MAT-CSC A67: Discrete Mathematics — Summer 2024

Assignmnet 2: Proof Methods

Due Date: Sunday, June 2, 11:59 PM, on Crowdmark

Q1. [2 pts] Formally prove that $\exists x \exists y P(x,y)$ and $\exists y \exists x P(x,y)$ are logically equivalent, *i.e.*, formally prove that $(\exists x \exists y P(x,y)) \rightarrow (\exists y \exists x P(x,y))$ and formally prove that $(\exists y \exists x P(x,y)) \rightarrow (\exists x \exists y P(x,y))$.
[**HINT:** Direct proof.]
[**What we expect:** Each step of your proofs and the reason for each step.]

Q2. [2 pts] Prove that $\forall x \exists y P(x,y)$ and $\exists y \forall x P(x,y)$ are not logically equivalent.
[**HINT:** Look for a counterexample.]
[**What we expect:** A predicate P and domains of x and y such that the two quantified statements have different truth values.]

Q3. [6 pts] For each of the following statements, determine whether it is true or false and then prove it (if it is true) or disprove it (if it is false).
[**What we expect:** For each part, first state if the statement is true or false. Then present a proof. You do not need to present your proof in formal logic format. An informal proof which is clear and correct would be sufficient.]

3.a. [1 pts] $n^2 - n + 41$ is prime for every non-negative integer n.

3.b. [2 pts] The ratio (result of division) of any two non-zero rational numbers is rational.
[**HINT:** Formally speaking, you must prove that

$$\forall x \forall y \Big(x \neq 0 \wedge y \neq 0 \wedge \exists p_1 \exists q_1 (q_1 \neq 0 \wedge x = p_1/q_1) \wedge \exists p_2 \exists q_2 (q_2 \neq 0 \wedge y = p_2/q_2) \Big) \rightarrow \exists p_3 \exists q_3 (q_3 \neq 0 \wedge x/y = p_3/q_3)$$

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3.c. [1 pts] Let f be a real-valued function of the real numbers (*i.e.*, $f : \mathbb{R} \rightarrow \mathbb{R}$). $f(x)$ is irrational if and only if x is irrational.

3.d. [2 pts] If a and b are irrational then a^b is also irrational.

Q4. [9 pts] Suppose a and b are two real numbers that have the same sign (in other words, a and b are both positive or they are both negative or they are both equal to 0). Their geometric mean is defined to be $\mathrm{sgn}(a) \cdot \sqrt{ab}$, where $\mathrm{sgn}(a)$ is the sign or signum function:

$$\mathrm{sgn}(x) = \begin{cases} -1 & \text{if } x < 0 \\ 0 & \text{if } x = 0 \\ 1 & \text{if } x > 0 \end{cases}$$

We want you to prove that the following three statements are equivalent:

(i) a is less than b.

(ii) the geometric mean of a and b is greater than a.

(iii) the geometric mean of a and b is less than b.

4.a. [2 pts] Prove that (i) $\rightarrow$ (ii).
[**What we expect:** An informal proof which is clear and correct.]
[**HINT:** Proof by cases.]

4.b. [2 pts] Prove that (ii) $\rightarrow$ (iii).
[**What we expect:** An informal proof which is clear and correct.]
[**HINT:** Proof by cases.]

4.c. [2 pts] Prove that (iii) $\rightarrow$ (i).
[**What we expect:** An informal proof which is clear and correct.]
[**HINT:** Proof by cases.]

4.d. **[1 pts]** Using the result from parts **4.a.**, **4.b.**, and **4.c.**, prove that (i) and (ii) are equivalent.
[**What we expect:** An informal proof which is clear and correct.]

4.e. **[1 pts]** Using the result from parts **4.a.**, **4.b.**, and **4.c.**, prove that (ii) and (iii) are equivalent.
[**What we expect:** An informal proof which is clear and correct.]

4.f. **[1 pts]** Using the result from parts **4.a.**, **4.b.**, and **4.c.**, prove that (iii) and (i) are equivalent.
[**What we expect:** An informal proof which is clear and correct.]