MAT-CSC A67: Discrete Mathematics — Summer 2024

Quiz 8

Due Date: Friday, July 12, 11:59 PM, on Crowdmark

Q1. Alice wants to prove $P(n)$ for all integers $n \geq 3$. She proves $\forall n \geq 3, P(n) \to P(n+2)$. Which base cases are needed to complete the induction?

Q2. Let $f : \mathbb{Z} \to \mathbb{Z}_{>0}$ have the following property: for any $n \in \mathbb{Z}$ we have that $f(n)$ is the average of $f(n-1)$ and $f(n+1)$. Prove f is constant.

Q3. Let $f : A \to B$ be a function. Prove that f is injective if and only if for any sets $S, T \subseteq A$ we have $f(S \cap T) = f(S) \cap f(T)$.

Q4. Suppose that $f : A \to B$ and $g : B \to C$ are functions such that $g \circ f$ is surjective. Prove that g is also.

Q5. Suppose that $f : A \to B$ and $g : B \to C$ are functions such that $g \circ f$ is injective. Prove that f is also.

Q6. **[Extra Question]** This question is for my fellow math enthusiasts who love discovering mathematics on their own. It has 0 points and won't be graded. It is just here for you to enjoy how useful the concepts that you study in this course are and how strangely surprising is the world of mathematics.

In this question, you prove that the set of transcendental numbers is infinite and uncountable.

A real number α is called algebraic if there exists a polynomial $p(x)$ with integer coefficients so that $p(\alpha) = 0$. A real number that is not algebraic is called transcendental. For centuries mathematicians were unsure whether transcendental numbers exist. Prove that transcendental numbers exist by following this outline (due to G. Cantor):

(a) Given $n \in \mathbb{Z}_{>0}$ let A_n denote the set of all algebraic numbers that satisfy a polynomial equation $a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n = 0$. Let B_{nm} denote those elements of A_n that satisfy a polynomial equation as above, but with $|a_k| \leq m$ for all k's. Prove that each B_{nm} is finite.

(b) Prove that each A_n is countable.

(c) Let A denote the set of all algebraic numbers. Prove that A is countable.

(d) Let T denote the set of all transcendental numbers. Prove that T is nonempty.