MAT-CSC A67: Discrete Mathematics — Summer 2024

Assignmnet 6: Probability

Due Date: Wednesday, August 7, 11:59 PM, on Crowdmark

[**NOTE:** In this class, you may assume the following unless otherwise stated.

- A coin is equally likely to land heads or tails, regardless of the results of other tosses.
- A die is equally likely to show any of its six sides, regardless of the results of other rolls.
- Choosing at random means that each element in the set of choices is equally likely.

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Q1. **(2 pt)** Let $S = \{1, 2, 3\}$. How many bijections $f : S \to S$ are there such that $f \circ f$ is equal to the identity? (*i.e.*, $(f \circ f)(x) = x$ for all $x \in S$)
[**HINT:** First, you need to prove that for every such bijection, there is an element $x \in S$ such that $f(x) = x$.]

Q2. **(8 pt)** In class, we have study the balls-in-bins model. This general model has countless applications. Throwing balls independently at random into bins can model any experiment with independent trials which have equally likely outcomes.
Answer the following problems. For each one, consider how the setting is like throwing some number of balls into some number of bins.

2.a. **(2 pt)** Erfan wakes up in the middle of night and can't go back to sleep. Naturally, he decides to sits at his desk and design a problem set for the course A67. It's late and he hasn't had enough sleep, so he starts typing letters at random. For each keystroke, Erfan types one of the 26 letters, independently of the other letters which he types. Erfan types out 10 letters.

Find the chance that the typed letters can be arranged to form the word "ASSIGNMENT".

2.b. **(2 pt)** Typing the assignment didn't help Erfan fall back asleep, and now he's bored. He picks up a fair six-sided die and starts playing a boring game. If he rolls the face with six spots, he will suddenly fall asleep. He rolls the die n times. Find the probability that the face with six spots never appears.

2.c. **(2 pt)** That game wasn't helpful, either. He is still awake and decides to pointlessly roll the fair six-sided die for $m \geq 6$ times. Find the chance that every face is seen.

2.d. **(2 pt)** A group of 4 students attend Erfan's office hours and each independently asks a question at random about one of the 4 parts of this problem. Find the chance that they all ask about different part.

Q3. **(6 pt)** A new variation of Covid is spreading across Canada. A Covid test correctly returns a positive among those who have Covid with probability 90% and correctly returns a negative among those who do not have Covid with probability 95%. Suppose that 4% of people in the population have Covid. Let C denotes the event that a person has Covid and T_1 denotes the event that the test result is positive.

3.a. **(2 pt)** A randomly selected person from the population takes the test and tests positive for Covid. Find the chance that they have Covid given their positive result.

3.b. **(2 pt)** This person wants to be extra certain that they have Covid, so they take an additional test and get a negative result. Let T_2 denotes the event that the second test result is positive. Find the chance that they have Covid given their two results. You may assume that test results are independent given the Covid status of the person.
[**HINT:** Test results are independent given the Covid status of the person. Hence, $\mathbb{P}(T_2 \mid C \cap T_1) = \mathbb{P}(T_2 \mid C)$.]

3.c. **(2 pt)** Further suppose that Covid is highly prevalent among children: in particular, 22% of people aged 5 or under have Covid, while only 1% of people aged over 5 have Covid. If it is possible, find the chance that a randomly selected person is aged 5 or under given that they have Covid. If it is not possible, explain why not.

Q4. **(2 pt)** In a lottery game, there is a bag of 14 numbered balls (from 1 to 14). You select 6 balls without replacement. You win if no two balls are consecutive numbers. For example, if the balls are numbered $1, 4, 6, 9, 11, 13$ you would win but if they were numbered $2, 5, 6, 8, 10, 12$ you would loose. What is the probability that you win the game?

Q5. **(6 pt)** For $N > 4$, let X and Y be independent random variables with possible values $\{1, \ldots, N\}$. For each $k \in \{1, \ldots, N\}$, let $p_k = \mathbb{P}(X = k)$ and $q_k = \mathbb{P}(Y = k)$. Find each of the following probabilities in terms of $p_1, \ldots, p_N$ and $q_1, \ldots, q_N$.

5.a. **(2 pt)** $\mathbb{P}(X = Y)$.

5.b. **(2 pt)** $\mathbb{P}(Y > X | X = 3)$.

5.c. **(2 pt)** $\mathbb{P}(X + Y \leq N | X \geq 3)$.

Q6. **(10 pt)** Find the requested expectation in each of the following settings.

6.a. **(2 pt)** For positive integers n and N, suppose n draws are made at random with replacement from the values $\{1, \ldots, N\}$. Find the expectation of the minimum value drawn.

6.b. **(2 pt)** Erfan has a coin which lands heads with probability $p \in [0, 1]$. Erfan tosses the coin until he sees heads. Let X be the number of tosses it takes for Erfan to see heads. Then, Erfan tosses the coin another X times. Find the expected number of heads he sees throughout all his tosses.

6.c. **(2 pt)** During his turn in a game, Erfan must roll a fair six-sided die $n \geq 2$ times. Erfan's score is the number of faces which appear exactly twice. For example, if $n = 8$ and Erfan rolls $1, 6, 1, 5, 4, 2, 6, 4$, then his score is 3. Find the expected value of Erfan's score.

6.d. **(2 pt)** Suppose I roll a single die repeatedly until three different faces have come up. What is the expected number of times I need to roll the die?
[**HINT:** First, figure out the expected number of rolls before two different faces have come up.]

6.e. **(2 pt)** Suppose I roll a standard 6-faced die 7 times. Let X be a random variable equal to the number of distinct faces that appear. What is the expectation of X?