# Week 11

- Combination with repetition
- Putting objects into boxes
- Pigeonhole Principle

# Summary So far:

Arrangement of $r$ objects chosen from $n$ distinct objects

Ex: arrangement of 3 objects from $\{1, \dots, 10\}$

## Terminologies

- r_permutation
- r_combination
- r_subset
- r_sequence

## Notations

$$P(n,r) = \frac{n!}{(n-r)!}$$

$$C(n,r) = \binom{n}{r} = \frac{n!}{(n-r)!\,r!}$$

**Repetition**

| Order | Repetition: Y | Repetition: N |
|---|---|---|
| Y | e.g: $(3,9,9)$<br>Name: r_sequence<br>#: $n^r$ | e.g: $(3,1,9) \neq (1,9,3)$<br>Name: r_perm.<br>#: $P(n,r) = \frac{n!}{(n-r)!}$<br>$= n \times (n-1) \times \dots \times (n-r+1)$ |
| N | e.g: $\{2 \times \text{"1"}, 1 \times \text{"9"}\}$<br>Name: r_collection<br>#: ? | e.g: $\{3,1,9\} = \{3,9,1\}$<br>Name: r_subset / r_comb.<br>#: $C(n,r) = \binom{n}{r} = \frac{n!}{(n-r)!\,r!}$<br>idea: division rule |

# Combination with Repetition: Motivating Example

- How many different 5-scoop ice creams can be made where each scoop is in {Mint, Choc, Van}?

Is repetition allowed? Yes

Does order matter? NO

Let's list all 2-scoop ice-creams:

| | | | |
|---|---|---|---|
| M M | {2xM} | 1xM, 1xC | M C |
| C C | 2xC | 1xM, 1xV | M V |
| V V | 2xV | 1xC, 1xV | C V |

$\}$ 6 ways.

Naive idea:

Count ordered ice creams and use division rule.

(Does it work?) NO

| ordered | | unordered |
|---|---|---|
| MMMMM | 1 to 1 mapping? $\longrightarrow$ | 5xM |
| MMMMC<br>MMMCM<br>MMCMM<br>MCMMM<br>CMMMM | 5 to 1 mapping? $\longrightarrow$ | 4xM, 1xC |
| CCMMM<br>⋮<br>CMCMM | 10 to 1 mapping? $\longrightarrow$ | 3xM, 2xC |

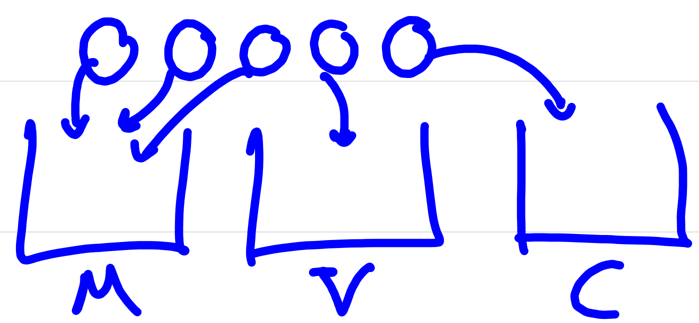
$\binom{5}{2} = 10$

{1,3}, {1,2}, {3,5}

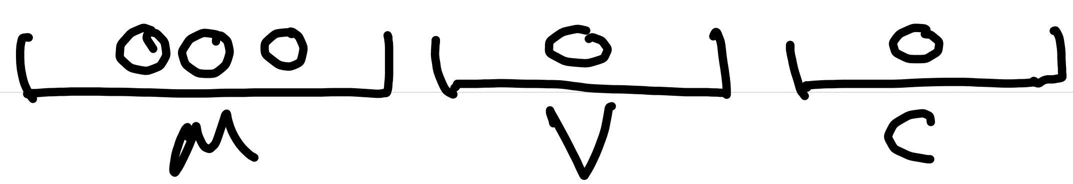
{1, 2, 3, 4, 5}

**Idea:** An icecream is completely determined by the **number of scoops of each flavor.**

3xM, 1xC, 1xV
- all we really care about is this numbers.
- These numbers should add up to 5.

**Important insight:** 

$\equiv$ allocating 5 (indistinguishable) scoops int 3 boxes

3xM, 1xV, 1xC $\equiv$ 

← an allocation of 5 token into 3 boxes

#Ice creams = # allocation of 5 token into 3 boxes.

Second Observation: Two more 1-1 correspondence which seem stupid but infact useful.

$$\underbrace{000}_{M}\ \underbrace{0}_{C}\ \underbrace{0}_{V} \equiv \underbrace{000}_{M}1\underbrace{0}_{C}1\underbrace{0}_{V} \equiv 0001010$$

7 bit string with exactly 2 ones.

$$\underbrace{\quad}_{M}\ \underbrace{00}_{C}\ \underbrace{000}_{V} \equiv 1001000$$

$$\underbrace{\quad}_{M}\ \underbrace{\quad}_{C}\ \underbrace{00000}_{V} \equiv 1100000$$

\# ice creams = # allocation of 5 tokens into 3 boxes

= # 7-bit string with exactly 2 one. $= \binom{7}{2}$

## General Theorem.

1- The number of ways to place $n$ "indistinguishable" objects into $K$ boxes is $\binom{n+K-1}{K-1}$

Proof: Such allocations are in 1-1 correspondence $n+K-1$ - bit string with exactly $K$-1.

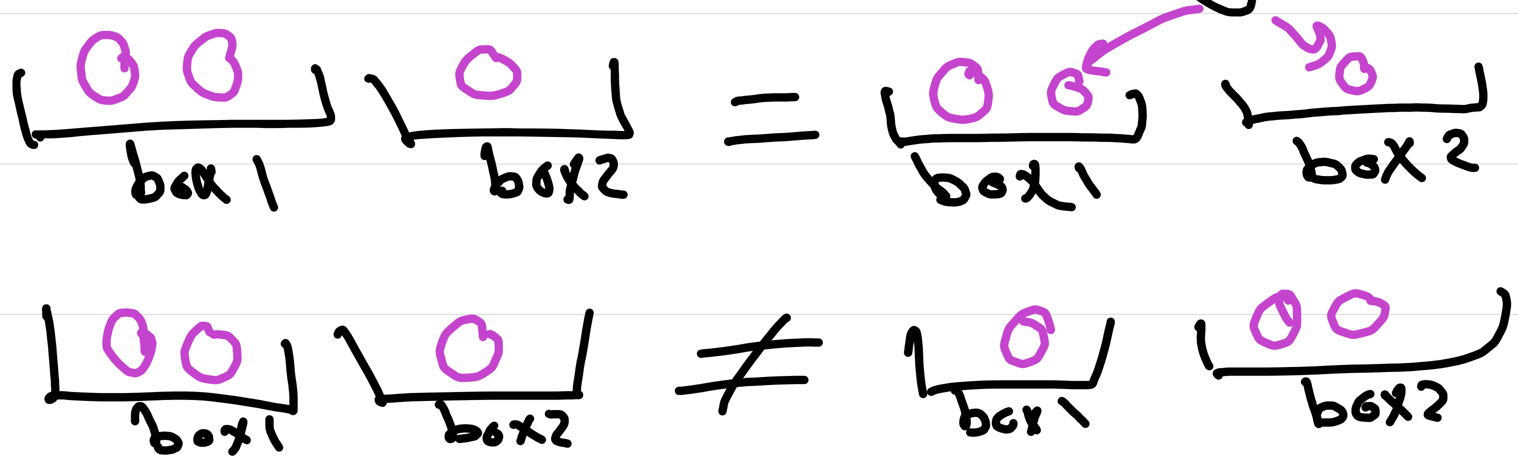
2- The number of r-combinations of $n$ objects with repetition (also known as K-collection) is $\binom{n+K-1}{K-1}$.

proof: Such K-collections are in 1-1 correspondence with allocation of $n$ tokens into $K$ boxes.

So far:

3 indistinguishable objects: ○ ○ ○

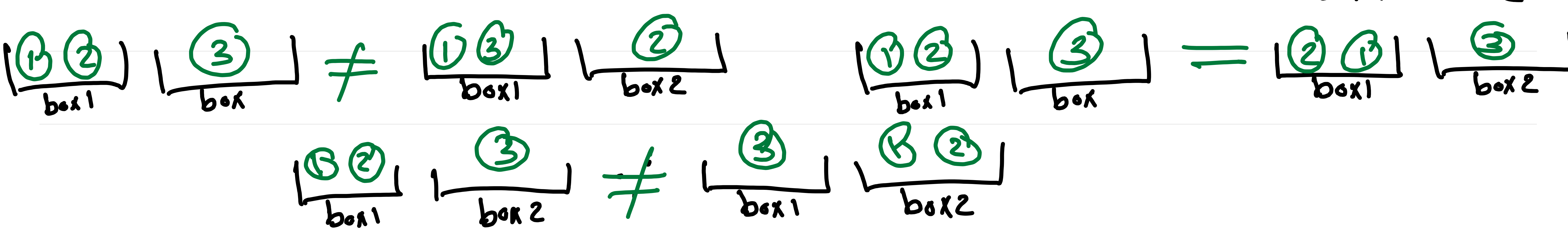
put them into 2 distinguishable boxes: box1 box2




Another variation: what if objects are distinguishable

3 distinguishable object: 1 2 3

put them into 2 distinguishable boxes: box1 box2

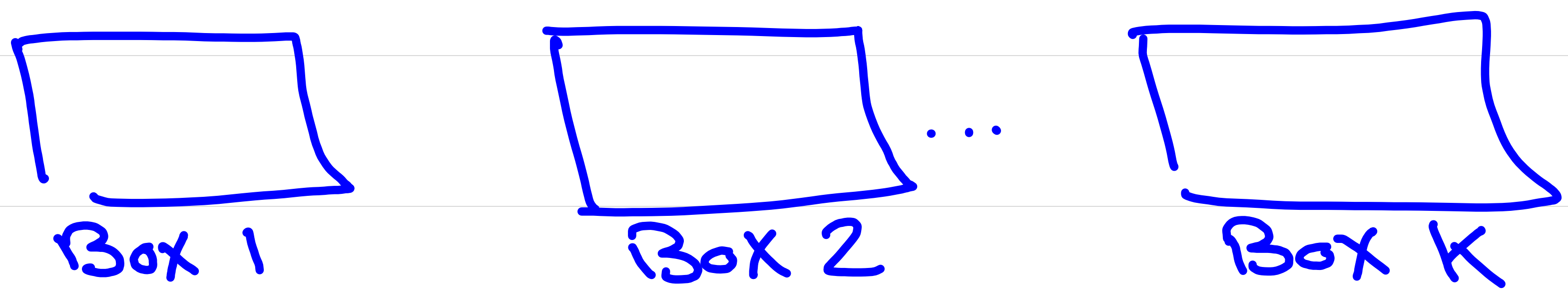


# Putting Objects Into Boxes

\# allocations of $n$ "distinguishable" objects into $K$ boxes

labeled

1 2 3 . . . $n$




$C_1$: choose a box for 1

$C_2$: given $C_1$, choose a box for 2

$\vdots$

$C_n$: choose a box for $n$

total # allocations $= K^n$

# allocations of $n$ "indistinguishable" objects into $k$ boxes.

$$= \binom{n+k-1}{k-1}.$$

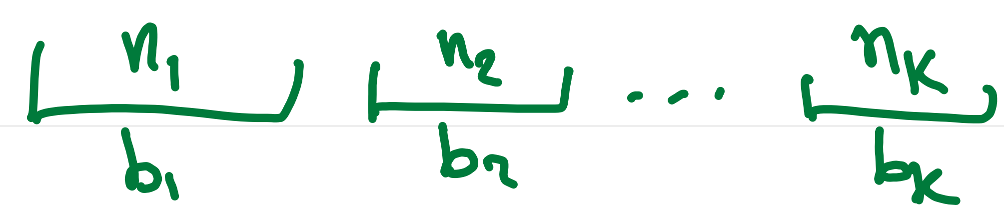
We had studied this previously.

**Another Variation:** Constraint that the $i^{th}$ box for $i = 1, \ldots, k$ must have $n_i$ objects.

**Indistinguishable objects:**

\# ways to allocate $n$ indistinguishable objects into $k$ boxes s.t. the number of objects in box $i$ is $n_i$ and $\sum_{i=1}^{k} n_i = n$.

Only 1 way




**What about distinguishable objects?**

# Motivating Example

EX: How many ways are there to deal 4 distinct unordered hands of 5 cards each from a deck of 52 cards ? Let's express this problem as an allocation problem. What are objects and what are the boxes.

$\boxed{1}\ \boxed{2}\ \dots\ \boxed{52}$

$\left.\begin{array}{c} \boxed{52}\ \boxed{7}\ \boxed{6}\ \boxed{1}\ \boxed{6} \\ \vdots \\ \boxed{\ }\ \boxed{\ }\ \boxed{1}\ \boxed{5}\ \boxed{\ } \end{array}\right\}$ 4 hands of cards

5-card hand

idea 1: 4x boxes, 20 cards (would work in some way)

idea2: 5x boxes, 52 disting. objects

$n_1 = n_2 = n_3 = n_4 = 5,\ n_5 = 32$

$C_1$: choose 5 cards from 52 cards

$C_2$: choose 5 cards from the remaining 47 cards

$C_3$: " " " " " " 42 "

$C_4$: " " " " " " 37 "

$C_5$: " 32 " " " " 32 "

#

How can we uniquely specify each allocation with a process of sequence of choices

$$\# \text{ways} = \binom{52}{5} \times \binom{47}{5} \times \binom{42}{5} \times \binom{37}{5} \times \cancel{\binom{32}{32}} \rightarrow 1$$

$$= \frac{52!}{5!\ \cancel{47!}} \times \frac{\cancel{47!}}{5!\ \cancel{42!}} \times \frac{\cancel{42!}}{5! \times \cancel{37!}} \times \frac{\cancel{37!}}{5!\ 32!} = \frac{52!}{5!\ 5!\ 5!\ 5!\ 32!}$$

In general: $\dfrac{n!}{n_1!\, n_2! \cdots n_k!}$ ways

to put $n$ "distinguishable" objects into $k$ boxes

with $n_i$ in $i^{th}$ box (provided $\sum_{i=1}^{k} n_i = n$)

**Summary:** Allocation of $n$ objects into $K$ boxes.
(order in the box doesn't matter)

Constraints?

| objects distinguishable | NO | $(n_1, n_2, \dots, n_K)$ (provided $\sum_{i=1}^{K} n_i = n$) |
|---|---|---|
| Y | $K^n$ | $\dfrac{n!}{n_1!\, n_2! \cdots n_K!}$ |
| N | $\binom{n+K-1}{K-1}$ | $1$ |

EX: How many distinct word can be made by rearranging the word ANAGRAM?

idea 1: $\frac{7!}{3!}$ ← total # Permutation of all letters if A's were distinguishable

← #way to permute those A's

idea 2: $\equiv$ allocating 7 letter posisions into 5 boxes (one for each distinct letter)

s.t. $n_A = 3$, $n_N = n_G = n_R = n_M = 1$.

$$\frac{n!}{n_A!\, n_N! \cdots n_M!} = \frac{7!}{3!\, 1!\, 1! \cdots 1!}$$

**Remark:** If boxes are indistinguishable, there are no closed form formula <skipped in this course>

$|0|\,|00| = |00|\,|0|$

eg.: how many ways to write 16 as a sum of 3 non-negative integers?

# Pigeonhole Principle (6.2)

**Theorem:** For any placement of $K+1$ objects into $K$ boxes, there is always at least one box with at least 2 objects.

at least

**Proof:** Prove by Contrapositive. Assume each box has at most one object. Then, we have at most $K$ items.

## Generalized PHP:

**Theorem:** If $N$ objects are placed into $k$ boxes, then there is at least one box containing at least $\lceil N/k \rceil$ objects.

## $\boxed{\text{How Could Something So Trivial be Useful?}}$

Let's look at some applications.

EX: Prove that in every group of 367 people, two people have the same birthday.

Proof: Because there are only 366 possible BD, and we have 367 people, by PHP, there must be atleast two people with same BD.

Ex: Show that for every integer $n$, there is a multiple of $n$ that has only 0's and 1's in its decimal expansion.

Proof: Assume $n$ is an arbitrary integer. Consider the following $n+1$ integers,

$$1,\ 11,\ 111,\ \ldots,\ \underbrace{11\ldots1}_{n+1}.$$

Since, there could be $n$ possible remainders when you divide an integer by $n$, by PHP, there must be two numbers in the sequence with the same remainder. Choose $i \neq j$ s.t. $a_j \equiv a_i \pmod{n}$. WLG, assume $a_i > a_j$. Observe that $a_i - a_j \equiv 0 \pmod{n}$. Hence, $n \mid a_i - a_j$, and $a_i - a_j$ is a positive integer with 1's & 0's.

Ex: If five distinct integers are chosen from $\{1, 2, \ldots, 8\}$ then, two of them must add up to 9.

Proof: Consider the following boxes

$\{1, 8\}, \{2, 7\}, \{3, 6\}, \{4, 5\}$

By PHP, by choosing 5 numbers, alway there exist two numbers that are in the same box.

EX: Assum integers $K$ and $\ell$ s.t. $\gcd(K, \ell)=1$.
Assume $0 \leq a < K$, and $0 \leq b < \ell$.
Prove that there exists an integer $0 \leq x < K\ell$
$x \equiv a \pmod{K}$ and $x \equiv b \pmod{\ell}$

Proof: Consider the integers
$S_0 = a$, $S_1 = a+K$, $S_2 = a+2K$, ..., $S_{\ell-1} = a+(\ell-1)K$
Assume by contradiction that $S_i$'s are not congruent to $b$ mod $\ell$.
Hence, there are $\ell-1$ possible remainders. By PHP, there exists $S_i$ and $S_j$ that have the same remainder divided by $\ell$. $S_i \equiv S_j \pmod{\ell}$. WLG $S_i > S_j$. Observe that

$\ell \mid S_i - S_j \longrightarrow \ell \mid (a+ik)-(a+jk) \longrightarrow \ell \mid k(i-j)$

Since $\gcd(\ell, k)=1$, then $\ell \mid (i-j)$ Since $0 \leq i-j \leq \ell-1$. $\times$