Welcome To The Counters Club!

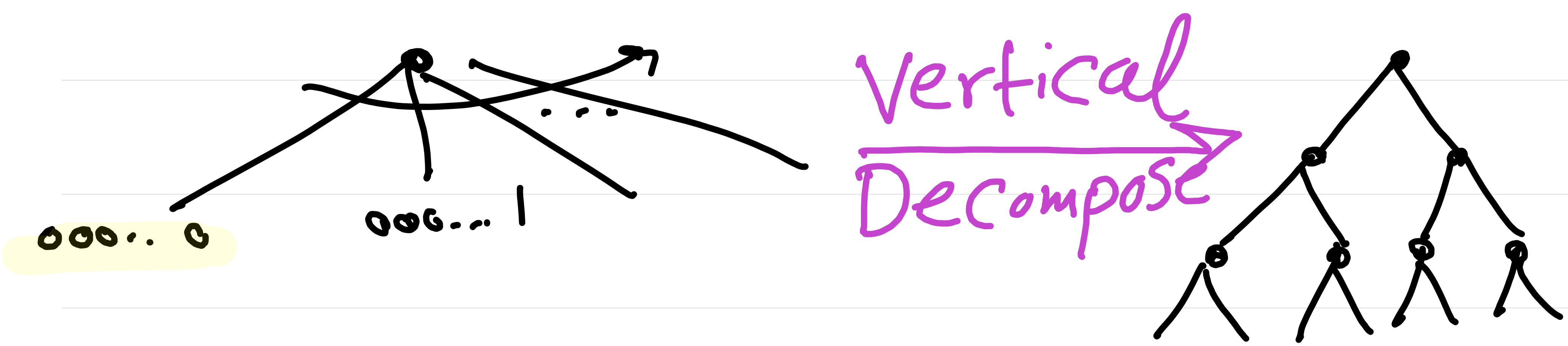
So far:

The Product Rule

(a very intuitive idea with just a name)

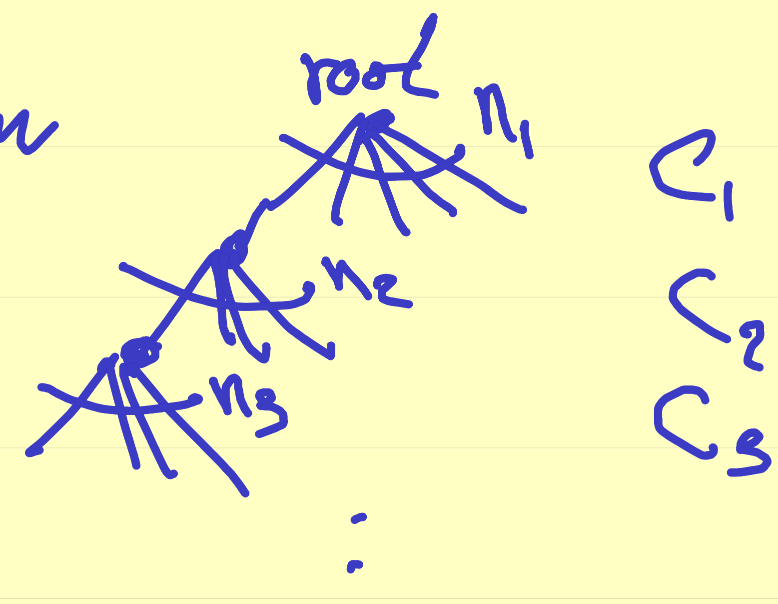
Idea:

Decompose a big choice into many little choices.



Product rule: Suppose each object with the desired property can be uniquely specified by a sequence of k choices $C_1, C_2, \ldots, C_k$ and the number of ways to make C_i is n_i for any $1 \leq i \leq k$. Then the total # of objects is $n_1 \times n_2 \times \cdots \times n_k$.

tree diagram



"vertical" decomposition of big choice into smaller choices

Given the Product rule, how do you solve a counting problem?

Step 1: Find a sequence of choices $C_1, \ldots, C_k$ uniquely specifying an object

Step 2: Count # ways to make each C_i given previous choices

$C_1 : n_1 \quad C_2 : n_2 \quad \ldots \quad C_k : n_k$

Step 3: Combine using product rule

$$\# \text{total} = n_1 \times n_2 \times \cdots \times n_k$$

$S \leftarrow$ set bitstring of length k $\quad f: S \longrightarrow \underline{\quad S' \quad}$ $\quad n_1 \times n_2 \times \cdots \times n_m$

$$\{1, \ldots, n_1 \times n_2 \times \cdots \times n_m\}$$

So far :

The Product Rule

(a very intuitive idea with just a name)

Today :

- More Examples
- New Roles (Sum Rule, Subtraction Rule, Division Rule)
- Binomial Theorem
- Combinatorial Identities
- Permutation & Combination with Repetition

EX3: Ranking of {UofT, UW, UBC, McGill}

Step 1: C_1: choose the 1^{st} rank from {UofT, UW, UBC, McGill}

C_2: Given the 1^{st} rank, choose 2^{nd} rank from {UofT, UW, ...}

C_3: Given 1^{st} & 2^{nd} rank, choose 3^{rd} rank from "

C_4: Given the previous choices, choose 4^{th} rank from "

Step 2: is there repetition or not? No

$n_1 = 4$, $n_2 = 3$, $n_3 = 2$, $n_4 = 1$

Step 3: By product rule, the total # rankings =

$n_1 \times n_2 \times n_3 \times n_4 = 4!$

This type of questions shows up so often that they have their own name.

Defn: An ordering of n distinct objects is called a <u>permutation</u>.

Theorem: The number of permutation of n distinct objects is $n!$.

- In our previous example:

  $n = 4$

Sum & Subtraction Rule

disjoint sets

Sum rule: If A and B are finite & $A \cap B = \emptyset$, then $|A \cup B| = |A| + |B|$.

How is this applicable to Counting?

EX 6: # of 2 digit numbers with no zeros s.t. both digits are even or both are odd.

$S = \{ (a,b) : a,b \in \{1, \dots, 9\}$ and (both a and b are even or both a and b are odd) $\}$

$S = A \cup B$ where $A = \{ (a,b) : a,b \text{ are odd} \}$

$B = \{ (a,b) : a,b \text{ are even \& non-zero} \}$

observe that $A \cap B = \emptyset$

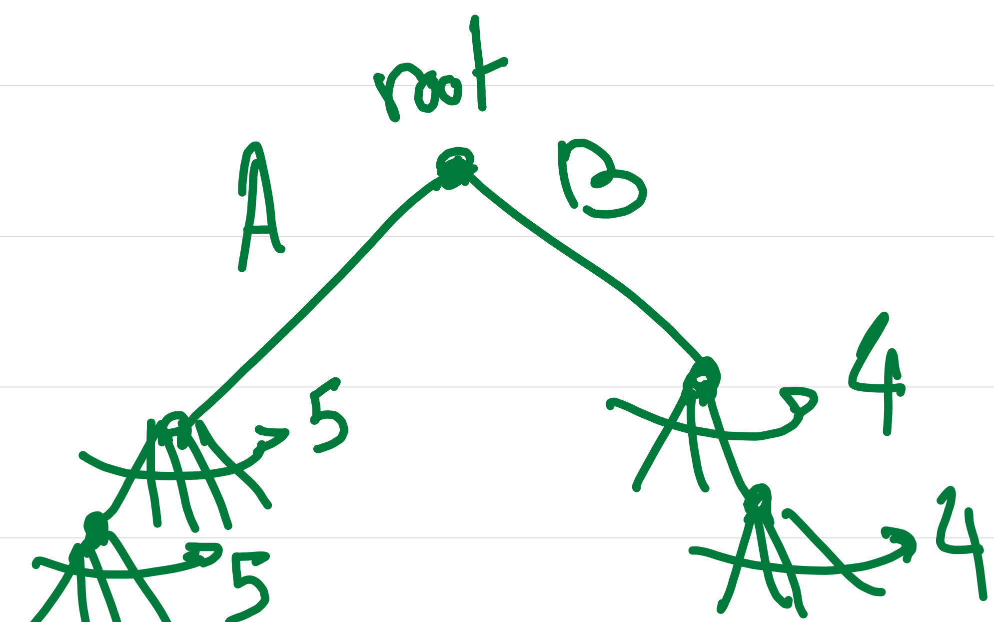
Observe that $|S| = |A| + |B|$

Is repetition allowed? Yes

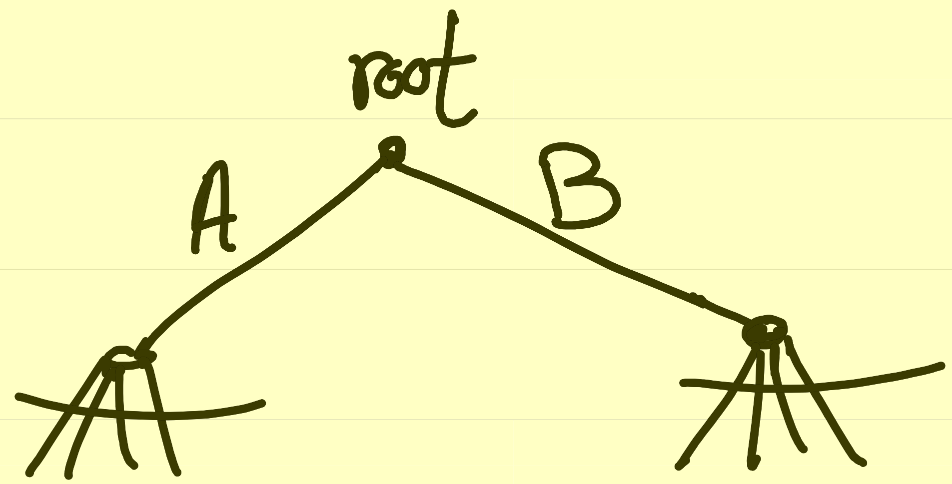
A: $|A| = 5^2$

B: $|B| = 4^2$

$|S| = 25 + 16.$



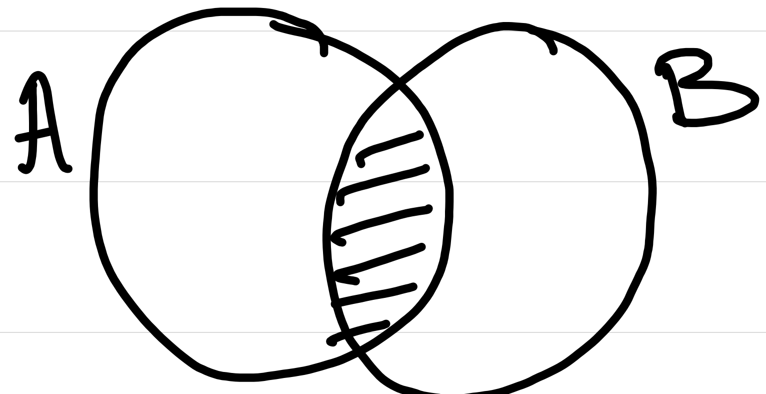
Sum Rule: If $A \cap B = \emptyset$ then $|A \cup B| = |A| + |B|$.



"horizontal" decomposition of choices in the tree diag.

Subtraction Rule:

If A and B are finite, $|A \cup B| = |A| + |B| - |A \cap B|$



How is this applicable to Counting?

EX2: Bit strings of length 5 ending in 00 or beginning with 1.

$S = \{b_1 b_2 \dots b_5 : b_i \in \{0,1\}\}$

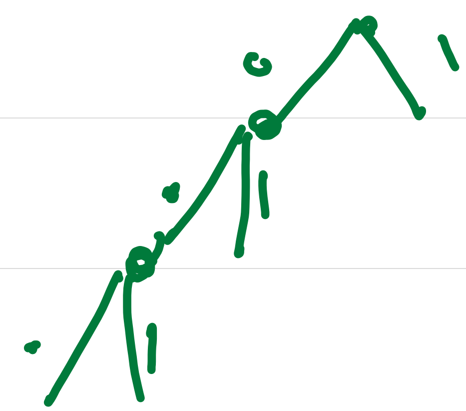
$S = A \cup B$

$A = \{b_1 b_2 b_3 00 : b_i \in \{0,1\}\}$

Observe that $A \cap B = \{1 b_2 b_3 00 : b_i \in \{0,1\}\}$

$B = \{1 b_2 b_3 b_4 b_5 : b_i \in \{0,1\}\}$

$|S| = |A| + |B| - |A \cap B|$



choose b_1 : $n_1 = 2$

choose b_2 : $n_2 = 2$

choose b_3 : $n_3 = 2$

by product rule $|A| = 2^3$

By similar argument $|B| = 2^4$

By similar argument $|A \cap B| = 2^2$

$|S| = |A| + |B| - |A \cap B|$

Subtraction Rule:

$$|A \cup B| = |A| + |B| - |A \cap B|$$

What if $S = A \cup B \cup C$? $\quad S = (A \cup B) \cup C$

$$|S| = |A \cup B| + |C| - |(A \cup B) \cap C|$$

$$= |A| + |B| + |C| - |A \cap B| - |(A \cap C) \cup (B \cap C)|$$

$$= |A| + |B| + |C| - |A \cap B| - \left(|A \cap C| + |B \cap C| - |A \cap B \cap C|\right)$$

Division Rule:

EX5: Unordered sets of 5 distict cards from a deck of 52

The difficulty with unordered sets is that it's generally hard to find the process that uniquely generates an unordered set. That's because inherently when you have a process there is an order by which you make the choices.

Let's consider a related (and easier) problem.

EX5': order hand of 5 distinct card from deck of 52.

EX5': # ordered 5 distinct cards from deck of 52.

C_1: choose the first card from the deck

C_2: given the first choice, choose the 2^{nd} card.

C_5: given previous choices, choose the last card

$n_1 = 52$, $n_2 = 51$, $n_3 = 50$, $n_4 = 49$, $n_5 = 48$

The total # ordered 5 distinct card $= 52 \times \dots \times 48 = \dfrac{52!}{47!}$.

Defn: An ordered sequence of r distinct objects from n objects is called an r-permutation. The # of r-permutation of n object is $\dfrac{n!}{(n-r)!} = P(n,r)$

$\langle$ In EX5', $n = 52$, $r = 5 \rangle$

How to answer EX 5?

Trick: # of ordered 5-card sequence =

# of unordered 5-cards × (# of ways to order a given set of 5 cards)

$\frac{52!}{47!}$

unordered	ordered
1, 2, 3, 4, 5	1, 2, 3, 4, 5
	1, 3, 2, 4, 5
	1, 5, 4, 2, 3

5!

Reason: Can uniquely specify an ordered 5-card hand as:

C_1: choose an unordered 5-card hand ← ?

C_2: given the 5-card hand, choose an ordering of them

5!

So, we can apply the product rule.

Therefore, # unordered 5-card hands

$$= \frac{\text{\# ordered 5-card hands}}{5!} = \frac{52!}{47! \times 5!}$$

Defn: An unordered set of r objects from a collection of n objects is called an r-combination.

The total # r-combination of n object is

$$\binom{n}{r} = C(n,r) = \frac{P(n,r)}{r!} = \frac{n!}{r!\,(n-r)!}$$

Read as "n choose r."

<In Ex 5, $n=52$, $r=5$>

Division Rule:

If $f: A \longrightarrow B$ is an m-to-one correspondence, i.e., $\forall b \in B$ $|f^{-1}(b)| = m$, then $|B| = \frac{|A|}{m}$.

⟨In Ex 5, $A = \{ (a_1, a_2, \ldots, a_5) : a_i \in \{1, \ldots, 52\}, a_i \neq a_j \ \forall i \neq j \}$

$B = \{ \{a_1, \ldots, a_5\} : a_i \in \{1, \ldots, 52\} \}$

implicitly unordered, and distinct.

$$f((a_1, a_2, \ldots, a_5)) = \{a_1, \ldots, a_5\}$$

Remember:

Always ask yourself if there is repetition,
if order matters

Binomial Theorem

EX: $(x+y)^2 = x^2+2xy+y^2$

$(x+y)^3 = (x+y)(x+y)(x+y) = x^3+3x^2y+3xy^2+y^3$

$(x+y)^{10} = (x+y)\cdots(x+y)$

These are seemingly algebraic questions which can be answered with counting technique.

Observe that $(x+y)^3 = \overset{S_1}{(x+y)}\overset{S_2}{(x+y)}\overset{S_3}{(x+y)}$

$= (x+y)\big((x+y)(x+y)\big) = x\big((x+y)(x+y)\big) +$
$y\big((x+y)(x+y)\big)$

$= xxx+xyx+xxy+xyy+yxx+yxy+yyx+yyy.$

$$(x+y)(x+y)(x+y) = xxx + xxy + xyx + xyy + yxx + yxy + yyx + yyy$$

What's the pattern? Let's give each group a name

$$\underbrace{(x+y)}_{S_1}\ \underbrace{(x+y)}_{S_2}\ \underbrace{(x+y)}_{S_3}$$

We can uniquely specify each term above:

C_1: choose an element from S_1

C_2: " " " " S_2

C_3: " " " " S_3

\# total term above $= 2^3 = 8$

$$(x+y)(x+y)(x+y) = xxx + xxy + xyx + xyy + yxx + yxy + yyx + yyy$$

Observation: $xxy = xyx = yxx = x^2y$

Observe that # terms with 3 X's = 1

terms with 2 X's = $C(3,2)$

terms with 1 X = $C(3,1)$

terms with 0 x's = 1

$(x+y)(x+y)(x+y)$ after grouping the term $=$

$$C(3,3)x^3 + C(3,2)x^2y + C(3,1)xy^2 + C(3,0)y^3$$

$=$

Notation: $\binom{n}{r} = C(n,r)$ read as "n choose r."

Theorem: If $n \geq 1$ then

$$(x+y)^n = \binom{n}{n}x^n + \binom{n}{n-1}x^{n-1}y + \binom{n}{n-2}x^{n-2}y^2 + \cdots + \binom{n}{1}xy^{n-1} + \binom{n}{0}y^n$$

$$= \sum_{r=0}^{n} \binom{n}{r} x^r y^{n-r}$$

Proof of the theorem:

Let $n \geqslant 1$ and $0 \leqslant r \leqslant n$ be arbitrary.

Observe that the coeficient of $x^r y^{n-r}$ in $(x+y)(x+y)\cdots(x+y)$ is the number of strings in x, y with n letters before grouping with exactly r x's.

$= \#$ r-subsets of $\{1, \ldots, n\} = \binom{n}{r}$

Combinatorial Identities

(Formulas proven using counting)

EX: If $n \geq 1$, $0 \leq r \leq n$ then $\binom{n}{r} = \binom{n}{n-r}$.

Proof 1 (algebraic and boring):

$\binom{n}{r} = \frac{n!}{(n-r)!\, r!}$ and $\binom{n}{n-r} = \frac{n!}{r! \times (n-r)!}$. Hence, $\binom{n}{r} = \binom{n}{n-r}$

Proof 2 (Combinatoric proofs, i.e., proof based on bijection):

Let $A = \{ S \subseteq \{1, .., n\} : |S| = r \}$ $\quad |A| = \binom{n}{r}$

$B = \{ S \subseteq \{1, .., n\} : |S| = n-r \}$ $\quad |B| = \binom{n}{n-r}$

Define $f: A \to B$ as $f(S) = \{1, ..., n\} - S$. Observe that f is a bijection. Hence, $|A| = |B|$. Therefore $\binom{n}{r} = \binom{n}{n-r}$.