MAT-CSC A67: Discrete Mathematics — Summer 2024

Assignmnet 5: Counting

Due Date: Wednesday, July 31, 11:59 PM, on Crowdmark

[**NOTE:** Bonus points earned in the assignments will only contribute to your total assignment grade and will not be applied to your quiz, term test, or final exam grades. We will be extra strict with grading bonus questions.]

Q1. Answer the following questions.

1.a. (1 pt) Show that any set of 76 distinct positive integers chosen from $\{1, \ldots, 100\}$ must contain 4 contiguous integers (*i.e.*, $n, n+1, n+2, n+3$ for some n).

1.b. (1 pt) In a room of 100 people, some pairs shake hands and some don't. No two people shake hands more than once and nobody shakes hands with herself. Prove some two people shook the same number of hands.

1.c. (1 pt) 5 points are chosen within an equilateral triangle of side length 1. Prove some two are no more than $\frac{1}{2}$ apart.

1.d. (1 pt) How many integers must be selected from the set of integers from 2 through 40, to ensure there are at least two integers with a common divisor greater than 1.

Q2. Let $F(n, k)$ denote the number of surjective functions from the set $\{1, 2, \ldots, n\}$ to the set $\{1, 2, \ldots, k\}$. There is no simple formula for computing $F(n, k)$ in general, but we have a recurrence relation:

$$F(n+1, k) = kF(n, k-1) + kF(n, k).$$

2.a. (2 pt) Figure out $F(n, n)$ and $F(2, 1)$ directly.

2.b. (2 pt) Use the recurrence relation and induction to prove for all integers $n \geq 2$, $F(n, n-1) = \binom{n}{2} \cdot (n-1)!$

Q3. For each of the following, invent a situation and count in two ways to prove the identity. Assume all variables are arbitrary positive integers.

3.a. (2 pt) $\sum\limits_{k=0}^{r} \binom{n}{k}\binom{m}{r-k} = \binom{n+m}{r}$.

3.b. (2 pt) $\binom{n}{k}\binom{k}{m} = \binom{n}{m}\binom{n-m}{k-m}$.

3.c. (2 pt) $\binom{k}{k} + \binom{k+1}{k} + \ldots + \binom{n}{k} = \binom{n+1}{k+1}$.

3.d. (2 pt) $\sum\limits_{k=0}^{n} k\binom{n}{k} = n2^{n-1}$.

3.e. (2 pt) $\sum\limits_{k=0}^{n} k\binom{n}{k}^2 = n\binom{2n-1}{n-1}$.

Q4. Let S_n be the number of ways to add up 1's and 2's to obtain n, where order matters. (For example, $S_3 = 3$, with the possible ways being $1+1+1$, $1+2$, and $2+1$.)

4.a. (2 pt) Give an expression involving a summation for S_n. (Hint: consider summing over cases.)

4.b. (2 pt) Give a recursive expression for S_n for $n \geq 2$ (You may assume that $S_0 = 1$, and $S_1 = 1$).

Q5. (2 pt) Given $n \in \mathbb{Z}_{>0}$, prove that $x = (n^2)!/(n!)^n$ is an integer by inventing a counting problem for which x is the answer.

Q6. Suppose p is a prime.

6.a. (2 pt) Prove that for any $1 \leq k \leq p-1$, $p \mid \binom{p}{k}$.

6.b. (2 pt) Prove Fermat's Little Theorem via induction, *i.e.*, prove that for all integers a,

$$a^p \equiv a \pmod{p}.$$

[**HINT:** You may find the Binomial Theorem helpful: $(x+y)^p = \sum_{i=0}^{p} \binom{p}{i} x^{p-i} y^i$.]

Q7. **(2 pt)** How many arrangements of the letters in the word SOCIOLOGICAL have no consecutive O's?

Q8. Bonus question.

8.a. Let S be a set with n elements. Evaluate each of these in closed form:

8.a. i. (1 pt) $\sum_{A \subseteq S} |A|$

8.a. ii. (1 pt) $\sum_{A,B \subseteq S} |A \cap B|$

8.b. (1 pt) Let S be a set of n real numbers. Given each nonempty subset of S we can compute its average; in this way we can produce $2^n - 1$ numbers. Prove that the average of these $2^n - 1$ numbers is the average of S.