Currents through inductances, capacitances and resistances

Introduction

This experiment illustrates voltage relations in circuits involving combinations of *inductance*, *capacitance* and *resistance.* The exercises deal with decay of voltages or currents in circuits momentarily disturbed but then left with constant (or zero) applied potentials. Further investigation of circuits is provided in a later experiment, "The Q of Oscillators".

AC circuit elements

In DC circuits, the electro-motive force pushes electrons along. Resistors remove the energy by converting it to heat.
In AC circuits, currents vary in time, so we have to consider variations in the energy stored in electric and magnetic fields of capacitors and inductors, respectively.

The Appendix at the end of this guide sheet provides details of AC circuit theory

Capacitance: whatever the configuration, the capacitance C between two opposite charged surfaces is defined by:

$$V = \frac{Q}{C} \qquad (1)$$

where Q is the magnitude of the charge distributed on either surface and V is the potential difference between the surfaces.

Inductance: The usual model for an inductor is a solenoid. By Faraday's Law of self-inductance, a changing current in a circuit induces a back electro-motive force (emf) that opposes the change in current:

$$V = L\frac{dI}{dt} \qquad (2)$$

where V is the back emf across the inductor, $\frac{dI}{dt}$ is the derivative of current through the inductor and L is the inductance.

Basic AC circuits

a) RC

Figure 1a) shows a circuit with no signal source, but we assume the capacitor to be initially charged. The voltage across the charged capacitor is:

$$V_0 = \frac{Q_0}{C} \qquad (3)$$

and produces an initial loop current $I_0 = \frac{V_0}{R}$ (4)

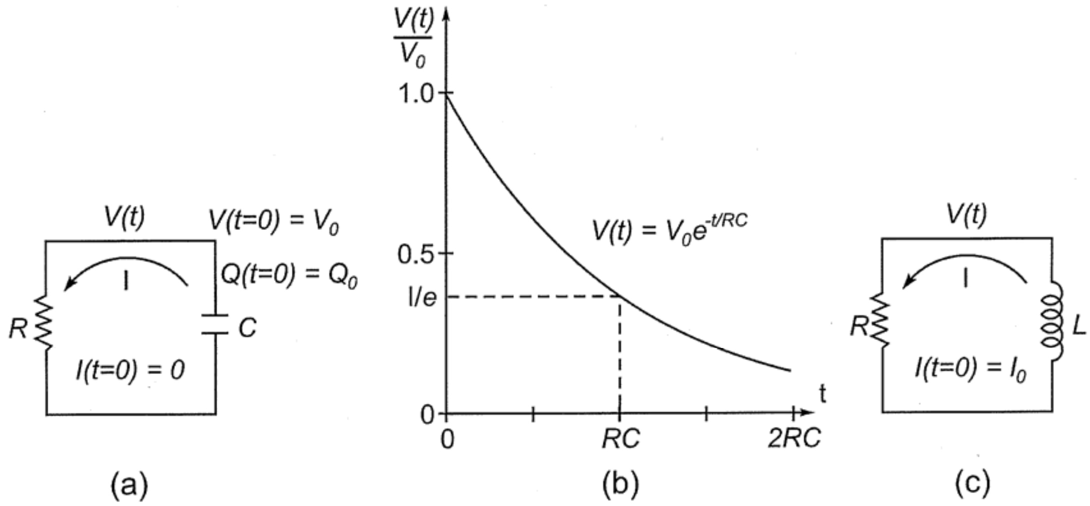


Figure 1 a) RC circuit; b) Exponential decay of voltage as a function of time in an RC circuit; c) RL circuit

By using Kirchhoff's loop rule and the derivative of Ohm's law, the voltage across the resistor (and capacitor) is given by (see Appendix):

$$V(t) = V_0 e^{-t/RC} = V_0 e^{-t/\tau} \tag{5}$$

The constant τ = RC is called the time constant of this circuit. It defines the time needed for V(t) to fall to 1/e of its initial value.
The voltage decay as a function of time is presented in Figure 1b).
The RL circuit (Figure 1c) produces a current expression with the same exponential time dependence as Equation (6) if we assume an initial loop current at t = 0. For this circuit, the time constant is found to be L/R.

b) LC

A series circuit with an inductor and a capacitor is presented in Figure 2a). We assume the capacitor fully charged at t = 0. Since the circuit has no resistor to remove energy from the electronic system, we can expect to find an oscillatory exchange of energy between the electric field of the capacitor and the magnetic field of the inductor.
By using the Kirchhoff's expression for the loop voltage sum and the differential Ohm's Law, and by solving the resulting differential equation (see References and Appendix for a detailed calculation), we obtain:

$$v(t) = \frac{q(t)}{C} = \frac{Q_0}{C}\cos(\omega_r t) = V_0 \cos(\omega_r t) \tag{6}$$

Where *v(t)* is the transient voltage drop across the capacitor (and across the inductor).

The specific frequency ω_r is called the natural frequency of the circuit:

$$\omega_r = \frac{1}{\sqrt{LC}} \tag{7}$$

Equation (6) above confirms the assumption of oscillatory behavior (see also the graph from Figure 2b)).

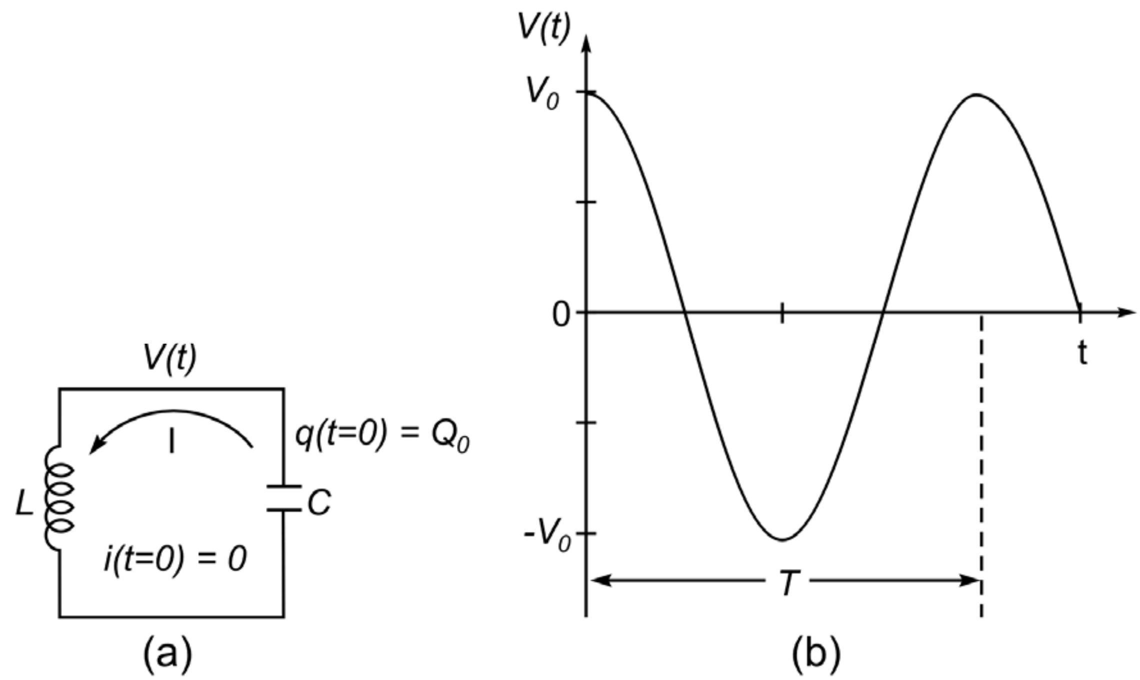


Figure 2 a) The LC circuit; b) The voltage across the capacitor as a function of time

The amplitude and phase of the oscillation are determined by the initial conditions, but the frequency is determined entirely by the circuit elements L and C.

c) LCR

The RLC circuit, presented in Figure 3a) includes a dissipative element (the resistor), so the differential equation resulting from the combination of Kirchhoff's expression for the loop voltage sum and the differential Ohm's Law will be more complicated (see Appendix). It has three possible solutions, depending on the value of the resistance R, relative to $\sqrt{L/C}$.

a) For: $R < 2\sqrt{L/C}$ the circuit is *underdamped* and the response function will be the product of a sinusoidal and an exponential (Figure 3b)).

b) For: $R = 2\sqrt{L/C}$ the circuit is *critically damped* with a non-oscillatory decay response (Figure 3c)).

c) For: $R > 2\sqrt{L/C}$ the circuit will be *overdamped* and the transient response will be given by the sum of two decaying exponentials (Figure 3d)).

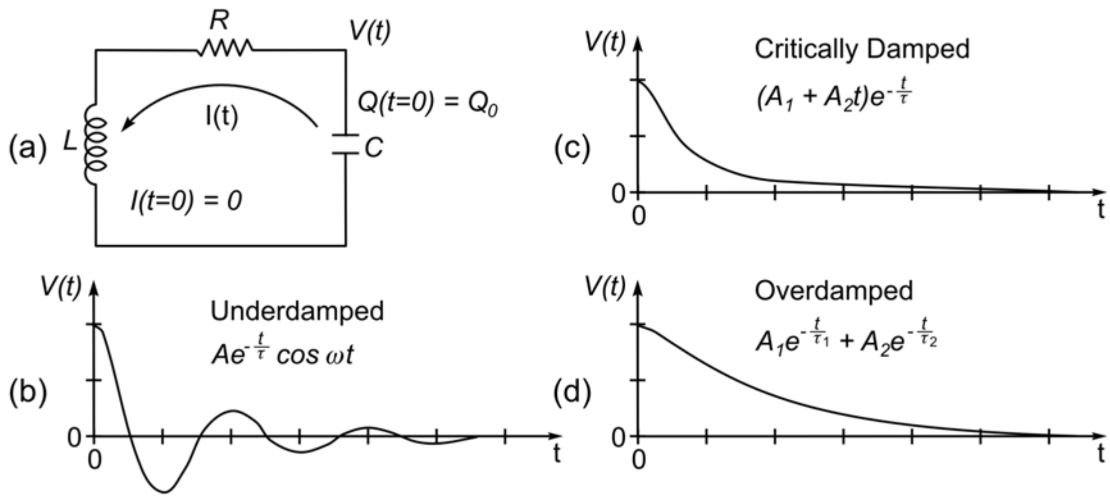


Figure 3 a) Initial conditions for the LCR circuit; b), c) d) Transient voltages

Techniques for observing transient decays of currents and voltages

If the decay times are sufficiently long to be followed slowly by eye (one second or longer) as is possible in the case of the CR circuit, the initial voltage V_0 can be set manually as in the following circuit:

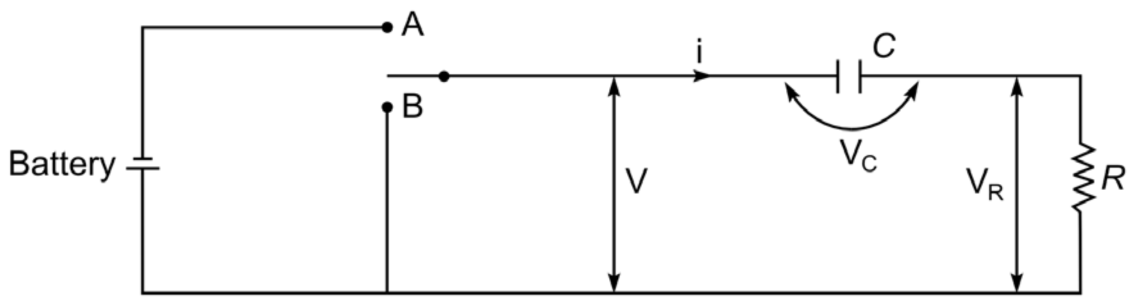


Figure 4. Circuit for studying slow transient decays

Each time the switch is acted upon, voltage V is changed but then stays constant. Behavior of the transient current i through the circuit could be observed with a meter, but is more easily observed on an oscilloscope connected across resistance R whose voltage drop V_R = iR is just proportional to the current i.
Voltage V_C may be observed with the oscilloscope connected across the capacitor C.
If the time constants are short, it is necessary to do the switching described above more rapidly. This is done by means of a pulse generator or a square wave signal generator. The square wave generator should be connected as illustrated, in order to observe the voltage across the resistor (V_R) or current i (see Figure 5).

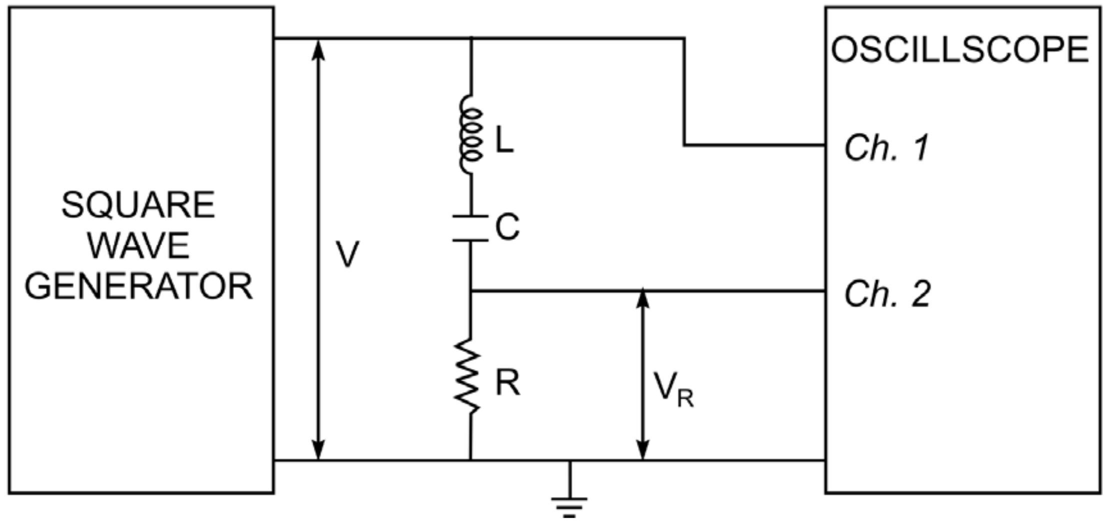


Figure 5 Setup for studying fast transient voltages

The above diagram is for a complete LCR circuit. It should be modified appropriately in order to study LR and CR loops. Triggering of the oscilloscope should be done internally. To observe V_C or V_L, capacitance or inductance should be interchanged with the resistance so that in all observations the ground terminal of the square wave generator is connected to the ground terminal of the oscilloscope.

NOTE: It is important that the signal generator should produce a square pulse independent of the current drawn from it. To improve the constancy of the output of your generator, two diodes should be connected in parallel in opposite directions across the generator output, and the output level should be turned to maximum.

Exercise 1 Current decay following a transient voltage disturbance

Connect the RC circuit to the battery through the switch, with values of R = 470 kΩ and C = 1.0 μF as in Figure 4. With oscilloscope sweep speed of > 25 ms/cm, first display V and then V_R. "Click" the switch just when the spot flies back to start and you will see most of the transient decay. Adjust the vertical sensitivity to a suitable setting, and adjust the Y position so that when the transient being examined is finished, the spot is moving along a convenient horizontal grid line. You may use the RUN/STOP function of the oscilloscope to freeze the signal. Estimate the time constant. Compare this to the calculated value of RC.

Note 1 (Very important) : THIS SECTION OF THE EXPERIMENT SHOULD BE DONE QUICKLY (IN LESS THAN 15 MINUTES!)
Note 2*:* Capacitors and resistors are not labeled. You may need to measure them.

Connect the RC circuit to the square wave generator, using the circuit of Figure 5 appropriately modified, and observe V and V_R for any value of R between 100Ω and 100 kΩ, and for C = 0.022 μF. What are the observed time constants? How do these compare with the value of RC?

Do the same as before observing V and V_R for the LR circuit. Use the smallest R and the coil provided. (L for this coil is between 30 mH and 300 mH.). From the observed time constant, estimate the inductance of the coil.

Note 3*:* The coil is not a pure inductance, but acts as if there were a perfect inductance in series with a resistance. This effective series resistance is called the internal series resistance of the coil. What effect does this resistance have on the observed results in this part of the experiment?

Do the same as before, observing V, V_L and V_C for the L-C circuit, using C = 0.022μF. From the frequency of the oscillations, again calculate the value of the inductance.

Note 4**:** You will have to get used to adjusting the oscilloscope sweep speed and the square wave generator repetition frequency in order to have an adequate full display of the voltages being observed.

Current-voltage relationships for sinusoidal AC

See the Appendix for definitions of: capacitive and inductive reactance, phase difference and circuit impedance.
For a series RC circuit, the phase difference between V_C and I can be determined by observing the phase relation between V and V_R as a function of frequency.
Observation: in AC measurements, V, V_R, V_C and V_L are amplitudes of v(t) (instantaneous value of voltage across a circuit element).
Similar measurements for an LR circuit can be used to check a similar relation for the inductance.

For the LCR circuit we can define the total impedance Z:

$$Z = \sqrt{(\omega L - 1/\omega C)^2 + R^2} \tag{8}$$

Z can be found by measuring the total voltage V, compared to the voltage $V_R = R \cdot I$ across the resistor, so:

$$Z = \frac{V}{V_R} R \tag{9}$$

Exercise 2: AC currents and voltages

The circuit in Figure 6 is useful for comparing V_C with V_R in magnitude and phase, in the RC circuit:

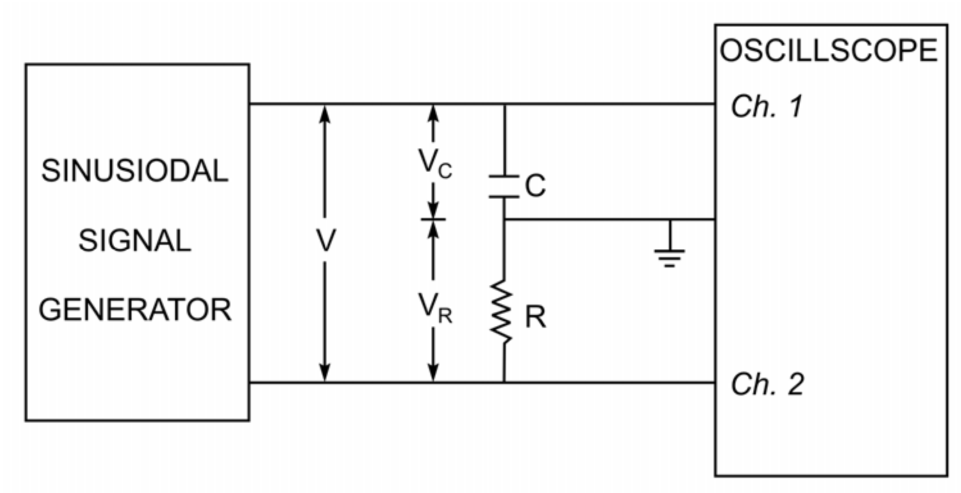


Figure 6 RC circuit (AC current)

Note that this circuit compares ($+V_C$) with ($-V_R$) because both voltages are measured relative to the SCOPE ground. Thus it is important to be careful in interpreting observed phase differences.
For comparing V with V_R in the L-C-R circuit, the circuit in Figure 7 is useful.

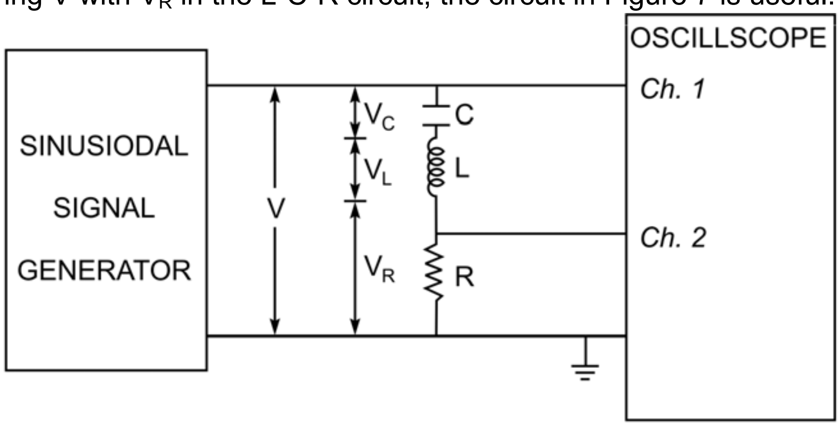


Figure 7 LR circuit (AC current)

For C = 0.022 μF and R between 100Ω and 100kΩ, measure V_C/V_R in magnitude and relative phase in the RC circuit for several frequencies between 10 Hz and 1.0 MHz. (Observe whether V_C leads or lags V_R in phase.)
Repeat for the LR circuit, using the coil from Exercise 1. Use R = 100Ω.
Measure V/V_R in both magnitude and relative phase for the LCR circuit. Cover a selection of frequencies that shows the resonance curve.

Note: in doing this exercise you may insert an isolation transformer (provided) between the function generator and the circuit. The transformer is used to decouple the signal generator from the circuit (allows AC signals, but blocks DC and interference effects). It also provides the needed ground for Exercise 2 (Figs. 6 and 7)

Data Analysis (PHY224/324 students: Python requirement)

1. Plot *Z vs frequency* curves and interpret the shapes and intercepts of the curves.
2. Plot *phase vs frequency* curves in semi-log coordinates and interpret the shapes of the curves.

3. Comment on and interpret any deviations you may observe from the theoretically predicted curves. How many resonances did you observe in the LCR circuit?
4. Optional: If you have already studied about the quality factor Q, calculate Q for the LCR circuit using the nominal values of R, L, C. from your plot of *Z vs frequency*. Do the two values agree within experimental errors? What is your value for Z at resonance? Is it equal to R? Account for any difference.

If you wish, you may borrow an impedance bridge from the Resource Centre to obtain the values of your R, L, and C to a precision of $\pm$ 0.1 % along with their loss factors.

Questions

You will probably find that your data displays features which cannot be explained on the assumption that R, L, and C are all pure elements. Some extraneous effects are: resistance in the inductor windings and capacitance between them, the resistance of the dielectric of the capacitor.
What then, are the simplest 'equivalent circuits' of these elements which might explain your data? Study the plots carefully for clues!

General note

The following will prove helpful in the execution of the experiment: Draw a diagram for each circuit you will be using for all sections of the experiment before coming to class. When you write up the experiment report, include with each set of graphs or oscilloscope printouts a small diagram of the basic circuit used. This makes identification of the quantity being measured easier. In your report, compare the shapes of the curves obtained and the time constants measured to the results predicted by the theory. Comment on the correspondence or lack of correspondence to the theory.

For a deeper investigation of LCR circuits, see the experiment "The Q of Oscillators".

This experiment was revised in 2007 by Ruxandra M Serbanescu and Luke Helt.

Appendix: AC circuit equations

RC loop (see Figure 1)

Kirchhoff's Law for a closed loop can be written as: $\sum_i V_i = 0$. Differentiating:

$$\sum_i \frac{dV_i}{dt} = 0 \qquad \text{(A1)}$$

For resistor: $\frac{dV}{dt} = R\frac{dI}{dt}$; for capacitor: $\frac{dV}{dt} = \frac{I}{C}$. Summing (clockwise loop):

$\frac{RdI}{dt} + \frac{I}{C} = 0$. Integrate from initial I_0 at t=0 to I at time t:

$$\int_{I_0}^{I} \frac{dI}{I} = -\frac{1}{RC}\int_0^t dt \;\Rightarrow\; \ln\frac{I}{I_0} = -\frac{t}{RC} \;\Rightarrow\; I(t) = I_0 e^{-t/RC} \qquad \text{(A2)}$$

Ohm's Law: $V(t) = RI(t) = V_0 e^{-\frac{t}{\tau}}$ (A3)

The time constant $\tau = RC$ defines the time needed for voltage to decrease to 1/e of its initial value.

LC loop (see Figure 2)

Kirchhoff's expression for the loop voltage sum is: $L\frac{di}{dt} + \frac{q}{C} = 0$ where $i = \frac{dq}{dt}$ is the instantaneous current value at t. Rewrite Kirchhoff's law:

$$L\frac{d^2q}{dt^2} + \frac{q}{C} = 0 \qquad \text{(A4)}$$

Note that variables cannot be separated and solution has to be assumed as a general sinusoidal expression:

$q(t) = Q_0\cos(\omega t + \theta)$ where q(t) is maximum at t=0: q(0) = Q_0, therefore the phase θ has to be 0.

The solution can be tested in equation (A4):

$$-LQ_0\omega^2\cos(\omega t) + \frac{Q}{C}\cos(\omega t) = 0 \qquad \text{(A4')}$$

Note that equation (A4') is satisfied only if $\omega^2 = \frac{1}{\sqrt{LC}} = \omega_r^2$

This defines ω_r as natural (resonant) frequency of the circuit.
The current through capacitor can be expressed as:

$$i(t) = \frac{dq}{dt} = -Q_0\omega_r\sin(\omega_r t) = -I_0\sin(\omega_r t) = -I_0\cos\left(\omega_r t + \frac{\pi}{2}\right) \qquad \text{(A5)}$$

The voltage drop across the capacitor can be written as:

$$v(t) = \frac{q(t)}{C} = \frac{Q_0}{C}\cos(\omega_r t) = V_0 \cos(\omega_r t) \qquad \text{(A6)}$$

The amplitude V_0 and phase θ are determined by the initial conditions (θ). Frequency ω_r is determined by the circuit elements.

<u>LRC loop</u> <u>(see Figure 3)</u>

Introduction of a dissipative element (R) will lead to different transient response functions, depending on the relative value of the resistor with respect to other circuit elements.
Using Kirchhoff's loop law, we can write:

$$L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{q}{C} = 0 \qquad \text{(A7)}$$

A possible solution is an exponential function:

$$q(t) = Q_0 e^{-\frac{R}{2L}t \pm \left(\frac{R^2}{4L^2} - \frac{1}{LC}\right)t} \qquad \text{(A8)}$$

The bracket $\left(\frac{R^2}{4L^2} - \frac{1}{LC}\right)$ can be positive, zero or negative, depending on the relative values of the terms. Each of the conditions leads to a solution of equation (A7) and describes a particular behavior of the circuit.

1) For $\left(\frac{R^2}{4L^2} > \frac{1}{LC}\right)$ the circuit will be overdamped (Fig. 3d):

$$q(t) = Q_0 e^{-\frac{R}{2L}t} \sinh\left(\frac{R^2}{4L^2} - \frac{1}{LC}\right)^{1/2} t \qquad \text{(A9)}$$

2) For $\left(\frac{R^2}{4L^2} = \frac{1}{LC}\right)$ the circuit is critically damped (Fig. 3c):

$$q(t) = (A_1 + A_2 t)e^{-\frac{R}{2L}t} \qquad \text{(A10)}$$

3) For $\left(\frac{R^2}{4L^2} < \frac{1}{LC}\right)$ the circuit is lightly damped and the transient response will be a combination between a sinusoidal function and an exponential (see Figure 3b).

The expression $\sqrt{\frac{R^2}{4L^2}-\frac{1}{LC}}$ is now an imaginary number (square root of a negative quantity). Therefore, a complex exponential solution has to be tested:

$q = Ae^{j(\omega_0 t+\theta)}$ where $\omega_0 = \frac{1}{\sqrt{LC}}$ is defined as "natural" angular frequency of the loop; A and θ are constants to be adjusted according to initial conditions in the loop.
Trying the complex solution in equation (A7) and choosing the real part only, results into:

$q(t) = Q_0 e^{-\frac{R}{2L}t}\sin(\omega t+\theta)$ where $\omega = \sqrt{\omega_0^2 - \frac{R^2}{4L^2}}$ is the angular frequency of the LCR loop. (A11)

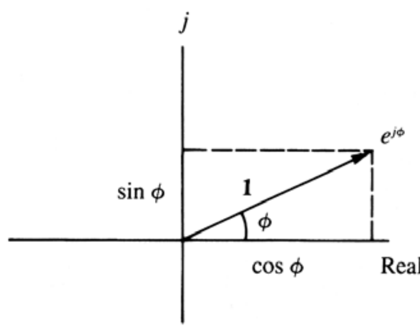
Phasors and complex exponentials

Understanding the AC circuits is very much simplified by the use of phasors (rotating vectors) in the complex plane to describe the relationships between current and voltage.

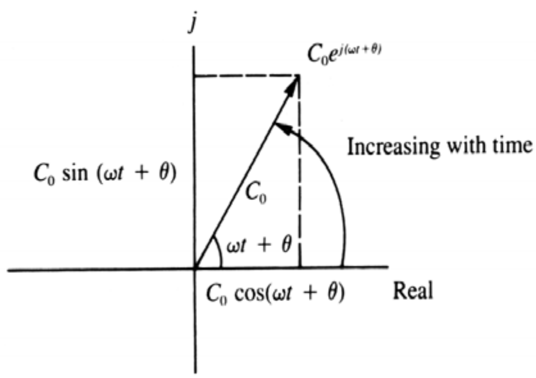
Euler's equation:

$e^{j\phi} = \cos\phi + j\sin\phi$ where $j=\sqrt{-1}$ (A12)

makes it possible to visualize the complex exponential $e^{j\phi}$ as a unit vector (phasor) on the complex plane:



Any complex number $c(t) = C_0 e^{j(\omega t+\theta)}$ can be visualized in the same way:



The complex phasor is a convenient mathematical tool. Whenever a comparison is to be made with physical measurements it is necessary to take the real part of the phasor to be the physical solution.

Circuits with sinusoidal sources

An AC signal source is defined as an electromotive force source capable of providing:

$$v(t) = V_0 \cos(\omega t + \theta) \; and \; i(t) = I_0 \cos(\omega t + \theta)$$

or: $v(t) = V_0 e^{j(\omega t + \theta)} \quad and \quad i(t) = I_0 e^{j(\omega t+\theta)}$ (A13)

Ohm's Law applies to AC circuit loops, only that complex expressions for v(ω, t) and i(ω, t) are used:

$$v(\omega, t) = Z(\omega) i(\omega, t) \tag{A14}$$

Equation (A14) defines the complex impedance of the loop: Z(ω).

Simple expressions for Z can be derived for each of the loops studied before.

<u>a) Resistive impedance</u>; use a simple loop with only one resistor R.

Assuming sinusoidal current of the form: $i(t) = i \, e^{j\omega t}$ and voltage given by:

$v(t) = V_R \, e^{j\omega t}$, combine them in Ohm's Law to get: $Z_R = R$ (A15)

Note that this impedance is always a real number.

<u>b) Inductive impedance</u>: use a simple loop with only one inductor L:

Kirchhoff's voltage law yields:

$$v(t) - L\frac{di(t)}{dt} = 0 \;\; or: v = j\omega L \, i$$

This defines the inductive impedance as:

$$Z_L = j\omega L \tag{A16}$$

Note that this impedance is zero for DC signals: only AC currents can produce a back *emf* and make the inductor "visible" (reactive) in the AC circuit loop.

<u>c) Capacitive impedance</u>: use a simple loop with only one capacitor C:

Writing again the loop equation: $v(t) - \frac{q}{C} = 0, \quad then: \frac{dv(t)}{dt} - \frac{i(t)}{C} = 0$

and assuming v(t) and i(t) of the form (A13), we get:

$v(t) = \frac{1}{j\omega C} i(t)$, which defines the capacitive impedance: $Z_C = \frac{1}{j\omega C}$ (A17)

Complex impedances in series or parallel can be combined using the same rules derived for resistor combinations:

$$Z_{eq,series} = Z_1 + Z_2$$

$$Z_{eq,parallel} = \frac{Z_1 Z_2}{Z_1 + Z_2} \tag{A18}$$

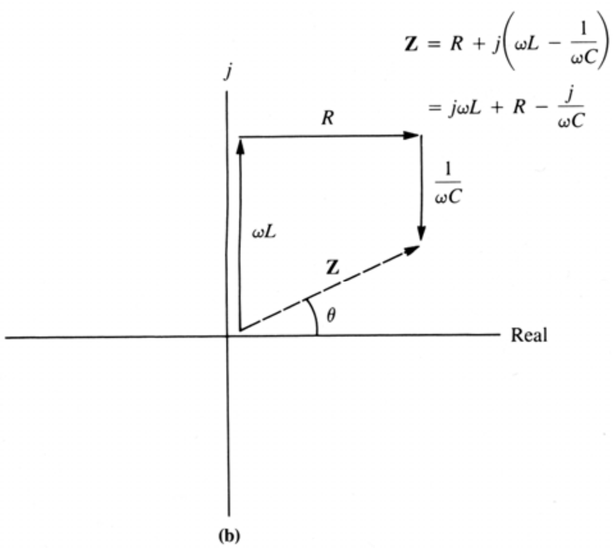
The equivalent impedance will have both real (resistive) and imaginary (reactive) parts:

$$Z_{eq} = R + jX \tag{A19}$$

X is called reactance. Reactance of a capacitor is defined as: $X_C = -\frac{1}{\omega C}$ and that of an inductor could be expressed as: $X_L = \omega L$.
The complex impedance for a series LCR loop can be written as:

$$Z = R + j\left(\omega L - \frac{1}{\omega C}\right) = j\omega L + R - \frac{j}{\omega C} \tag{A20}$$

and the phasor representation in the complex plane is presented below:



The phase angle θ is actually the angle displacement of voltage versus current. Circuit elements ultimately define this angle:

$$\theta = \arctan\left(\frac{X}{R}\right) \tag{A21}$$

Note that at frequency $\omega = \frac{1}{\sqrt{LC}}$, the effects due to inductance and capacitance cancel and the impedance is purely resistive.

Phase angles between voltage and current

a) Simple resistor: current and voltage are in phase: $v_R = Ri_R$ (A22)

b) Inductive impedance: voltage is ahead of current by π/2:

$$v_L = j\omega L\, i_L = e^{j\pi/2}\omega L\, i_L \quad \text{(A23)}$$

c) Capacitive impedance: voltage lags behind the current by π/2

$$v_C = \frac{1}{j\omega C} i_C = \frac{1}{\omega C} e^{-j\pi/2}\, i_C \quad \text{(A24)}$$

Resonance

In a series LCR circuit, the observed oscillation is due to energy flow between the magnetic and electric fields. We noticed before (Eq. A.21) that when the frequency is $\omega = \frac{1}{\sqrt{LC}}$, the reactance term is zero. This means that if we **drive** the circuit at that frequency, the phase angle between current and voltage will be zero and there will be a maximum transfer of power between the driving source and the circuit. This condition is called resonance.