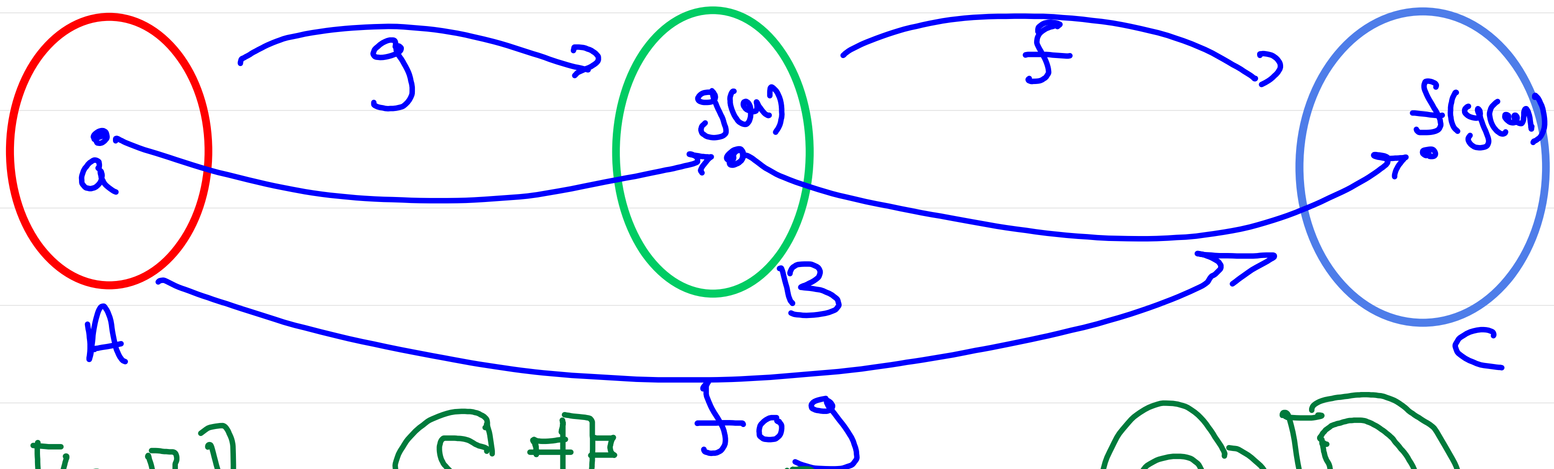
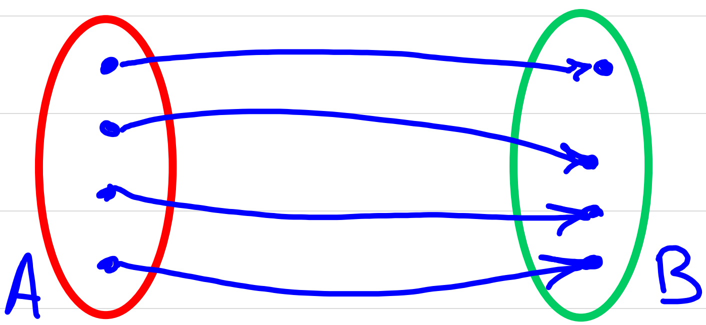
Defn: If $f: B \to C$ and $g: A \to B$ are functions, the function $f \circ g : A \to C$ defined by $f \circ g(a) = f(g(a))$ is the Composition of f and g.



Will Study ON MONDAY

Defn: If a function is one-to-one and onto, it is called a bijection/one-to-one correspondence.

Remark: If f is a bijection, for every $b \in B$ there is *exactly one element* $a \in A$ such that $f(a) = b$.



inverse of f

Therefore, we can define a new function $f^{-1} : B \to A$ by the role $f^{-1}(b) = a$ if $f(a) = b$

Remark:

$$f^{-1} \circ f(a) = f^{-1}(f(a)) = a \quad \forall a \in A.$$

$$f \circ f^{-1}(b) = f(f^{-1}(b)) = b \quad \forall b \in B.$$

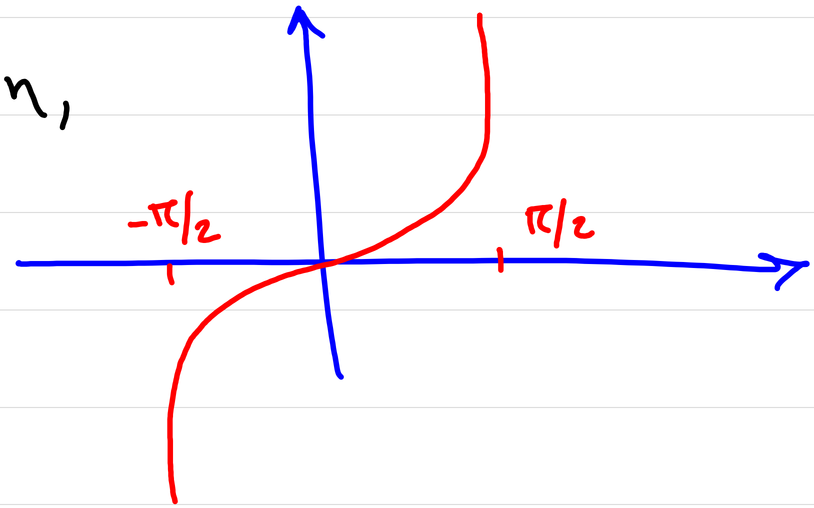
Easy fact: f has an inverse $\longleftrightarrow$ f is a bijection.

e.g., $f : (-\pi/2, \pi/2) \longrightarrow \mathbb{R}$ by $f(x) = \tan(x)$

f is a bijection,

with inverse

$f^{-1}(x) = \arctan(x)$



Prop: For any finite sets $|A| = |B|$ if and only if there is a bijection $f: A \longrightarrow B$.

Proof:

Cardinality of Infinite Sets

$\mathbb{Z}$, $\mathbb{Z}_{>0}$, $\mathbb{R}$, $\mathbb{Z}\times\mathbb{Z}$

Defn: To sets A and B have the same cardinality denoted $|A| = |B|$ if there is a bijection $f: A \to B$

Defn: A set A is countable if it is finite or $|A| = |\mathbb{Z}_{>0}|$
(i.e., if it is finite or there is a bijection $f: \mathbb{Z}_{>0} \to A$)

Defn: An infinite sequence is a bijection from $\mathbb{Z}_{>0}$ to a set A (also called an "enumeration" of A) Key: each element appears exactly once.

Defn: An infinite sequence is a function from set $\mathbb{Z}_{>0}$ to A.

$$1, 2, 3, 4, \ldots \xrightarrow{f} f(1) = -12,\ f(2) = 0,\ f(3) = 0, \ldots$$

Defn: An infinite sequence with a bijection function from set $\mathbb{Z}_{>0}$ to A is called an "enumeration" of A. The key property here is bijection, hence each element of A must appear exactly once in the sequence.

e.g., $\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\} \supseteq \mathbb{Z}_{\geq 0}$

Claim: $\mathbb{Z}$ is countable.

Proof 1: Consider the sequence $0, +1, -1, +2, -2, \ldots$
observe that each $n \in \mathbb{Z}$ appears exactly once in the sequence. Therefore, $\mathbb{Z}$ is countable $\square$.

Proof 2:

Define $f: \mathbb{Z}_+ \longrightarrow \mathbb{Z}$ by

$$f(n) = \begin{cases} n/2 & \text{if } n \text{ is even} \\ -\left(\frac{n-1}{2}\right) & \text{if } n \text{ is odd.} \end{cases}$$

$\left(f(1) = 0, \quad f(2) = 1, \quad f(3) = 1, \ldots \right)$

Proof 2 Continue:

We will show that f is one-to-one and onto.

one-to-one: Assume arbitrary $n_1, n_2 \in \mathbb{Z}_+$ and $f(n_1) = f(n_2)$

observe that $f(n_1)$ and $f(n_2)$ have the same sign. So either n_1 & n_2 are even.

or n_1 & n_2 are odd.

Case 1: n_1 & n_2 are even.

Since, $\frac{n_1}{2} = \frac{n_2}{2}$, $n_1 = n_2$ as desired.

Case 2: n_1 & n_2 are odd.

Since, $-\frac{(n_1-1)}{2} = -\frac{(n_2-1)}{2}$, $n_1 = n_2$

as desired.

Onto: Assume m is an arbitrary integer. if $m > 0$, observe that $f(2m) = m$. if $m \leq 0$, observe that $f(-2m+1) = m$. $\square$

e.g., $\mathbb{Z}_{>0} \times \mathbb{Z}_{>0}$

Claim: $\mathbb{Z}_{>0} \times \mathbb{Z}_{>0}$ is countable

Proof:

$$\begin{array}{ccc} D_2 & D_3 & D_4 \\ (1,1) & (1,2) & (1,3) \\ (2,1) & (2,2) & (2,3) \\ (3,1) & (3,2) & (3,3) \\ (4,1) & (4,2) & (4,3) \\ \vdots & \vdots & \end{array}$$

Let $D_s = \{(a,b) \in \mathbb{Z}_{>0} \times \mathbb{Z}_{>0} : a+b=s\}$ be the diagonal with sum s.

Consider the sequence of elements of D_2, left to right, D_3 left to right, elements of D_4, left to right, ...

Proof Continue:

Notice that every $(a,b) \in \mathbb{Z}_{>0} \times \mathbb{Z}_{>0}$ appears exactly once in the sequence, specifically in D_{a+b}

Thus the sequence is an enumeration of $\mathbb{Z}_{>0} \times \mathbb{Z}_{>0}$, so $|\mathbb{Z}_{>0} \times \mathbb{Z}_{>0}| = |\mathbb{Z}_{>0}|$. $\square$

Prop: If A and B are countable, so is $A \times B$.

Remark: The positive rational numbers are countable.

e.g., $(0,1)$

Remark: every $x \in (0,1)$ has a decimal expansion

$x = 0.x_1 x_2 x_3 \ldots$

Claim: $(0,1)$ is uncountable (i.e., $\forall f: \mathbb{Z}_{>0} \to (0,1)$, f is not a bijection)

Proof: Assume arbitrary $f: \mathbb{Z}_{>0} \to (0,1)$, is a function

We will show that f is not onto.

Consider the sequence of decimal expansion.

$$f(1) = 0.\, d_{11}\, d_{12}\, d_{13} \ldots$$
$$f(2) = 0.\, d_{21}\, d_{22}\, d_{23} \ldots$$
$$f(3) = 0.\, d_{31}\, d_{32}\, d_{33}$$

We will Construct some $x \in (0,1)$ s.t. $f(n) \neq x$ for every $n \in \mathbb{Z}_{>0}$.

Let $x = 0.x_1 x_2 x_3$ where

$$x_i = \begin{cases} 4 & \text{if } d_{ii} = 5 \\ 5 & \text{if } d_{ii} \neq 5 \end{cases}$$

By construction, x differs from $f(i)$ on the ith digit.

$$f(1) = 0.\, d_{11} \quad d_{12} \quad d_{13} \quad \cdots$$
$$f(2) = 0.\, d_{21} \quad d_{22} \quad d_{23} \quad \cdots$$
$$f(3) = 0.\, d_{31} \quad d_{32} \quad d_{33} \quad \cdots$$

Defn: $|A| \leq |B|$ if there is a one-to-one function $f: A \to B$.

$|A| < |B|$ if $|A| \leq |B|$ and $|A| \neq |B|$