Reviewing Divisibility, Prime, Sigma, and Pi

Divisibility:

Let $n, d \in \mathbb{Z}$ be two arbitrary integers.

n is divisible by d if $\exists k \in \mathbb{Z}\ (n = dk)$.

We use the notation $d \mid n$, read as "d divides n" to denote that n is divisible by d.

$6 \mid 30$

Remark: $\forall n \in \mathbb{Z}\ 0 \nmid n$ (note: $0 \mid 0$ is defined as false)

Remark: $\forall n \in \mathbb{Z}^{\neq 0}\ n \mid 0$ (since you can choose $k = 0$, i.e., $0 = n \times 0$)

Remark: $d \nmid n$ denotes that d does not divide n

$7 \nmid 30$

Prime Numbers:

Defn: $p \in \mathbb{Z}^{>1}$ is a prime number if p is only divisible by p, $(-p)$, 1, and (-1).

Fundamental Theorem of Arithmitic (FTA): Every $n \in \mathbb{Z}^{>1}$ is prime itself or is the product of unique combination of prime numbers.

(Everybody Knows this Theorem & how to use it. Today, we will prove part of it by Strong induction)

Theorem: There are infinite number of Primes

(We will prove it by Contradiction)

Sigma (Σ) and Pi (Π) Notation

$$\sum_{i=n}^{m} a_i = a_n + a_{n+1} + \cdots + a_m$$

$$\prod_{i=n}^{m} a_i = a_n \times a_{n+1} \times \cdots \times a_m$$

- $$\sum_{i=n}^{m+1} a_i = \sum_{i=n}^{m} a_i + a_{m+1}$$
- $$\prod_{i=n}^{m+1} a_i = \prod_{i=n}^{m} a_i \times a_{m+1}$$
- $$\overbrace{\sum_{i=n+1}^{m} a_i}^{a_{n+1} + \cdots + a_m} = \overbrace{\sum_{i=n}^{m} a_i}^{a_n + a_{n+1} + \cdots + a_m} - a_n$$
- $$\prod_{i=n+1}^{m} a_i = \prod_{i=n}^{m} a_i \times \frac{1}{a_n}$$

EX: $\sum_{i=1}^{n} \frac{1}{i} = \frac{1}{1} + \frac{1}{2} + \cdots = \sum_{i=0}^{n-1} \frac{1}{i+1}$

EX: $\prod_{i=1}^{n} \sqrt{i^3} = \sqrt{1^3} \cdot \sqrt{2^3} \cdots = \prod_{i=0}^{n-1} \sqrt{(i+1)^3}$

- $$\sum_{i=m}^{n} a_i = \sum_{i=m+k}^{n+k} a_{i-k}$$
- $$\prod_{i=m}^{n} a_i = \prod_{i=m+k}^{n+k} a_{i-k}$$
- $$\sum_{i=m}^{n} a_i + \sum_{i=m}^{n} b_i = \sum_{i=m}^{n} (a_i + b_i)$$
- $$\prod_{i=m}^{n} a_i \times \prod_{i=m}^{n} b_i = \prod_{i=m}^{n} (a_i \cdot b_i)$$

Mathematical Induction

Principle of Mathematical Induction:

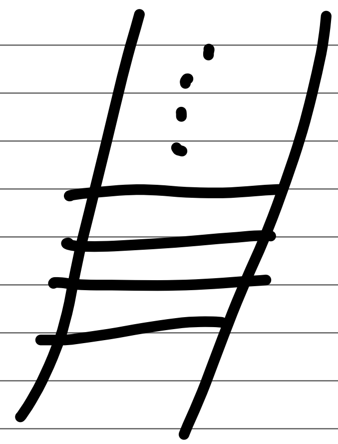
Let $P(n)$ be a statement defined for integer n, and let k be a fixed integer.
Suppose the following are true.

1. $P(k)$ is true
2. For all $m \in \mathbb{Z}$ with $m \geq k$, if $P(m)$ is true, then $P(m+1)$ is true.

Then, for all $n \geq k$ $P(n)$ is true.

- Metaphor:

Ladder



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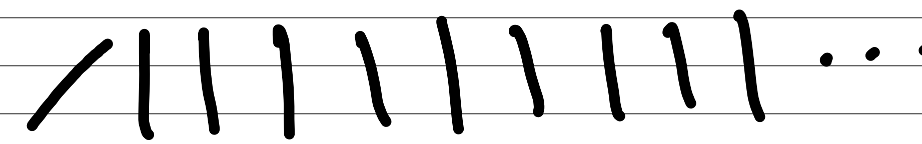

$$P(1) \longrightarrow P(2) \longrightarrow P(3) \longrightarrow \cdots$$

Structure of Mathematical Induction Proof:

Goal: prove that $P(n)$ is true $\forall n \geq n_0$.

$P(n)$ is called the induction hypothesis

We proceed by induction.

- Base case $(n = n_0)$
  Show that $P(n_0)$ is true.

- Inductive step:
  Assume $K \geq n_0$ is arbitrary. Assume $P(K)$ is true.
  Show that $P(K+1)$ is true.
  This completes the inductive step $\square$

EX: Prove that $\sum_{i=1}^{n} i^3 = \left(\frac{n(n+1)}{2}\right)^2$ for $n \geq 1$.

Proof: We proceed by induction. Let $P(n)$ denote the statement that $\sum_{i=1}^{n} i^3 = \left(\frac{n(n+1)}{2}\right)^2$.

- Base Case $(n=1)$:

Observe that $\sum_{i=1}^{1} i^3 = 1^3 = 1 = \left(\frac{1(1+1)}{2}\right)^2$.

- Inductive step: Assume $K \geq 1$ is arbitrary.

Assume, $P(K)$ is true, i.e., $\sum_{i=1}^{K} i^3 = \left(\frac{K(K+1)}{2}\right)^2$.

Observe that $\sum_{i=1}^{K+1} i^3 = \sum_{i=1}^{K} i^3 + (K+1)^3 = \frac{K^2(K+1)^2}{4} + (K+1)^3$

$= \frac{K^2(K+1)^2 + 4(K+1)^3}{4} = \frac{(K+1)^2(K^2+4K+4)}{4} = \frac{(K+1)^2(K+2)^2}{4}$

Hence, $P(K+1)$ holds. This completes the inductive step.

EX: Prove that $\sum_{i=1}^{n} \frac{1}{i^2} < 2 - \frac{1}{n}$ for $n > 1$.

Strong Induction

Principle of Strong Induction.

Let $P(n)$ be a statement defined for integer n and let k be a fixed integer. Suppose the following is true.

1. $P(K)$
2. for all $m \geq k$, if $P(\ell)$ is true for all $K \leq \ell \leq m$, then $P(K+1)$ is true.

Then we can conclude that $P(n)$ is true for all $n \geq K$.

The process of Strong Induction:

$P(1) \to P(2)$

$P(1) \wedge P(2) \to P(3)$

$\vdots$

Or equivalently:

$P(1) \to P(1) \wedge P(2)$

$P(1) \wedge P(2) \to P(1) \wedge P(2) \wedge P(3)$

$\vdots$

$$\begin{cases} Q(1) \to Q(2) \\ Q(2) \to Q(3) \\ \vdots \end{cases} \quad \left(Q(n) = P(1) \wedge \dots \wedge P(n) \right)$$

Strong induction is in fact a regular induction in which the hypothesis of the induction is

$Q(n) = P(1) \wedge \dots \wedge P(n)$

Structure of Strong Induction Proof.

We proceed by strong induction.

- Base case ($n=n_0$): show that $p(n_0)$ holds
- Inductive step: Assume $k \geq n_0$ is arbitrary.

Assume $p(l)$ is true for all $n_0 \leq l \leq k$.

Show that $P(k+1)$ holds.

That completes the inductive step $\square$.

EX. Any integer $n \geq 2$ is a prime or can be written as product of primes.

We proceed by strong induction.

- Base case $(n=2)$: observe that 2 is a prime

- Inductive step: Assume arbitrary $K \geq 2$.

Assume all integers $2 \leq l \leq K$ are either prime or can be written as product of primes.

We show that $K+1$ is a prime or product of prime through proof by

Cases.

Case 1 ($K+1$ is prim). In this case, $K+1$ is a prime.

Case 2 ($K+1$ is composit): Choose $2 \leq a < K+1$ and $2 \leq b < K+1$ s.t. $K+1 = ab$. By induction step assumption, a and b are either primes or can be writen as product of primes.

Choose $p_1, \dots, p_i$ and $q_1, \dots, q_j$ s.t.

$a = p_1 \times \dots \times p_i$ and $b = q_1 \times \dots \times q_j$.

Observe that $k+1 = ab = p_1 \times \cdots \times p_i \times q_1 \times \cdots \times q_j$.

So, $k+1$ can be writen as product of primes. This Completes the inductive step.