University of Toronto Scarborough
Sample Term Test 1, MAT-CSCA67: Discrete Mathematics, Summer 2024
Section: LEC01, Erfan Meskar
Aids: No aid-sheet is permitted. No electronic or mechanical computing devices are permitted.
Date: June 7, Time: 2:10 PM, Duration: 50 minutes

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- There are 4 questions and 4 pages in this exam, including this one. When you receive the signal to start, please make sure that your copy of the examination is complete.
- Answer each question directly on the examination paper, in the space provided.

	Q1	Q2	Q3	Q4	Total
Max	9	4	7	10	30
Score					

Q1. (9 pts) Mark either True or False to indicate the truth value of each of the following statements and provide a brief one or two sentence explanation.

1.a. (3 pts) The compound propositions $(p \leftrightarrow q) \rightarrow r$ and $r \vee (\neg p \leftrightarrow \neg q)$ are logically equivalent.

☐ TRUE ☑ FALSE

Let p, q, and r have the truth values T, F, and F, respectly. observe that $(p \leftrightarrow q) \rightarrow r$ would have a truth value of T and $r \vee (\neg p \leftrightarrow \neg q)$ would have a different truth value (i.e., F)

1.b. (3 pts) The proposition $(\exists x P(x)) \wedge (\forall y \forall z (P(y) \wedge P(z) \rightarrow y = z))$ means that there is exactly one element x in the domain such that $P(x)$ is true.

☑ TRUE ☐ FALSE

The first part (i.e. $\exists x\ P(x)$) ensures that there is at least one element, and the second part (i.e. $\forall y \forall z \ldots$) guartees that if there are more than one element, it will be a false statement.

1.c. (3 pts) The compound proposition $(p \rightarrow \text{F}) \vee (p \rightarrow \text{T})$ is a tautology, where T and F are true and false.

☑ TRUE ☐ FALSE

$p \rightarrow T$ is always true and $(p \rightarrow F) \vee T$ is always T.

Q2. **(4 pts)** Use existential and universal quantifiers to express the statement "Everyone has exactly two biological parents" using the propositional function $P(x, y)$, which represents "x is the biological parent of y."

$$\forall y \; \exists x \; \exists z. \; P(x,y) \wedge P(z,y) \wedge (x \neq z) \wedge \left(\forall a \; P(a,y) \to (a = x \vee a = z)\right)$$

Q3. **(7 pts)** Prove that there exists some integer k such that $\sum_{i=1}^{n} i^2 < n^3$ for any $n \geq k$.

We will prove that for $K=2$, the proposition $\sum_{i=1}^{n} i^2 < n^3$ for any $n \geq K$ is true.

We want to prove that $\sum_{i=1}^{n} i^2 < n^3$ for any $n \geq 2$.

We proceed by induction. The induction hypothesis is $P(n)$ stating that $\sum_{i=1}^{n} i^2 < n^3$.

– Base case $(n=2)$: observe that $\sum_{i=1}^{2} i^2 = 5 < 2^3$

– Inductive step: Assume arbitrary $\ell \geq 2$. Assume $P(\ell)$, i.e. $\sum_{i=1}^{\ell} i^2 < \ell^3$.

Observe that
$$\sum_{i=1}^{\ell+1} i^2 = \sum_{i=1}^{\ell} i^2 + (\ell+1)^2 < \ell^3 + \ell^2 + 2\ell + 1 < \ell^3 + 3\ell^2 + 3\ell + 1 = (\ell+1)^3,$$

where the last inequality is due to $\ell^2 < 3\ell^2$ and $2\ell < 3\ell$

Q4. (10 pts) The arithmetic mean of two numbers x and y is defined as $A(x, y) = \frac{(x+y)}{2}$. The geometric mean of two nonnegative numbers x and y is defined as $G(x, y) = \sqrt{xy}$. Prove that $G(a, b) \leq A(a, b)$ for any positive a and b, and $G(a, b) < A(a, b)$ unless $a = b$.

Assume arbitrary $a \geq 0$ and $b \geq 0$. observe that $(a-b)^2 \geq 0$. Therefore,

$$a^2 - 2ab + b^2 \geq 0 \Rightarrow a^2 + 2ab + b^2 \geq 4ab$$

$$\Rightarrow (a+b)^2 \geq 4ab \Rightarrow \frac{(a+b)^2}{4} \geq ab$$

$$\xRightarrow[\text{(since } a \geq 0,\ b \geq 0)]{} \sqrt{\frac{(a+b)^2}{4}} \geq \sqrt{ab} \Rightarrow \frac{|a+b|}{2} \geq \sqrt{ab}$$

$$\xRightarrow[\text{(since } a \geq 0,\ b \geq 0)]{} \frac{a+b}{2} \geq \sqrt{ab}$$

Therefore, $G(a,b) \leq A(a,b)$ for any positive a and b.

observe that if $a \neq b$, then $(a-b)^2 > 0$ by using the same implications as above, we can show that in this case $\frac{a+b}{2} > \sqrt{ab}$. Therefore, $G(a,b) < A(a,b)$ for any $a \geq 0$, $b \geq 0$, and $a \neq b$, as desired.