# Welcome To The Counters Club!

So far: The Product / Sum / Subtract / Division Rule

- **Permutation:** An ordered arrangement of $n$ "distinct" objects from $n$
  $(1,3,5,9,7,2,4,8,6,10)$: a permutation of objects $1,\dots,10$.
  # permutations: $n!$
- **r-permutation:** An "ordered" sequence of $r$ "distinct" objects from $n$.
  $(3,2,1,10)$: 4-permutation of 10 objects
  $\#\text{r-permutations} = P(n,r) = \frac{n!}{(n-r)!}$
- **r-Combination:** An "unordered" set of $r$ "distinct" objects from $n$.
  $\{3,2,1,10\}$: 4-Collection of 10 objects.
  $\#\text{r-combinations} = C(n,r) = \binom{n}{r} = \frac{n!}{r!\,(n-r)!}$

(also known a r-subset, i.e., a subset of size $r$)

Today:
- Finish combinatorial identities with more examples.
- Permutation/Combination with repetition (if time permits)

EX: If $n \geq 1$, $2^n = \binom{n}{n} + \binom{n}{n-1} + \cdots + \binom{n}{1} + \binom{n}{0} = \sum_{r=0}^{n} \binom{n}{r}$.

Proof 1 (algebraic): Observe that $(x+y)^n = \binom{n}{n}x^n + \binom{n}{n-1}x^{n-1}y + \cdots + \binom{n}{0}y^n$

Set $x=1$ and $y=1$. This would result in the above equality.

Proof 2 (counting in two ways): # of subset of set $\{1, \ldots, n\}$

Each subset can be uniquely specified by the process.

$C_1$: choose if 1 is contained in the subset

$C_2$: given $C_1$, choose if 2 is in " "

$\vdots$

$C_n$: given previous $C_i$'s, choose if $n$ is in the subset.

By product rule, total # subsets is $2^n$

Observe that

$\mathcal{P}(\{1, \ldots, n\}) = A_0 \cup A_1 \cup \cdots \cup A_n$, where

$A_i$ is the set of subset of $\{1, \ldots, n\}$ with size $i$. Observe that $A_i \cap A_j \neq \emptyset$ for any $i \neq j$. Thus, by sum rule, we have $|\mathcal{P}(\{1, \ldots, n\})| = |A_0| + |A_1| + \cdots + |A_n|$

$= \binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{n}$

## Ex (Pascal's Identity)

If $n \geq 1$ and $0 < r < n$ then $\binom{n}{r} = \binom{n-1}{r-1} + \binom{n-1}{r}$.

$\binom{n}{0} = 1 = \frac{n!}{0!\, n!}$ $\qquad$ $\binom{n}{n} = 1 = \frac{n!}{n!\, 0!}$

## Pascal's Triangle:

$$
\begin{array}{lccccccc}
n=0 & \binom{0}{i} & & & & 1 & & & \\
n=1 & \binom{1}{i} & & & 1 & & 1 & & \\
n=2 & \binom{2}{i} & & 1 & & 2 & & 1 & \\
n=2 & \binom{3}{i} & 1 & & 3 & & 3 & & 1
\end{array}
$$

**Proof:** We count the number of $r$-subsets of $\{1, \dots, n\}$ in two ways. Let $A$ be the set of all subsets of size $r$.

Way 1: we know it is $\binom{n}{r}$.

Way 2: observe that $A = B \cup B'$, where $B$ is the set of subsets of size $r$ in which element $1$ is contained, and $B'$ is the set of subset of size $r$ that do not contain the element $1$. Observe $B \cap B' = \emptyset$. So by sum rule, $|A| = |B| + |B'|$

Observe that $|B'| = \binom{n-1}{r}$ = # ways to choose $r$ element from the set $\{2, \dots, n\}$.

Observe that each member of $B$ can be uniquely specified by

$C_1$: choose element $1$

$C_2$: choose $(r-1)$ elements from $\{2, \dots, n\}$.

Hence, $|B| = \binom{n-1}{r-1}$. Thus, $|A| = \binom{n-1}{r-1} + \binom{n-1}{r}$. Hence,

$$\binom{n}{r} = \binom{n-1}{r-1} + \binom{n-1}{r}.$$

There are many other Combinatorial identities.

<Read Theorem 3 and 4 in the book 6.4>

Very interesting.

# Permutation & Combination with Repetition?

So far, We studied Counting arrangements where repetitions were not allowed.

EX: how many 7-digit phone numbers can be made using the digits $\{0, \dots, 9\}$? (We can choose the same digit multiple times, The order matters)

Answer: A number is uniquely specified by the following choices:

$C_1$: choose the first digit from $\{0, \dots, 9\}$ $n_1 = 10$

$C_2$: given $c_1$, choose the $2^{nd}$ digit. $n_2 = 10$

$\vdots$

$C_7$: given all $c_i$'s, choose the $7^{th}$ digit. $n_7 = 10$

total # ways $= 10^7$

**In general:** An $r$-permutation with repetitions of $n$ object is an $r$-sequence.
The # of them is $n^r$.

A more challenging example:

How many different ice creams can be made with 5 scoops where each scoop is in {Mint, choc, Van}?

Is repetition allowed? Yes

Does order matter? NO

Let's list all 2-scoop ice-creams:

MM    MC
CC    MV
VV    CV

} 6 ways.

Naive idea:

Count ordered ice creams and use division rule.

(Does it work?) NO

| ordered | | unordered |
|---|---|---|
| MMMMM | $\longrightarrow$ | 5xM |
| MMMMC | | |
| MMMCM | | |
| MMCMM | $\longrightarrow$ | 4xM, 1xC |
| MCMMM | | |
| CMMMM | | |
| CCMMM | $\longrightarrow$ | 3xM, 2xC |

$\binom{5}{2}$

$\vdots$

CMCMM $\leftarrow$ $\{1,2,3,4,5\}$

$\binom{4}{1}$ $\{1,3\}, \{1,2\}, \{3,5\}$

10

Idea: An icecream is completely determined by the number of scoops of each flavor.

3xM, 1xC, 1xV

- all we really care about is this numbers.
- These numbers should add up to 5.

Important insight: