

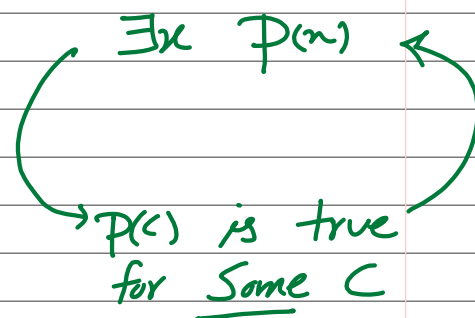
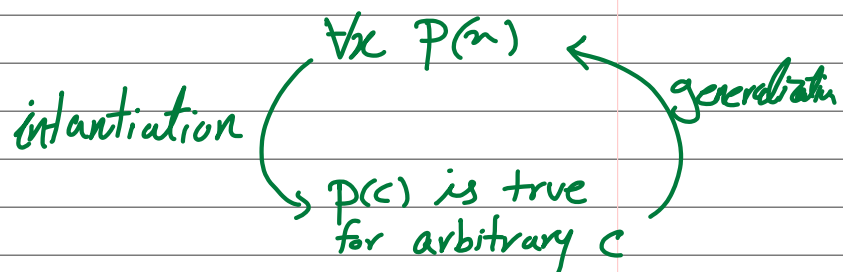
Week 3 - Friday

Last time: predicates, Quantifiers, Rules of inference

$P(x)$

$\forall \exists$

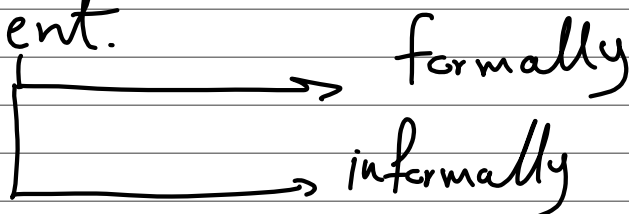
$$\frac{P \rightarrow q \quad P}{\therefore q}$$



Valid arguments:

an argument is valid if every statement follows from the previous ones via rules of inference.

Today: Proof is a valid argument used to establish the truth of a mathematical statement.



Different types of proof. I) Direct proof II) Contraposition III) Contradiction

Proof techniques

The ones that we already know.

Goal: $\forall x P(x)$

Structure:

Assume a is arbitrary

\vdots

Therefore, $P(a)$

Therefore, $\forall x P(x)$ (by universal gen.)

Example: prove $\forall n \text{ odd}(n) \rightarrow \text{odd}(n^2)$.

Structure: Assume c is arbitrary

\vdots

$\text{odd}(c) \rightarrow \text{odd}(c^2)$

Therefore, $\forall x \text{ odd}(x) \rightarrow \text{odd}(x^2)$

Proof Techniques

I) Direct Proof

Goal: $P \rightarrow Q$

Structure:

Assume P

\vdots
 Q

Therefore, $P \rightarrow Q$

II) Proof by Contraposition

Goal: $P \rightarrow Q$

Structure:

- Assume $\neg Q$

\vdots

- $\neg P$

Therefore, $P \rightarrow Q$

III) Proof by Contradiction:

Goal: P

Structure:

Goal: $P \rightarrow Q$

Structure:

I) Direct Proof

Informal

Formal

Dfn: An integer n is even if there exists an integer K s.t. $n=2K$

$$\forall n \text{ even}(n) \longleftrightarrow \exists K (n=2K)$$

Dfn: An integer n is odd if there exists an integer K s.t. $n=2K+1$

$$\forall n \text{ odd}(n) \longleftrightarrow \exists K (n=2K+1)$$

Remark: in definitions, iff means iff (if and only if)

Remark: can easily prove that $\forall n \text{ odd}(n) \longleftrightarrow \neg \text{even}(n)$

Proposition: If an integer n is odd, then n^2 is odd.

$$\forall n \text{ odd}(n) \longrightarrow \text{odd}(n^2)$$

We will prove this proposition informally and formally on the next page.

Prop. A: If an integer n is odd, then n^2 is odd

Proof:

Assume n is odd.

Choose k s.t. $n = 2k + 1$.

Observe that

$$n^2 = (2k_0 + 1)^2 = 2(2k_0^2 + 2k_0) + 1$$

Therefore, n^2 is odd
as desired \square

$$\forall n (\text{odd}(n) \rightarrow \text{odd}(n^2))$$

~~Assume~~ n is arbitrary

~~Assume~~ $\text{odd}(n)$

$$\exists k (n = 2k + 1) \quad (\text{by defn.})$$

$$n = 2k_0 + 1 \quad (\exists \text{ inst.})$$

$$n^2 = (2k_0 + 1)^2 = 4k_0^2 + 4k_0 + 1 \quad (\text{by Arith.})$$
$$= 2(2k_0^2 + 2k_0) + 1$$

$$n^2 = 2l + 1 \quad \text{for } l = 2k_0^2 + 2k_0$$

$$\exists l \ n^2 = 2l + 1 \quad (\exists \text{ gen.})$$

$$\text{odd}(n^2) \quad (\text{by defn.})$$

$$\text{odd}(n) \rightarrow \text{odd}(n^2) \quad \text{by } \star$$

$$\forall n \ \text{odd}(n) \rightarrow \text{odd}(n^2)$$

Remark: There are many different phrases in English to say the same statement

- Assume n is ...; Let n ...
- Therefore, thus, it follows, ...

II) Proof by Contraposition

prop B: For every integer n , if n^2 is odd, then n is odd.

Note: prop B is the Converse of prop A.

$$\forall n \text{ odd}(n^2) \rightarrow \text{odd}(n)$$

|||

$$\forall n \neg \text{odd}(n) \rightarrow \neg \text{odd}(n^2)$$

|||

$$\forall n \text{ even}(n) \rightarrow \text{even}(n^2)$$

proof:

Assume n is even.

Choose k st. $n = 2k$

observe that $n^2 = 2(2k^2)$

Thus, n^2 is even as desired \square

~~**~~ Assume n is arbitrary

~~*~~ Assume $\neg \text{odd}(n)$

$$\exists k \ n = 2k$$

$$n^2 = 4k^2 = 2(2k^2)$$

$$n^2 = 2l \text{ for } l = 2k^2$$

$$\exists l \ n^2 = 2l$$

$$\text{even}(n^2)$$

$$\neg \text{odd}(n^2)$$

$$\neg \text{odd}(n) \rightarrow \neg \text{odd}(n^2)$$

$$\text{odd}(n^2) \rightarrow \text{odd}(n) \text{ by } *$$

$$\forall n \text{ odd}(n^2) \rightarrow \text{odd}(n) \text{ by } **$$

III) Proof by Contradiction

Defn: A real number x is rational if $x = \frac{p}{q}$ for some integers $p, q \neq 0$.

Defn: A real number x is irrational if it's not rational

Fact (we don't prove this fact. You learned this in Elementary School.):

If x is rational, then $x = \frac{p}{q}$ for some integers $p, q \neq 0$ where p and q have no common factors.

(reason: can simplify/cancel common factors in a fraction)

prop. C: $\sqrt{2}$ is irrational.

irrational ($\sqrt{2}$)

$$\equiv \forall p \forall q (q \neq 0 \rightarrow \sqrt{2} \neq \frac{p}{q})$$

Proof:

[illegible]

[illegible]

[illegible]

[illegible]

[illegible]