MAT-CSC A67: Discrete Mathematics — Summer 2024

Assignmnet 3: Sets, Functions, Divisibility

Due Date: Sunday, July 7, 11:59 PM, on Crowdmark

Q0. **[0 pts, Warm-up]** This is a warm-up question. Make sure you are comfortable with these questions before you start working on the remaining ones. Feel free to openly discuss your solutions to the warm-up question on Piazza or during the office hours.

0.a. Reviewing core concepts.

0.a. i. **[Sequence]** Find an explicit formula for the sequence with the initial terms $\frac{1}{3}, \frac{3}{9}, \frac{9}{27}, \frac{16}{81}, \frac{25}{243}, \frac{36}{729}$.

0.a. ii. **[Change of Variable in Sum]** Transform the following sum by making the change of variable $j = i - 1$.

$$\sum_{i=1}^{n+1} \frac{(i-1)^2}{i.n}$$

0.a. iii. **[factorial]** Simplify $\frac{((n+1)!)^2}{(n!)^2}$ and $\frac{n!}{(n-k+1)!}$.

0.a. iv. **[Quotient-Remainder Theorem]** Suppose d is a positive integer and n is any integer. If $d \mid n$, what is the remainder when the Quotient-Remainder Theorem is applied to n with divisor d?

0.b. Proof methods review.

0.b. i. Use the definition of divisibility to prove the following statements.

- For all integers a, b, and c, if $a \mid b$ and $a \mid c$ then $a \mid (b + c)$.
- For all integers a, b, and c, if $ab \mid c$ then $a \mid c$ and $b \mid c$.

0.c. Induction review.

0.c. i. Use mathematical induction to prove for all integers $n \geq 1$,

$$\sum_{i=1}^{n} i(i!) = (n+1)! - 1.$$

0.c. ii. Observe that

$$\begin{aligned} \frac{1}{1 \cdot 3} &= \frac{1}{3}, \\ \frac{1}{1 \cdot 3} + \frac{1}{3 \cdot 5} &= \frac{2}{5}, \\ \frac{1}{1 \cdot 3} + \frac{1}{3 \cdot 5} + \frac{1}{5 \cdot 7} &= \frac{3}{7}, \\ \frac{1}{1 \cdot 3} + \frac{1}{3 \cdot 5} + \frac{1}{5 \cdot 7} + \frac{1}{7 \cdot 9} &= \frac{4}{9}. \end{aligned}$$

Conjecture a general formula and prove it by mathematical induction.

0.c. iii. Evaluate the sum $\sum_{i=1}^{n} \frac{i}{(i+1)!}$ for all $n = 1, 2, 3, 4, 5$. Make a conjecture for a formula for the sum for a general n, and prove your conjecture by induction.

0.c. iv. Use mathematical induction to prove $4 \mid 5^n - 1$, for integers $n \geq 0$.

0.c. v. Use mathematical induction to prove $2^n < (n+1)!$, for integers $n \geq 2$.

0.d. Use set-roster notation to list the elements in each of the following sets.

0.d. i. $S = \{n \in \mathbb{Z} : n = (-1)^k, \text{ for some integer } k\}$.

0.d. ii. $T = \{m \in \mathbb{Z} : m = 1 + (-1)^k, \text{ for some integer } k\}$.

0.d. iii. $U = \{r \in \mathbb{Z} : 2 \leq r \leq -2\}$.

0.d. iv. $V = \{s \in \mathbb{Z} : s > 2 \text{ or } s \leq 3\}$. [**NOTE:** It's OK to use "..." for this part. However, in general, avoid using ... in your set-roster notation and use a set-builder instead.]

0.d. v. $T \times S$, where T and S are defined as above.

0.d. vi. $V \times S$, where V and S are defined as above. [**NOTE:** It's OK to use "..." for this part.]

Q1. **(2 pts)** Pretend you are the TA for A67 and grade each of these attempted proofs. Give each one a score out of 5 points and justify your decisions; your explanations should be convincing but brief. Here is the rubric you should use:

(5 **pts)** Full credit: Proof is clear, complete, and correct: the proof is written in complete English sentences, variables are clearly declared, proof strategy is correctly identified and applied (*e.g.* for induction base case and induction step each are labelled properly), assumptions and goals are clearly articulated, and calculations are well supported and explained.

(4 **pts)** Partial credit: Proof is mostly clear, complete, and correct with minor errors or imprecision that do not affect the correctness of the conclusion.

(3 **pts)** Partial credit: Proof has many required components and is missing major component or exhibits conceptual error.

(2 **pts)** Partial credit: Some indication of structure of proof but main arguments missing or irrelevant.

(1 **pts)** Partial credit: Does not match other partial credit items but demonstrates some correct and relevant knowledge.

(0 **pts)** Incorrect or blank.

1.a. **Statement:** "the sum of the first n positive odd integers is n^2"
Attempted Proof:

Base Case ($n = 1$**):** First odd number is 1. Observe that $1^2 = 1$, as desired.
$(n+1)^2 = n^2 + 2n + 1$
n^2 is the sum of the first n odd numbers, and $2n + 1$ is the next odd number in the sequence, therefore $(n+1)^2 =$ the sum of the first $n + 1$ odd numbers.

1.b. **Statement:** "For every nonnegative integer n, $3 \mid n$."
Attempted Proof:

We proceed by strong mathematical induction.
Basis step: Indeed, $3 \mid 0$ because there is an integer, namely 0 such that $0 = 30$.
Induction step: Let $k \geq 0$ be arbitrary. Assume for all nonnegative integers j with $0 \leq j \leq k$, that $3 \mid j$. Write $k + 1 = m + n$, where m, n are integers less than $k + 1$. By the induction hypothesis, $3 \mid m$ and $3 \mid n$. Choose integers a and b such that $m = 3a$ and $n = 3b$. Therefore, $k + 1 = (3a) + (3b) = 3(a + b)$. Hence, $3 \mid (k + 1)$, as required.

Q2. **(5 pts)** We now consider a simplified version of the famous four color theorem. The four color theorem states that any map can be colored with four colors such that any two adjacent countries[1] have different colors. The four color theorem is very difficult to prove, and several bogus proofs were claimed since the problem was first posed in 1852. In fact, it was not until 1976 that a computer-assisted proof of the theorem was finally given by Appel and Haken. (For an interesting history of the problem and a state-of-the-art proof, which is nonetheless still very challenging, see www.math.gatech.edu/ thomas/FC/fourcolor.html.)

In this assignment, we consider a simpler version of the theorem in which our "map" is given by a rectangle which is divided into regions by drawing straight lines, such that each line divides the rectangle into two regions. Figure 1 shows an example of coloring a map divided into regions by drawing 2 straight lines.

Prove that we can color any "map" given by a rectangle which is divided into regions by drawing any number of any straight lines using no more than two colors (say, red and blue) such that no two bordering regions have the same color.

[1] Countries are defined to be adjacent if they share any non-trivial part of a border, i.e., anything more than a point.

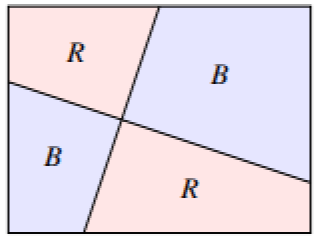

Figure 1: An example of coloring a map divided into regions by drawing 2 straight lines.

Q3. **(5 pts) [Based on a similar question from "Discrete Mathematics with Applications, 4th Edition," by Susanna Epp]** Suppose that n grown-up and n baby are distributed around the outside of a circular table. Use mathematical induction to prove that for any integer $n \geq 1$, given any such seating plan, it is always possible to find a starting point so that if you travel around the table in a clockwise direction the number of grown-ups you pass is never less than the number of babies you have passed. For example, in Figure 2, we use g to denote a grown-up and b to denote a baby. In this setup, you should start at the grown-up in red.

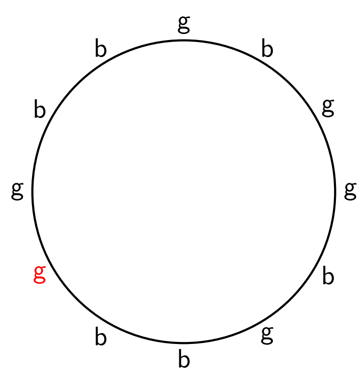

Figure 2: An example of a seating plan for 6 grown-ups and 6 babies.

Q4. **(5 pts)** Let x be an arbitrary integer. Assume $x \equiv 3 \pmod 5$ and $x \equiv 2 \pmod{11}$. What is $x \mod 55$? Prove your answer.

Q5. **(5 pts)** Prove that if sets A and B are countable, then their union $A \cup B$ is countable.

Q6. **(4 pts)** Prove or disprove the following statements. The statements in this question relate to Asymptotic Runtime Analysis, which is used to calibrate the efficiency of algorithms and is at the heart of the million dollar P vs. NP problem.:

6.a. $\exists C \in \mathbb{Z}\ \exists n_0 \in \mathbb{Z}^{>0}\ \forall n \in \mathbb{Z}^{\geq n_0}\ (\ n^2 < C(n^2 - 1)\ )$

6.b. $\exists C \in \mathbb{Z}\ \exists n_0 \in \mathbb{Z}^{>0}\ \forall n \in \mathbb{Z}^{\geq n_0}\ (\ (n^2 - 1) < C(n^2)\ )$

Q7. **(4 pts)** Prove or disprove each of the following claims. In your justifications, you may refer to any of the facts about cardinality we mentioned in class or that appear in the relevant sections in the textbook. Credit will be given for *correctness of the conclusion* of each part, *i.e.* whether claim is true or false, and *fair effort in the justification*.

7.a. $\forall X \in \mathcal{P}(\mathbb{R})\ \forall Y \in \mathcal{P}(\mathbb{R})\ (\ (|X| = |Y|) \rightarrow (X = Y)\ )$

7.b. $\exists A \in \mathcal{P}(\mathbb{R})\ \exists B \in \mathcal{P}(\mathbb{R})\ (\ \mathbb{Z} \subseteq A\ \wedge\ \mathbb{Z} \subseteq B\ \wedge\ \neg(|A| = |B|)\ )$