MAT-CSC A67: Discrete Mathematics — Summer 2024

Quiz 3

Due Date: Friday, May 31, 11:59 PM, on Crowdmark

Q1. Consider the following predicate.

$F(a, b)$: "a is a factor of b", a's domain: non-zero integers (i.e., $\mathbb{Z}^{\neq 0}$), b's domain: integers (*i.e.*, $\mathbb{Z}$).

Consider the following proof.

> Assume e to be an arbitrary integer. By definition of disjunction, $(e = 0) \vee \neg(e = 0)$ is true. Thus, we proceed by cases.
>
> - **Case 1:** assume $e = 0$. Consider the witness $x = 1$, a nonzero integer, and then we need to show that $F(1, 0)$. By the definition of the predicate F, we can rewrite this goal as $\exists c(0 = c \cdot 1)$. We pick the witness $c = 0$, which is an integer and therefore in the domain. Calculating, $c \cdot 1 = 0 \cdot 1 = 0$, as required. Since the predicate $F(1, 0)$ evaluates to true, the witness $x = 1$ proves the existential claim and the subgoal has been proved.
> - **Case 2:** assume $\neg(e = 0)$. Consider the witness $x = e$, a nonzero integer, and then we need to show that $F(e, e)$. By the definition of the predicate F, we can rewrite this goal as $\exists c(e = c \cdot e)$. We pick the witness $c = 1$, which is an integer and therefore in the domain. Calculating, $c \cdot e = 1 \cdot e = e$, as required. Since the predicate $F(e, e)$ evaluates to true, the witness $x = e$ proves the existential claim and the subgoal has been proved.
>
> The proof by cases is now complete for the arbitrary element e as desired. □

From the statements below, choose the one that is proved by the proof above.

1. $\forall x F(1, x)$
2. $\forall x F(x, 1)$
3. $\forall y \forall x F(x, y)$
4. $\forall y \exists x F(x, y)$
5. $\forall x \exists y F(x, y)$

Q2. Prove that for any integer n and prime number p, if $p \mid n$, then $p \nmid (n + 1)$. Before you start, you may find the following definitions and lemma helpful.

- **Definition 1:** Let $n, d \in \mathbb{Z}$. Then n is divisible by d if $n = dk$ for some $k \in \mathbb{Z}$. We use the notation $d \mid n$, read as d divides n, to denote that n is divisible by d.
- **Definition 2:** The integer $p > 1$ is a prime if it is only divisible by p, $-p$, 1, and -1.
- **Definition 3:** We use the notation $d \nmid n$, read as d does not divide n, to denote that n is not divisible by d.

Now you are ready to prove the following proposition.

> For any integer n and prime number p, if $p \mid n$, then $p \nmid (n + 1)$.