Week 3 - Friday

Last time: predicates, Quantifiers, Rules of inference

$P(x)$ $\forall$ $\exists$

$$\begin{array}{l} p \to q \\ \underline{p} \\ \therefore q \end{array}$$

$\forall x\, P(x)$

instantiation   generalization

$P(c)$ is true for arbitrary c

$\exists x\, P(x)$

$P(c)$ is true for <u>some</u> c

Valid arguments: an argument is valid if every statement follows from the previous ones via rules of inference.

Today: **Proof.** is a valid argument used to establish the truth of a mathematical statement.

- formally
- informally

Different types of proof. I) Direct proof   II) Contraposition   III) Contradiction

Proof techniques

The ones that we already know.

Goal: $\forall x\, P(x)$

Structure:

Assume a is arbitrary

$\vdots$

Therfore, $P(a)$

Therefore, $\forall x\, P(x)$ (by universal gen.)

Example: prove $\forall n\ odd(n) \longrightarrow odd(n^2)$.

Structure: Assume c is arbitrary

$$\left\{ \begin{array}{l} \vdots \\ odd(c) \longrightarrow odd(c^2) \end{array} \right.$$

Therefore, $\forall x\ odd(x) \rightarrow odd(x^2)$

Proof Techniques

I) Direct Proof

Goal: $P \to Q$

Structure:

Assume P

$\vdots$

Q

Therefore, $P \to Q$

II) Proof by Contraposition

Goal: $P \to Q$

Structure:

- Assume $\neg Q$

  $\vdots$

- $\neg P$

Therefore, $P \to Q$

III) Proof by Contradiction:

Goal: P

Structure:

Goal: $P \to q$

Structure:

I) Direct Proof

Informal	Formal
Dfn: An integer n is even if there exists an integer K s.t. $n=2K$	$\forall n \; even(n) \longleftrightarrow \exists K \, (n=2K)$
Dfn: An integer n is odd if there exists an integer K s.t. $n=2K+1$	$\forall n \; odd(n) \longleftrightarrow \exists K \, (n=2K+1)$

Remark: in definitions, if means iff (if and only if)

Remark: can easily prove that $\forall n \; odd(n) \longleftrightarrow \neg even(n)$

Proposition: If an integer n is odd, then n^2 is odd.	$\forall n \; odd(n) \longrightarrow odd(n^2)$

We will prove this proposition informally and formally on the next page.

Prop. A: If an integer n is odd, then n^2 is odd

$\forall n\,(\mathrm{odd}(n) \to \mathrm{odd}(n^2))$

Proof:

Assume n is odd.

Choose k s.t. $n=2k+1$.

Observe that

$n^2=(2k_0+1)^2 = 2(2k_0^2+2k_0)+1$

Therefore, n^2 is odd as desired $\square$

★★ Assume n is arbitrary

★ Assume $\mathrm{odd}(n)$

$\exists k\,(n=2k+1)$ (by defn.)

$n=2k_0+1$ ($\exists$ inst.)

$n^2=(2k_0+1)^2 = 4k_0^2+4k_0+1$ (by Arith)

$= 2(2k_0^2+2k_0)+1$

$n^2=2l+1$ for $l=2k_0^2+2k_0$

$\exists l\; n^2=2l+1$ ($\exists$ gen.)

$\mathrm{odd}(n^2)$ (by defn.)

$\mathrm{odd}(n) \to \mathrm{odd}(n^2)$ by ★

$\forall n\; \mathrm{odd}(n) \to \mathrm{odd}(n^2)$

Remark: There are many different phrases in English to say the same statement

- Assume n is ...; Let n ...
- Therefore, Thus, it follow, ...

II) Proof by Contraposition

Prop B: For every integer n, if n^2 is odd, then n is odd.

Note: prop. B is the converse of prop. A.

$\forall n \; odd(n^2) \to odd(n)$

$|||$

$\forall n \; \neg odd(n) \to \neg odd(n^2)$

$|||$

$\forall n \; even(n) \to even(n^2)$

Proof:

Assume n is even.

Choose k s.t. $n = 2k$

observe that $n^2 = 2(2k^2)$

Thus, n^2 is even as desired $\square$

** Assume n is arbitrary

* Assume $\neg odd(n)$

$\exists k \; n = 2k$

$n^2 = 4k^2 = 2(2k^2)$

$n^2 = 2\ell$ for $\ell = 2k^2$

$\exists \ell \; n^2 = 2\ell$

$even(n^2)$

$\neg odd(n^2)$

$\neg odd(n) \to \neg odd(n^2)$

$odd(n^2) \to odd(n)$ by *

$\forall n \; odd(n^2) \to odd(n)$ by **

III) Proof by Contradiction

Defn: A real number x is rational if $x = \frac{p}{q}$ for some integers $p, q \neq 0$.

Defn: A real number x is irrational if it's not rational

Fact (we don't prove this fact. You learned this in Elementary School.):

If x is rational, then $x = \frac{p}{q}$ for some integers $p, q \neq 0$ where p and q have no common factors.

(reason: can simplify/cancel common factors in a fraction)

Prop. C: $\sqrt{2}$ is irrational.

$$\text{irrational}(\sqrt{2})$$
$$\equiv \forall p \forall q \left(q \neq 0 \longrightarrow \sqrt{2} \neq \frac{p}{q}\right)$$

Proof: