

# Motion through fluids

## References:

1. J. P. Owen, W.S. Ryu: *The effects of linear and quadratic drag on falling spheres: an undergraduate laboratory*, Eur. J. Phys. 26 (2005) 1085-1091
2. P. Nelson: Biological Physics, Updated 1<sup>st</sup>. ed., Freeman (2008), Ch. 5.

## Introduction

In swimming bacteria or diffusing proteins, viscous rather than inertial forces dominate the dynamics of motion. A common measure of the ratio of the inertial to viscous forces is known as the Reynolds number:

$$R_e = \frac{\rho l v}{\eta} \quad (1)$$

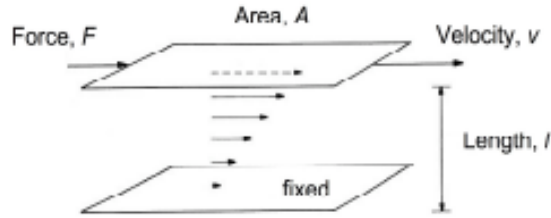
Where  $\rho$  is the fluid density,  $v$  is the velocity of the object,  $l$  is a characteristic length of the object, and  $\eta$  is the fluid viscosity.

Our everyday experience is mostly with high Reynolds number environments where inertial forces dominate. Swimming, for example, is a high Reynolds number activity. We propel ourselves through the water by accelerating the fluid behind us; the inertial force from a single stroke lets us glide meters before we come to a stop. Low Reynolds number activities are less common, but stirring a jar of honey with a spoon is one example. It's the viscosity of the honey and not the mass of the honey that makes the stirring difficult. When you let go of the spoon, does it continue to swirl around the jar? No, the spoon stops moving fairly quickly. The viscous force dominates the inertial force.

Swimming can be a low Reynolds number activity when the length scale of the swimmer is small. Microorganisms fit this category. A bacterium such as *E. coli*, is about one micron ( $10^{-6}$  meters) in diameter and travels around 20  $\mu\text{m}$  per second, so swimming bacteria have a Reynolds number much less than one and the viscous forces dominate inertial forces. To us, this is a very alien hydrodynamic world. For you to swim at an equivalent Reynolds number, you would need to "swim" in something viscous like honey, at speeds of about a foot a day, while cycling our arms at about 1 stroke per hour.

## Theoretical background

Inertial forces ( $F = ma$ ) are familiar, but what are viscous forces? Imagine you have a fluid between two plates. Intuitively you know that as the viscosity of the fluid increases, it requires more force to slide the plates apart (think water versus honey). Now assume that the bottom plate is fixed while the top plate, at some distance  $l$ , is free to move parallel to the fixed plate (see Figure 1).



**Figure 1: Fluid between moving plates: Force/area = viscosity\*velocity/length**

If a force  $F$  is applied to the top plate, it will move at some velocity  $v$ , forming a velocity gradient between the top and bottom plates. As the viscosity increases, it will take a larger force to form the same velocity gradient. It is this proportionality between the force per area (also known as shear stress) and the velocity per length (shear rate) that is known as viscosity. From Figure 1, we can define the viscous force ( $F_v$ ):

$$F_v = \frac{\eta A v}{l} \quad (2)$$

Inertial forces are in the form  $F_i = ma$  and for a volume of fluid  $V$  with density  $\rho$ , this can be written as:

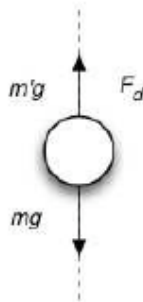
$$F_i = \rho V a = \frac{\rho l A v}{t} \quad (3)$$

Taking the ratio gives the Reynolds number:

$$R_e = \frac{F_i}{F_v} = \frac{\rho l v}{\eta} \quad (4)$$

### Equations of motion

In this experiment, you will be following the motion of objects falling in fluids. Check out the force diagram for a sphere falling in a fluid (Fig 2).



**Figure 2: A falling sphere, where  $mg$  is the weight of the object,  $m'g$  is the buoyant force, and  $F_d$  is the drag force.**

The equation of motion is:

$$m \frac{dv}{dt} = (m - m')g - F_d = M'g - F_d \quad (5)$$

Where  $F_d$  is the drag force and  $M'$  is the effective mass of the object corrected for buoyancy. When the object reaches its **terminal velocity** ( $a = dv/dt = 0$ ) and:

$$M'g = F_d \quad (6)$$

The form of  $F_d$  depends on the Reynolds number.

At a **high Reynolds number**, the drag force is commonly written as:

$$F_d = \frac{1}{2} \rho C_d A v^2 \quad (7)$$

The drag coefficient  $C_d$  is measured empirically,  $\rho$  is density of fluid and  $A$  is cross-sectional area of the object, perpendicular to the direction of motion.

Terminal velocity in this regime can be obtained from the equation of motion:

$$m \frac{dv}{dt} = mg - \frac{1}{2} \rho C_d A v^2 \quad (8)$$

The equation of motion can be solved analytically by separation of variables. The general form of the equation is:

$$m \frac{dv}{dt} = mg - cv^2 \quad (9)$$

Solution takes the form:

$$v(t) = v_{term} \tanh\left(\frac{gt}{v_{term}}\right) \quad (10)$$

Where terminal velocity  $v_{term}$  is given by:

$$v_{term} = \left(\frac{2mg}{\rho C_d A}\right)^{1/2} \quad (11)$$

➔ Integrate (9) and solve for  $v(t)$ . Verify that solution from (10) is correct. To solve for the constant of integration, use the initial condition,  $v(0) = 0$ .

**Terminal velocity in the high Reynolds number regime** depends on the bead radius as:

$$v_{term} = \left(\frac{2mg}{\rho C_d A}\right)^{1/2} \cong \left(\frac{r^3}{r^2}\right)^{1/2} \cong r^{1/2} \quad (12)$$

The derivation of drag force at a **low Reynolds number** is not as straightforward as the high Reynolds number case.

However, from the discussion of viscous forces above, one can see that  $F_d$  is generally proportional to  $\eta$ ,  $l$ , and  $v$ . Some specific cases have been solved exactly, such as the drag on a sphere of radius  $r$  moving at velocity  $v$  in a fluid with viscosity  $\eta$ . This is known as Stokes drag:

$$F_d = 6\pi\eta vr \quad (13)$$

In the following experiments we will be working with objects of constant density but of different size. How should the terminal velocity vary as a function of linear dimension?

From the force equation at low Reynolds number, the terminal velocity of the sphere is:

$$v_{term} = \frac{M'g}{6\pi\eta r} \quad (14)$$

The assumption here is that the sphere's velocity is relatively constant throughout its fall. Let's check if this is expected. For an object falling at low Reynolds number, the equation of motion is (ignoring the buoyant force for now):

$$m \frac{dv}{dt} = mg - 6\pi\eta rv \quad (15)$$

The general form of the equation of motion is:

$$m \frac{dv}{dt} = mg - bv \quad (16)$$

Separate variables and integrate. The solution is:

$$v(t) = \frac{mg}{6\pi\eta r} (1 - e^{-t/\tau}) \quad (17)$$

Where  $\tau = \frac{m}{6\pi r \eta}$

➔ Integrate and solve for  $v(t)$ . Verify that solution from eq. (16) is correct.

Qualitatively, we can see that the solution has the correct initial and asymptotic behaviour:

At  $t = 0$ ,  $v = 0$ . At  $t = \infty$ ,  $v = mg/6\pi\eta r$ , the expected terminal velocity.

How quickly the system reaches the terminal velocity depends on the time constant  $\tau$ .

➔ Estimate  $\tau$  for a 1 mm diameter aluminum sphere in glycerine,  $r = 0.5 \times 10^{-3}$  m,  $\eta = 1500\text{cp}$ , (1cp = 1centipoise =  $10^{-2}$ Poise),  $\rho = 2.7 \text{ g/cm}^3$  (Aluminum).

**Terminal velocity in the low Reynolds number regime** depends on the particle radius as:

$$v_{term} = \frac{M'g}{6\pi\eta r} \cong \frac{r^3}{r} \cong r^2 \quad (18)$$

## The experiment

### Notes about the procedure

By measuring the vertical speed of different sized spherical particles in glycerol or water, you will determine how the terminal velocity scales with the radius of the sphere and the viscosity of the fluid. In order to find out how fast the terminal velocity is setup, estimate  $\tau$ .

You will use a video tracking method.

Open the LabView application 'Motion through Fluids' (shortcut on the computer desktop).

Confirm the default camera.

The program asks you to select the number of frames to be recorded. Do the trial experiments with 120 frames; adjust the number according to your experimental conditions later on.

The frame rate means how many frames per second. Try 20 (this will cover 6 seconds when combined with 120 frames), adjust it later.

You have to provide a location for saving the \*.avi file (movie of the falling particle).

Be ready to drop the bead: submerge the tweezers + bead in glycerol in order to avoid surface tension problems. Click on "Start the avi movie capture". You'll hear two short warning beeps followed by a long one: at the long one release the bead.

The application will output a text file with time (in seconds) and position (in mm) of the falling bead.

### Correction for the wall effect\*

We have already discussed the forces acting on a sphere falling in a liquid. There is a gravity force, a buoyancy force, and a drag force. For the calculation of the drag force, we used Stokes law, which assumes the sphere to be moving in an unbound or infinite fluid. Objects falling near a boundary (like the wall of a container) fall more slowly than an object falling far from a wall. You can see this for yourself by simultaneously dropping two identical spheres at the same time: one near the wall and one at the center of the container (try it!).

To account for the effect of the container wall on the motion of the falling sphere, a correction factor has been determined. The velocity correction for a sphere falling in the center of a fluid-filled cylinder is:

$$v_{corr} = \frac{v_m}{1 - 2.104 \left( \frac{d}{D} \right) + 2.089 \left( \frac{d}{D} \right)^2} \quad (19)$$

Where  $v_m$  is the experimentally measured mean velocity,  $d$  is the diameter of the sphere,  $D$  is the dimension of the container, perpendicular to the fall direction, and  $v_{corr}$  is the expected velocity of the sphere if it were falling in an unbounded fluid.

*\*This is derived from experimental measurements and taken from Lommatzsch, T., et al. "Conceptual study of an absolute falling-ball viscometer", Metrologia, 2001, 38 (531-534).*

➔ Estimate the correction for all sizes.

### Exercise 1: Low Reynolds number

You'll receive a box marked 'Glycerine' with five different sizes of spherical beads made of Aluminum or Teflon. Beads range from 1.94mm to 6.35mm in diameter.

A rectangular container (9.5cm×9.5cm×31.0cm) filled with glycerol will be provided. The container is placed in front of a dark chamber. The video camera is located at the rear of the chamber.

Test your setup by dropping beads and timing the fall using a stopwatch. This would give you a hint about the number of frames and frames/second.

Do 2-3 measurements for each bead size. Measure the diameter of each size, using a caliper.

→ Calculate the mean velocity for each bead size (this is the mean terminal velocity). Correct the data for the wall effect. Calculate the Reynolds number. Does the value match the "low  $Re$ " condition of your problem?

→ Using Python, plot the mean terminal velocity of the spheres in glycerine as a function of radius. Fit the data using eq. (18). The slope of the plot should be close to 1. Did you notice any discrepancy?

### Exercise 2: High Reynolds Number

You will also receive a box with Nylon beads marked Water. Measure the diameter of each bead. Replace the large container marked 'Pure Glycerine' with the container marked 'Water'. Please take care not to spill glycerine (it is messy and slippery).

Make sure you remove all the air bubbles from the container walls.

Test your setup again by dropping beads without the tracking software to estimate the fall time. As you will notice, the fall of larger particles is not in a straight line. Sometimes, they wobble due to water turbulence. Practice until you get the best trajectory.

Take 5 measurements for each bead size using the tracking program.

→ Calculate the mean velocity for each bead size. Can this be interpreted as 'terminal velocity'? Correct the data for the wall effect. Calculate the Reynolds number in each case. Is there a critical velocity (or a critical bead size) that confirms the 'high  $Re$ ' number ( $Re > 200$ )?

→ Using Python, plot mean velocity of the spheres in water, as a function of the radius. Fit the data using eq. (12). Do you notice any discrepancy with theory?

### Appendix (Physical Reference Data):

- Glycerol viscosity = 934 centipoises (cp) or 9.34 g/(cm s) at 25°C
- Glycerol density = 1.26 (g/cm<sup>3</sup>)
- Viscosity of water ≈ 1 cp @ 25°C Density of water ≈ 1 g/cm<sup>3</sup> @ 25°C
- Aluminum density = 2.7 g/cm<sup>3</sup>, Teflon density = 2.2 g/cm<sup>3</sup>, Nylon density 1.12 g/cm<sup>3</sup>.

*The experimental setup was built by Larry Avramidis. Larry also designed the LabView tracking program.*

*This guide was written by Ruxandra Serbanescu in 2013. It includes material from Reference 1, which was part of an experiment designed by W. Ryu at Princeton University in 2005.*