Week 11

- Classical probability
- Probability Spaces
- Conditional Probability
- Independence

Summary So far:

Arrangement of r objects chosen from n distinct objects

Ex: arrangement of 3 objects from $\{1, \dots, 10\}$

Terminologies

- r-permutation
- r-combination
- r-subset
- r-sequence

Notations

$$P(n,r) = \frac{n!}{(n-r)!}$$

$$C(n,r) = \binom{n}{r} = \frac{n!}{(n-r)!\, r!}$$

Order \ Repetition	Y	N
Y	e.g: $(3,9,9)$ Name: r-sequence #: n^r	e.g: $(3,1,9) \neq (1,9,3)$ Name: r-perm. #: $P(n,r) = \frac{n!}{(n-r)!}$ $= n \times (n-1) \times \dots \times (n-r+1)$
N	e.g: $\{2\times\text{"1"}, 1\times\text{"9"}\}$ Name: r-collection #: $\binom{n+r-1}{r-1}$	e.g: $\{3,1,9\} = \{3,9,1\}$ Name: r-subset / r-comb. #: $C(n,r) = \binom{n}{r} = \frac{n!}{(n-r)!\, r!}$ idea: division rule

Summary: Allocation of n objects into K boxes.

(order in the box doesn't matter)

	Constraints? NO	$(n_1, n_2, \dots, n_K)$ (provided $\sum_{i=1}^{K} n_i = n$)
objects distinguishable Y	K^n	$\dfrac{n!}{n_1!\, n_2! \cdots n_k!}$
N	$\binom{n+K-1}{K-1}$	1

Summary: Generalized PHP

Theorem: If N objects are placed into k boxes, then there is at least one box containing at least $\lceil N/k \rceil$ objects.

Today's Quiz: $(x+y+z)^n = (x+y+z)(x+y+z)\cdots(x+y+z)$

Q1) # terms before grouping?

Q2) # term after grouping?

Q3) # of times $x^{n_1} y^{n_2} z^{n_3}$ appears before grouping (i.e. the coefficient of $x^{n_1} y^{n_2} z^{n_3}$ after grouping)

No. It will be in your assignment.

Probability (ch. 7)

Q) What is Probability?

A) To study the chance/likelihood/Risk/Certainty/uncertainty of some event happening.

Let's make some statement involving probability:

The chance of you getting A_+ in course is 10%.

The chance of rolling an even number on a fair die is 0.5.

**** We will be Precise ****

- In this course, as with the previous topics, we first define what probability is.

- We will be precise as always. We are now formal Mathematician / Scientists.

Study of Prob. is useful in many fields including medicine, Computer Science, Statistics, AI, ML

randomized algorithm

Complexity analisys

Handwavy / Not formal probability

- Let's first answer a probability question by our intuition
  - This helps us develope a precise definition of probability

EX1: Suppose you flip two fair coins. What is the chance they are different?

Possible outcomes: $\{HH, HT, TH, TT\} \leftarrow S$

Successful outcomes: $\{HT, TH\} \leftarrow E$

Thus, by intuition, The success prob. $= \dfrac{\#\text{ successful outcomes}}{\#\text{ total outcomes}} = \dfrac{|E|}{|S|}$

$$\mathbb{P}(E) = \frac{2}{4} = 1/2$$

Precise (classical) Probability

Defn:

1- An experiment is a well defined procedure with a set of possible outcomes.

- An outcome is a complete description of the result of an experiment

2- The set of all outcomes is called the sample space S.

3- An event E is a subset of sample space

Two assumptions:

① The sample space is finite.

② All outcomes are equally likely.

Defn Cont'd:

4. The <u>Probability</u> of an event E, denoted by $\mathbb{P}(E)$ is

$$\mathbb{P}(E) = \frac{|E|}{|S|}.$$

Q) The probabilities is always between zero and 1. why?

Proof: Since, $|E| \geq 0$ and $|S| > 0$, $\mathbb{P}(E) = \frac{|E|}{|S|} \geq 0$.

Since $E \subseteq S$, we have $|E| \leq |S|$.

Then $\mathbb{P}(E) = \frac{|E|}{|S|} \leq 1$.

EX2: Flip 10 fair Coins. What is the Probability of getting exactly 5 heads?

Experiment? flipping 10 coins.

Sample Space S? $S = \{(c_1, \dots, c_{10}) : \forall i \; c_i \in \{H, T\}\}$

Event E? $E = \{(c_1, \dots, c_{10}) \in S : \text{exactly 5 of } c_i\text{'s are } H\}$

$$\mathbb{P}(E) = \frac{|E|}{|S|} = \frac{\binom{10}{5}}{2^{10}} \approx 0.25$$

EX3: In poker, a flush is 5 cards of the same suit. What is the probability of being dealt a flush?

D(♦)	H(♥)	C(♣)	S(♠)
1			
2			
⋮			
13			

(labeled cards)

Experiment: picking 5 unordered cards

Sample space S: $S = \{\{c_1, c_2, \dots, c_5\} : \forall i \; c_i \in \underbrace{A_D \cup A_H \cup A_C \cup A_S}_{52 \text{ cards.}}\}$

Event E: $E = \{X \in S : X \subseteq A_D \vee X \subseteq A_H \vee X \subseteq A_C \vee X \subseteq A_S\}$

Probability E: $\dfrac{|E|}{|S|} = \dfrac{4 \times \binom{13}{5}}{\binom{52}{5}}$

Note that, $E = E_D \cup E_H \cup E_C \cup E_S$, where $E_D = \{X \in S : X \subseteq A_D\}$

Remark: We could also model this probability problem as ordered hands and the result would have been the same. Can you argue why?

Remark: Choose S so that it has all the relevant information/features (to express E), and for which counting is easy.

Ex 4: Birthday Problem

What is the probability that two people in class (of 30 people) have the same birthday?

Experiment: Assign each person in class a birthday from $\{1, \dots, 365\}$ uniformly at random.

Sample space S: $S = \{(a_1, \dots, a_{30}) : a_i \in \{1, \dots, 365\}\} = \{1, \dots, 365\}^{30}$

Event E: $E = \{(a_1, \dots, a_{30}) \in S : \exists i, j \ (j \neq i \wedge a_i = a_j)\}$

$\mathbb{P}(E)$? $= \dfrac{|E|}{|S|}$

$|S| = 365^{30}$

Note that $|E| = |S| - |\overline{E}|$. $|\overline{E}| = P(365, 30) = \dfrac{365!}{335!}$

$\mathbb{P}(E) = 1 - \mathbb{P}(\overline{E}) = 1 - \dfrac{365!/335!}{365^{30}} \geq 90\%$

upshot: Sometime it is easier to find $\mathbb{P}(\overline{E})$ instead of $\mathbb{P}(E)$.

$$\mathbb{P}(E) = 1 - \mathbb{P}(\overline{E})$$

because $|E| = |S| - |\overline{E}|$.

Probability Theory. **Motivation:** what if all outcomes are not equally likely

Defn: A <u>Probability space</u> consists of a sample space S with a <u>probability distribution</u>

$P: S \to \mathbb{R}$ satisfying ① $P(s) \in [0,1]$ $\forall s \in S$

② $\sum_{s \in S} P(s) = 1$

dice

1	2	3
0.1	0.1	0.1
4	5	6
0.5	0.1	0.1

$\mathbb{P}(\{4,6\}) = 0.6$

" $P(s)$ = likelihood of outcome s."

The <u>Probability of an event E</u> is

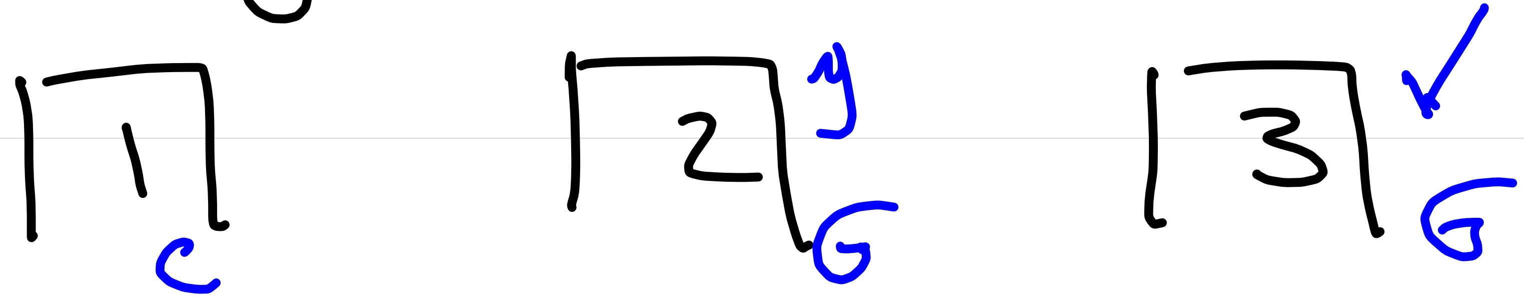
$$\mathbb{P}(E) = \sum_{s \in E} P(s)$$

Remark: Previously we studied "classical probability" in which all outcomes are equally likely. Then, the distribution function would be:

$$p : S \to \mathbb{R} \qquad p(s) = \frac{1}{|S|}$$

Monty Hall Problem

The game show has three doors.



1- a Car is randomly placed behind one door uniformly at random. Goats placed behind the other doors

2- you choose a door randomly.

3- Monty opens one of the other doors with a goat behind it uniformly at random and reveals a goat.

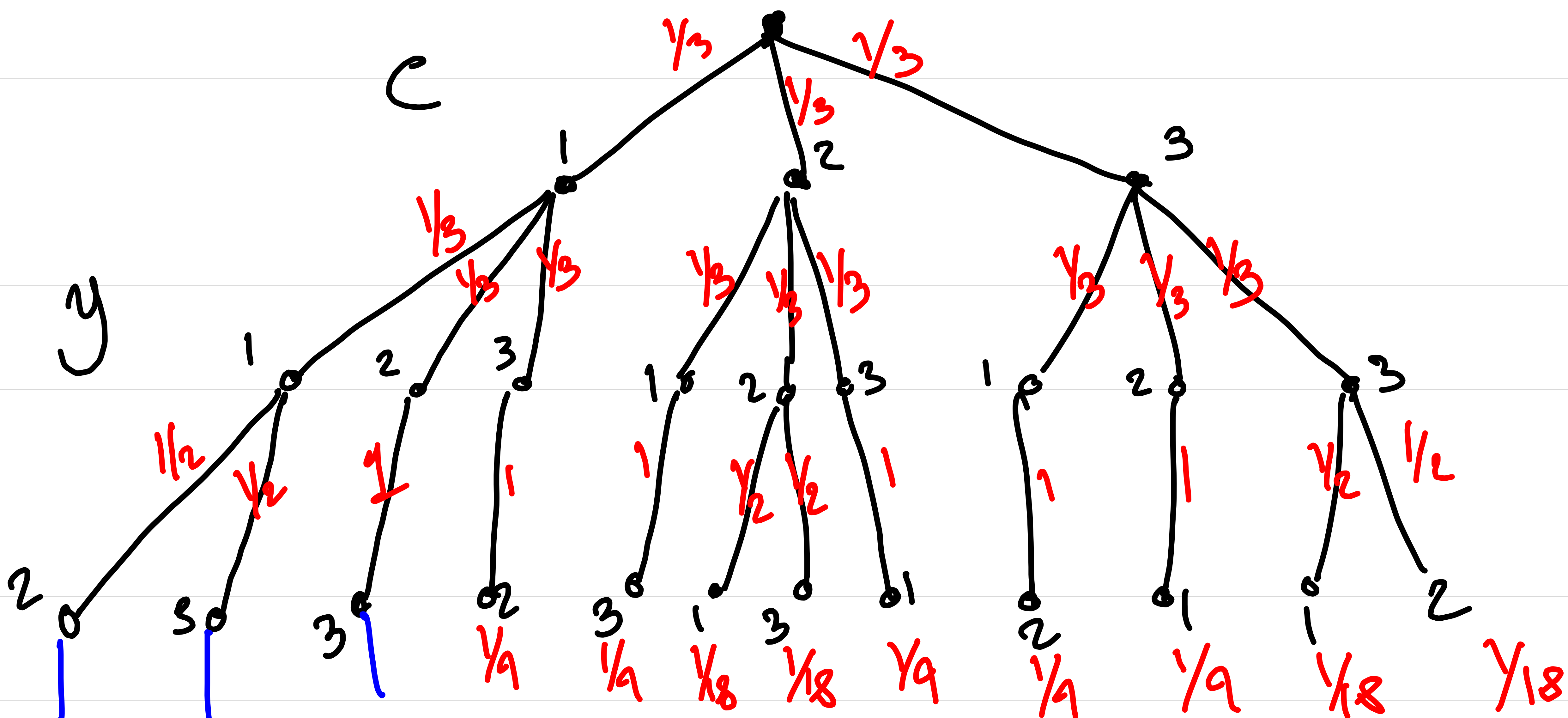
- You are allowed to now switch the door or not. What should you do?

Experiment: Steps 1–3.

Sample space: $S = \{(c,y,m) : m,y,c \in \{1,2,3\} \wedge m \neq y \wedge m \neq c\}$

E: $E = \{(c,y,m) \in S : c = y\}$

The event in which you win without switching



Outcome $(1,1,2)$ $(1,1,3)$ $(1,2,3)$

$P(s)$ $\frac{1}{3}\times\frac{1}{3}\times\frac{1}{2}=\frac{1}{18}$ $\frac{1}{18}$ $\frac{1}{9}$ ← will explain why.

$E=\{(1,1,2),(1,1,3),(2,2,1),(2,2,3),(3,3,1),(3,3,2)\}$

$\mathbb{P}(E)=6\times\frac{1}{18}=\frac{1}{3}$ and $\mathbb{P}(\bar{E})=1-\frac{1}{3}=\frac{2}{3}$. So, switching is better.

So far: Given sample space, probability distribution function, and event: calculate $\mathbb{P}(E) = \sum_{s \in E} p(s)$

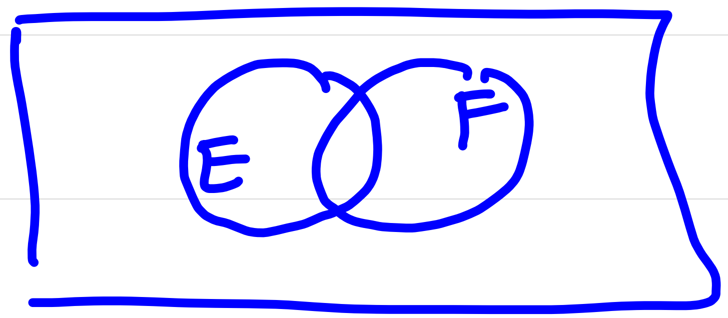
In the end, we are not doing anything but Counting.

Now: Given the Probabilities of some events $E_1, E_2, \dots, E_n$ what can you say about Probability of another event F?

Let's develope some tools for relating probabilities of different events

Sum Rule: $\mathbb{P}(E\cup F) = \mathbb{P}(E) + \mathbb{P}(F) - \mathbb{P}(E\cap F)$

<proof is the same as the proof for sets>



Special cases:

- If $E\cap F = \emptyset$, then $\mathbb{P}(E\cup F) = \mathbb{P}(E) + \mathbb{P}(F)$

  "disjoint" or "mutually exclusive" events

- $\mathbb{P}(E) + \mathbb{P}(\bar{E}) = 1 = \mathbb{P}(S)$ sample space.