MAT-CSC A67: Discrete Mathematics — Summer 2024

Common Mistakes When Writing Proofs

1 Common Errors When Writing Proofs

The ability to write clean and concise proofs is a remarkable skill and is among the highest forms of intellectual enlightenment one can achieve. It requires your mind to critically reflect on its inner workings (*i.e.*, your thought processes) and reorganize them into a coherent and logical sequence of thoughts. In other words, your mind is improving itself at a very fundamental level, far transcending the boundaries of computer science or any particular area of study. The benefits of this training will touch every aspect of your life as you know it; indeed, it will shape the way you approach life itself.

As with any such fundamental achievement, developing the ability to write rigorous proofs is likely among the most difficult learning challenges you will face in university, so do not despair if it gives you trouble; you are not alone. There is simply no substitute here for lots and lots of practice. To help get you started on your way, we now raise some red flags regarding common pitfalls in composing proofs. Let us begin with a simple but common error.

> **Claim:** $-2 = 2$.
>
> **Proof?:** Assume $-2 = 2$. Squaring both sides, we have $(-2)^2 = 2^2$, or $4 = 4$, which is true. We conclude that $-2 = 2$, as desired. □

The theorem is obviously false, so what did we do wrong? Our arithmetic is correct, and each step rigorously follows from the previous step. So, the error must lie in the very beginning of the proof, where we made a brazen assumption: That $-2 = 2$. But wait, wasn't this the very statement we were trying to prove? Exactly. In other words, to prove the statement $P \equiv$ "-2 = 2", we just proved that $P \to \mathsf{T}$ which is not the same as proving P.

Lesson #1: When writing proofs, do not assume the claim you aim to prove!

Anyone can fall into this pitfall. None of us is above this common proof error. Do you not believe me? Let us take a look at a similar error made by one of the students on Quiz 4.

> **Proof of the Inductive Step?:** Assume for some $k \geq 3$, $P(k)$ holds, *i.e.*, $3k < k^3$. We want to show that $P(k+1)$ would be true, *i.e.*, $3(k+1) < (k+1)^3$.
>
> $$3(k+1) < (k+1)^3$$
> $$3(k+1) < k^3 + 3k^2 + 3k + 1$$
> $$3 < k^3 + 3k^2 + 1$$
>
> It holds when $k \geq 3$ cause $k^3 + 3k^2 + 1$ grows when k grows. □

So, what did we do wrong? Again, our arithmetic is correct, and each step rigorously follows from the previous step. Similar to the previous example, to prove the statement $\big((k \geq 3) \wedge \big(3k < k^3\big)\big) \to \big(3(k+1) < (k+1)^3\big)$, we just proved that $(k \geq 3) \wedge \big(3k < k^3\big) \wedge \big(3(k+1) < (k+1)^3\big) \to \mathsf{T}$ which is not the same.

How many students out of 70 do you think made this error on Quiz 4?

Do you see any other error in this proof? Let us study the first statement in the proof more closely. "Assume for some $k \geq 3$, $P(k)$ holds." Why "some?" Let us rewrite this statement in propositional logic language.

The statement asks us to assume that $\exists k \geq 3 \; (P(k))$. Is this really what we should be assuming? We want to prove the inductive step, which is to prove $\forall k \geq 3 \, (P(k) \rightarrow P(k+1))$. So, we need to show that for **any** value of k greater than equal to 3, if $P(k)$ holds, then $P(k+1)$ holds. Therefore, we should consider **any arbitrary** $k \geq 3$. Then, for that considered k, we should assume that $P(k)$ holds. Finally, logically conclude that $P(k+1)$ holds.

Can you rewrite the first statement in the proof to fix it? It can be done in many different ways:

- Assume arbitrary $k \geq 3$. Assume $P(k)$ holds.
- Assume $k \geq 3$ is arbitrary. Assume $P(k)$ holds.
- Consider an arbitrary $k \geq 3$. Assume $P(k)$ holds.
- Consider any $k \geq 3$. Assume $P(k)$ holds.
- Let $k \geq 3$ be arbitrary. Assume $P(k)$ holds.

Lesson #2: When writing proofs, use "some", "any", and "arbitrary" in their right place!

How many students out of 70 do you think made this second error on Quiz 4?

The third error that we are going to study is even worse. It is the error of introducing/using a variable in your proof without even defining it. Consider the following proof submitted by one of the students for Quiz 3.

Proof by Contradictions?: Suppose $p \mid n$, then $p \mid (n+1)$.

Given: $p \mid n$, which means $n = kp$ for some integer k

$p \mid (n+1)$, which means $n+1 = rp$ for some integer r

$$\begin{cases} n = kp \\ n+1 = rp \end{cases}$$

$$\rightarrow n+1 = kp+1 = rp$$

$$kp+1 = rp$$

$$\rightarrow kp+1 = rp$$

$$1 = rp - kp$$

$$1 = (r-k)p$$

This equation implies p divides 1, which is impossible because p is a prime number greater than 1. $\square$

Look at the first statement in the proof. Do you see the error of not properly defining the variables? What are p and n? Are p and n integer, prime, natural, or real numbers? Even more importantly, are p and n any arbitrary numbers we can pick, or are they specific numbers we choose? Should it be "**choose integer** n and **prime** number p **such that** ..." or "**assume arbitrary integer** n and **prime** number p ...?" No one can tell, because no one is certain of the writer's intention. The whole purpose of learning how to write a proof is to ensure that no one needs to guess what we mean. If proof requires people to guess, then it will be open to different interpretations. Not cool in math!

Lesson #3: Properly define and quantify your variables before using them!

How many students out of 70 do you think made this third error on Quiz 3 and 4?

Now, let us take another look at the previous proof. Do you see any other errors? It is not easy to follow the logic behind this proof. Using **unconventional mathematical notations** and representations makes it a pain to go through this proof. Furthermore, the statements are **not complete sentences**. Let us go over this proof line by line to get a better understanding of the errors.

- The statement "Suppose $p \mid n$, then $p \mid (n+1)$." is vague. What is the mathematical meaning of "**suppose** $p \mid n$, **then** $p \mid (n+1)$." Do they want us to "**suppose** that $p \mid n$ implies $p \mid (n+1)$?" Did they mean "**suppose** if $p \mid n$, **then** $p \mid (n+1)$?" Or did they simply want us to "**suppose** $p \mid n$ **and** $p \mid (n+1)$?" This could be a simple typo. But simple typos like this can make a proof entirely invalid. Why? Because now we have to guess what was the real intention of the writer. Mathematicians don't do that.

- The same error occurs in the second statement (*i.e.*, "Given: $p \mid n$, which means $n = kp$ for some integer k"). It is not a sentence. Even if we rewrite it as a sentence (*e.g.*, "Given $p \mid n$, which means $n = kp$ for some integer k, and $p \mid (n+1)$, which means $n+1 = rp$ for some integer r"), it will not be a **complete sentence**. This sentence is not concluded. The intention of this statement is not clear unless it is a complete sentence.

- Now, consider the statement $\begin{cases} n = kp \\ n+1 = rp \end{cases}$. This is an example of unconventional mathematical notation. What is the purpose of the notation $\Bigg\{$? Is it the start of the definition of a multi-case function? Is it for declaring a matrix? Is it to denote a set of equations that we are supposed to solve? Could it be that the writer simply wanted to say "$n = kp$ and $n+1 = rp$?" Or, could it be that they meant to say $\left.\begin{array}{l} n = kp, \\ n+1 = rp \end{array}\right\} \to n+1 = kp+1 = rp$? This unconventional use of mathematical notation makes the reader confused. A confused reader would stop reading the proof and label it as an invalid one. This error could have been easily avoided by saying "$n = kp$ and $n+1 = rp$ imply that $n+1 = kp+1 = rp$," which is a **complete sentence** with **conventional notations**.

- Now, let's examine the last part of the proof, *i.e.*,

$$\begin{aligned} \text{“}\!\to n+1 = kp+1 &= rp \\ kp+1 &= rp \\ \to kp+1 &= rp \\ 1 &= rp - kp \\ 1 &= (r-k)p.\text{”} \end{aligned}$$

  All you statements, including the mathematical statements like what we have above, should form complete sentences. As you can see, in this example, multiple statements are repeated without any reason, and the statements are not connected to each other. For instance, consider the first line of equations to the second one. These two lines are not connected to each other. The second one is not connected to first one by implication (*i.e.*, $\to$) or with conjunctive words (*e.g.*, "and", "or", "furthermore", "moreover", "additionally", "thus", "hence", "so", "but", etc.). This entire statement could have been expressed as "$n = kp$ and $n+1 = rp$ imply that $n+1 = kp+1 = rp$. Therefore, $1 = (r-k)p$." Or if we were adamant about using merely math notation, we could express it as

$$\left.\begin{array}{l} n = kp, \\ n+1 = rp \end{array}\right\} \to n+1 = kp+1 = rp \to 1 = (r-k)p.$$

  In case the mathematical statement is too long, we can break it into multiple lines. But, we have to do it carefully to make sure that the reader can easily follow the logical flow of our long statement. For instance, the above staement could have been expressed as

$$\left.\begin{aligned} n &= kp, \\ n+1 &= rp \end{aligned}\right\} \to n+1 = kp+1 = rp$$
$$\to 1 = (r-k)p.$$

As you can see in the statement above, the two line are connected to each other with an implication (*i.e.*, $\to$). Furthermore, note that in your proof, a mathematical statement (*e.g.*, equation, inequality, implication, etc.) should be either a complete sentence by itself, or be part of a complete sentence. Hence, we must use proper punctuation and connective words for our mathematical statement. For instance, observe how we use "that", ".", and "," in to express the following statement.

Observe that

$$\left.\begin{aligned} n &= kp, \\ n+1 &= rp \end{aligned}\right\} \to n+1 = kp+1 = rp \to 1 = (r-k)p.$$

Lesson #4: Write your proof in complete sentences. Use conventional mathematical notations to express your statements. Note that a math expression in your proof should be either a complete sentence or part of a complete sentence. Don't be afraid of using punctuation in mathematical expressions.

The next set of common mistakes are more arithmetical errors rather than logical ones. Consider the following excerpt from a proof submitted by a student.

Proof by Cases?: Case 1 $(a, b < 0)$:

Since a and b are negative and $a < b$:

$$\to \text{sgn}(a)\sqrt{ab} = -\sqrt{ab}$$

$\to$ since $a < b, -ab > a^2 \to -\sqrt{ab} > a$. $\square$

Let us first rewrite the proof in complete sentences.

In this case, since s and b are negative and $a < b$, $\text{sgn}(a)\sqrt{ab} = -\sqrt{ab}$. Furthermore, observe that since $a < b$ and $-ab > a^2$, we would have $-\sqrt{ab} > a$. $\square$

We observe that there are two major arithmetic errors in this proof. Why do we have $-ab > a^2$? This statement is obviously incorrect as $-ab$ is negative and a^2 is positive. Furthermore, is the statement "$-ab > a^2 \to -\sqrt{ab} > a$" true? Lets rewrite this statement in terms of two real numbers s and t. Can we claim that "$-s > t^2 \to -\sqrt{s} > t$?" What if s is negative? then $\sqrt{s}$ is not defined.

Let's consider the statement "for any real number x and any real number y, if $x^2 > y^2$, then $x > y$." Even this statement is not true. For instance, consider $x = -2$ and $y = 1$. What did we do wrong? It is because of haphazard use of square root. Note that

$$\sqrt{x^2} = |x| = \begin{cases} x, & \text{if } x \geq 0, \\ -x, & \text{if } x < 0. \end{cases}$$

The true statement would be "for any real number x and any real number y, if $x^2 > y^2$, then $|x| > |y|$."

Now it's your turn. Determine the truth value of the following statements and provide your reasoning.

- $\forall a \forall x \forall y\, (x < y \longrightarrow x \cdot a < y \cdot a)$ **[HINT:** False]

- $\forall a > 0 \forall x \forall y \, (x < y \longrightarrow x \cdot a < y \cdot a)$ [**HINT:** True]
- $\forall a < 0 \forall x \forall y \, (x < y \longrightarrow x \cdot a < y \cdot a)$ [**HINT:** False]
- $\forall a < 0 \forall x \forall y \, (x < y \longrightarrow x \cdot a > y \cdot a)$ [**HINT:** True]
- $\forall a \leq 0 \forall x \forall y \, (x < y \longrightarrow x \cdot a \geq y \cdot a)$ [**HINT:** True]
- $\forall x \forall y \left(x \leq y \longrightarrow x^2 \leq y^2\right)$ [**HINT:** False]
- $\forall x \geq 0 \forall y \geq 0 \left(x \leq y \longrightarrow \sqrt{x} \leq \sqrt{y}\right)$ [**HINT:** True]
- $\forall x \forall y \left(x^2 \leq y^2 \longrightarrow x \leq y\right)$ [**HINT:** False]
- $\forall x \forall y \left(x^2 \leq y^2 \longrightarrow |x| \leq |y|\right)$ [**HINT:** True]
- $\forall x \geq 0 \forall y \geq 0 \left(x^2 \leq y^2 \longrightarrow x \leq y\right)$ [**HINT:** True]
- $\forall x \leq 0 \forall y \leq 0 \left(x^2 \leq y^2 \longrightarrow x \geq y\right)$ [**HINT:** True]

Lesson #5: Be very careful when using inequalities with *multipliers*, *squares*, *square roots*, and *absolute values*!

When writing a proof, think of these 5 lessons. First, start writing the proof on a scratch note. Proofs can be tricky. We may need to try different techniques to find the one suitable for proving a proposition. Moreover, we may need to study the conclusion/proposition to understand it and get some idea on how to prove it. However, this process belongs to the scratch note, not the official proof. The reader is not interested in our thought process in the official proof. Instead, the reader wants to see if we can write a logically sound, correct, and clear proof. Ensure the official proof is not based on an argument like "proposition $\rightarrow \ldots \rightarrow$ True." This is a severe mistake which nullifies the proof. On the corner of the scratch note, keep track of the variables/objects used in the proof and check if they are defined and appropriately quantified. Finally, be careful when working with inequalities, absolute values, squares, and square roots.