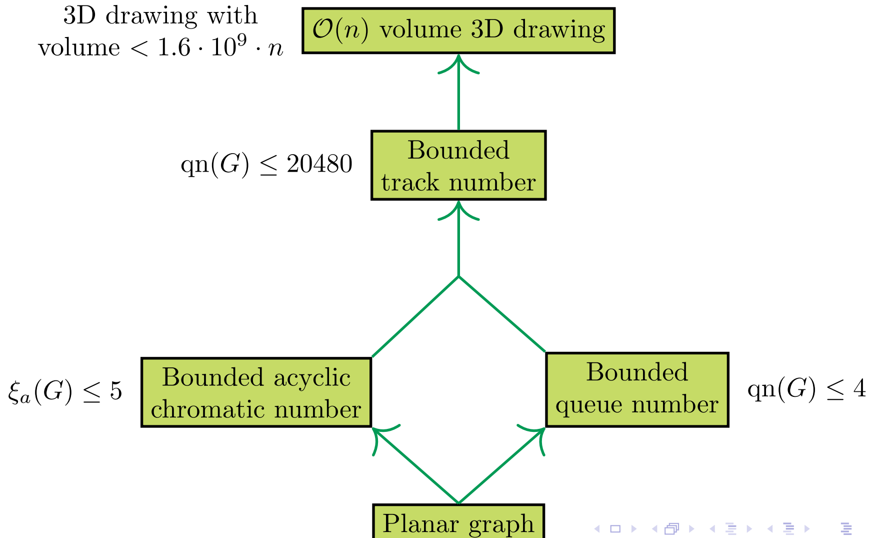
Best known upper bounds



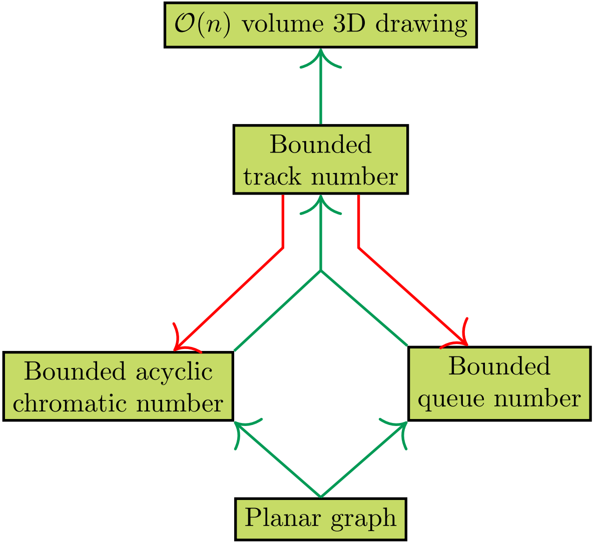
Best known lower bound of queue number



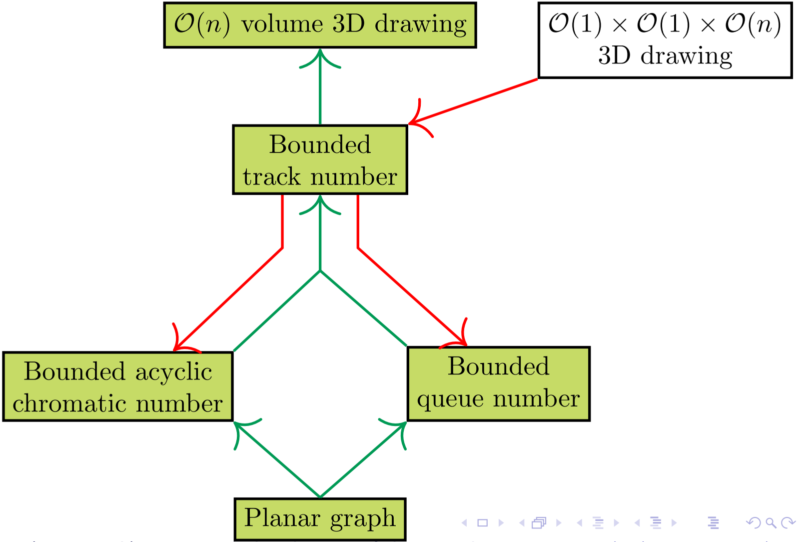
Further results



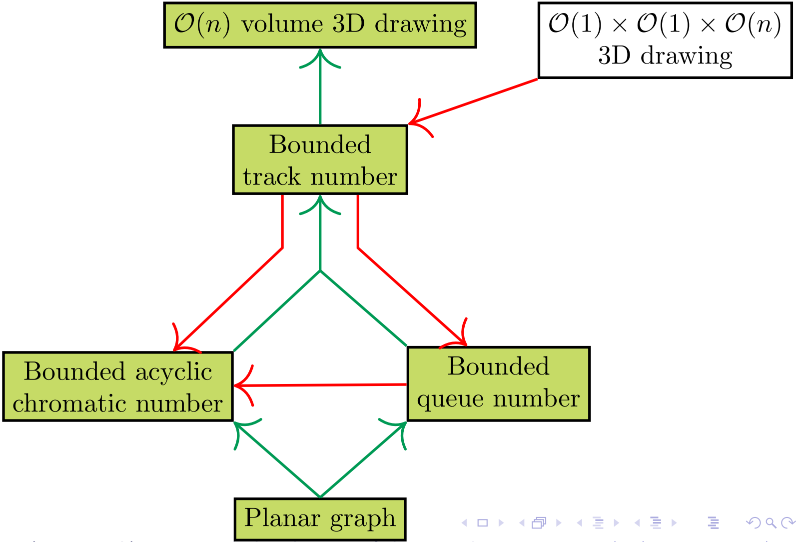
Further results



Further results



Further results



Open problems

Open problems

- Can we improve the upper bound on

Open problems

- Can we improve the upper bound on
  - the volume of n-vertex 3D drawings?

Open problems

- Can we improve the upper bound on
  - the volume of n-vertex 3D drawings?
  - the track number?

Open problems

- Can we improve the upper bound on
  - the volume of n-vertex 3D drawings?
  - the track number?
  - the queue number?
- Can we construct linear-volume drawings with

Open problems

- Can we improve the upper bound on
  - the volume of n-vertex 3D drawings?
  - the track number?
  - the queue number?
- Can we construct linear-volume drawings with
  - low aspect ratio?

Open problems

- Can we improve the upper bound on
  - the volume of n-vertex 3D drawings?
  - the track number?
  - the queue number?
- Can we construct linear-volume drawings with
  - low aspect ratio?
  - adjacent vertices being close?

Open problems

- Can we improve the upper bound on
  - the volume of n-vertex 3D drawings?
  - the track number?
  - the queue number?
- Can we construct linear-volume drawings with
  - low aspect ratio?
  - adjacent vertices being close?
  - other additional properties?