Week 2 - Monday

So far: We had seen how to mathematically show and work with statements like "$2+4>3$", which have either true or false truth value, but not both.

We know how to combine them with negation, disjunction, conjunction, conditional, biconditional, etc.

We also know when two propositions are logically equivalent.

Week 2:
+ We learn how to work with statements like "$x+2>3$", for which their truth value depends on the value of x. [propositional functions, Sec 1.4]
+ We will learn how to make propositions like "for all x, $x^2>0$." [quantifiers, Sec 1.4]
+ We will learn how to make propositions like "for all days of the week, there exists a TA such that the TA is available for office hour" [Nested quantifiers, sec 1.5]
+ Rules of Inference. How to make an argument in logic? (from the truth of some statements, we derive the truth of another statement, called conclusion.) [rules of inference, sec 1.6]

We need to understan valid arguments to be able to prove things

I) Propositional Functions (also known as predicate)

Is an statement containing one or more Variables from a domain which becomes a proposition when all of the variables are instantiated.

[The domain must be specified, often implicitly]

EX. "$\underbrace{x}_{\text{Variable}}$ $\underbrace{\text{is greater than 3}}_{\text{predicate, or Propositional function}}$" domain: Integers

+ Can be denoted by $P(x)$
+ once a value has been assigned to function P, the statement $P(x)$ becomes a Proposition and has a truth value.

EX. Propositional function with a single variable

"x is greater than 3" domain: integer

$P(x)$

$P(10)$ w/ truth value T

EX. Propostional function with mutiple variables

"$x > y$" domain = real, real

$Q(x, y)$

$Q(10, 100)$ w/ truth value F

Quantifiers

- When the variables in a propositional function are assigned values (i.e., intantiated), the resulting statement becomes a proposition with a certain truth value.
- However, there is another important way to create proposition out of a propositional function, called

quantification

Two types of quantification:

1) universal quantification 2) Existential quantification

Defn: Univeral Quantifier

The universal quantification of the propositional function P is the statement

"for every x in the domain, $P(x)$"

, and is denoted by

$\forall x\, P(x)$

, and read as for all x $P(x)$

EX. $\forall x$ (x is a prime) domain: $\mathbb{N}$

bound

Defn. A variable appearing in a quantifier is called bound

$Q(x,y)$

$P(y) \leftarrow \underbrace{\forall x\,(x > y)}$ domain: integer

* A statement in which all variables are bound, is a proposition (else not)

Defn: Existential quantifier

The existential quantification of a Propositional function P is the statement

"There exists x in the domain, s.t. $P(x)$"

, and is denoted by $\exists x \, P(x)$

, and read as " there exists x s.t. $P(x)$

EX: ① $\exists x \, (x \text{ is even})$ domain: integer T

② $\exists x \, (x^2 = 2)$ domain = Integers F

③ $\exists x \, (x^2 = 2)$ domain = real T

Remark: Generally, an implicit assumption is made that all domains for quantifiers are non-empty. Note that if the domain is empty, then $\forall x \, P(x)$ is true for any propositional function P because there is no element x in the domain for which $P(x)$ is false.

Similarly, if domain is empty, $\exists x \, P(x)$ is false.

How would these quantifiers interact with logical operators?

Negation of quantifiers:

EX. "Every integer is a Prime" $\forall x\,(x \text{ is Prime})$

negation: "It's not the case that every integer is a prime"

"There exists some integer such that the integer is composite"

General Pattern: (known as DeMorgan's law for quantifiers)

$$\begin{cases} \neg(\forall x\, P(x)) \equiv \exists x\, \neg P(x) \\ \neg(\exists x\, P(x)) \equiv \forall x\, \neg P(x) \end{cases}$$

$\neg$ flips the quantifier

Defn: Two statements involving quantifiers are logically equivalent if and only if they have the same truth value no matter which predicate is subsituted into these statements and which domain is used for the variables in these propositional functions. We use the notation $S \equiv T$ to indicate that two statements S and T involving predicates and quantifiers are logically equivalent.

Nesting of quantifiers

We can have multiple variables and multiple quantifiers.

EX. $\forall x \forall y \, (x+y=y+x)$ dom: integers

"for every x in the domain, for all y in the domain, $x+y=y+x$"

$\forall y \forall x \, (x+y=y+x)$

$\forall x \forall y \, Q(x,y) \equiv \forall y \forall x \, Q(x,y)$

In general

EX: $\exists x \exists y \; x^2+y^2=5^2$ dom: integers

"there exists x in the domain, s.t there exists y in the domain, s.t. $x^2+y^2=5^2$."

$\exists y \exists x \; x^2+y^2=5^2$

EX. $\forall x \exists y (y = x^2)$ domain: integers

"for all x in the domain, there exists y in the domain, s.t. $y = x^2$"

Truth value: T

$\exists y \forall x (y = x^2)$

"There exist y in the domain, s.t. for all x in the domain $y = x^2$"

Remark: note that $\forall x \exists y Q(x,y)$ is not logically equiv. to $\exists y \forall x Q(x,y)$

EX. $\forall x \forall y \forall z Q(x,y,z) \equiv \forall y \forall z \forall x Q(x,y,z)$

EX. $\exists x (\forall y \forall z Q(x,y,z)) \equiv \exists x \forall z \forall y Q(x,y,z)$

$\exists x \forall y \exists z Q(x,y,z) \not\equiv \exists z \forall y \exists x Q(x,y,z)$

Negation of nested quantifiers:

EX. $\neg(\exists x \forall y\, Q(x,y)) \equiv \forall x(\neg \forall y\, Q(x,y))$

$\equiv \forall x \exists y(\neg Q(x,y))$

EX. $\neg(\forall x \exists y((x \neq 0) \rightarrow xy=1)) \equiv$

$\exists x(\neg \exists y(x \neq 0 \rightarrow xy=1)) \equiv$

$\exists x \forall y \neg(x \neq 0 \rightarrow xy=1) \equiv$

$\exists x \forall y \neg(\neg(x \neq 0) \vee xy=1) \equiv$

$\exists x \forall y(\neg(\neg(x \neq 0)) \wedge xy \neq 1) \equiv$

$\exists x \forall y(x \neq 0 \wedge xy \neq 1)$

Quantifiers with restricted domains

An abbreviated notation is often used to restrict the domain of a quantifier.

EX. $\forall x<0\ (x^2>0)$, domain: real numbers.

What does this statement express?

"for all with $x<0$, $x^2>0$"

Can we state it without restricted domain?

$\forall x\ (x<0 \rightarrow x^2>0)$

$\cancel{\forall x\ ((x<0) \land (x^2>0))}$

EX. $\exists z>0\ (z^2=2)$ domain: integers.

$\exists z\ ((z>0) \land (z^2=2))$

$\cancel{\exists z\ ((z>0) \rightarrow z^2=2)}$

EX. Translating from English to logic.

"Some students in this class has visited Mexico"

domain: people

$p(x)$: "x is a student in this class"

$q(x)$: "x has visited Mexico"

$$\exists x\,(p(x) \land q(x))$$

EX. "Every student in this class has visited either US or Mexico, but not both" domain: people

$p(x)$: "x is a student in this class"

$q(x)$: "x has visited Mexico"

$R(x)$: "x has visited US"

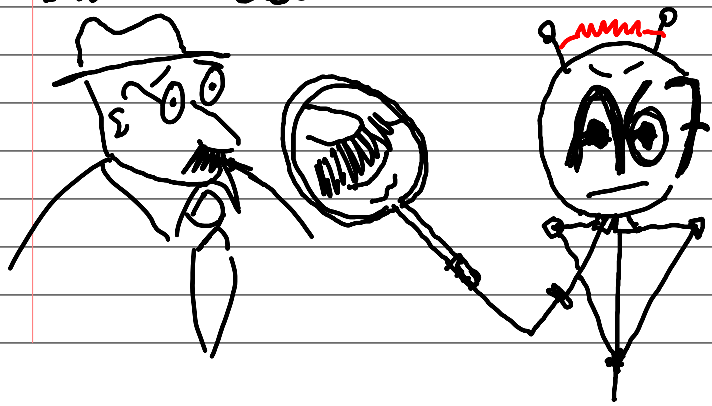
$$\forall x\,(p(x) \rightarrow (q(x) \oplus R(x)))$$

$$\forall x\,(p(x) \rightarrow ((\neg q(x) \land R(x)) \lor (q(x) \land \neg R(x))))$$

In class activity

Agent A67: Investigate!

- Holmes owns two suits: one black and one tweed.
- He always wears either a tweed suit or sandals
- Whenever he wears his tweed suit and purple shirt, he chooses not to wear a tie
- He never wears the tweed suit unless he is also wearing either a purple shirt or sandals.
- whenever he wears sandal, he also wears a purple shirt
- yesterday, holmes wore a bow tie
- what else did he wear?



Rules of Inference

Consider the following sequence of statements.

"If I'm studying hard, then I'll get A+"

"I'm studying hard"

Therefore, I'll get A+

This is an argument which has two premises (hypothesis) and after putting them together, it makes a conclusion.

When would you say that this argument is valid?

This argument would be a valid argument if whenever all hypothesis are true, the conclusion is true.

We can rewrite this argument in logical argument form

$$\begin{array}{l} p \longrightarrow q \\ p \\ \hline \therefore q \end{array}$$

p	q	$p \to q$	$(p) \wedge (p \to q)$	q
T	T	T	T	T
T	F	F	F	F
F	T	T	F	T
F	F	T	F	F

Defn: Argument

An argument is a sequence of propositions, the last of which is called Conclusion and the rest are called hypotheses (premises)

An argument is valid if truth of all its premises implies that the conclusion is true.

An argument form in propositional logic is a sequence of compound propositions involving propositional variables.

$$\begin{array}{l} p_1 \\ \vdots \\ p_n \\ \hline \therefore q \end{array}$$

An argument form is valid if no matter which particular propositions are subsituted for the propositional variables in its premises, the conclusion is true if all premises are true.

$$\begin{array}{l} p_1 \\ \vdots \\ p_n \\ \hline \therefore q \end{array}$$ is valid if $(p_1 \wedge p_2 \wedge \cdots \wedge p_n) \rightarrow q$ is a tautology

We can always use truth tables to show that an argument form is valid, i.e.,

$$(p_1 \wedge \ldots \wedge p_n) \longrightarrow q \text{ is a tautology.}$$

However, it is tedious

Instead we can use some simple valid argument forms (tautology) as building blocks to construct more complicated valid argument forms.

These simple valid arguments are called rules of inference.

TABLE 1 Rules of Inference.		
Rule of Inference	*Tautology*	*Name*
p $p \to q$ $\therefore\ q$	$(p \wedge (p \to q)) \to q$	Modus ponens
$\neg q$ $p \to q$ $\therefore\ \neg p$	$(\neg q \wedge (p \to q)) \to \neg p$	Modus tollens
$p \to q$ $q \to r$ $\therefore\ p \to r$	$((p \to q) \wedge (q \to r)) \to (p \to r)$	Hypothetical syllogism
$p \vee q$ $\neg p$ $\therefore\ q$	$((p \vee q) \wedge \neg p) \to q$	Disjunctive syllogism
p $\therefore\ p \vee q$	$p \to (p \vee q)$	Addition
$p \wedge q$ $\therefore\ p$	$(p \wedge q) \to p$	Simplification
p q $\therefore\ p \wedge q$	$((p) \wedge (q)) \to (p \wedge q)$	Conjunction
$p \vee q$ $\neg p \vee r$ $\therefore\ q \vee r$	$((p \vee q) \wedge (\neg p \vee r)) \to (q \vee r)$	Resolution

Table from the textbook, Sec 1.6.

Ex. Let's do Modus Ponens w/ truth table.

p	q	$p \to q$	$p \wedge (p \to q)$	q

EX. Show that the following argument form is valid.

(Show that $p \wedge q \rightarrow p \vee q$ is a tautology)

$$\begin{array}{l} p \wedge q \\ \hline \therefore\ p \vee q \end{array}$$

Step	Reason
1. $p \wedge q$	hypothesis
2. p	by simplification using (1)
3. $p \vee q$	by addition using (2)

EX.

Show that the premises "It is not sunny this afternoon and it is colder than yesterday," "We will go swimming only if it is sunny," "If we do not go swimming, then we will take a canoe trip," and "If we take a canoe trip, then we will be home by sunset" lead to the conclusion "We will be home by sunset."

p: "It's sunny this afternoon"

q: "It's colder than yesterday"

r: "We will go swimming"

s: "We'll take a canoe trip"

t: "We'll be home by sunset"

$$\neg p \wedge q$$
$$r \rightarrow p$$
$$\neg r \rightarrow s$$
$$s \rightarrow t$$
$$\therefore t$$

Step	Reason
1. $\neg p \wedge q$	hypo
2. $\neg p$	using (1)
3. $r \rightarrow p$	hypo
4. $\neg r$	using (2), (3)
5. $\neg r \rightarrow s$	hypo
6. s	by (4), (5)
7. $s \rightarrow t$	by hypo
8. t	by (6), (7)