Week 14: This Is The End

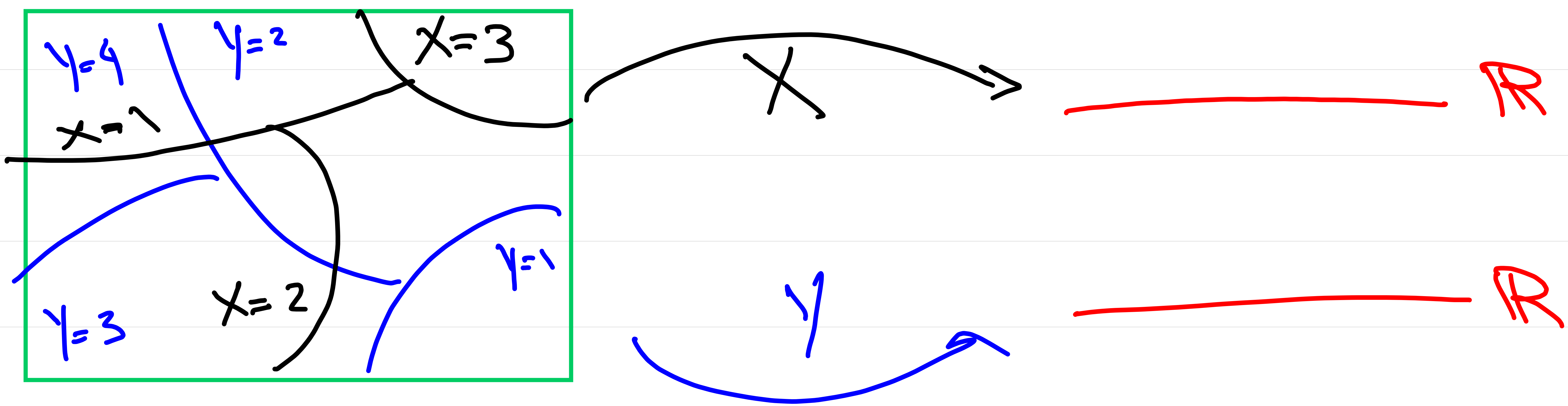
- Finishing Algebra of Random Variables
- "Advanced Counting"
  - Recurrence Relation
  - Inclusion - Exclusion
- Bye, bye, my friend, goodbye

Defn: Two random variables defined on the same sample space $X, Y : S \to \mathbb{R}$ are independent if for all $r_1, r_2 \in \mathbb{R}$

$$\mathbb{P}(X = r_1 \text{ and } Y = r_2) = \mathbb{P}(X = r_1)\,\mathbb{P}(Y = r_2)$$

$\cap$

these events are independent



Ex: Roll two independent dice

X = first dice face number

Y = second dice face number

$\forall r_1 \in \{1, \dots, 6\}\ \forall r_2 \in \{1, \dots, 6\}\quad \mathbb{P}(X = r_1 \text{ and } Y = r_2)$

$$= \mathbb{P}(X = r_1) \cdot \mathbb{P}(Y = r_2)$$

So, X and Y are independent random variables

However, let $Z = \mathbb{1}_{\{\text{first die is } 1\}}$

$\mathbb{P}(X = 1 \text{ and } Z = 1) = \mathbb{P}(X = 1) = 1/6$

$\mathbb{P}(X = 1)\, \mathbb{P}(Z = 1) = \frac{1}{6} \cdot \frac{1}{6} \neq \frac{1}{6}$

(Z and X are Not independent)

Theorem: If X and Y defined on the same sample space are independent random variable, then

$$\mathbb{E}XY = \mathbb{E}X \, \mathbb{E}Y.$$

<read the proof in the textbook>

EX: Roll two independent dice

X = face number of first dice, Y = face number of second dice

Find the average value of the product of the two dice.

Answer: We already proved that X and Y are independent.

Thus, $\forall r_1 \in \{1,..,6\}\ \forall r_2 \in \{1,...,6\}\ \mathbb{P}(X=r_1 \text{ and } Y=r_2) = \mathbb{P}(X=r_1)\,\mathbb{P}(X=r_2)$.

Since X and Y are independent, $\mathbb{E}XY = \mathbb{E}X\,\mathbb{E}Y$

Observe that $\mathbb{E}X = \mathbb{E}Y = \frac{1+2+\cdots+6}{6} = 7/2$

Thus, $\mathbb{E}XY = (7/2)^2$.

Independence allows us to decompose complicated random variables into simpler ones.

Remark: for linearity of expectation, does not require independence. So, no matter what $X_1, \dots, X_n$ are we always have $\mathbb{E}(X_1 + \dots + X_n) = \mathbb{E}X_1 + \dots + \mathbb{E}X_n$

Remark: If X_1 and X_2 are independent,

n

$$\mathbb{E}(X_1 X_2) = \mathbb{E}X_1 \, \mathbb{E}X_2$$

Defn: Random Variables $X_1, \dots, X_n$ are *mutually independent* if for any $r_1, r_2, \dots, r_n \in \mathbb{R}$,

$$\mathbb{P}\left(\bigcap_{i=1}^{n} X_i = r_i\right) = \prod_{i=1}^{n} \mathbb{P}(X_i = r_i)$$

Theorem: If random variables $X_1, X_2, \dots, X_n$ are mutually independent, then

$$\mathbb{E}(X_1 X_2 \cdots X_n) = \prod_{i=1}^{n} \mathbb{E} X_i$$

Remark: It is perfectly valid to have constant random variable $X : S \to \mathbb{R}$ $X(s) = c$ $\forall s \in S$ and $c \in \mathbb{R}$.

$$\mathbb{E}X = \sum_{s \in S} X(s) \cdot p(s) = \sum_{s \in S} c \cdot p(s) = c \sum_{s \in S} p(s)$$
$$= c.$$

Remark: We can combine sum, product, scalar multiplier, and constants to define polynomials in random variables, e.g.

$$(X+Y)^2 = X^2 + 2XY + Y^2$$

$$(X+1)^2 = X^2 + 2X + 1 \quad \leftarrow \text{algebra with functions}$$

Constant random variable

Advanced Counting (ch 8.1, 8.5_6

We will develope some "new" techniques to answer the following questions.

EX: How many words in the letters $\{L, A\}$ are there of length 100 such that no two L's are adjacent?
(Recurrence Relation, 8.1)

EX: Suppose there are 260 ticketed (assigned seats) passengers flight. How many seating arrangement are there such that no passenger sits in their assigned seat?
(Inclusion - Exclusion, 8.5-6)

Recurrence Relation

A recurrence relation is simply a recursive definition of a sequence $s_0, s_1, s_2, \ldots$

EX: How many words in the letters $\{L, A\}$ are there of length 100 such that no two L's are adjacent?

Answer: Let S_n be the set of such words of length n. Let s_n denote the size of S_n, i.e., $s_n = |S_n|$.

Observe that $S_1 = \{A, L\}$. Thus, $s_1 = |S_n| = 2$

and $S_2 = \{AL, LA, AA\}$. Thus, $s_2 = |S_2| = 3$

Observe that any such n-letter word starts with either "A" or "L".

Let A_n and L_n denote the set of such n-letter words that start with "A" and "L", respectively.

Observe that $S_n = A_n \cup L_n$ and $A_n \cap L_n = \emptyset$.
Hence, $s_n = |S_n| = |A_n| + |L_n|$.

Any element of A_n can be uniquely specified by the following choices.

C_1: Set the first letter to be "A"

C_2: Choose an $(n-1)$-letter word from S_{n-1}

$A \underbrace{\_\ \_\ \cdots\ \_}_{n-1}$

can be any word that does not have any adjacent "Z".

Hence, by product rule

$$|A_n| = 1 \times S_{n-1} = S_{n-1}$$

Any element of L_n can be uniquely specified by the following choices.

C_1: Set the first letter to be "L"

C_2: Set the second letter to be "A"

C_3: choose an element from S_{n-2}

By product rule, $|L_n| = S_{n-2}$

By the sum rule, $S_n = |A_n| + |L_n| = S_{n-1} + S_{n-2}$

(This is an implicit solution of the problem)

$S_1 = 2,\ S_2 = 3,\ S_3 = 5,\ S_4 = 8,\ \ldots$

EX: Find a recurrence relation for C_n, the number of ways to parenthesize the product of $n+1$ numbers, $x_0 \cdot x_1 \cdot \dots \cdot x_n$, to specify the order of multiplication.

$C_0 = 1, \quad C_1 = 1, \quad C_2 = 2.$

For example, $C_3 = 5$ because there are five ways to parenthesize $x_0 \cdot x_1 \cdot x_2 \cdot x_3$

$x_0 \cdot ((x_1 \cdot x_2) \cdot x_3), \quad x_0 \cdot (x_1 \cdot (x_2 \cdot x_3))$

$(x_0 \cdot x_1) \cdot (x_2 \cdot x_3)$

$((x_0 \cdot x_1) \cdot x_2) \cdot x_3, \quad (x_0 \cdot (x_1 \cdot x_2)) \cdot x_3$

Answer: The Key observation is that for each parenthization, we should choose one of the dots to be the outmost multiplication, i.e. the last multiplication to perform.

for example

$$\Pi = \underbrace{(x_0 \cdot x_1)}_{\pi_1} \cdot \underbrace{(x_2 \cdot x_3)}_{\pi_2} \qquad \pi = \overbrace{(x_0 \cdot (x_1 \cdot x_2))}^{\pi_1} \cdot \overbrace{x_3}^{\pi_2}$$

the outmost multiplication

observe that each of the outmost product specifies the length of π_1 and π_2

Let A_i denote the set of parenthesization for which the i-th dot is the outmost multiplication.

Observe that $S_n = \sum_{i=1}^{n} |A_i|$.

Each element of A_i can be uniquely specify by the following sequence of choices

C_1: choose a parenthesization π_1 for $x_0 \cdot x_1 \cdot \ldots \cdot x_{i-1}$

C_2: " " " " π_2 $x_i \cdot x_{i+1} \cdot \ldots \cdot x_n$

$x'_0 \cdot x'_1 \cdot \ldots \cdot x'_{n-i}$

C_3: set $\pi = (\pi_1) \cdot (\pi_2)$

By product rule, $|A_i| = S_{i-1} \times S_{n-i}$

Hence, by sum rule, $S_n = \sum_{i=1}^{n} |A_i| = \sum_{i=1}^{n} S_{i-1} S_{n-i}$

S_n is the well-known Catalan sequence.

optional reading: see chapter 8.2 to learn how to solve recurrence relations. For instance, one can prove that the solution of the fibonacci numbers (i.e., $f_n = f_{n-1} + f_{n-2}$ with initial condition $f_0 = 1$ and $f_1 = 1$ is

$$f_n = \frac{1}{\sqrt{5}} \left(\frac{1+\sqrt{5}}{2} \right)^n - \frac{1}{\sqrt{5}} \left(\frac{1-\sqrt{5}}{2} \right)^n .$$

Also, one can solve the recurrence relation of Catalan Numbers and prove that

$$C_n = \frac{1}{n+1} \binom{2n}{n}$$

<Solving recurrence with eigen-vectors.>

Inclusion - Exclusion

Recall 1: For arbitrary finite sets A and B,

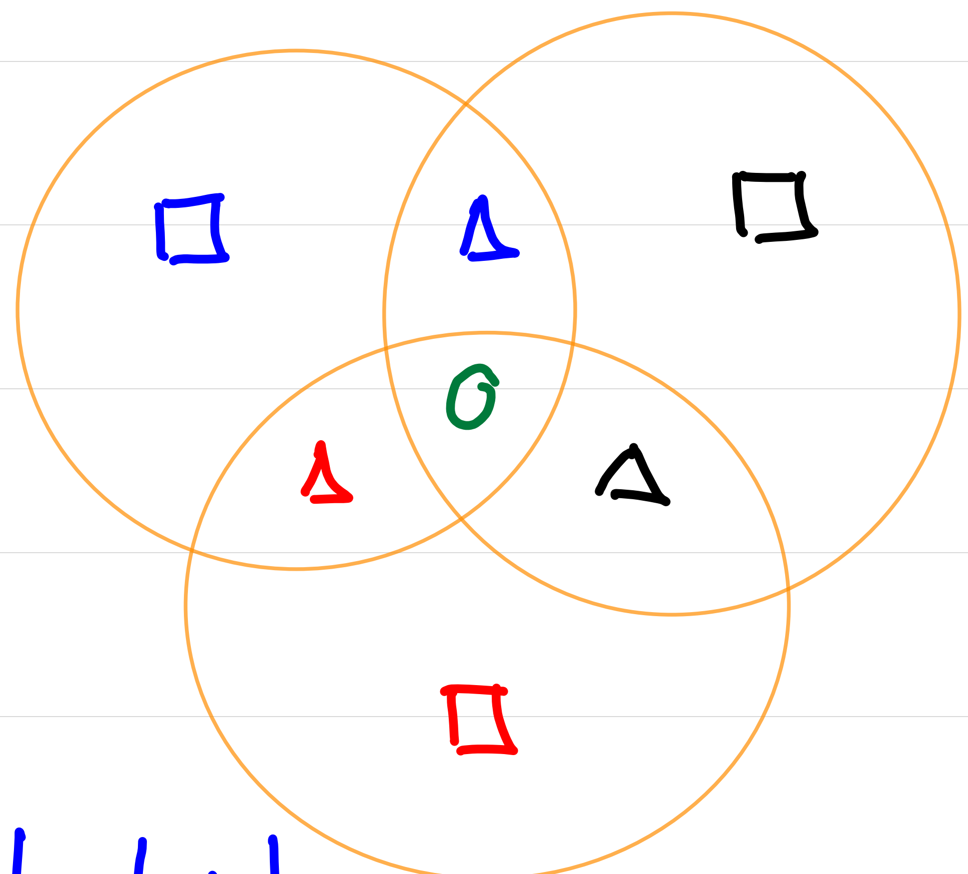
$$|A \cup B| = |A| + |B| - |A \cap B|.$$

Recall 2: For arbitrary finite sets A, B, and C,

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|.$$

proof idea:

Each term is counted the same # of times on both sides.



How many time count "o" on LHS $= 1$

# of times count "o" on RHS $= 1 + 1 + 1 - 1 - 1 - 1 + 1 = 1$

General Theorem: For arbitrary finite sets $A_1, \dots, A_n$

$$|A_1 \cup A_2 \cup \dots \cup A_n| = |A_1| + |A_2| + \dots + |A_n|$$

$$- |A_1 \cap A_2| - |A_1 \cap A_3| - \dots \quad \text{(all pairs)}$$

$$+ |A_1 \cap A_2 \cap A_3| + \dots \quad \text{(all triples)}$$

$$\vdots$$

$$+ (-1)^{n-1} |A_1 \cap A_2 \cap \dots \cap A_n|.$$

$$= \sum_{k=1}^{n} (-1)^{k-1} \sum_{\substack{S \subseteq \{1, \dots, n\} \\ |S| = k}} \left| \bigcap_{i \in S} A_i \right|$$

Summation over S with this property

Proof: Assume arbitrary finite sets $A_1, \dots, A_n$.
Assume an arbitrary element $x \in A_1 \cup A_2 \cup \dots \cup A_n$.
We prove the equality in the Theorem by showing that x is counted the same number of times on both sides of the equality

$$|A_1 \cup A_2 \cup \dots \cup A_n| = \sum_{k=1}^{n} (-1)^{k-1} \sum_{\substack{S \subseteq \{1,\dots,n\} \\ |S|=k}} \left| \bigcap_{i \in S} A_i \right|.$$

x is counted once on the LHS.
Let $B = \{ i \in \{1, \dots, n\} : x \in A_i \}$. Assume $t = |B|$.

The # of time that x is counted on the RHS is

# of times on RHS = (the # of sets contain x)

$- ($ the # of pairs of sets that contain x)

$+ ($ the # of triples of sets that contain x)

$\vdots$

$+ (-1)^{t-1}($ the # of t-sets that contain x)

$$= t - \binom{t}{2} + \binom{t}{3} - \cdots + (-1)^{t-1}\binom{t}{t}$$

$$= 1.$$

Recall $0 = (1+(-1))^t = \underbrace{\binom{t}{0}(-1)^0}_{1} + \binom{t}{1}(-1)^1 + \cdots + \binom{t}{t}(-1)^t$

Hence, $1 = \binom{t}{1} - \binom{t}{2} + \binom{t}{3} - \cdots + (-1)^{t-1}\binom{t}{t}$.

Defn: A permutation $\pi = (\pi_1, \pi_2, \dots, \pi_n)$ of $\{1, \dots, n\}$ is a derangement if $\pi_i \neq i$ for all $i \in \{1, \dots, n\}$

Ex: Suppose there are 260 ticketed (assigned seats) passengers flight. How many seating arrangement are there such that no passenger sits in their assigned seat?

Answer: We should find the # of derangement of $\{1, \dots, n\}$ (in EX, $n = 260$)

Let $U_n = \{\pi : \pi \text{ is permutation of } \{1, \dots, n\}\}$

$$D_n = \{\pi \in S_n : \pi \text{ is derangement of } \{1, \dots, n\}\}$$

Observe that $|D_n| = |U_n| - |\overline{D}_n| = n! - |\overline{D}_n|$

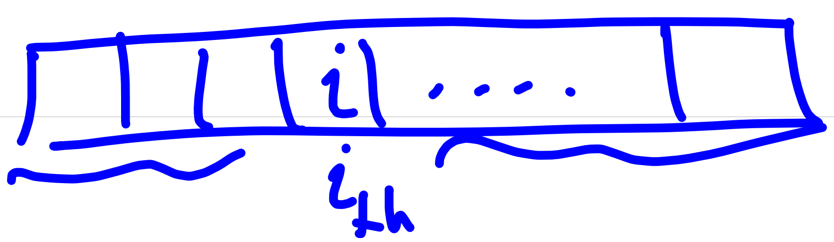
where $\overline{D}_n$ is the set of permutation that are not derangement.

Let $A_i = \{\pi \in U_n : \pi_i = i\}$ $\leftarrow$ A_i set of permutation with i as its fixed point.

$$\overline{D}_n = A_1 \cup A_2 \cup \dots \cup A_n$$

$$|\overline{D}_n| = |A_1 \cup \dots \cup A_n| = \sum_{k=1}^{n} (-1)^{k-1} \sum_{\substack{S \subseteq \{1,\dots,n\} \\ |S| = k}} \left| \bigcap_{i \in S} A_i \right|$$

Observe that $|A_i| = (n-1)!$ and $|A_i \cap A_j| = (n-2)!$



for any $i, j \in \{1, \dots, n\}$ and $i \neq j$.

In general, $\left|\bigcap_{i\in S} A_i\right| = (n-|S|)!$ for any $S \subseteq \{1,\dots,n\}$ and $S \neq \emptyset$.

Thus, $|A_1 \cup A_2 \cup \dots \cup A_n| = \displaystyle\sum_{k=1}^{n} (-1)^{k-1} \sum_{\substack{S\subseteq\{1,\dots,n\}\\ |S|=k}} \left|\bigcap_{i\in S} A_i\right|$

$$= \sum_{k=1}^{n} (-1)^{k-1} \sum_{\substack{S\subseteq\{1,\dots,n\}\\ |S|=k}} (n-k)! = \sum_{k=1}^{n} (-1)^{k-1} \binom{n}{k} (n-k)!$$

$$= \sum_{k=1}^{n} (-1)^{k-1} \frac{n!}{k!} = n! \sum_{k=1}^{n} (-1)^{k-1} \frac{1}{k!}$$

Hence, $|D_n| = n! - |\bar{D}_n| = n! \left(1 + \sum_{k=1}^{n} (-1)^k \frac{1}{k!}\right)$

$\approx n! \times \frac{1}{e} \approx$