Three-dimensional straight-line drawings of planar graphs

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ETH Zürich

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Why 3D drawings?

- "Nice" drawings

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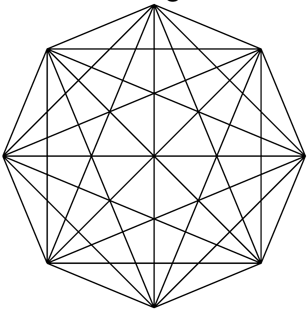

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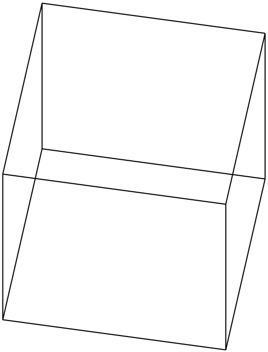

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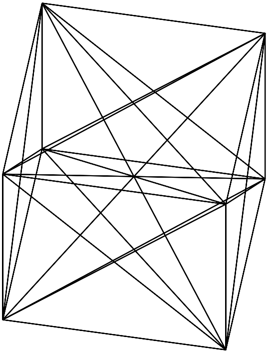

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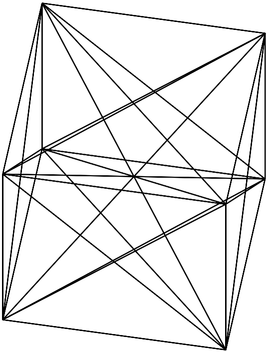

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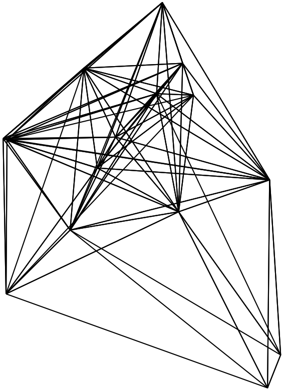

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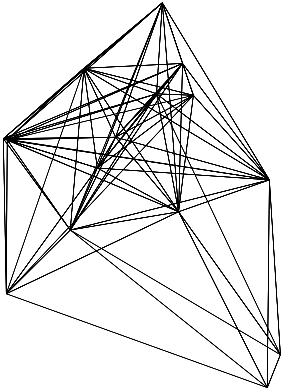

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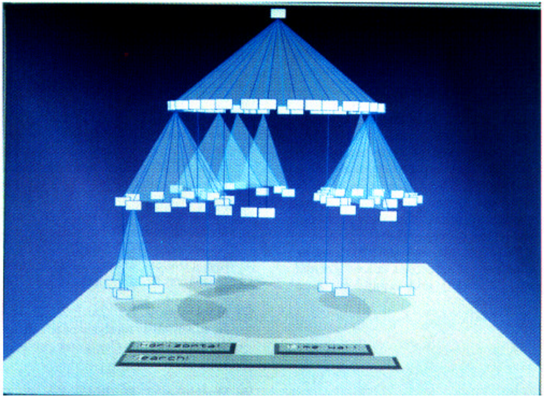


Figure from: George, Jock and Stuart (1991): *Cone Trees: Animated 3D Visualizations of Hierarchical Information*

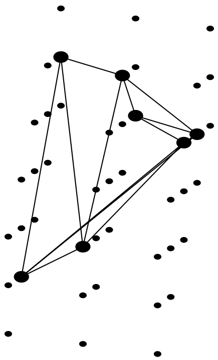
Why 3D drawings?

- "Nice" drawings
- Visualization of data with spacial information
- User Interfaces
  - Virtual Reality?

Main result

Theorem (Dujmović et al. 2020)

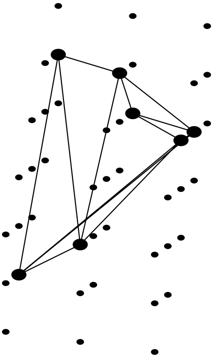
Every planar graph with n vertices has a 3-dimensional crossing-free straight-line drawing on the integer grid



Main result

Theorem (Dujmović et al. 2020)

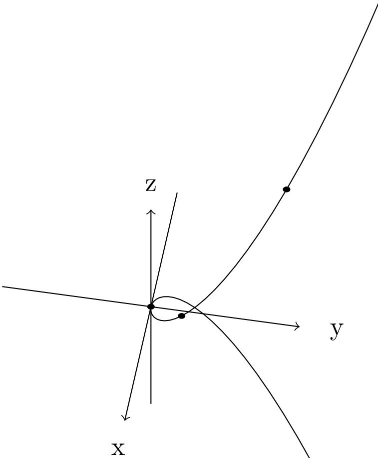
Every planar graph with n vertices has a 3-dimensional crossing-free straight-line drawing on the integer grid with $\mathcal{O}(n)$ Volume.



Why is this interesting?

The Moment Curve

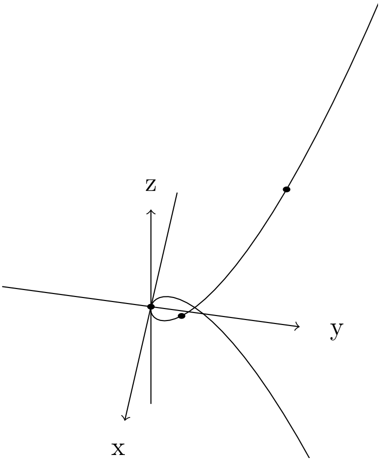
- Place i-th vertex at (i, i^2, i^3)



The Moment Curve

- Place i-th vertex at (i, i^2, i^3)
- Four vertices are coplanar iff this matrix has determinant zero:

$$M = \begin{pmatrix} & i_0 & i_0^2 & i_0^3 \\ & i_1 & i_1^2 & i_1^3 \\ & i_2 & i_2^2 & i_2^3 \\ & i_3 & i_3^2 & i_3^3 \end{pmatrix}$$



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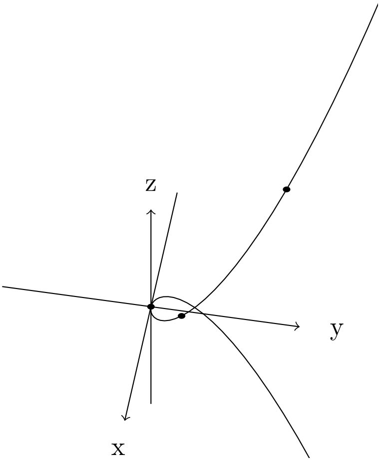

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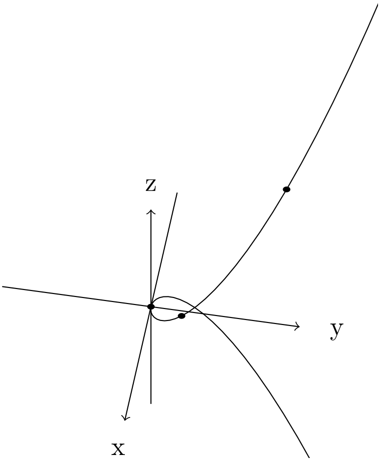
- This is a Vandermonde Matrix with

$$\det(M) = \prod_{0 \le k < l < 4} (i_l - i_k)$$



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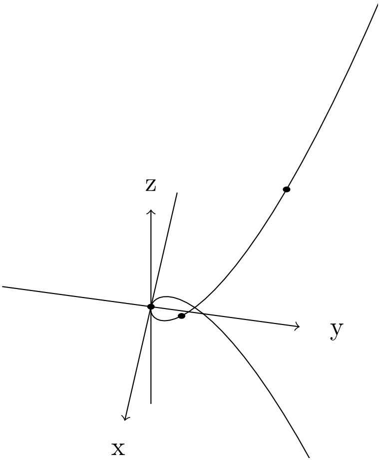

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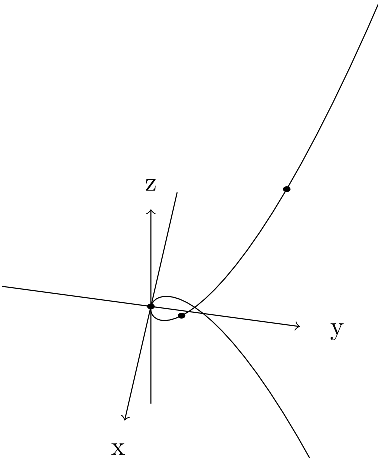

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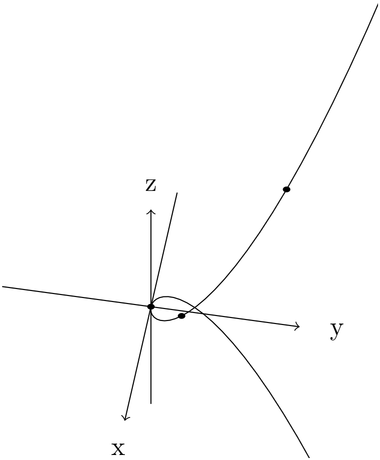

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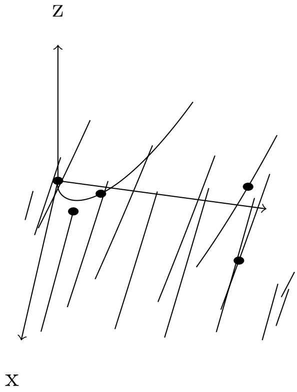

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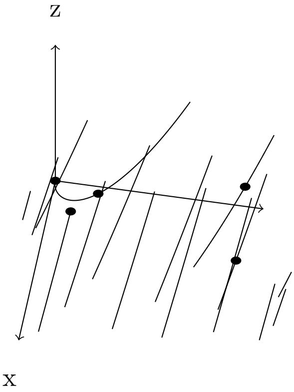

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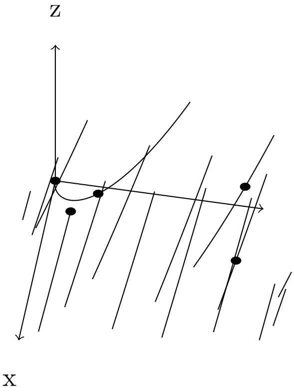

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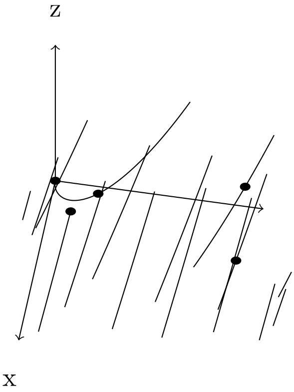

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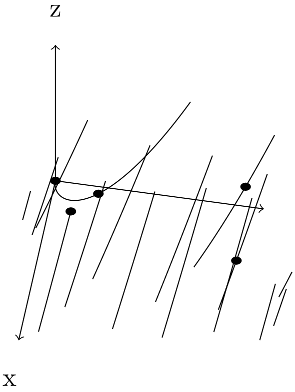

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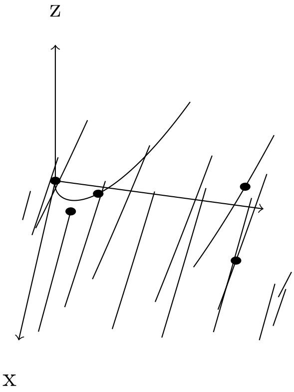

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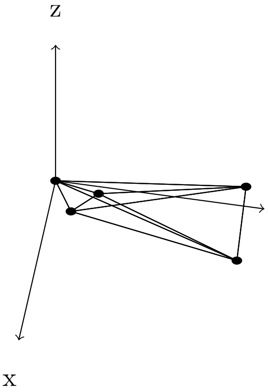
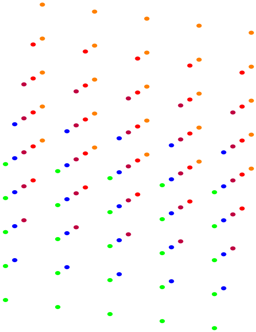


Upper bound for general graphs

Theorem (Cohen et al. 1996)

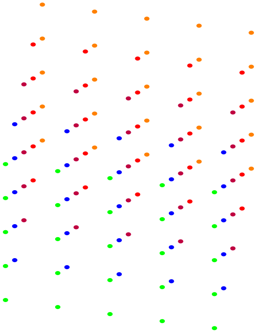
Every graph on n vertices has a $\mathcal{O}(n^3)$ 3D crossing-free straight-line drawing.

Can we do better?



Can we do better?

We try drawing a complete graph



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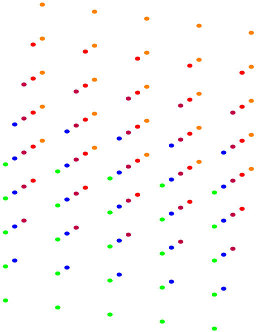

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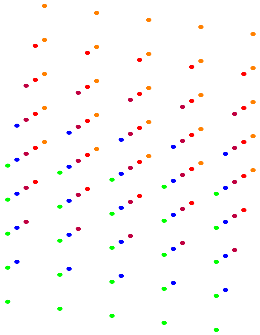
We try drawing a complete graph

- Cut grid into planes
- At most four vertices per plane: K_5 doesn't fit



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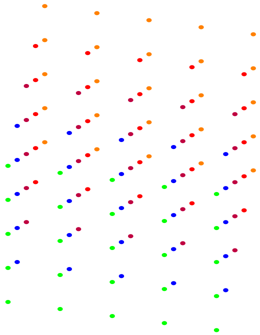

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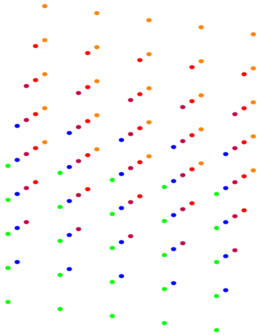
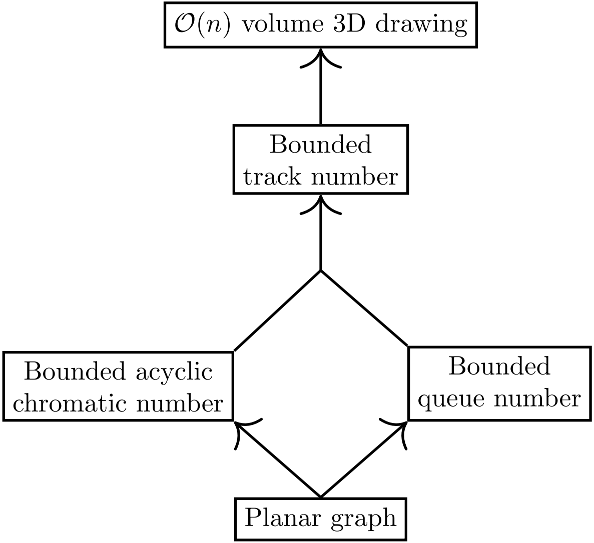
- At least $\lceil \frac{n}{4} \rceil^3 \in \Omega(n^3)$ Volume



Outline of the proof



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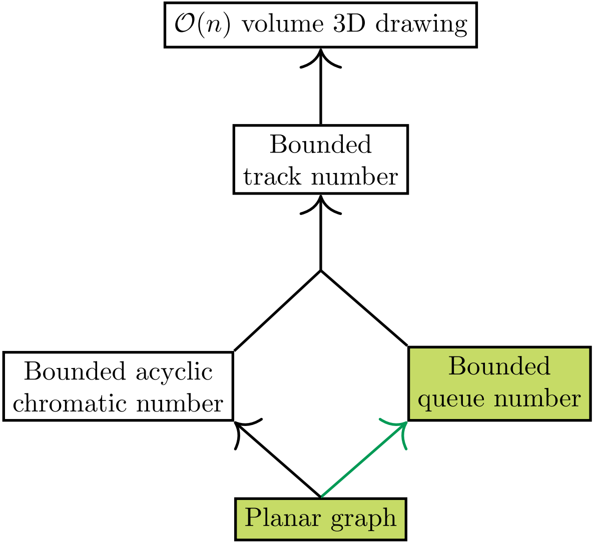

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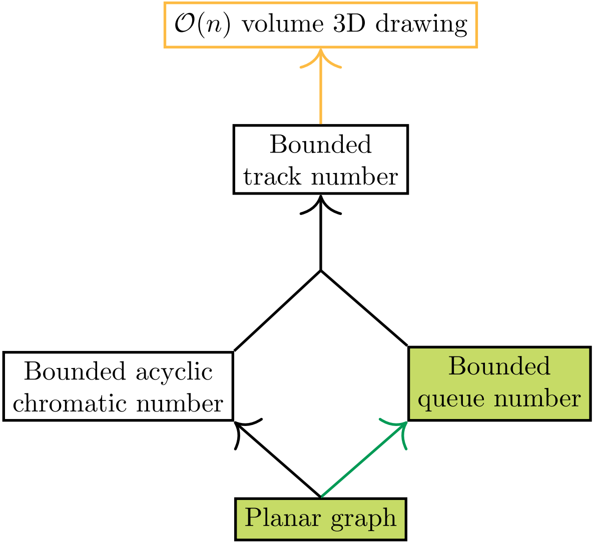
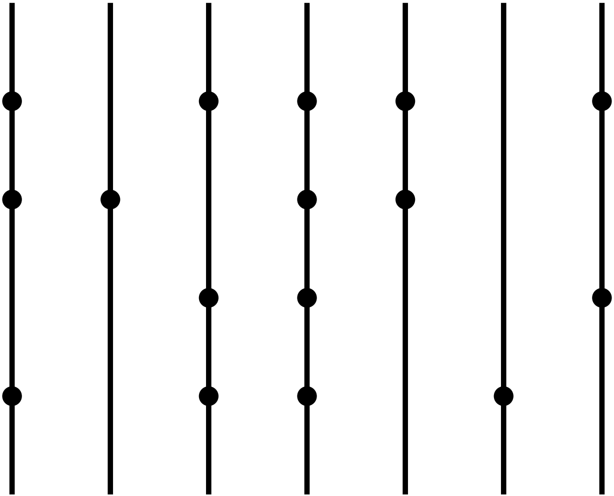


Track Layouts



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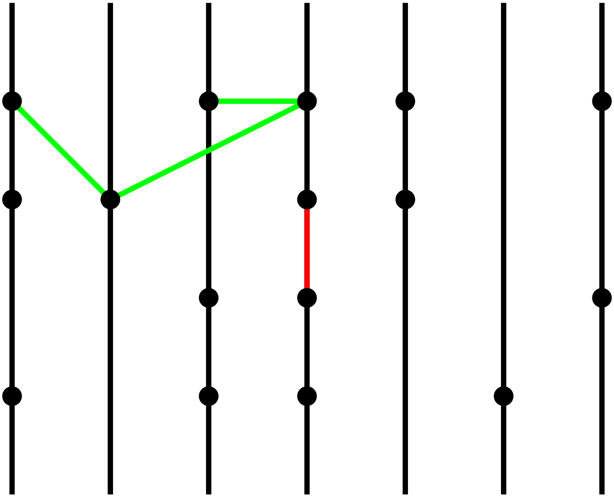

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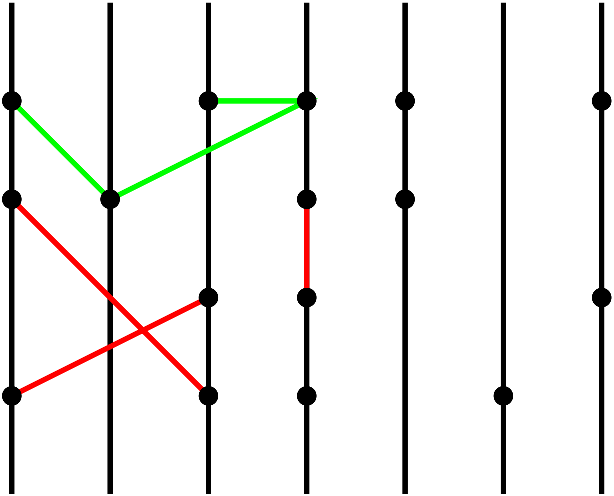

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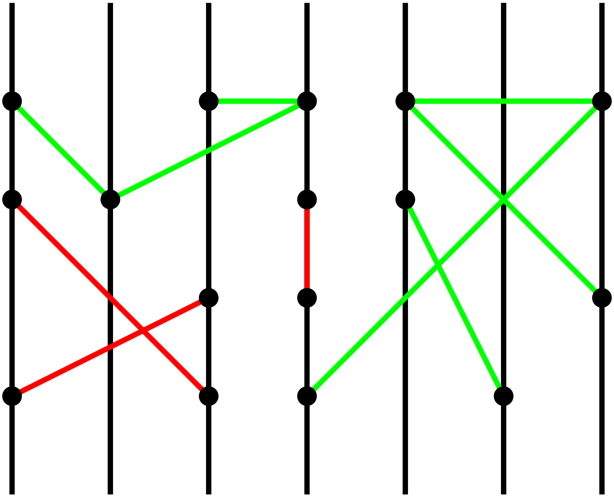
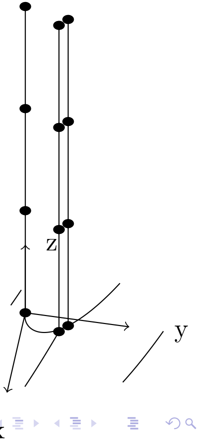


The Modular Moment Curve with tracks

- Let p prime, $t \leq p < 2t$
- Place j-th vertex of i-th track at $(i, i^2 \mod p, i^3 \mod p + jp)$



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Case 1: All i's different

- Four vertices are coplanar iff this matrix has determinant zero:

$$M = \begin{pmatrix} 1 & i_0 & i_0^2 \mod p & i_0^3 \mod p + j_0 p \\ 1 & i_1 & i_1^2 \mod p & i_1^3 \mod p + j_1 p \\ 1 & i_2 & i_2^2 \mod p & i_2^3 \mod p + j_2 p \\ 1 & i_3 & i_3^2 \mod p & i_3^3 \mod p + j_3 p \end{pmatrix}$$

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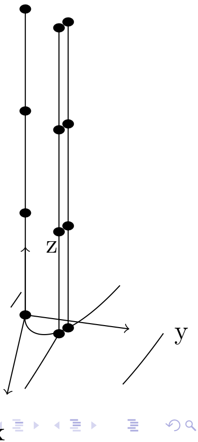

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Case 2: Two i's the same, WLOG $i_0 = i_1$

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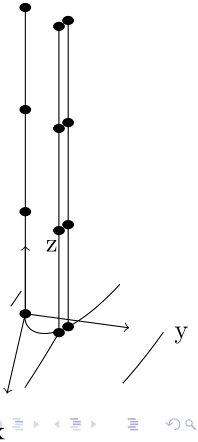

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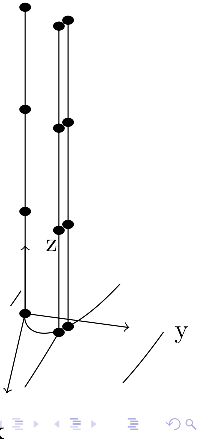
$$M = \begin{pmatrix} 1 & i_1 & i_1^2 \mod p & i_1^3 \mod p + j_0 p \\ 1 & i_1 & i_1^2 \mod p & i_1^3 \mod p + j_1 p \\ 1 & i_2 & i_2^2 \mod p & i_2^3 \mod p + j_2 p \\ 1 & i_3 & i_3^2 \mod p & i_3^3 \mod p + j_3 p \end{pmatrix}$$

- $\det(M) =$

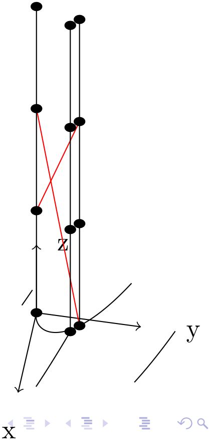
$$\underbrace{(j_1 p - j_0 p)}_{\neq 0} \cdot \det \underbrace{\begin{pmatrix} 1 & i_1 & i_1^2 \mod p \\ 1 & i_2 & i_2^2 \mod p \\ 1 & i_3 & i_3^2 \mod p \end{pmatrix}}_{\neq 0} \neq 0$$



The Modular Moment Curve with tracks

- Let p prime, $t \leq p < 2t$
- Place j-th vertex of i-th track at $(i, i^2 \mod p, i^3 \mod p + jp)$

Case 3: Two pairs of is are the same,
e.g. $i_0 = i_2$ and $i_1 = i_3$



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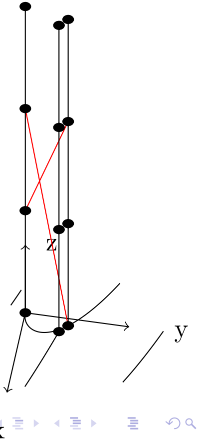

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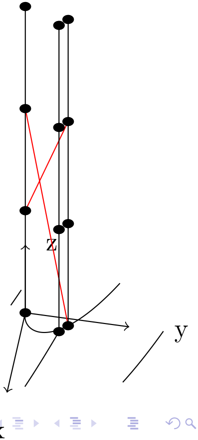
- Crossing in the 3D drawing would also be crossing in track layout



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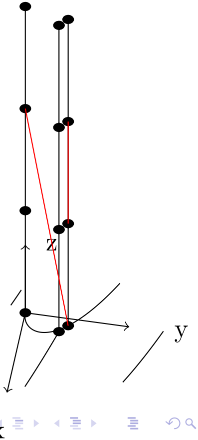

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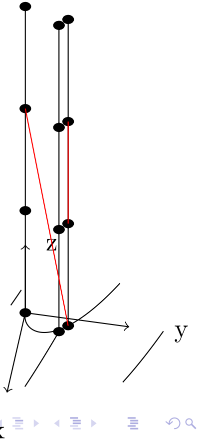
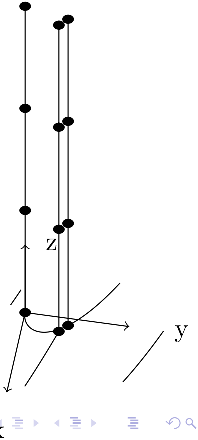
- Impossible: requires intertrack edges



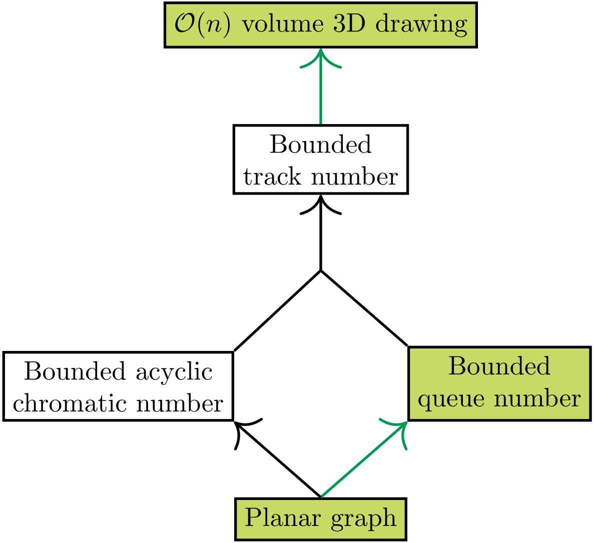
The Modular Moment Curve with tracks

Theorem (Dujmović et al. 2005)

Every graph on n vertices with track number t has a 3-dimensional crossing-free straight-line drawing on the integer grid with $\mathcal{O}(t^3 n)$ volume.



Outline of the proof



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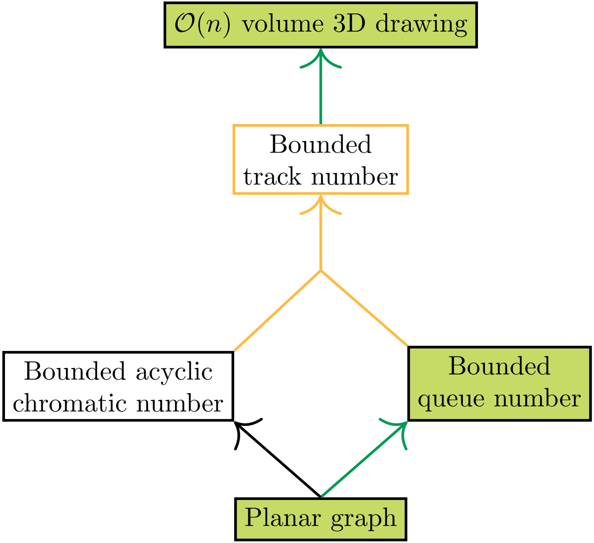

Definition (Acyclic coloring)

An acyclic coloring is a proper coloring such that every bichromatic subgraph is a forest, or equivalently, such that every cycle receives at least three colors.

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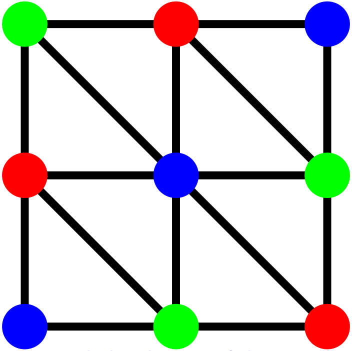

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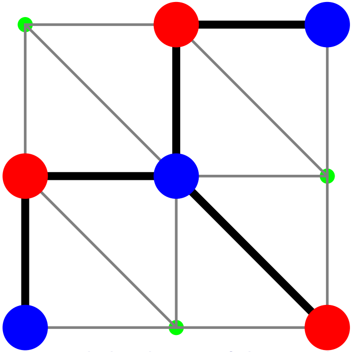

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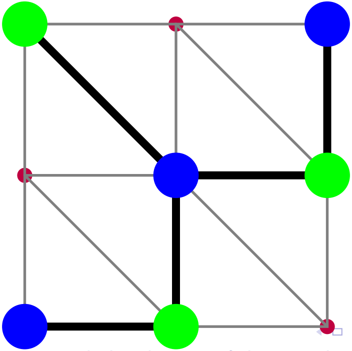

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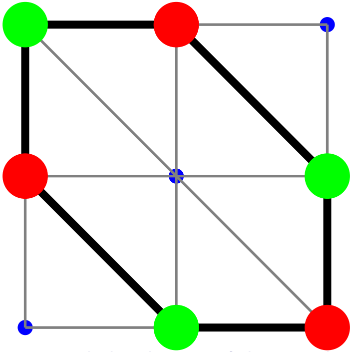

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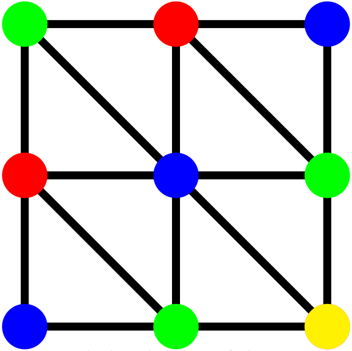
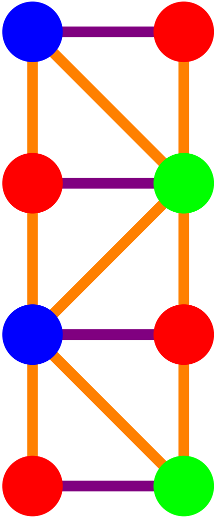


The Refinement Lemma

Lemma (Alon and Marshall 1998)

Given a graph with



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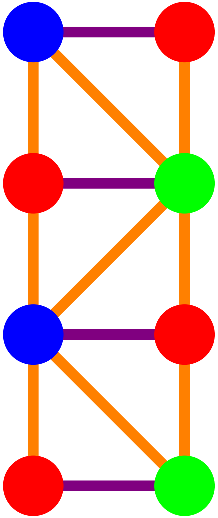

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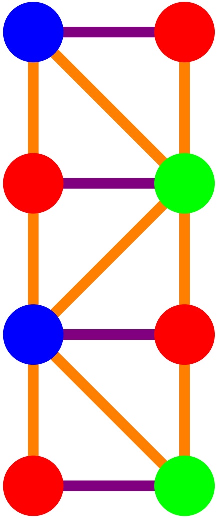

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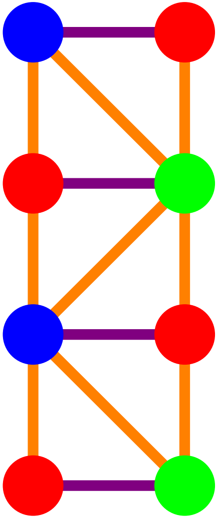

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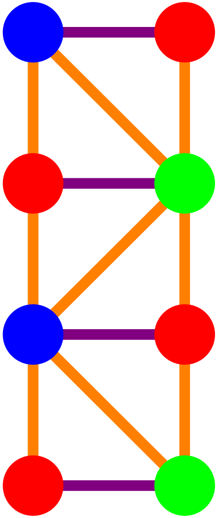

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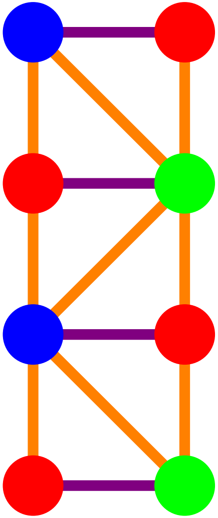

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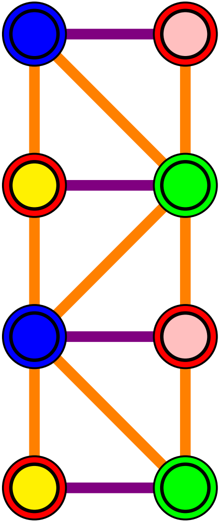

Theorem (Dujmović et al. 2005)

Every graph on n vertices with queue number q and acyclic chromatic number c has track number at most $c(2q)^{c-1}$

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- Partition the edges of each queue into upwards and downwards edges, then use the resulting $2q$ sets as a $2q$-edge-coloring.
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- Yes (we fixed them!)

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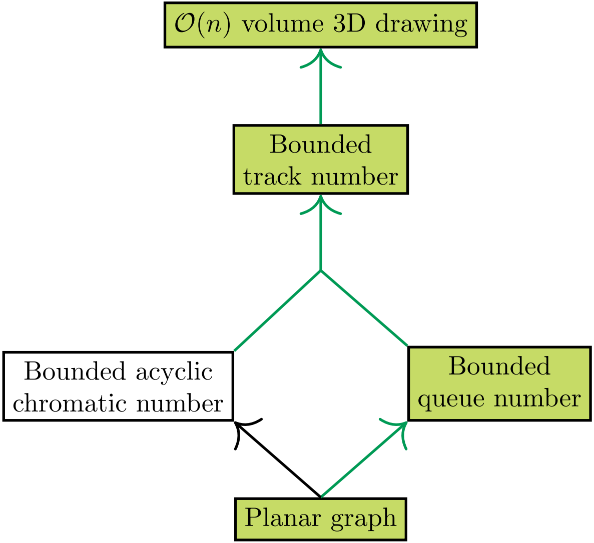
Outline of the proof



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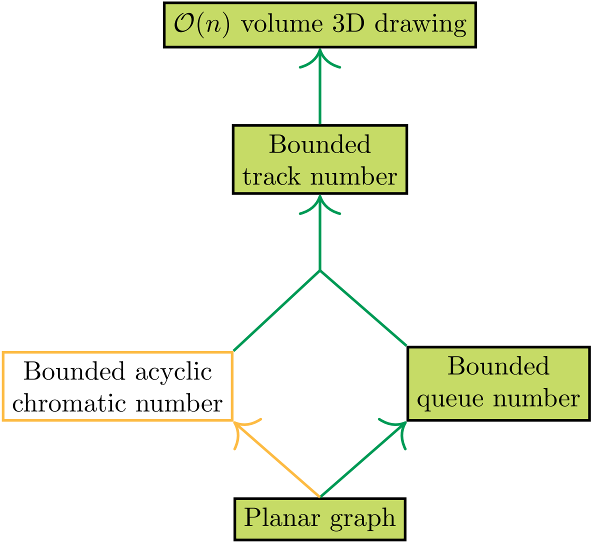

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- The addition of v can't create a new bichromatic cycle: Otherwise, u and w would have the same color, but they are neighbors.

□

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- Observe: Edges of G are within a single shell or between two adjacent shells.
- We will now color each shell S_i such that it cannot be the innermost shell of a bichromatic cycle.

Coloring the shells

Task

Color the shell S_i such that it cannot be the innermost shell of a bichromatic cycle.

Concluding the proof

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Proof (cont.)

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Proof (cont.)

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Better bounds

Theorem (Borodin 1979)

Planar graphs have acyclic chromatic number at most five.

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Same as our proof for outerplanar graphs (i.e. reducing any graph to another graph with less vertices), except with 450 cases. □

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Better bounds

Theorem (Borodin 1979)

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Proof.

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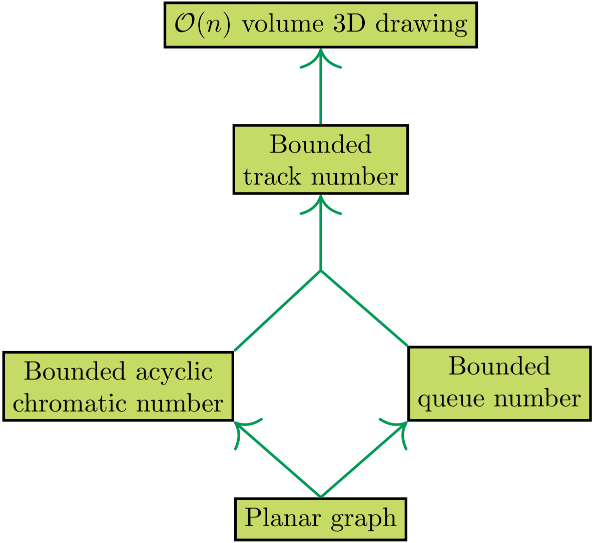
Theorem (Grünbaum 1972)

Some planar graphs have acyclic chromatic number five.

Summary


$\mathcal{O}(n)$ volume 3D drawing
Bounded track number
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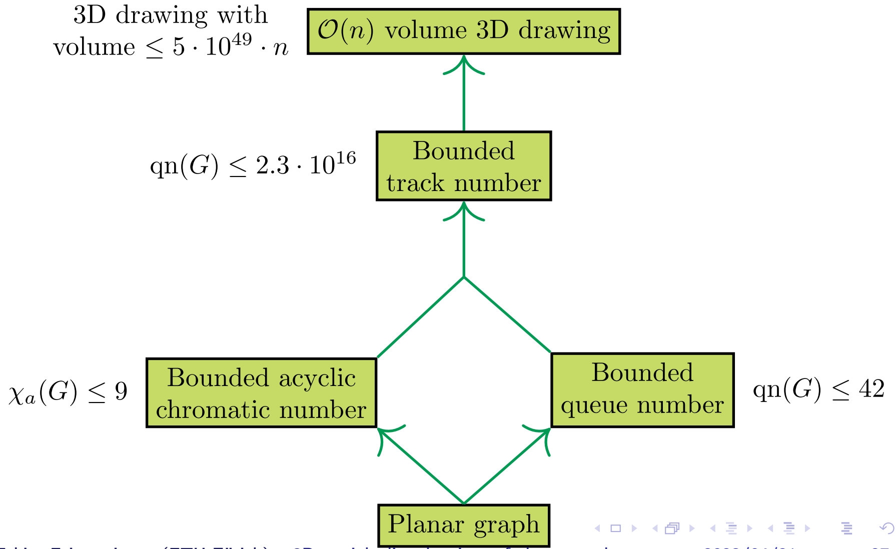
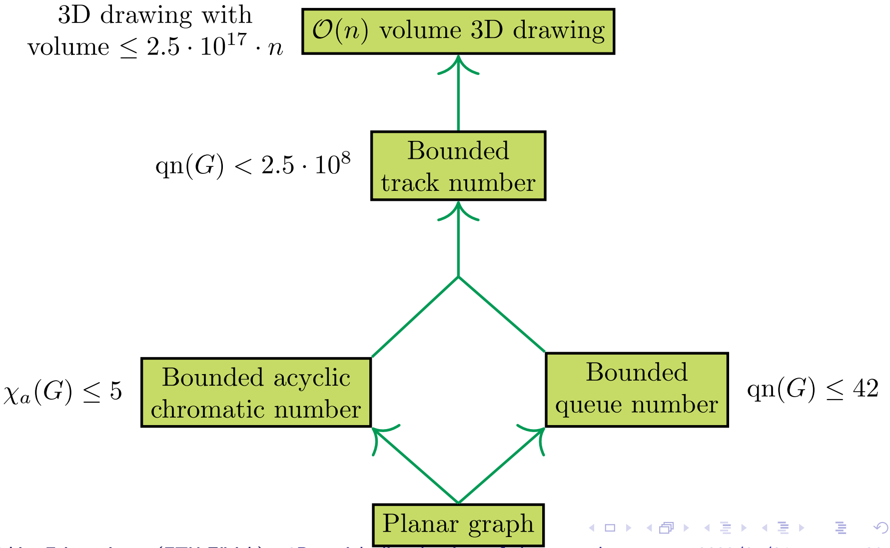
Bounded queue number
Planar graph

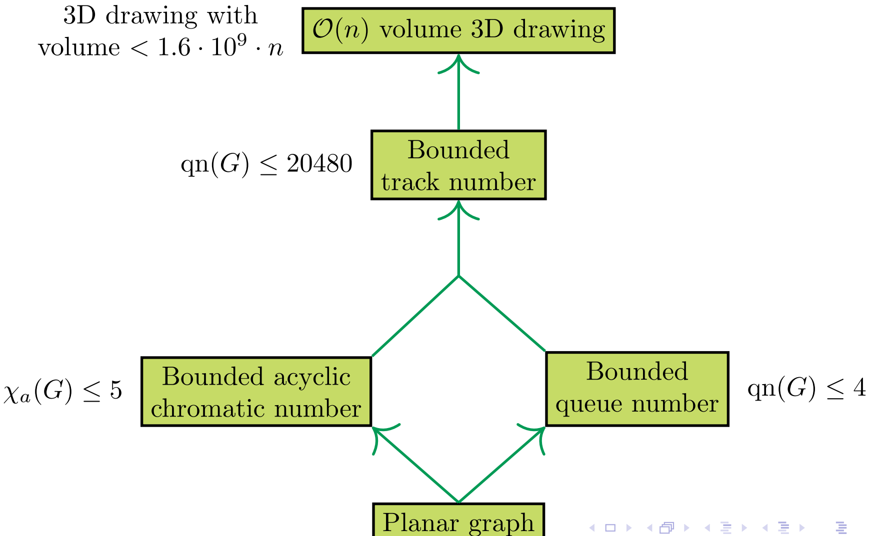
Summary


3D drawing with volume $\leq 5 \cdot 10^{49} \cdot n$
$\mathcal{O}(n)$ volume 3D drawing
$\mathrm{qn}(G) \leq 2.3 \cdot 10^{16}$
Bounded track number
$\chi_a(G) \leq 9$
Bounded acyclic chromatic number
Bounded queue number
$\mathrm{qn}(G) \leq 42$
Planar graph

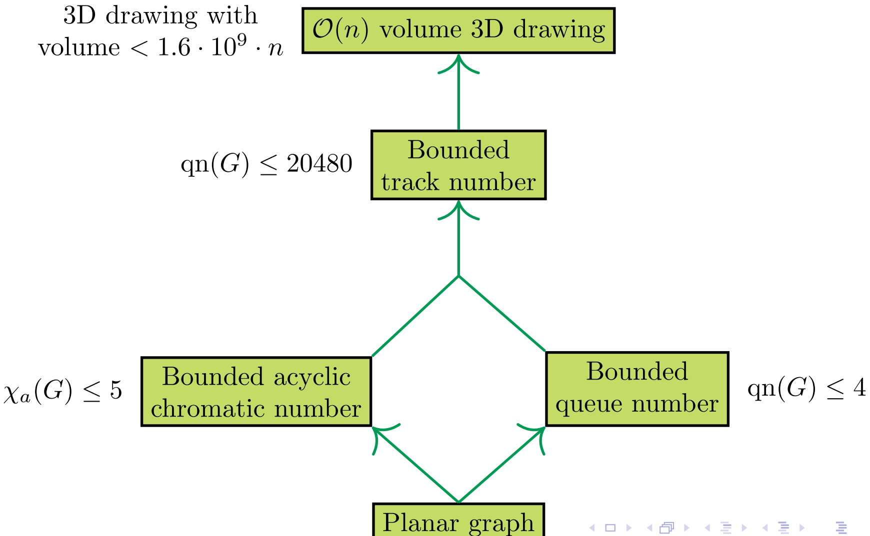
Best known upper bounds



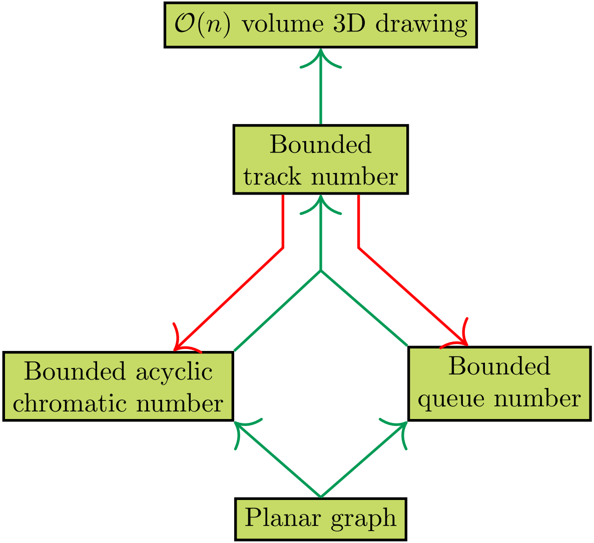
Best known lower bound of queue number



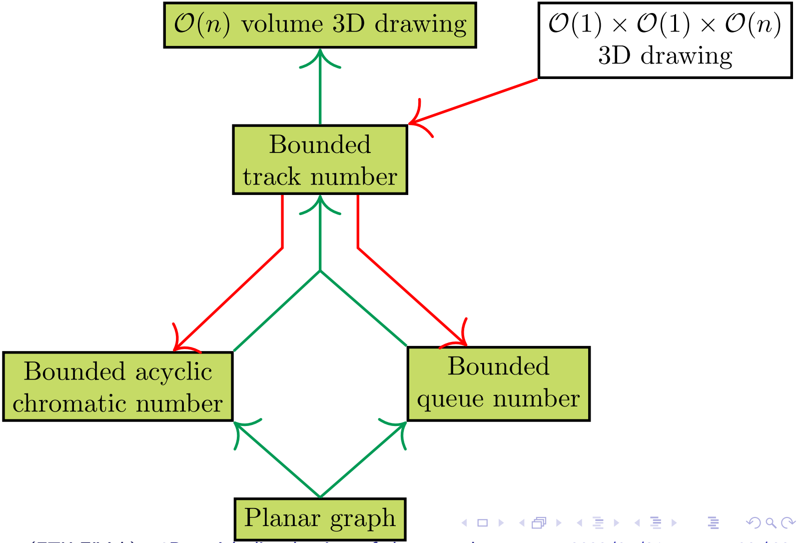
Further results



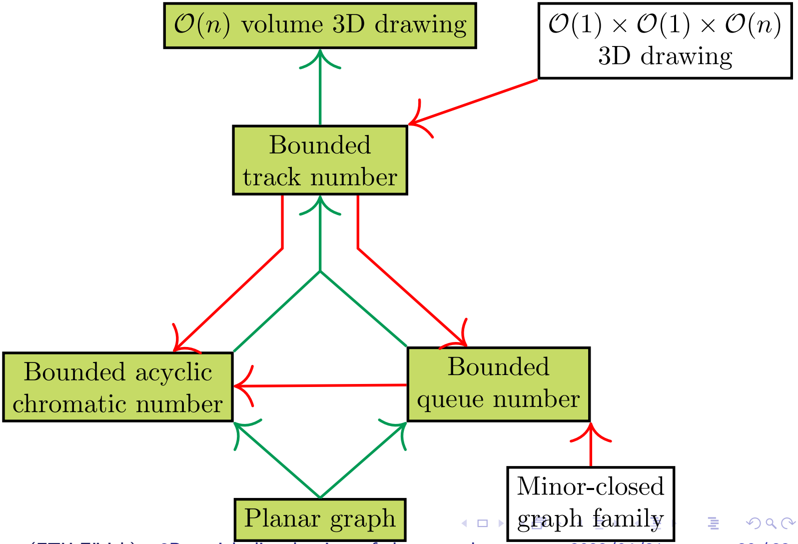
Further results



Further results



Further results



Open problems

Open problems

- Can we improve the upper bound on

Open problems

- Can we improve the upper bound on
  - the volume of n-vertex 3D drawings?

Open problems

- Can we improve the upper bound on
  - the volume of n-vertex 3D drawings?
  - the track number?

Open problems

- Can we improve the upper bound on
  - the volume of n-vertex 3D drawings?
  - the track number?
  - the queue number?
- Can we construct linear-volume drawings with

Open problems

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  - the queue number?
- Can we construct linear-volume drawings with
  - low aspect ratio?

Open problems

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  - low aspect ratio?
  - adjacent vertices being close?

Open problems

- Can we improve the upper bound on
  - the volume of n-vertex 3D drawings?
  - the track number?
  - the queue number?
- Can we construct linear-volume drawings with
  - low aspect ratio?
  - adjacent vertices being close?
  - other additional properties?