PHYS 599 Final Report

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January 11, 2022

Abstract

We investigate, by analytical means, how the convective instability due to a vortex in a corner causes flow separation. We connect this flow separation to the parameters of the vortex, such as circulation and distance between the walls. The obstacle to overcome would be how to model the vortex near a wall and derive equations that govern the vortex dynamics in this case, which are done using the method of images. The second obstacle would be to determine the point of flow separation, which is done using the Thwaites' approximation [6]. We were able to develop a group of MATLAB programs that can take in the input parameters, such as circulation and the coordinate of the center of the vortex, and output the separation point.

Keywords

Fluid dynamics; Vortex Flows; Vortex dynamics; Thwaites' approximation; Method of images

1 Introduction

Vortices in general are not a hard concept to grasp, a circular motion of fluid about a center point, where there's low pressure. Therefore, anything at the edge will eventually gravitate towards the center. However, when a vortex is in a cavity, it is difficult to predict how the vortex will react with the pressure distribution that it itself creates. This prediction is important as it would dictate what would happen to these walls in a real-life scenario. Nevertheless, if the conditions are just right enough then the vortex would create an adverse pressure distribution on walls, which would cause the flow to separate, and exploring this flow separation is the primary objective of this research. From experimental evidence[3], it has been established that these flow separations do exist in real-life vortex interactions, but predicting them using the parameters of the primary vortex, in a circumstance where the vortex is in a corner has not been done extensively.

2 Background

To establish relatability let a look at an everyday example, while passing by a building on a windy day, you will notice there are stronger gusts to wind at certain parts of the building depending on the wind direction. You will notice these gusts are mainly situated at the opposite side of the building to where the wind is directly hitting. This seems counter-intuitive, as the gusts should be where the wind is hitting the building rather than the opposite side, which is shaded from the wind. This phenomenon is due to the interaction of a certain type of vortex, a horseshoe vortex, and the building. This is demonstrated in Figure 1:

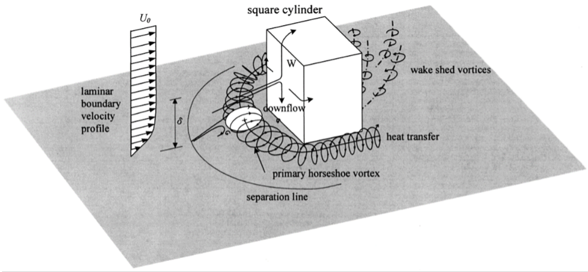

Figure 1: Diagram showing the flow pattern in front of a horse-shoe vortex, the cuboid is a building for our example [7]

The laminar boundary velocity profile shows the wind direction, but also assumes the wind is flowing a linear laminar flow. The building in this example is represented by the cuboid, as the wind comes in contact with the building the wind disperses in different directions. Some flow past the side of the build, some of it goes over the top, however some flow down the building. This downflow then interacts with the original wind flow and creates a vortex with a horizontal axis. This vortex with a horizontal axis then extends to the sides of the building, however, as it moves pasts the corner, the flow separates and interacts with the already present wind flow and creates a disruption, and in large amounts and with enough wind speed these gusts of wind can become a matter of public safety. Hence, when building such structures, it is important to take the flow separation into account, and predicting aspects of the flow separation in such cases can be quite useful.

3 Methodology

3.1 Potential Flow

Before we are able to implement the Method of Images [1], Potential flow [5] has to be understood, as that is what defines a fluid's motion using equations. In Potential flows, there are already known equations for some of the basic fluid motions, where we assume the fluid is irrotational (Curl of the particular equation is zero), incompressible (divergence is zero), and inviscid fluid (Viscosity of the fluid is almost zero). Hence, we are creating a very idealized situation where some of the more complexities of fluid motion are not affecting the model. For example, Figure 2 shows Uniform flow[2]:

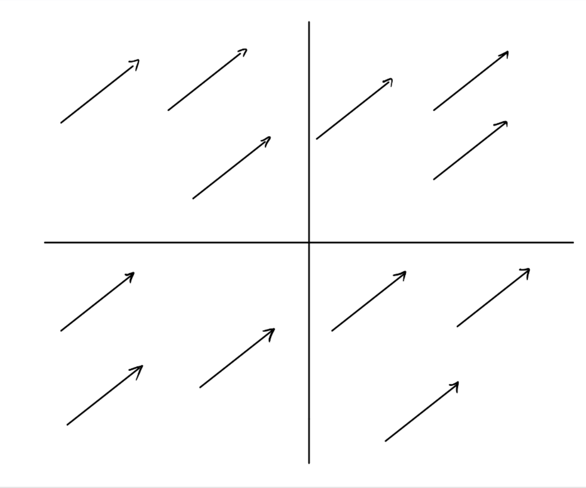

Figure 2: Flow pattern of a Uniform flow [2]

The flow in this example is in a 2D plane, where the direction of flow is making an angle α with respect to the x-axis. Here the horizontal and vertical lines are the x and y-axis respectively. Note that the flow speed is constant throughout the plane. There is one other canonical fluid flow that we will use to demonstrate potential fluid flow, Source flow [2]. Figure 3 shows a visual representation of a source flow.

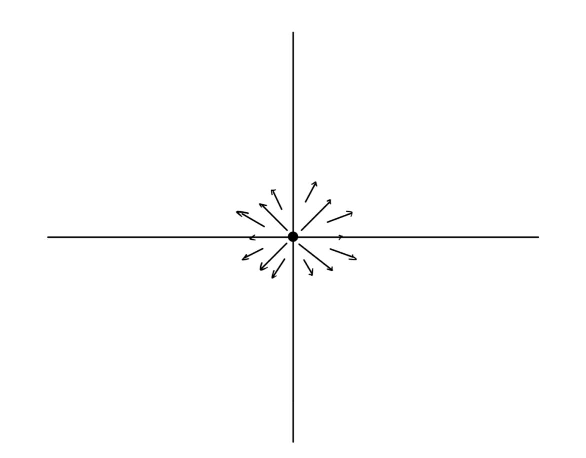

Figure 3: Flow pattern of a Source flow [2]

As we can see from the pattern this flow evenly dissipates in all directions. There are more simple flows that can be combined to form more and more complex flows. The important feature of these simple flows is that not only can they be added together, but the equations governing the flow patterns can also be added together. This is because of the assumptions we have made about these fluid flows: inviscid, irrotational, and incompressible.

Now what if a uniform flow and a source flow happen to be on the same 2D plane, what would the flow pattern look like? Note that the α, in this case, is zero when the uniform and source flow are on the same plane. Figure 4 demonstrates this.

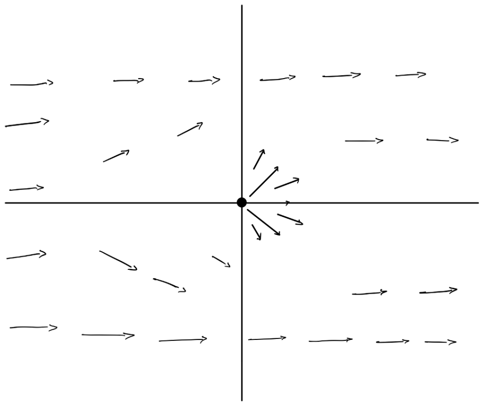

Figure 4: Flow pattern of Uniform flow and Source flow on the same plane [2]

The flow pattern seems to suggest, and as we have already established, they add up linearly. However, without proper equations, it is hard to demonstrate in terms of math. Both the uniform and source flow have equations governing the fluid flow pattern which we call potential functions (streamline functions also define flow patterns, however, they are perpendicular to potential functions, therefore the same flow has streamline and potential functions). For uniform flow the potential is:

$$w = Ue^{-i\alpha}z \tag{1}$$

Where U is the constant speed of the fluid flow, and α is the angle between the fluid flow and the x-axis in the anticlockwise direction. How this relates to the fluid flow is the velocity components of the fluid flow are defined by differentiating w in terms of z. Potential flow theory [5] says:

$$\frac{dw}{dz} = u - iv \tag{2}$$

Where u is the x velocity component and v is the y velocity component. Now if we differentiate our potential equation for uniform flow, we get:

$$\frac{dw}{dz} = Ue^{-i\alpha} = U(cos(\alpha) - isin(\alpha)) \tag{3}$$

Therefore,

$$u = Ucos(\alpha) \tag{4}$$

$$v = Usin(\alpha) \tag{5}$$

Now the velocities are recognizably what we would expect. However, for the Source, it is a bit more complicated to deduce from a glance at the fluid flow diagram, but the potential is [2]:

$$w = \frac{Q}{2\pi} log(z) \tag{6}$$

The appearance of 2π is due to the angular dependence from the source being circular, the Q is an integration constant, or volume flux which determines the strength of the Source.

As aforementioned for the case where the uniform and source flows are on the same plane, the potential functions are added together, and the new potential function is:

$$w = Uz + \frac{Q}{2\pi} log(z) \tag{7}$$

Note that the value for α is set to zero, as the Uniform flow is from left to right and parallel to the x-axis.

3.2 Method of Images

We know potential functions can be added together. However, the simple flows are boundary-less, but the vortex we are trying to model has walls. To simulate walls we use the Method of Images [1], for example, if we have a Source in front of a wall, the flow pattern is demonstrated in Figure 5.

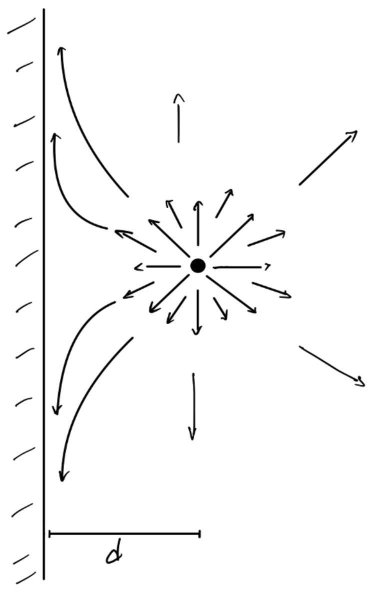

Figure 5: Diagram showing the flow pattern of a Source near a wall.

To get the potential equation, let us assume that the center of the source lies in the x-axis and the wall is the y-axis. Therefore, the Source lies at (d,0), then the potential for just the source (if there was no wall) would be:

$$w = \frac{Q}{2\pi} log(z - d) \tag{8}$$

However, to simulate the wall we can view the wall as a mirror, then there is going to be another source on the x-axis at x=-d, which will have the potential:

$$w = \frac{Q}{2\pi} log(z + d) \tag{9}$$

Nevertheless, if we add the potentials then the flow pattern would then look like what we would expect:

$$w = \frac{Q}{2\pi}log(z-d) + \frac{Q}{2\pi}log(z+d) \quad (10)$$

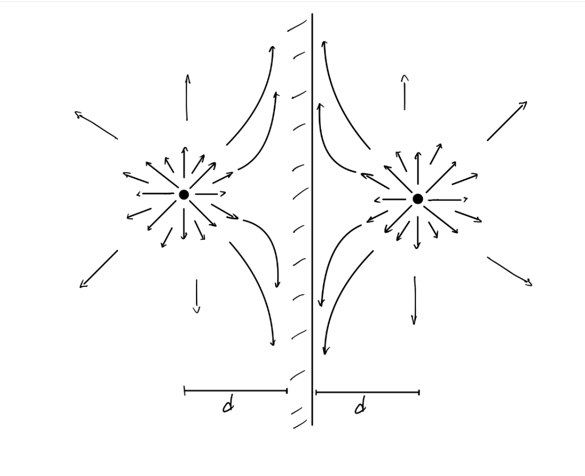

Figure 6: Diagram showing the flow pattern of a Source near a wall using the Method of Images

This potential then follows all the rules that we have established so far, as well as it's derivative will give us the velocity components.

3.3 Vortex in a Corner

The procedure for a vortex near a corner is very similar. Using the Method of Images the vortex near a wall looks like Figure 7.

For the physical vortex in the first quadrant, the streamline function is:

$$\psi_1 = -\frac{\Gamma}{4\pi}ln[(x-a)^2 + (y-b)^2] \quad (11)$$

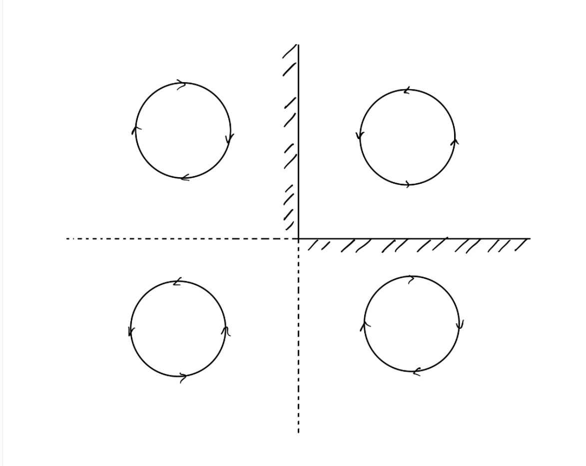

Figure 7: Diagram showing the flow pattern of a vortex near a L-shaped wall using the Method of Images

Where Γ is the circulation or strength of the vortex, a is the horizontal distance from the wall to the center of the vortex, and b is the vertical distance from the wall to the center of the vortex. The streamline functions for the rest of the vortices in the anti-clockwise direction are as follows:

$$\psi_2 = \frac{\Gamma}{4\pi}ln[(x+a)^2 + (y-b)^2] \quad (12)$$

$$\psi_3 = -\frac{\Gamma}{4\pi}ln[(x+a)^2 + (y+b)^2] \quad (13)$$

$$\psi_4 = \frac{\Gamma}{4\pi}ln[(x-a)^2 + (y+b)^2] \quad (14)$$

After all of these streamline functions are added, we can determine the horizontal and vertical velocity distributions using partial derivatives of the streamline functions:

$$u = \frac{\delta\psi}{\delta y} \tag{15}$$

$$v = -\frac{\delta\psi}{\delta x} \tag{16}$$

Solving for u and v gives us the velocity distributions:

$$u = \frac{\Gamma}{\pi}[\frac{b}{((x-a)^2+b^2)} - \frac{b}{((x+a)^2+b^2)}] \tag{17}$$

$$v = \frac{\Gamma}{\pi}[\frac{a}{((y+b)^2+a^2)} - \frac{a}{((y-b)^2+a^2)}] \tag{18}$$

Using which we can now determine the pressure distribution along the walls, using the formula:

$$P + \frac{1}{2}\rho\overrightarrow{u}.\overrightarrow{u} = P_0 \tag{19}$$

Then for the horizontal and vertical wall, the pressure distribution comes out to be (Appendix A shows all the steps to determine the pressure distribution):

$$P_H = P_0 - \frac{1}{2}\rho(\frac{\Gamma}{\pi})^2[\frac{b}{((x-a)^2+b^2)} - \frac{b}{((x+a)^2+b^2)}]^2 \tag{20}$$

$$P_V = P_0 - \frac{1}{2}\rho(\frac{\Gamma}{\pi})^2[\frac{a}{((y+b)^2+a^2)} - \frac{a}{((y-b)^2+a^2)}]^2 \tag{21}$$

4 Results

Using the pressure distribution, we can determine the pressure gradient for each of the walls, where the pressure gradient for the horizontal wall would be dp/dx, and for the vertical wall, it would be dp/dy. When pressure distribution along with the pressure gradient was plotted for each of the walls, where the circulation was set 7000 m^2s^{-1} and the center was at (10,10), the results were as follows (Appendix C shows the MATLAB sample code used to plot these graphs):

The pressure on both the walls seems to reach its maximum when the center of the vortex is at the smallest distance from the wall (i.e., the perpendicular distance from the wall to the center of the vortex), which is expected.

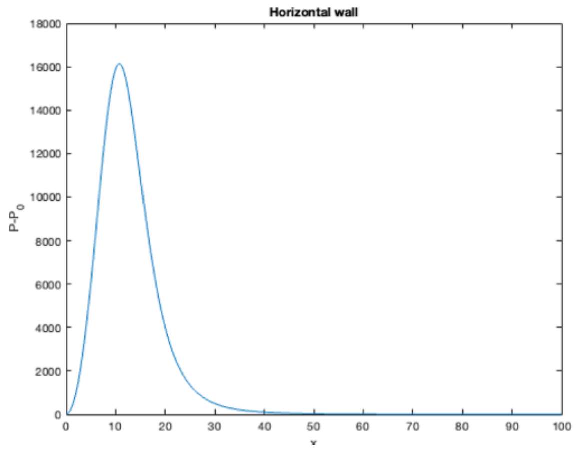

Figure 8: Pressure distribution along the horizontal wall, $\Gamma = 7000$ and center (10,10)

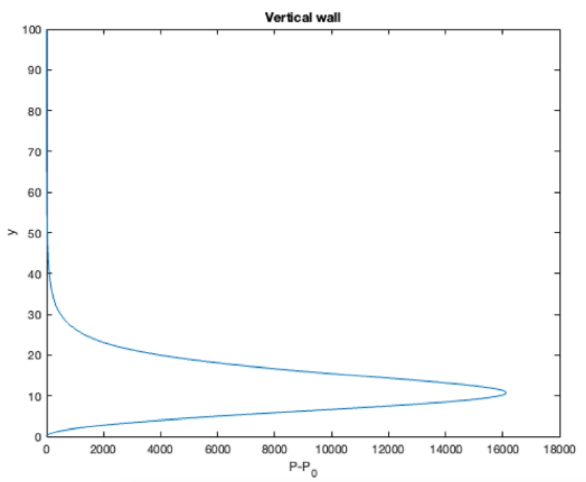

Figure 9: Pressure distribution along the vertical wall, $\Gamma = 7000$ and center (10,10)

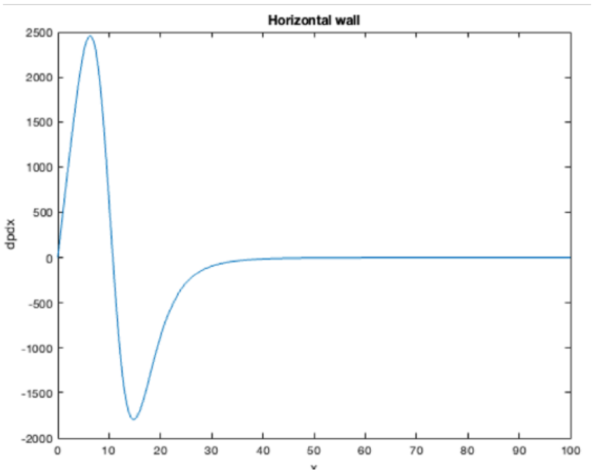

Figure 10: Pressure gradient along the horizontal wall, $\Gamma = 7000$ and center (10,10)

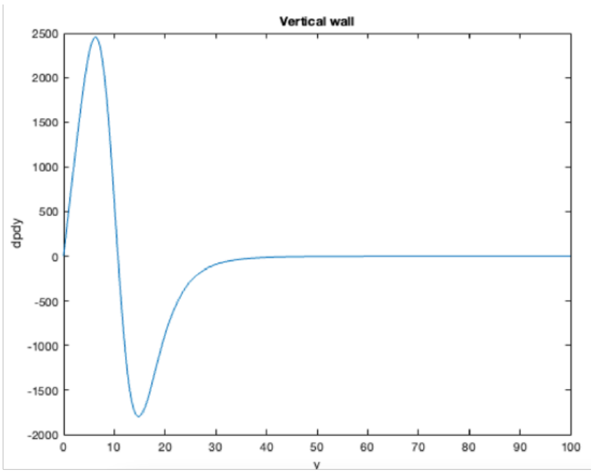

Figure 11: Pressure gradient along the vertical wall, $\Gamma = 7000$ and center (10,10)

To see the effect of change in circulation, the center was kept constant, and the circulation of the vortex was changed to 2000 m^2s^{-1}. The change in circulation seems to scale pressure distribution and gradient but maintains the overall shape of the graph. This is due to where Γ lies in the equation for pressure. However, to investigate how the center of the vortex affects the pressure distribution and gradient, these graphs were plotted:

As evident the change in the center of the vortex being moved closer to the vertical wall, causes drastic changes in the pressure distribution and the pressure gradient, which intuitively makes sense. The vortex being closer to the wall should impose more pressure on the wall and vice versa. Here as seen in the graph the pressure gradient of

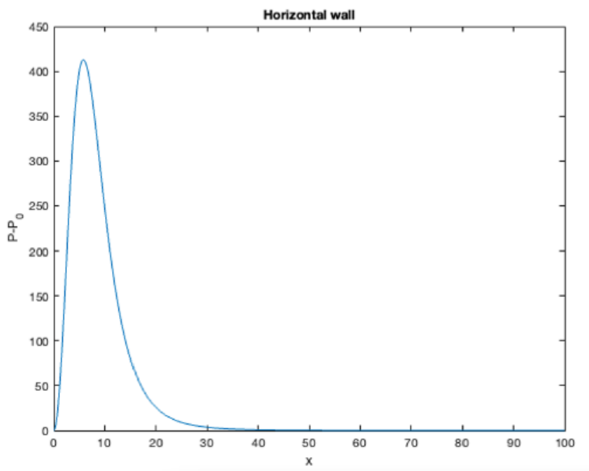

Figure 12: Pressure distribution along the horizontal wall, $\Gamma = 7000$ and center (1,10)

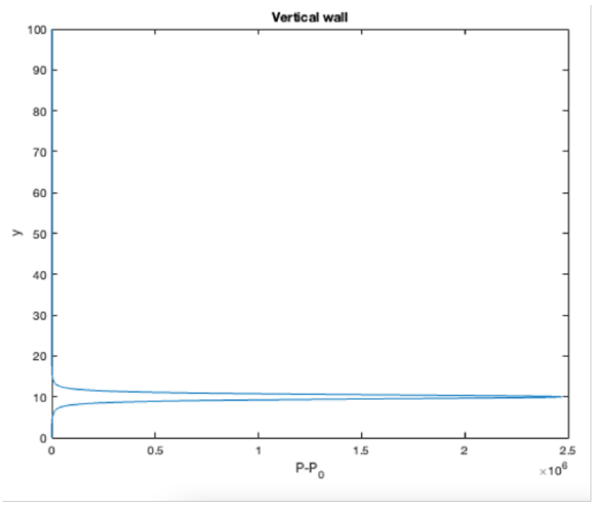

Figure 13: Pressure distribution along the vertical wall, $\Gamma = 7000$ and center (1,10)

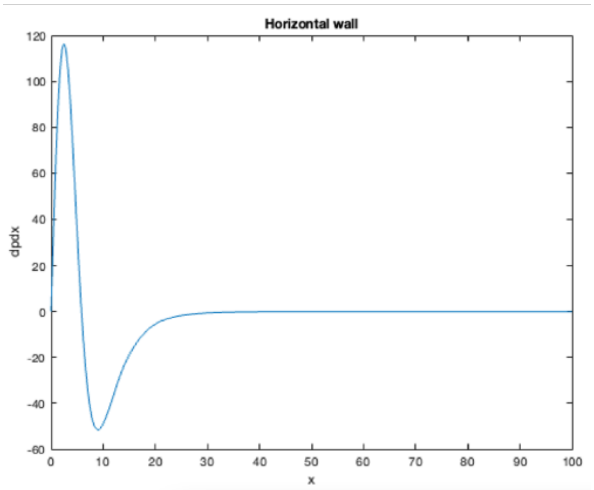

Figure 14: Pressure gradient along the horizontal wall, $\Gamma = 7000$ and center (1,10)

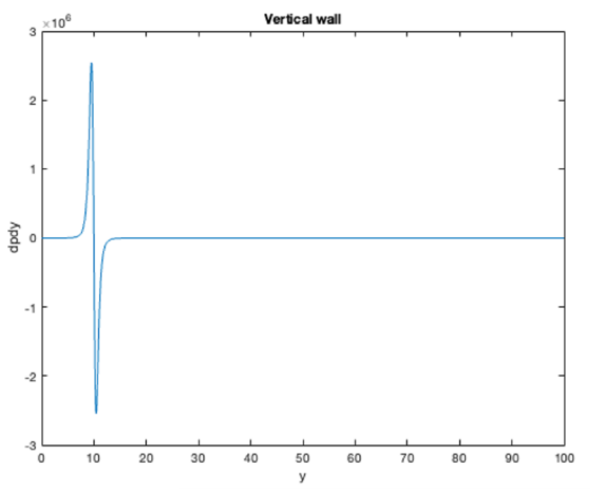

Figure 15: Pressure gradient along the vertical wall, $\Gamma = 7000$ and center (1,10)

the flow changes, there is an acceleration in the fluid, therefore it is not a steady flow, which causes the boundary layer to separate.

To determine the point of separation Thwaites' method [6] was used, which is an approximate solution to the momentum boundary layer equation:

$$\frac{\rho_\infty}{\mu_\infty}\frac{d}{dx}[V^6 Q^2] = 0.45V^5 \tag{22}$$

Where ρ is the fluid density, μ is the viscosity of the fluid, V is the velocity distribution and Q is the momentum thickness. Using this equation in combination with the velocity distribution equations of u and v, we can determine the point of separation (Appendix B shows all the steps to determine the point of separation):

$$\lambda_x = 0.075\frac{du}{dx} \tag{23}$$

$$\lambda_y = 0.075\frac{dv}{dy} \tag{24}$$

In this method, an approximation is made which states separation occurs when the value of λ is -0.0898 [6]. So, for different values of a and b, we get different values of x and y, which show us where the separation is occurring. MATLAB was used to solve the equation, which took in the parameters such as a and b(which define the center of the vortex) and Γ (circulation of the vortex) and gives us the point of separation for these parameters.

The center of the vortex was set to (10,10) and circulation $7000m^2s^{-1}$, it was observed that there were no points of separation as the roots of the equation came out to be imaginary. When the equation was solved using MATLAB for the values of a and b to be (3,3), the separation on the horizontal wall was at x=0.26 and on the vertical wall, the separation was at 2.71. Next, the circulation value changed to $2000m^2s^{-1}$ and the separation changed to 0.48 for the horizontal wall and 2.67 for the vertical wall (Appendix C shows the sample code used to determine the points of separation).

Conclusion

For a bridge pier over a river, there a similar formation of a horseshoe vortex. The interaction of the separated flow and the wake of the pier, soil and rocks at the base will be displaced which will weaken the pier's structural integrity[4]. We know this displacement of rocks and soil will depend on the strength and position of the vortex, which was not quantifiable. Therefore, quantitatively predicting when the flow will separate and minimizing its effects will allow us to build better bridge piers.

However, there's more to it. Including viscous effects and having another parameter for the angle between the walls, will allow making even better modeling for the effects of a vortex in a corner.

Acknowledgements

Thanks to Sir Robert Martinuzzi for pointing out mistakes and providing me with feedbacks, as well as, providing the lectures notes which gave me a summarized digestive resource compared to the online lecture

resources.

References

[1] 3.8 - method of images.

[2] Fluids - lecture 15 notes.

[3] Allen, J. R. Chapter 3 an outline of flow separation. *Developments in Sedimentology 30* (1 1982), 101–131.

[4] Sakib, N. Cfd techniques for simulation of flow in a scour hole around a bridge pier, 2013.

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[6] Thwaites, B., and Ae, A. F. R. S. On the momentum equation in laminar boundary-layer flow a new method of uniparametric calculation.

[7] Uchida, T., Araya, R., Uchida, T., and Araya, R. Practical applications of the large-eddy simulation technique for wind environment assessment around new national stadium, japan (tokyo olympic stadium). *Open Journal of Fluid Dynamics 9* (10 2019), 269–291.

Appendix A

Derivation of the velocity and pressure distributions along the horizontal and vertical walls

$$u = \frac{\delta\psi}{\delta y} = \frac{\delta\psi_1}{\delta y} + \frac{\delta\psi_2}{\delta y} + \frac{\delta\psi_3}{\delta y} + \frac{\delta\psi_4}{\delta y}$$

$$= \frac{-\Gamma}{\pi}[\frac{y-b}{((x-a)^2+(y-b)^2)} - \frac{y-b}{((x+a)^2+(y-b)^2)} - \frac{y+b}{((x-a)^2+(y+b)^2)} + \frac{y+b}{((x+a)^2+(y+b)^2)}$$

Along the horizontal wall, y=0:

$$u = \frac{-\Gamma}{\pi}[\frac{-b}{((x-a)^2+b^2)} + \frac{b}{((x+a)^2+b^2} - \frac{b}{((x-a)^2+b^2)} + \frac{b}{((x+a)^2+b^2)}$$
$$= \frac{\Gamma}{\pi}[\frac{b}{((x-a)^2+b^2)} - \frac{b}{((x+a)^2+b^2)}]$$

Along the vertical wall, x=0:

$$v = -\frac{\delta\psi}{\delta x}$$

$$v = \frac{\Gamma}{\pi}[\frac{x-a}{((y-b)^2+(x-a)^2)} - \frac{x+a}{((y-b)^2+(x+a)^2)} - \frac{x-a}{((y+b)^2+(x-a)^2)} + \frac{x+a}{((y+b)^2+(x+a)^2)}]$$

As x=0:

$$v = \frac{\Gamma}{\pi}[\frac{a}{((y+b)^2+a^2)} - \frac{a}{((y-b)^2+a^2)}]$$

Determining pressure: For the horizontal wall, y=0, v=0:

$$\begin{aligned} P &= P_0 - \frac{1}{2}\rho(u^2+v^2) \\ &= P_0 - \frac{1}{2}\rho(\frac{\Gamma}{\pi})^2[\frac{b}{((x-a)^2+b^2)} \\ &\qquad - \frac{b}{((x+a)^2+b^2)}]^2 \end{aligned}$$

For the vertical wall, x=0, u=0:

$$\begin{aligned} P = P_0 - \frac{1}{2}\rho(\frac{\Gamma}{\pi})^2[&\frac{a}{((y+b)^2+a^2)} \\ &- \frac{a}{((y-b)^2+a^2)}]^2 \end{aligned}$$

Appendix B

Using Thwaites' method to determine the point of flow separation:

$$\begin{aligned} P_V = P_0 - \frac{1}{2}\rho(\frac{\Gamma}{\pi})^2[&\frac{a}{((y+b)^2+a^2)} \\ &- \frac{a}{((y-b)^2+a^2)}]^2 \end{aligned}$$

$$\frac{\rho_\infty}{\mu_\infty}\frac{d}{dx}[V^6Q^2] = 0.45V^5 \tag{25}$$

For the horizontal wall:

$$\frac{\rho_\infty}{\mu_\infty}\frac{d}{dx}[u(x)^6Q^2] = 0.45u(x)^5 \tag{26}$$

$$\frac{\rho_\infty}{\mu_\infty}[u(x)^6Q^2] = \int 0.45u(x)^5dx \tag{27}$$

$$\frac{\rho_\infty}{\mu_\infty}Q^2 = \frac{0.45}{6}\frac{dV}{ds} + C \tag{28}$$

We can determine $\frac{du}{dx}$, which gives us:

$$\begin{aligned} \frac{du}{dx} = \frac{2\Gamma}{\pi}&(\frac{2b(x+a)}{(x+a)^2+b^2)^2} + (\frac{2b(x-a)}{(x-a)^2+b^2)^2}) \\ &(\frac{b}{(x-a)^2+b^2)^2} - \frac{b}{(x+a)^2+b^2)^2}) \end{aligned}$$

Assuming Q=0 at x=0 gives us C=0, therefore:

$$Q_x = \sqrt{0.075\frac{\mu_\infty}{\rho_\infty}\frac{du}{dx}} \tag{29}$$

For the vertical wall:

We have the same steps but with v(y):

$$\begin{aligned} \frac{dv}{dy} = \frac{2\Gamma}{\pi}&(\frac{2a(y+b)}{(y+b)^2+a^2)^2} + (\frac{2a(y-b)}{(y-b)^2+a^2)^2}) \\ &(\frac{a}{(y-b)^2+a^2)^2} - \frac{a}{(y+b)^2+a^2)^2}) \end{aligned}$$

$$Q_y = \sqrt{0.075\frac{\mu_\infty}{\rho_\infty}\frac{dv}{dy}} \tag{30}$$

The boundary layer approximation is related to Q using the equation:

$$\lambda_x = \frac{\rho_\infty}{\mu_\infty}(Q_x)^2\frac{du}{dx} \tag{31}$$

$$\lambda_y = \frac{\rho_\infty}{\mu_\infty}(Q_y)^2\frac{dv}{dy} \tag{32}$$

Therefore:

$$\lambda_x = 0.075\frac{du}{dx} \tag{33}$$

$$\lambda_y = 0.075\frac{dv}{dy} \tag{34}$$

Appendix C

MATLAB code used to plot the graphs of pressure distribution and gradient:

```
x = linspace(0,100,1000);
a=10;
b=1;
gamma=7000;
rho=1;
scale=(1/2)*rho*((gamma/pi).^2);

y = scale*((b./((x-a).^2+b.^2)-b./((x+a).^2+b.^2)).^2);
dpdx=2*scale*((2.*b.*(x+a))./((x+a).^2+b.^2).^2-(2.*b.*(x-a))./((x-a).^2+b.^2).^2).*(b./((x-a).^2+b.^2)-b./((x+a).^2+b.^2));
plot(x,y)
xlabel('x')
ylabel('P-P_0')
title('Horizontal wall')

figure()
plot(x,dpdx)
xlabel('x')
ylabel('dpdx')
title('Horizontal wall')

figure()
y1 = linspace(0,100,1000);
x1 = scale*((a./((y1+b).^2+a.^2)-a./((y1-b).^2+a.^2)).^2);
dpdy=2*scale*((2.*a.*(y1-b))./((y1-b).^2+a.^2).^2-(2.*a.*(y1+b))./((y1+b).^2+a.^2).^2).*(a./((y1+b).^2+a.^2)-a./((y1-b).^2+a.^2));
plot(x1,y1)
xlabel('P-P_0')
ylabel('y')
title('Vertical wall')

figure()
plot(y1,dpdy)
xlabel('y')
ylabel('dpdy')
title('Vertical wall')
```

MATLAB code used to determine the point of separation:

```
syms x y
a=3;
b=3;
c=-1;
gamma=2000;

%eqn = a*x^2 + b*x + c == 0;
eqnx= (0.075)*(2*gamma./pi)*((2*b*(x + a))./((x + a).^2 + b.^2).^2 + (2*b*(x - a))./((x - a).^2 + b.^2).^2)*(b./((x - a).^2 + b.^2) - b/((x + a).^2 + b.^2))==-0.0898;
assume (x>=0)
L_x = vpa(solve(eqnx,x))

eqny= (0.075)*(2*gamma./pi)*((2*a*(y + b))./((y + b).^2 + a.^2).^2 + (2*a*(y - b))./((y - b).^2 + a.^2).^2)*(a./((y - b).^2 + a.^2) - a/((y + b).^2 + a.^2))==-0.0898;
assume (y>=0)
L_y = vpa(solve(eqny,y))

fprintf('The point(s) of seperation in the x-axis is/are: %f.\n', min(L_x));
fprintf('The point(s) of seperation in the y-axis is/are: %f.\n', max(L_y));
```

```
L_x =

0.48455670726983094665232900874751
 2.6717235164723807234897594585798


L_y =

0.48455670726983094665232900874751
 2.6717235164723807234897594585798

The point(s) of seperation in the x-axis is/are: 0.484557.
The point(s) of seperation in the y-axis is/are: 2.671724.
```