Optimizing resource allocation under envy constraints

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Final Presentation

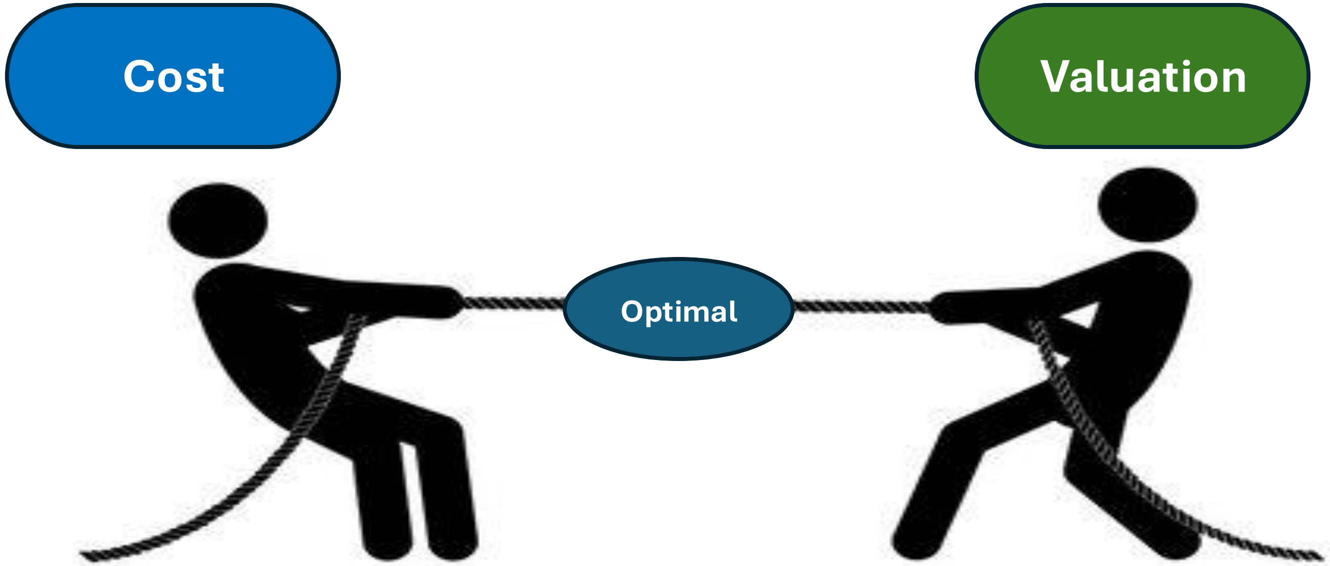
Introduction

- **Problem Statement:** Efficiently allocating resources to strike a balance between minimizing costs and maximizing overall satisfaction among agents

- **Significance:** Improving productivity and employee morale while simultaneously reducing costs

Premise

- Several real-world situations arise where an overseer should **assign a list of tasks** to her employees.
- Overseer would want to **minimize the overall cost**, while also **appeasing the employees** with their preferred tasks
- The **2 objectives** often act as **trade-offs** to one another



Assumptions

- **Cost variables :** Time required by each agent to complete a task
- **Valuation variables :** Degree of preference of each agent for a task
- Cost and valuations are **independent** of each other
- Cost and Valuation matrices consists of **random value between 0 and 1**
- **Number of tasks >= Number of agents**
- Valuations and costs are **additive**

Notations

- Let N be the set of agents and M be the set of goods.
- Let c_{ij} be the cost of allocating good $j \in M$ to agent $i \in N$
- Let each agent $i \in N$ evaluate good $j \in M$ at v_{ij}.
- Let A_i be the bundle allocated to agent i.

Formulation

- The problem can be stated as:
  - $\min \Sigma_{i\in N, j\in M}\ c_{ij}x_{ij}$
  - s.t. $\Sigma_{i\in N}\ x_{ij} = 1 \quad \forall\ j \in M$
  - $\exists\ g \in A_k: v_i(A_i) \geq v_i(A_k \backslash g)\ \forall\ k \in N$
  - $x_{ij} \in \{0,1\}\ \forall\ i \in N, j \in M$

- **Note :** We can directly solve the above problem using an optimization solver (or using a greedy approach) if the EF1 constraint is omitted.

Approaches

Random assignment

Minimum Cost assignment

Round-robin assignment

Min Cost assignment with Envy Cycle Elimination

Bang for Buck Round-robin assignment

Approaches

- **Min Cost assignment with Envy Cycle Elimination**
  - Use optimization solver (Gurobi) to **assign min cost allocation**
  - Construct **envy graph**
  - If there are **envy cycles**, **swap** the bundles along the cycle
  - Repeat until there is no cycle

- **Bang for Buck Round-robin assignment**
  - Start with empty allocations
  - Arbitrarily order the agents
  - Let the remaining goods be **R**
  - Repeat while there are remaining goods:
    - Let each agent pick one good from the remaining goods
    - Assign to each agent **$i \in N$,** the following good : $\mathbf{argmax_{j \in R}\, v_{ij} / c_{ij}}$

Experiment Setup

Agents and tasks

- Consider number of agents : **|N| = {10,20,30,40,50,60}**
- Consider number of tasks : **|M| = {60,80,100,120,140}**
- Iterate over every combination of agents and goods

Cost and values

For each **(|N|, |M|)** combination :

- Randomly sample each cost **c_{ij}~U(0,1)**
- Randomly sample valuations **v_{ij}~U(0,1)**
- For each of the 5 approaches, run **1000 trials**

Simulations

Cost Analysis

- **<u>Benchmark</u> : Min cost assignment** (through BIP optimization)
- Cost implies total cost incurred after assignment
- Compare other approaches against benchmark :

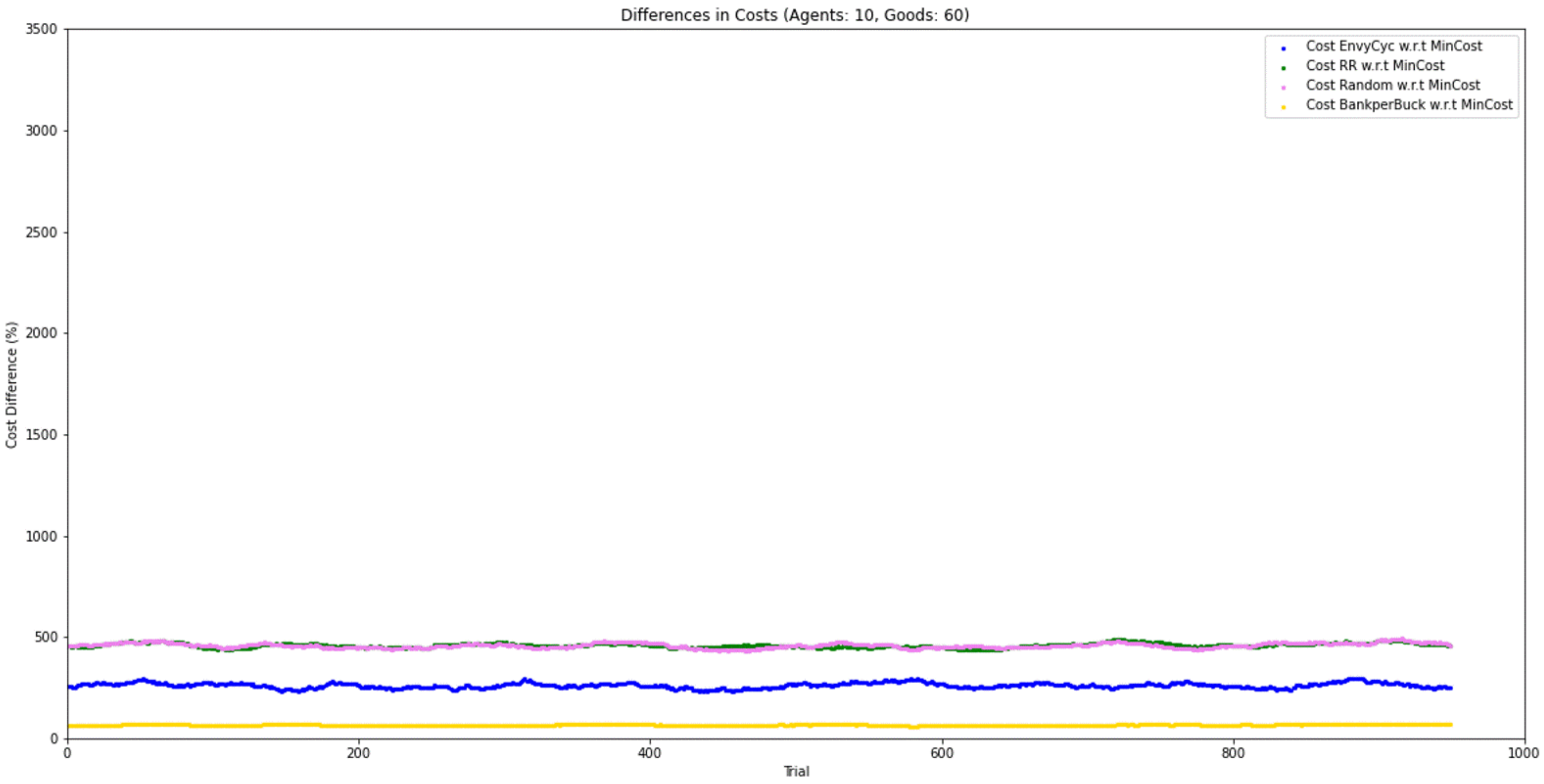
$$\left(\frac{\sum_i \text{Cost}_{\text{other approach,i}}}{\sum_i \text{Cost}_{\text{min,i}}} - 1\right) \times 100\%$$

Value Analysis

- Consider value as the valuation of every agent's own goods
- In a relative comparative setting, higher self-valuation implies less envy
- Valuation implies total self-valuation of all agents
- **<u>Benchmark</u> : Round-Robin assignment** (EF1 guarantee)
- Compare other approaches against benchmark :

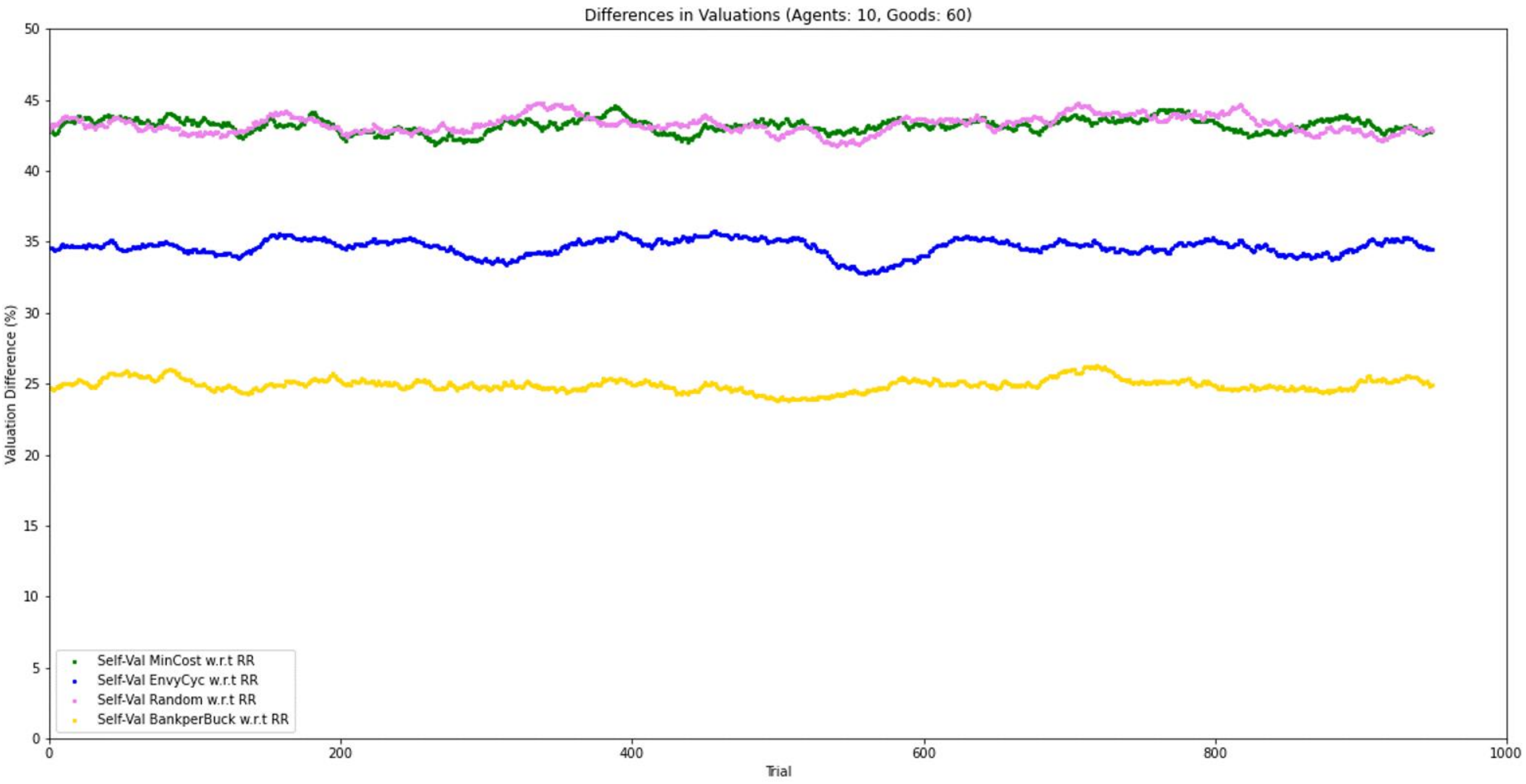
$$\left(1 - \frac{\sum_i \text{Value}_{\text{other approach,i}}}{\sum_i \text{Value}_{\text{round-robin,i}}}\right) \times 100\%$$

Result : Cost

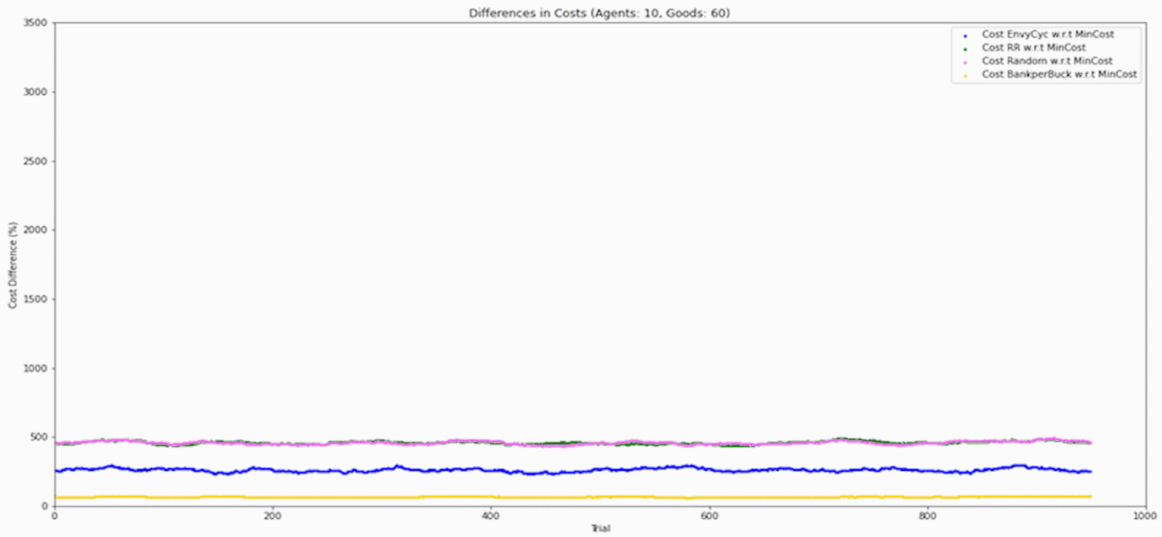


Differences in Costs (Agents: 10, Goods: 60)
Cost EnvyCyc w.r.t MinCost
Cost RR w.r.t MinCost
Cost Random w.r.t MinCost
Cost BankperBuck w.r.t MinCost
Cost Difference (%)
3500
3000
2500
2000
1500
1000
500
0
0
200
400
600
800
1000
Trial

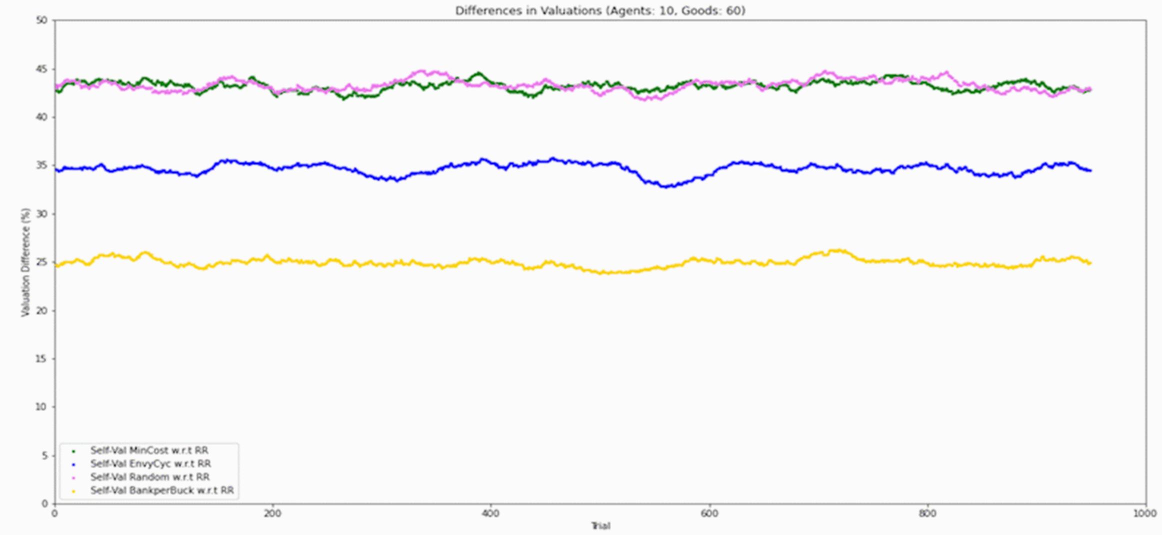
Result : Value



Cost


Differences in Costs (Agents: 10, Goods: 60)
Cost EnvyCyc w.r.t MinCost
Cost RR w.r.t MinCost
Cost Random w.r.t MinCost
Cost BankperBuck w.r.t MinCost
Cost Difference (%)
Trial

Value


Differences in Valuations (Agents: 10, Goods: 60)
Self-Val MinCost w.r.t RR
Self-Val EnvyCyc w.r.t RR
Self-Val Random w.r.t RR
Self-Val BankperBuck w.r.t RR
Valuation Difference (%)
Trial

Results

|N|	|M|	Mean % cost difference w.r.t. min cost allocation				Standard deviation of % cost difference w.r.t. min cost allocation			
		Modified envy cycle	Round robin	Random	Bang per buck	Modified envy cycle	Round robin	Random	Bang per buck
10	60	267.67	457.61	455.15	66.76	115.04	74.56	75.29	17.09
10	80	268.21	453.15	451.85	59.08	109.45	61.98	61.56	13.01
10	100	265.33	453.79	453.95	54.65	115.80	58.05	57.71	10.84
30	60	1142.78	1477.02	1477.87	184.66	259.22	226.97	236.50	43.85
30	80	1143.28	1476.27	1470.48	152.54	240.79	200.50	197.70	33.55
30	100	1146.28	1469.98	1471.08	132.89	215.47	173.48	176.32	26.28
50	60	2081.53	2518.57	2517.86	322.78	416.79	394.87	392.26	75.15
50	80	2085.99	2506.19	2514.22	262.15	368.83	339.57	335.52	55.83
50	100	2097.85	2488.21	2501	216.87	323.51	302.87	298.06	44.51

Results

|N|	|M|	Mean % valuation difference w.r.t. min cost allocation				Standard deviation of % valuation difference w.r.t. min cost allocation			
		Modified envy cycle	Min cost	Random	Bang per buck	Modified envy cycle	Min cost	Random	Bang per buck
10	60	34.48	43.32	43.28	25.22	3.57	4.25	4.17	3.56
10	80	36.25	43.95	43.9	25.6	2.91	3.48	3.52	2.96
10	100	37.2	43.94	43.99	25.57	2.68	3.14	3.12	2.55
30	60	25.13	46.37	46.19	29.03	2.99	4	4.01	3.36
30	80	27.7	46.53	46.54	29.41	2.54	3.46	3.53	2.86
30	100	29.64	46.84	47.12	29.79	2.3	3.1	3.26	2.52
50	60	19.02	46.7	46.57	29.68	2.92	3.88	3.96	3.45
50	80	21.3	47.03	47.17	30.29	2.49	3.39	3.29	2.92
50	100	23.39	47.33	47.48	30.58	2.15	2.94	3.07	2.51

Observations : Based on Cost

- **Round-Robin** gives a cost as **bad** as **Random assignment**
- **Bang for Buck** consistently **performs the best**
- **Min-cost assignment with Envy cycle elimination** performs slightly better than random assignment, yet **considerably bad at higher number of agents**
- All approaches worsen with increasing number of agents.

Observations : Based on Value

- **Min cost assignment** gives a valuation as **bad** as **Random assignment**
- **Bang for Buck** is the **most robust** among the other assignments and performs relatively well
- **Min-cost assignment with Envy cycle elimination** performs the best with high number of agents and low number of tasks (better than Bang for Buck)
- However, for low number of agents and high number of tasks cause Envy cycle elimination approach is out-performed by Bang for Buck
- Envy cycle elimination approach's performance :
    - ***directly proportional to the number of agents***
    - ***inversely proportional to the number of goods***

Conclusion

- **Min Cost assignment** and **Round Robin approaches** are the **most imbalanced**
- **Envy-Cycle elimination approach** is not suitable due to the **massive cost repercussions** for high number of agents
- **Bang for Buck variation on Round Robin approach** seems to **perform the best** in both settings and is the **most robust and balanced** among all

Thank you