University of Toronto Scarborough
MAT/CSCA67: Discrete Mathematics, Summer 2024, Erfan Meskar
Term Test 2: July 19, 2:10 PM
Duration: 50 minutes
Aids: No aid-sheet is permitted. No electronic or mechanical computing devices are permitted.

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- The University of Toronto Scarborough and you, as a student, share a commitment to academic integrity. You are reminded that you may be charged with an academic offence for possessing any unauthorized aids during the writing of an exam. Clear, sealable, plastic bags have been provided for all electronic devices with storage, including but not limited to: cell phones, smart watches, SMART devices, tablets, laptops, and calculators. Please turn off all devices, seal them in the bag provided, and place the bag under your desk for the duration of the examination. You will not be able to touch the bag or its contents until the exam is over. If, during an exam, any of these items are found on your person or in the area of your desk other than in the clear, sealable, plastic bag, you may be charged with an academic offence. A typical penalty for an academic offence may cause you to fail the course.
- There are **3 questions** and **8 pages** in this exam, including this one. When you receive the signal to start, please make sure that your copy of the examination is complete.
- Question **3.d** is an optional bonus question, worth 2 points. Please attempt this question only after completing all other questions, as it is particularly challenging. The maximum achievable score on this exam is 40 out of 38 points.
- Answer each question directly on the examination paper, in the space provided. **Please write carefully and clearly, in complete English sentences.**
- This exam includes an extra page for your scratch notes or in case you need extra space for any question, which must be submitted. Do not remove any page, including the scratch note page from your exam booklet.

Q1 **[7 pts]** Prove that for any function $f : A \to B$ and any sets $C \subseteq A$ and $D \subseteq A$,

$$f(C \cap D) \subseteq f(C) \cap f(D).$$

[**Write your answer to Q1 here.**]

Q2 **[11 pts]** Use induction to prove that $\mathbb{Z}^k = \underbrace{\mathbb{Z} \times \mathbb{Z} \times \ldots \times \mathbb{Z}}_{k}$ is countably infinite for any integer $k \geq 2$, where $\mathbb{Z}$ denotes the set of integers.

[HINT 1: We know that for any countably infinite sets A and B, the set $A \times B$ is also countably infinite. You can use this fact without proving it.**]**
[HINT 2: $(A \times B) \times C \neq A \times B \times C$, but an obvious bijection exists between the two.**]**

[**Write your answer to Q2 here.**]

Q3 **[20 pts + 2 bonus pts]** Note that this question has 3 parts plus 1 extra bonus part.

3.a [8 pts] Prove that for any positive integers a and k and any positive integer $b > ka$,

$$\gcd(a, b) = \gcd(a, b - ka).$$

[**Write your answer to 3.a here.**]

3.b [5 pts] Prove that for any positive integers s, t, and q, if $\gcd(s,t) = 1$ and $q \mid s$, then $\gcd(q,t) = 1$.

[**Write your answer to 3.b here.**]

3.c **[7 pts]** Consider the following recurrence for $n \geq 2$:

$$f(n) = f(n-1) + 2f(n-2),$$

and $f(0) = f(1) = 1$. Prove that $\gcd(f(n+1), f(n)) = 1$ for any $n \geq 0$.

[HINT 1: Observe that $f(n)$ is odd for all $n \geq 0$. You can use this fact without proving it.**]**
[HINT 2: You may use the fact stated in **3.a** without proving it.**]**
[HINT 3: You can use the following theorem without proving it:

Theorem: for any positive integers k, s, and t, if $\gcd(s,t) = 1$, then $\gcd(s, kt) = \gcd(s, k)$.**]**

[**Write your answer to 3.c here.**]

3.d [2 bonus pts] Prove that for any positive integers k, s, and t, if $\gcd(s,t) = 1$, then

$$\gcd(s, kt) = \gcd(s, k).$$

[**Write your answer to 3.d here.**]

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