Week 13_Mon

likelihood / prob. / distribution function

Last time:

Experiment, outcomes, Sample space, $p : S \to [0,1]$

Event $E \subseteq S$, $\mathbb{P}(E) = \sum_{s \in E} p(s)$

- Conditional Probability: $\mathbb{P}(E|F) = \mathbb{P}(E \cap F) / \mathbb{P}(F)$
- Product Rule: $\mathbb{P}(E \cap F) = \mathbb{P}(E|F)\,\mathbb{P}(F)$
- Sum Rule: $\mathbb{P}(E \cup F) = \mathbb{P}(E) + \mathbb{P}(F) - \mathbb{P}(E \cap F)$
- Total Probability: $\mathbb{P}(E) = \mathbb{P}(E|F)\,\mathbb{P}(F) + \mathbb{P}(E|\overline{F})\,\mathbb{P}(\overline{F})$
- Bayes' Theorem: $\mathbb{P}(E|F) = \dfrac{\mathbb{P}(F|E)\,\mathbb{P}(E)}{\mathbb{P}(F)}$

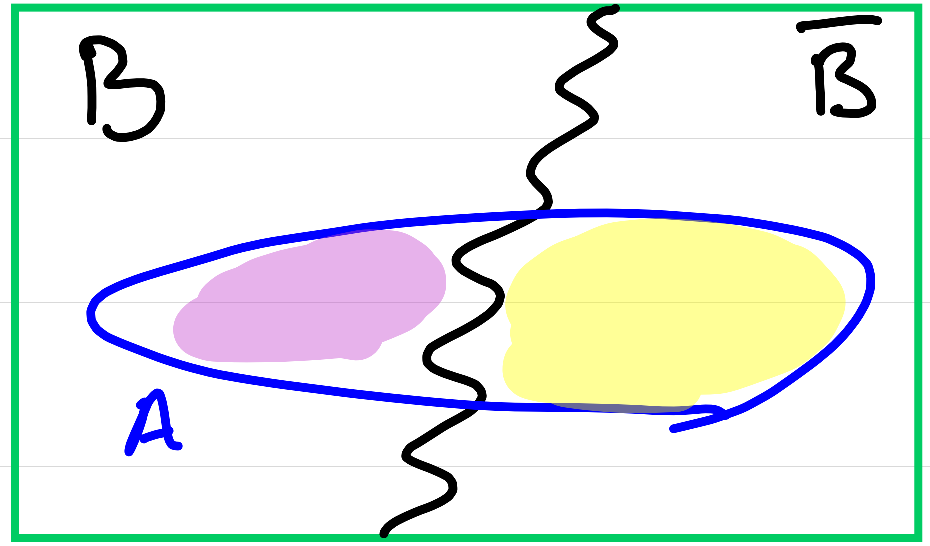
$\mathbb{P}(F \cap \overline{E}) = \mathbb{P}(F) - \mathbb{P}(F \cap E)$

$$= \frac{\mathbb{P}(F|E)\,\mathbb{P}(E)}{\mathbb{P}(F|E)\,\mathbb{P}(E) + \mathbb{P}(F|\overline{E})\,\mathbb{P}(\overline{E})}$$

$\mathbb{P}(F|E) \neq 1 - \mathbb{P}(F|\overline{E})$ in general

Bayes' Theorem:

$$P(B|A) = \frac{P(A|B)P(B)}{P(A)}$$



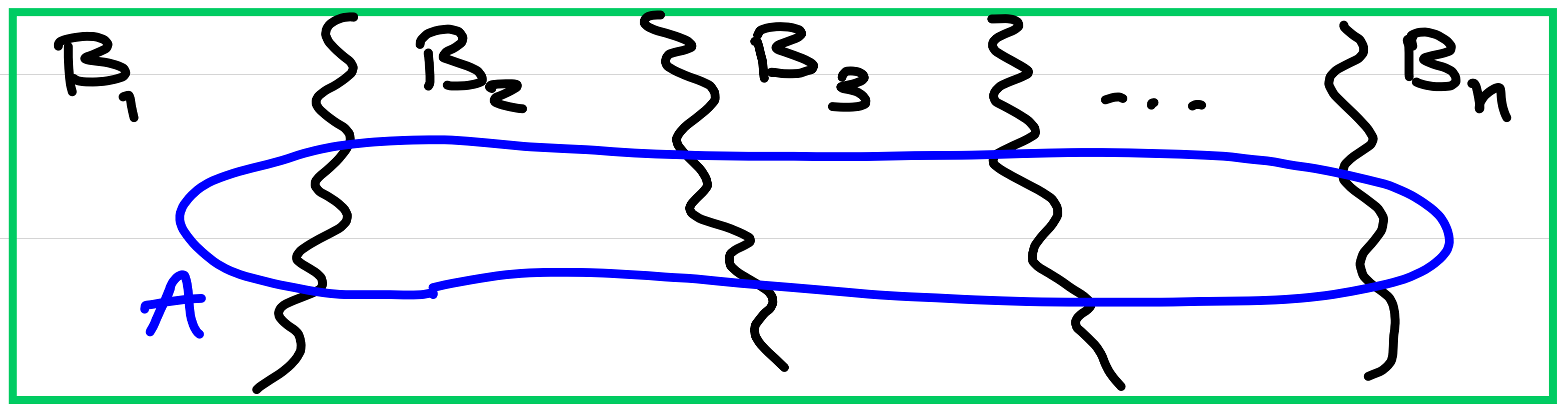
$$= \frac{P(A|B)P(B)}{P(A|B)P(B) + P(A|\bar{B})P(\bar{B})}$$ ← Total probability

Generalized Bayes rule:

Let sample space be disjoint union of events $B_1, \ldots, B_n$.

$P(A) = P(A|B_1)P(B_1) + \cdots + P(A|B_n)P(B_n)$ ← Total probability

So, $$P(B_K|A) = \frac{P(A|B_K)P(B_K)}{\underbrace{P(A|B_1)P(B_1) + \cdots + P(A|B_n)P(B_n)}_{\text{Total Probability}}}$$



EX: Medicine Testing.

C: Person has PC.   T: Person tested positive.

1) If patient has PC, test is positive 80% time. $\mathbb{P}(T|C)=0.8$

What is false negative rate? $\mathbb{P}(\overline{T}|C) = \frac{\mathbb{P}(\overline{T}\cap C)}{\mathbb{P}(C)} = \frac{\mathbb{P}(C) - \mathbb{P}(T\cap C)}{\mathbb{P}(C)}$

$= 1 - \mathbb{P}(T|C) = 0.2$

2) If patient does not have PC, test is negative 90% time.

$\mathbb{P}(\overline{T}|\overline{C}) = 0.9$

What is false positive rate?

$\mathbb{P}(T|\overline{C}) = 1 - \mathbb{P}(\overline{T}|\overline{C}) = 0.1$

3) 0.16% of the male population has PC.

$\mathbb{P}(T) = ?$ $\mathbb{P}(C|T) = ?$

Today:

- Independence of Events
- Random Variable
- Linearity of Expectation
- Distribution
- Independence of Random Variables
- Algebra of Random Variables

Independence of Events

Defn: Two events E and F are independent if

$$\mathbb{P}(E \cap F) = \mathbb{P}(E) \times \mathbb{P}(F)$$

In terms of conditional probability:

given that $\mathbb{P}(F) > 0$, $\mathbb{P}(E|F) = \mathbb{P}(E)$

given that $\mathbb{P}(E) > 0$, $\mathbb{P}(F|E) = \mathbb{P}(F)$

$$\mathbb{P}(E|F) = \frac{\mathbb{P}(E \cap F)}{\mathbb{P}(F)} = \frac{\mathbb{P}(E)\,\mathbb{P}(F)}{\mathbb{P}(F)} = \mathbb{P}(E)$$

EX: flip two independent fair coin, let H_1 be first coin is head and H_2 be the second coin is head $S = \{H, T\}^2$

$$\mathbb{P}(H_1 \cap H_2) = \frac{1}{4} = \mathbb{P}(H_1)\,\mathbb{P}(H_2).$$

Hence, H_1 and H_2 are independent.

$\mathbb{P}(H_1) = 1/2$ $\quad \mathbb{P}(H_2) = 1/2$

Ex: Roll a die. E_1 = outcome is 1

E_6 = outcome is 6

Are they independent?

☐ yes ☑ No

$$\mathbb{P}(E_1 \cap E_6) = \mathbb{P}(\{s \in S : \text{the outcome } s \text{ is } 1 \text{ and } 6\})$$

$$= \mathbb{P}(\emptyset) = 0$$

$\mathbb{P}(E_1) = 1/6$

$\mathbb{P}(E_6) = 1/6$

Since, $\mathbb{P}(E_1 \cap E_6) \neq \mathbb{P}(E_1)\,\mathbb{P}(E_6)$, E_1 and E_6 are not independent.

Ex: flip 3 independent fair Coins. e.g., $HTT \in E_{odd}$

$S = \{H, T\}^3$

$E_i = \{s \in S : i^{th} \text{ coin is } H\}$ $E_{odd} = \{s \in S : \text{an odd \# coins are } H\}$

Independent fair coin. So,

- $\mathbb{P}(E_1 | E_2) = \mathbb{P}(E_1) = 1/2$
- $\mathbb{P}(E_1 | E_3) = \mathbb{P}(E_1) = 1/2$
- $\mathbb{P}(E_1 \cap E_2) = \mathbb{P}(E_1)\mathbb{P}(E_2) = 1/4$
- $\mathbb{P}(E_1 \cap E_2 \cap E_3) = 1/8$

Q) Are E_1 and E_{odd} independent?

$\mathbb{P}(E_1) = 1/2$ $E_{odd} = F_1 \cup F_3$ where $F_i = \{s \in S : \text{there are } i \text{ H's}\}$

$$\mathbb{P}(E_{odd}) = \mathbb{P}(F_1) + \mathbb{P}(F_3) = \binom{3}{1} \cdot \frac{1}{8} + \binom{3}{3} \cdot \frac{1}{8} = 1/2$$

$\underbrace{\mathbb{P}(E_1 \cap E_{odd})}_{HHH \ \ HTT} = \frac{2}{8} = 1/4$. observe that $\mathbb{P}(E_1 \cap E_{odd}) = \mathbb{P}(E_1)\,\mathbb{P}(E_{odd})$

So, independent.

The Tragic Case of Sally Clark:

Sally clark was a nurse.

She had two child. The first one died in 1996 and the second child died in 1997.

Due to SIDS: happens in $\frac{1}{8500}$ of Population.

Prosecution made this claim that

$$\mathbb{P}(C_1 \text{ death} \cap C_2 \text{ death}) = \frac{1}{8500} \times \frac{1}{8500} = \frac{1}{73 \text{ Million}}$$

Verdict: they decided that she was guilty.

In 2001: Royal Statistical Society claimed that these event are not independent.

They showed that actual prob. of C_1 and $C_2 \sim \frac{1}{6000}$

As a result, she was released.

Random Variables

So far: We talked about events, either happen or not

Now: outcomes with a numerical value

Ex: flip 3 independent biased coins with

$\mathbb{P}(H) = q, \quad q \in [0,1]$.

What is the average number of heads?

What do you mean by average?

Intuition: go through all outcomes, count the number of heads for each outcome. Then find the average over the number of heads. By average, we mean find the weighted sum.

Let's introduce some new notions to be able to rigorously study this notion of average

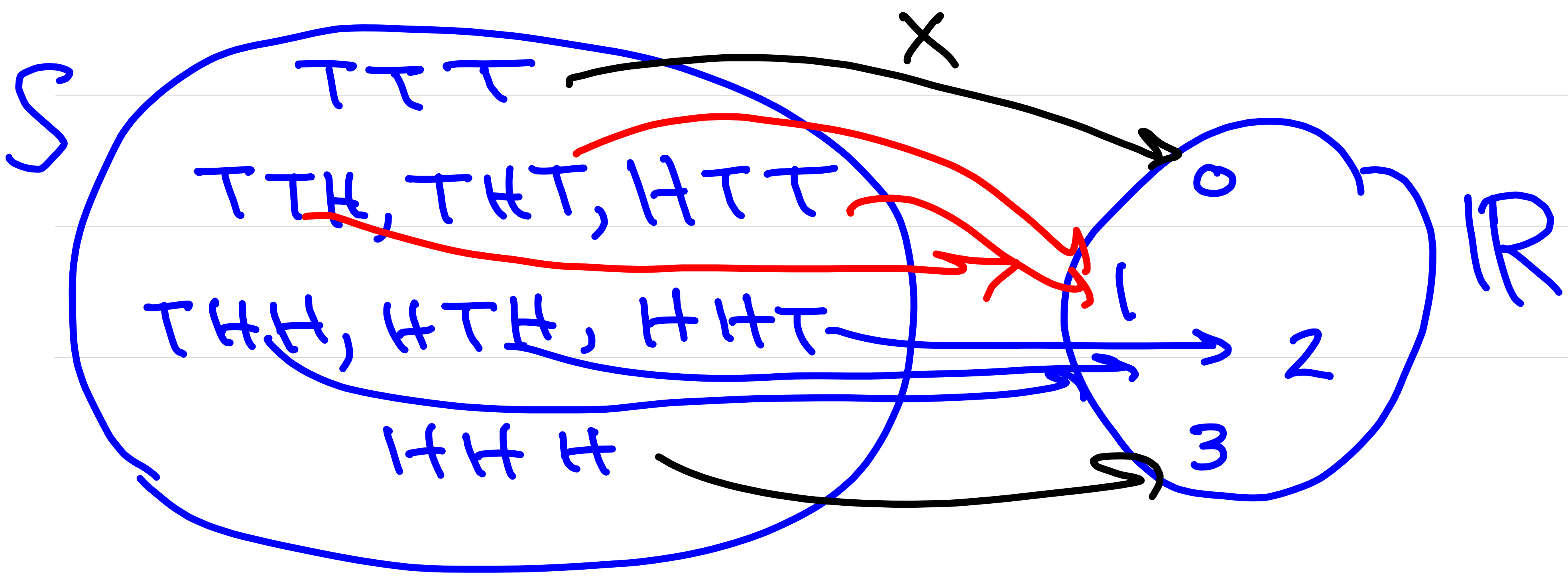
Defn: A random variable on a probability space is a function $X : S \to \mathbb{R}$.

In the previous example: $S = \{H,T\}^3$

$$P(s) = q^{H(s)} \cdot (1-q)^{T(s)}$$

- where $H(s)$ is the number of heads in s. and $T(s)$ is # tails in s.

Random Variable $X : S \to \mathbb{R}$



$$X(s) = \# \text{ of heads in } s.$$

Defn: The expectation of a random variable is

$$\mathbb{E}X = \sum_{s\in S} p(s)\, X(s)$$

weighted average of X with respect to $p(s)$

In the previous example:

$$\mathbb{E}X = \sum_{s\in S} p(s)\, X(s) = p(TTT)\, X(TTT) + p(TTH)\, X(TTH) + \cdots$$

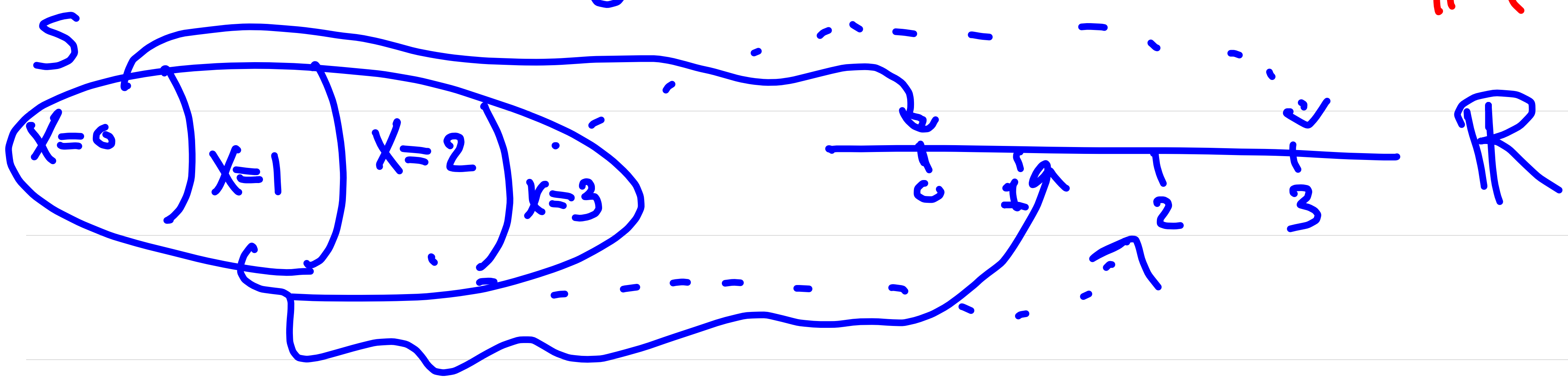
$$= \sum_{r\in \text{range}(X)} r\cdot \mathbb{P}(X=r) = \sum_{r=0}^{3} r\cdot \mathbb{P}(X=r) = 0\cdot(1-q)^3 + 1\cdot\binom{3}{1}\cdot q\cdot(1-q^2) + 2\binom{3}{2}q^2(1-q) + 3\binom{3}{3}q^3.$$

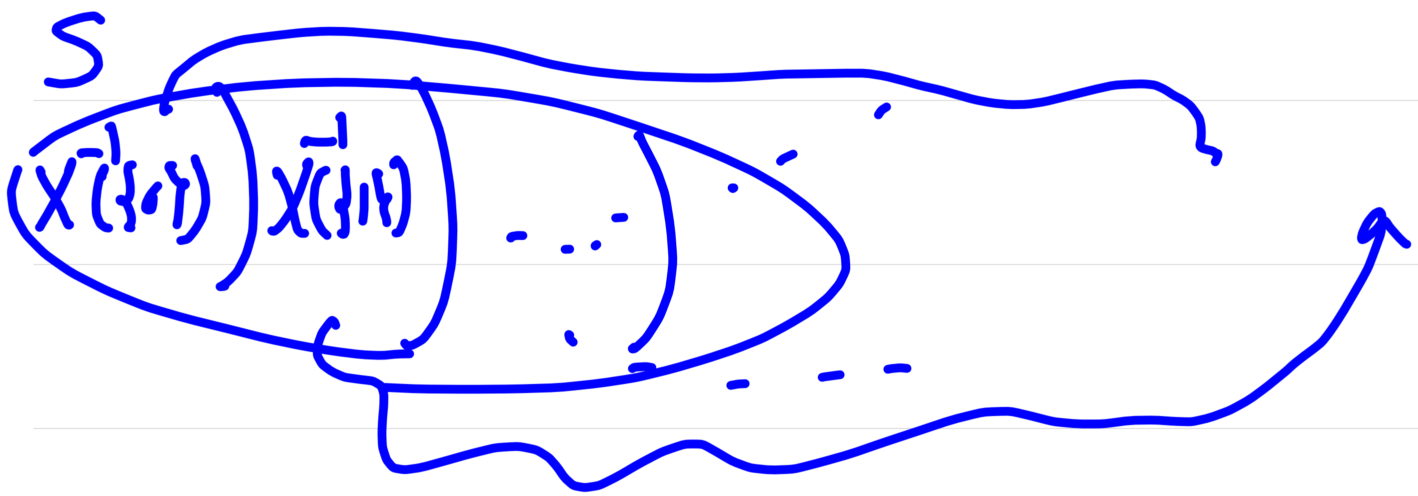
$X=r$ is a shorthand for the set $\{s\in S : X(s)=r\}$

Equivalent Definition of Expectation:

$$\mathbb{E}X = \sum_{r \in \mathrm{range}(X)} r \cdot \mathbb{P}(X=r)$$

$$\mathbb{P}(X=r) = \mathbb{P}(\{s \in S : X(s)=r\})$$





Bernoulli trials: Independent repeated trials of an experiment with 2 possible outcome. Flip n independent biased coin each with $P(H) = q$ and $P(T) = 1-q$.

Let X be the number of heads.

$S = \{H,T\}^n$ $\qquad P(s) = q^{H(s)} \cdot (1-q)^{n-H(s)}$

$$\mathbb{P}(X=k) = \binom{n}{k} \cdot q^k (1-q)^{n-k}$$

binomial

Notice that $\sum_{k=0}^{n} \mathbb{P}(X=k) = \sum_{k=0}^{n} \binom{n}{k} q^k (1-q)^{n-k} = (q+(1-q))^n = 1$ ✓

Let's find $EX = \sum_{k=0}^{n} k\,\mathbb{P}(X=k)$ on next page.

$$\mathbb{E}X = \sum_{k=0}^{n} k\, \mathbb{P}(X=k) = \sum_{k=0}^{n} k \cdot \frac{n!}{k!\,(n-k)!}\, q^{x} (1-q)^{n-k}$$

$$= \sum_{k=1}^{n} \frac{n!}{(k-1)!\,(n-k)!}\, q^{k} (1-q)^{n-k} = n \sum_{k=1}^{n} \frac{(n-1)!}{(k-1)!\,(n-k)!}\, q^{k} (1-q)^{n-k}$$

$$= nq \sum_{k=1}^{n} \binom{n-1}{k-1} q^{k-1} (1-q)^{(n-1)-(k-1)}$$

$$= nq \sum_{j=0}^{n-1} \binom{n-1}{j} q^{j} (1-q)^{(n-1)-j} = nq \times \left(q+(1-q)\right)^{n-1} = nq$$

Linearity of Expectation

You can add two random variables defined on the same prob. space $X_1 : S \to \mathbb{R}$ $X_2 : S \to \mathbb{R}$, by defining

$$(X_1 + X_2)(s) = X_1(s) + X_2(s)$$

Linearity of Expectation:

If X_1, X_2 are random variables on the same prob. space, then

$$\mathbb{E}(X_1 + X_2) = \mathbb{E}X_1 + \mathbb{E}X_2$$

"The average of the sum, is the sum of averages"

Proof:

Assume X_1 and X_2 are any arbitrary random variable defined on the same arbitrary probability space with sample space S and probabilities $P: S \to [0,1]$. Observe that

$$\mathbb{E}(X_1+X_2) = \sum_{s\in S}(X_1+X_2)(s)\cdot P(s) = \sum_{s\in S}\big(X_1(s)+X_2(s)\big)P(s)$$

$$= \sum_{s\in S} X_1(s)P(s)+X_2(s)P(s) = \sum_{s\in S} X_1(s)P(s) + \sum_{s\in S} X_2(s)P(s)$$

$$= \mathbb{E}X_1 + \mathbb{E}X_2,$$

as desired. $\square$

Let's revisit Bernoulli trials example.

Recall that X is the number of heads.

Define a random variable for each coin

for any $i \leq n$ and $s \in S$, $X_i(s) = \begin{cases} 1 & \text{if } i^{th} \text{ coin is Head} \\ 0 & \text{if } i^{th} \text{ coin is Tail} \end{cases}$

n Coins ① ② . . . ⓝ

X_1 X_2 . . . X_n

$X(s) = X_1(s) + X_2(s) + \cdots + X_n(s)$

$X(s) = (X_1 + \cdots + X_n)(s)$

$\mathbb{E}X_1 = 0 \cdot (1-q) + 1 \cdot q = q \quad (\mathbb{E}X_i = q \ \forall i)$

$\mathbb{E}X = \mathbb{E}(X_1 + \cdots + X_n) = \mathbb{E}X_1 + \cdots + \mathbb{E}X_n = nq.$

EX: Suppose π is a random permutation of $\{1, \dots, n\}$.

e.g., $\pi = 3, 2, 1, n, n-1, \dots, 4$.

What is the average number of fixed points, i.e., the average number of numbers in their correct position?

(in the example above, only number 2 is in its correct position)

$S = \{\pi : \pi \text{ is a permutation of } \{1, \dots, n\}\}$

$X : S \to \mathbb{R}$, where $X(\pi) = \#$ of fixed points of π.

$X_i : S \to \mathbb{R}$, $X_i(\pi) = \begin{cases} 1 & \text{if } i^{th} \text{ position in } \pi \text{ is } i \\ 0 & \text{otherwise} \end{cases}$.

Observe that $X(\pi) = (X_1 + \dots + X_n)(\pi)$

$\mathbb{P}(X_i = 1) = \frac{(n-1)!}{n!} = \frac{1}{n}$. Hence, $\mathbb{E}X_i = 1 \times \frac{1}{n} = 1/n$

$\mathbb{E}X = \mathbb{E}(X_1 + \dots + X_n) = \sum_{i=1}^{n} \mathbb{E}X_i = n \times 1/n = 1$

Example 7 in section 7.4 of the textbook is interesting.
make sure to check it out.

Remark:

Event

$E \subseteq S$ is a subset of S.

Indicator function of event E is defined as

$\mathbb{1}_E : S \to \mathbb{R}$,

$$\mathbb{1}_E(s) = \begin{cases} 1 & \text{if } s \in E \\ 0 & \text{if } s \notin E \end{cases}$$

$$\mathbb{E}\, \mathbb{1}_E = 0 \cdot \mathbb{P}(\bar{E}) + 1 \cdot \mathbb{P}(E)$$
$$= \mathbb{P}(E)$$

Random Variable

$X : S \to \mathbb{R}$ is function.

Defines events corresponding to r in the range(X)

$E_r = \{ s \in S : X(s) = r \}$,

i.e., $X = r$

We can also define other events based on X, e.g., $F = \{ s \in S : X(s) < 1.0 \}$

Ex: Suppose I throw n (indepedent) balls into n bins.

$\mathbb{P}(\text{Ball } i \text{ end up in bin } j) = \frac{1}{n}$, for all i and j.

How many bins remain empty on average?

$1 \qquad 2 \qquad \dots \qquad n$

Define $X_i = \begin{cases} 1 & \text{if bin } i \text{ is empty} \\ 0 & \text{otherwise} \end{cases}$ for $i = 1, \dots, n$

Observe that $X = X_1 + \dots + X_n$

$\mathbb{E}X_1 = 0 \cdot \mathbb{P}(X_1 = 0) + 1\,\mathbb{P}(X_1 = 1) = \left(1 - \frac{1}{n}\right)^n$ (for large $n \approx \frac{1}{e}$)

$\mathbb{E}X = \mathbb{E}(X_1 + \dots + X_n) = \sum_{i=1}^{n} \mathbb{E}X_i = n\left(1 - 1/n\right)^n$ (for large $n \approx \frac{n}{e}$)

$\approx 37\%$ of bins.