L05 - Fourier transforms

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Author: Nico Grisouard, nicolas.grisouard@physics.utoronto.ca
Supporting textbook chapters for week 5: Chapter 7.
Lecture 5, topics: * Discrete Fourier Transforms (DFTs) * Fast Fourier Transforms (FFTs)

1 Fourier series: reminders

We can express a periodic function on an interval $[0, L]$ as a Fourier series.

$$
\begin{aligned}
f(x) &= \sum_{k=0}^{\infty}\left[\alpha_k \cos\left(\frac{2\pi k x}{L}\right) + \beta_k \sin\left(\frac{2\pi k x}{L}\right)\right] \\
&= \sum_{k=-\infty}^{\infty} \gamma_k \exp\left(i\frac{2\pi k x}{L}\right),
\end{aligned}
$$

with

$$
\begin{aligned}
\gamma_k &= \frac{1}{2}\left(\alpha_{-k} + i\beta_{-k}\right) \quad \text{if} \quad k<0, \\
\gamma_k &= 0 \quad \text{if} \quad k=0, \\
\gamma_k &= \frac{1}{2}\left(\alpha_{k} - i\beta_{k}\right) \quad \text{if} \quad k>0,
\end{aligned}
$$

and

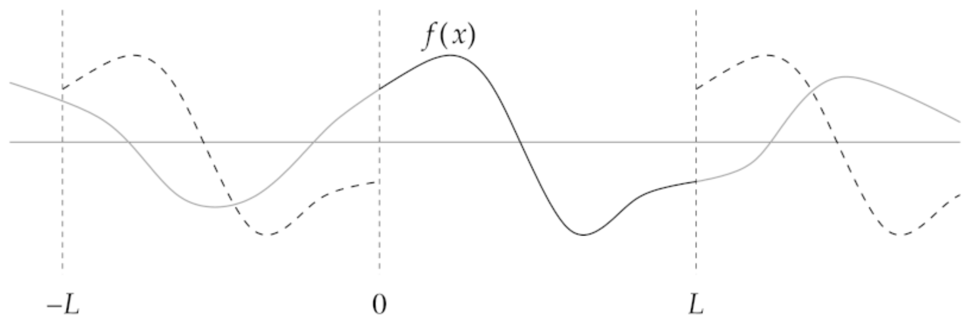
$$
\forall k, \quad \gamma_k = \frac{1}{L}\int_0^L f(x)\exp\left(-i\frac{2\pi k x}{L}\right)dx \quad \text{from orthogonality of sin functions.}
$$

Orthogonality of the sine functions:

$$
\begin{aligned}
\int_0^L \sin\left(\frac{\pi n x}{L}\right)\sin\left(\frac{\pi m x}{L}\right)dx &= \frac{L}{2}\delta_{nm}, \\
\int_0^L \cos\left(\frac{\pi n x}{L}\right)\cos\left(\frac{\pi m x}{L}\right)dx &= \frac{L}{2}\delta_{nm}, \\
\int_0^L \sin\left(\frac{\pi n x}{L}\right)\cos\left(\frac{\pi m x}{L}\right)dx &= 0
\end{aligned}
$$

Even for non-periodic functions, we can repeat the function over the portion of interest and discard the rest. Below are two pictures, one representing a plucked string, and the other representing the same from, as if it was part of a bigger, infinitely periodic function.

Here is Newman's take on it:



2 Discrete Fourier Transform

2.1 Principle

- Now let's think about the integral used for obtaining the γ_k's.
- We divide $[0, L]$ up into N segments and use the trapezoidal rule and periodicity of the function:

$$\begin{aligned}
\gamma_k &= \frac{1}{L}\int_0^L f(x)\exp\left(-i\frac{2\pi kx}{L}\right)dx \\
&\approx \frac{1}{L}\frac{L}{N}\left[\frac{1}{2}f(0)+\frac{1}{2}f(L)+\sum_{n=1}^{N-1}f(x_n)\exp\left(-i\frac{2\pi kx_n}{L}\right)\right] \\
&= \frac{1}{N}\left[\sum_{n=0}^{N-1}f(x_n)\exp\left(-i\frac{2\pi kx_n}{L}\right)\right] \quad \text{because } f(0)=f(L)
\end{aligned}$$

- Now define the Discrete Fourier Transform (DFT) as follows:

$$y_k = f(x_k); \qquad c_k = N\gamma_k;$$

$$\begin{aligned}
\text{DFT:} \quad & c_k = \sum_{n=0}^{N-1} y_n \exp\left(-i\frac{2\pi kn}{N}\right) \Rightarrow c_k = c^*_{N-k} \text{ for } y(x)\in\mathbb{R}. \\
\text{iDFT:} \quad & y_n = \frac{1}{N}\sum_{k=0}^{N-1} c_k \exp\left(i\frac{2\pi kn}{N}\right).
\end{aligned}$$

- The inverse DFT follows from the definition of the DFT and properties of exponential sums.

Note how

$$\text{DFT:} \quad c_k = \sum_{n=0}^{N-1} y_n \exp\left(-i\frac{2\pi kn}{N}\right)$$
$$\Rightarrow c_k = c^*_{N-k} \text{ for } y(x)\in\mathbb{R}.$$

- If $y(x) \in \mathbb{R}$, then we only need $N/2+1$ (N even) or $(N+1)/2$ (N odd) points to actually know the DFT.

- Python's N//2+1 will give you this number.

```
In [11]: N = 3
         print('N//2+1 = ' + str(N//2 + 1))
```

```
N//2+1 = 2
```

2.2 An implementation of the DFT

See the script dft_ts.py, adapted from Newman's online material.

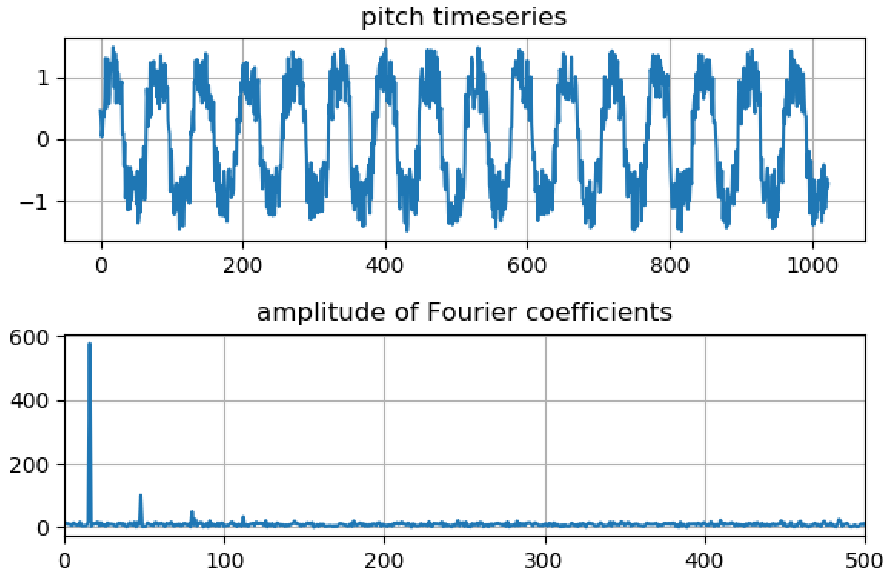
```
In [ ]: # %load dft_ts.py
        # Adapted from Newman's dft script
        import numpy as np
        import pylab as plt

        def dft(y):
            N = len(y)
            c = np.zeros(N//2+1, complex)
            for k in range(N//2+1):
                for n in range(N):
                    c[k] += y[n]*np.exp(-2j*np.pi*k*n/N)
            return c
```

```
In [ ]: y = plt.loadtxt("pitch.txt",float)
        plt.subplot(211)
        plt.plot(y)
        plt.title('pitch timeseries')
        plt.grid()

        c = dft(y)
        plt.subplot(212)
        plt.plot(abs(c))
        plt.title('amplitude of Fourier coefficients')
        plt.xlim(0, 500)
        plt.grid()

        plt.tight_layout()
        plt.savefig('timeseries.png', dpi = 100)
        plt.close()
```

3 Fast Fourier Transforms

3.1 Can we speed up the DFT?

$$\text{Recall} \qquad c_k = \sum_{n=0}^{N-1} y_n \exp\left(-i\frac{2\pi kn}{N}\right)$$



The physical signal (top) and its DFT (bottom)

- The `dft` snippet below requires $O(N^2)$ operations.

```
In [ ]: for k in range(N//2+1):
            for n in range(N):
                c[k] += y[n]*np.exp(-2j*np.pi*k*n/N)
```

- To run a billion operations, $N \sim$ 32,000: too few to be practical.
- The Fast Fourier Transform (FFT) overcomes this (Cooley & Tukey in 1960's, first found by Gauss in 1805).

3.2 FFTs divide and conquer

Assume $N = 2^M$ (other prime numbers in the decomposition are possible, but they will slow down the execution).

$$\text{Split} \qquad c_k = \sum_{n=0}^{N-1} y_n \exp\left(-i\frac{2\pi kn}{N}\right) = E_k + \omega O_k,$$

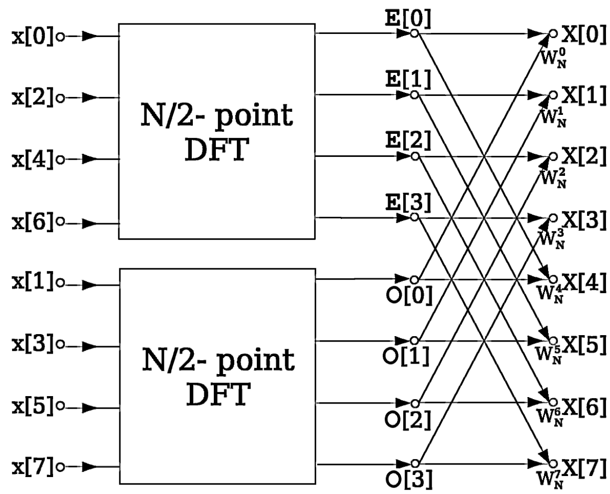
with

$$E_k = \sum_{p=0}^{N/2-1} y_{2p} \exp\left(-i\frac{2p\pi k}{N/2}\right) \qquad \text{the even indices,}$$

$$O_k = \sum_{p=0}^{N/2-1} y_{2p+1} \exp\left(-i2p\frac{\pi k}{N/2}\right) \qquad \text{the odd indices, and}$$

$$\omega = \mathrm{e}^{-i2\pi k/N} \qquad \text{the "twiddle factor".}$$

$$\text{Split} \qquad c_k = \sum_{n=0}^{N-1} y_n \exp\left(-i\frac{2\pi kn}{N}\right) = E_k + \omega O_k,$$



Decimation in time of a length-N DFT into two length-N/2 DFTs followed by a combining stage (By Virens, CC BY 3.0 (https://creativecommons.org/licenses/by/3.0), from Wikimedia Commons).

* E_k and O_k represent DFTs over points sampled twice as far apart as the original interval. * If we stopped here: # of operations would be $2 \times (N/2)^2 + 2 \approx N^2/2 + 2$ (bisection + twiddle factor): **a lot less operations for large N!**. * keep going: E_k and O_k can be split in two (bisected) themselves. * If $N = 2^M$, after $M = \log_2(N)$ bisections, we get N DFTs of a single sample:

$$c_0 = \sum_{n=0}^{1-1} y_n \mathrm{e}^{-i2\pi kn/N} = y_0,$$

the value at the sample point!

So, you actually go back: * take all sample points: that's the 1st set of samples. * pair them according to the last FFT bisection in the process above, * multiply the odd pairs by appropriate twiddle factors, * use these results to work the next set of samples.

- j-th set of samples at the m-th stage:

$$E_k^{(m,j)} = \sum_{p=0}^{N/2^m - 1} y_{2^m p + j} \exp\left(-i\frac{2\pi kp}{N/2^m}\right), \quad j \in \{0 \dots 2^m - 1\}$$

  Note: all E_k and O_k of previous slides are now some $E_k^{(m,j)}$.
- $2ˆm = $ # of FTs at each level (indexed by j),
- $N/2ˆm = $ # of samples per intermediate FT (indexed by k),
- Recursively, working from $M = \log_2 N$:

  - First step: $E_k^{(M,j)} = y_j$ (no k dependence), **ops:** N
  - Next steps: $E_k^{(m,j)} = E_k^{(m+1,j)} + \omega^{2^m k} E_k^{(m+1,j+2^m)}$, **ops:** $N/2^m \times 2^m = N$
  - Last step: $E_k^{(0,0)} = c_k$, the desired DFT coefficients. **ops:** $N \times 1 = N$

- We end up with N terms in each of the $\log_2 N$ bisections, so the number of operations is $N \log_2 N$.

- Huge speed increase for large N
- For $N = 10^6$, old DFT algorithm is $O(N^2) = 10^{12}$ ops, but FFT is $O(N \log_2 N) \sim 2 \times 10^7$ ops.
- Opens door to a wide range of calculations.
- Note that the same reasoning applies to the inverse FT: the algorithm is called the inverse FFT (iFFT).
- You can write your own FFT (see Exercise 7.7).
- But there are good tricks for saving memory that are implemented in packages like `numpy.fft`:

https://docs.scipy.org/doc/numpy-1.13.0/reference/routines.fft.html

- For this lab, we will be using FFTs to calculate DFTs.

See script `fft_ts.py`, which is derived from `dft_ts.py`.

In [ ]:
```
# %load fft_ts.py
# various exercises with dft and fft
import numpy as np
import pylab as plt
from numpy.fft import rfft, irfft
from time import time

pi = np.pi
# function to calculate the dft


def dft(y):
    N = len(y)
    c = np.zeros(N//2+1, complex)
    for k in range(N//2+1):
        for n in range(N):
            c[k] += y[n]*np.exp(-2j*pi*k*n/N)
    return c
```

In [ ]:
```
# plot time series and dft
y = plt.loadtxt("pitch.txt", float)

plt.figure(1, figsize=(8, 10))
plt.subplot(3, 1, 1)
plt.plot(y)
plt.title('pitch timeseries')
plt.grid()

dft_time = time()
c = dft(y)
dft_time = time() - dft_time

plt.subplot(3, 1, 2)
plt.plot(abs(c))
```

```
plt.title('amplitude of Fourier coefficients')
plt.xlim(0, 500)
plt.grid()

# ------------------
# now do it again with FFT
fft_time = time()
c2 = rfft(y)
fft_time = time() - fft_time

# compare home made dft with fft performance
print('dft time {0:10.2e} and fft time {1:10.2e}'.format(dft_time, fft_time))

# plot
plt.subplot(3, 1, 3)
plt.plot(abs(c2))
plt.title('amplitude of Fourier coefficients using FFT')
plt.xlim(0, 500)
plt.grid()

plt.tight_layout()

print('maximum  |c2-c|: ', max(abs(c2-c)))
```

In [ ]:
```
# now do things with proper time dimensions and filter out desired frequencies
# sampling frequency for audio signal
f = 44100.0   # Hz
# related temporal sample
dt = 1/f   # s
# length of vector
N = len(y)
# length of interval
T = N*dt
# convert to (angular) frequency
freq = np.arange(N/2+1)*2*pi/T
# dimensional time axis
t = np.arange(N)*dt
# sort on maximum frequency
MaxFreqs = np.argsort(abs(c2)**2)  # get indexes of largest three frequencies
MaxFreqs = MaxFreqs[-1:-4:-1]  # retain only top three
print('top three frequencies and their amplitudes:')
print('{0:10.2f} {1:10.2f} {2:10.2f} Hz'.format(
    freq[MaxFreqs[0]]/(2*pi),
    freq[MaxFreqs[1]]/(2*pi),
    freq[MaxFreqs[2]]/(2*pi)))
print('{0:10.2f} {1:10.2f} {2:10.2f}'.format(
    abs(c2[MaxFreqs[0]]), abs(c2[MaxFreqs[1]]), abs(c2[MaxFreqs[2]])))
```

```
# create a filtered array
c2_filt = np.copy(c2[:])
# zero out desired indices
c2_filt[MaxFreqs] = 0.0
# transform back to time domain
y_filt = irfft(c2_filt)

# now plot things dimensionally
plt.figure(2, figsize=(10, 10))
plt.subplot(2, 1, 1)
plt.plot(t, y, t, y_filt)
plt.xlabel('t(s)')
plt.title('pitch timeseries')
plt.grid()

plt.subplot(2, 1, 2)
plt.plot(freq/(2*pi), abs(c2), freq/(2*pi), abs(c2_filt))
plt.title('amplitude of fourier coefficients')
plt.xlim((0, 3600))
plt.xlabel('f (Hz)')
plt.grid()

plt.tight_layout()
plt.show()
plt.savefig('filtering_lab5.pdf')
```

```
In [ ]: plt.figure(3)
        # let's plot the cleaned up time series too - just for fun
        plt.plot(t, y-y_filt)
        plt.xlabel('t(s)')
        plt.title('pitch timeseries after removing lower amplitude signals')
        plt.grid()
```

4 2D DFTs

- Suppose we have a sample grid that is $M \times N$ with sample values y_{mn}.
- The 2D DFT works as follows:

- Fourier transform the M rows:

$$c'_{m\ell} = \sum_{n=0}^{N-1} y_{mn} \exp\left(-i\frac{2\pi\ell n}{N}\right)$$

Fourier transform the N cols to get 2D DFT:

$$c_{k\ell} = \sum_{m=0}^{M-1} c'_{m\ell} \exp\left(-i\frac{2\pi km}{M}\right) = \sum_{k=0}^{M-1} \sum_{n=0}^{N-1} y_{mn} \exp\left[-i2\pi\left(\frac{km}{M} + \frac{\ell n}{N}\right)\right].$$

Inverse 2D DFT:

$$y_{mn} = \frac{1}{MN} \sum_{k=0}^{M-1} \sum_{\ell=0}^{N-1} c_{kl} \exp \left[i2\pi \left(\frac{km}{N} + \frac{\ell n}{N} \right) \right].$$