MAT-CSC A67: Discrete Mathematics — Summer 2024

Quiz 10

Due Date: Friday, July 26, 11:59 PM, on Crowdmark

Q1. There are 5 distinguishable bins labeled $\{1, 2, 3, 4, 5\}$. How many ways are there of placing 100 indistinguishable balls into the bins, if each bin must have at least as many balls as its label (*i.e.*, bin 1 must have at least 1 ball, bin 2 must have at least 2 balls, etc)? Be sure to explain your reasoning.

Q2. Find the number of 7-letter words formed using the 26 letter alphabet A-Z, subject to each of the following restrictions. "Word" means any sequence of 7 letters.

2.a. Words with no restrictions.

2.b. Words with all letters distinct, such as BEARFUL.

2.c. Words with the letters in alphabetical order, such as ACCRUXX.

2.d. Words with one A, 2 E's, 2 R's and 2 S's, such as ERASERS.

Q3. **[Extra Questions]** Here are some extra counting for you to practice. These questions won't be graded.

3.a. How many permutations of the 26 letters A-Z are there with A and Z not next to each other?

3.b. How many permutations of the word ABRACADABRA are there?

3.c. How many ways are there to split n_1 indistinguishable apples and n_2 indistinguishable bananas among k people?

3.d. How many 13-card unordered hands have a 4-4-3-2 distribution – meaning the hand contains 4 cards from each of two suits, 3 cards from a third suit, and 2 cards from the remaining suit?

3.e. You have a drunken sailor walking along the real line starting at 0 and ending at n and the sailor takes steps forward or backward of size 1. The sailor uses $n + 2k$ steps in total. How many possible ways could this happen? For example, for $n = 2$ and $k = 1$, one of the possible ways is "backward, forward, forward, forward" or "forward, forward, forward, backward". (Note that the sailor went below 0 in the first example and past 2 in the second which is allowed.)

3.f. In class, we studied the binomial theorem, in which we studied $(x + y)^n$. For instance, Consider $(x + y)^2 = (x + y)(x + y)$. before grouping, this polynomial looks like $xx + xy + yx + yy$. Thus, the number of terms in this polynomial before grouping is 4. After grouping, it looks like $x^2 + 2xy + y^2$. Hence, the number of terms in this polynomial after grouping is 3. Furthermore, the number of times xy appears before grouping (*i.e.*, the coefficient of xy after grouping) is 2.

Now, consider the polynomial $(x + y + z)^n = (x + y + z)(x + y + z) \ldots (x + y + z)$.

3.f. i. What is the number of terms before grouping?

3.f. ii. What is the number of terms after grouping?

3.f. iii. What is the number of times $x^{n_1}y^{n_2}z^{n_3}$ appears before grouping (i.e., the coefficient of $x^{n_1}y^{n_2}z^{n_3}$ after grouping)?

3.g. In a perfect world, any pair of people are either close friends or total strangers. Ramsey finds himself in a room with 5 other people. Prove that there are either 3 mutual friends or 3 mutual strangers among them.