Week 13_Friday

likelihood / prob. / distribution function

So far:

Experiment, outcomes, Sample space, $p : S \to [0,1]$

Event $E \subseteq S$, $\mathbb{P}(E) = \sum_{s \in E} p(s)$

- Conditional Probability: $\mathbb{P}(E|F) = \mathbb{P}(E \cap F) / \mathbb{P}(F)$
- Product Rule: $\mathbb{P}(E \cap F) = \mathbb{P}(E|F)\,\mathbb{P}(F)$
- Sum Rule: $\mathbb{P}(E \cup F) = \mathbb{P}(E) + \mathbb{P}(F) - \mathbb{P}(E \cap F)$
- Total Probability: $\mathbb{P}(E) = \mathbb{P}(E|F)\,\mathbb{P}(F) + \mathbb{P}(E|\overline{F})\,\mathbb{P}(\overline{F})$
- Bayes' Theorem: $\mathbb{P}(E|F) = \dfrac{\mathbb{P}(F|E)\,\mathbb{P}(E)}{\mathbb{P}(F)}$

$\mathbb{P}(F \cap E) = \mathbb{P}(F) - \mathbb{P}(F \cap \overline{E})$

$$= \frac{\mathbb{P}(F|E)\,\mathbb{P}(E)}{\mathbb{P}(F|E)\,\mathbb{P}(E) + \mathbb{P}(F|\overline{E})\,\mathbb{P}(\overline{E})}$$

$\mathbb{P}(F|E) \neq 1 - \mathbb{P}(F|\overline{E})$ in general

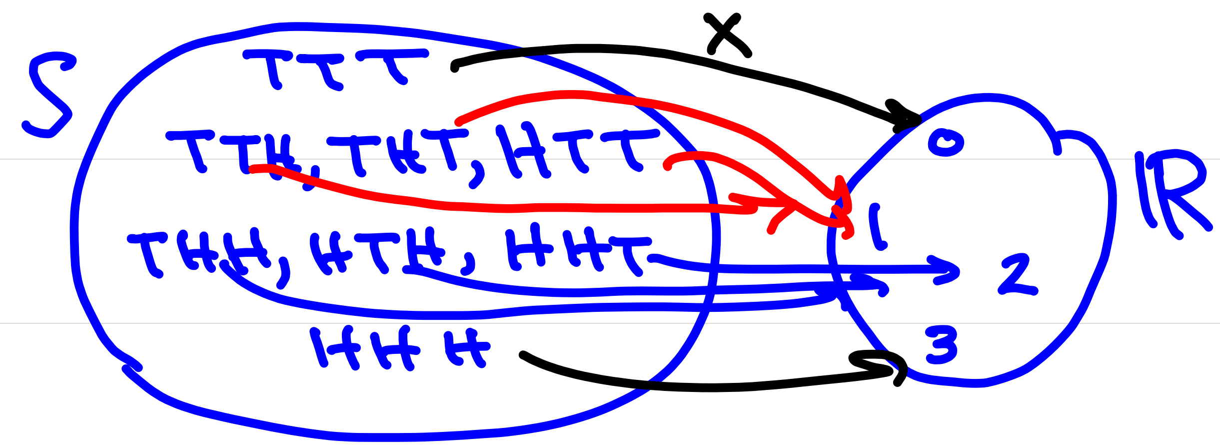
So far:

- **Independence of Events** $\mathbb{P}(E \cap F) = \mathbb{P}(E)\,\mathbb{P}(F)$

$$\mathbb{P}(E \mid F) = \mathbb{P}(E)$$

$$\mathbb{P}(F \mid E) = \mathbb{P}(F)$$

- **Random Variable** $X : S \to \mathbb{R}$



- **Expectation**

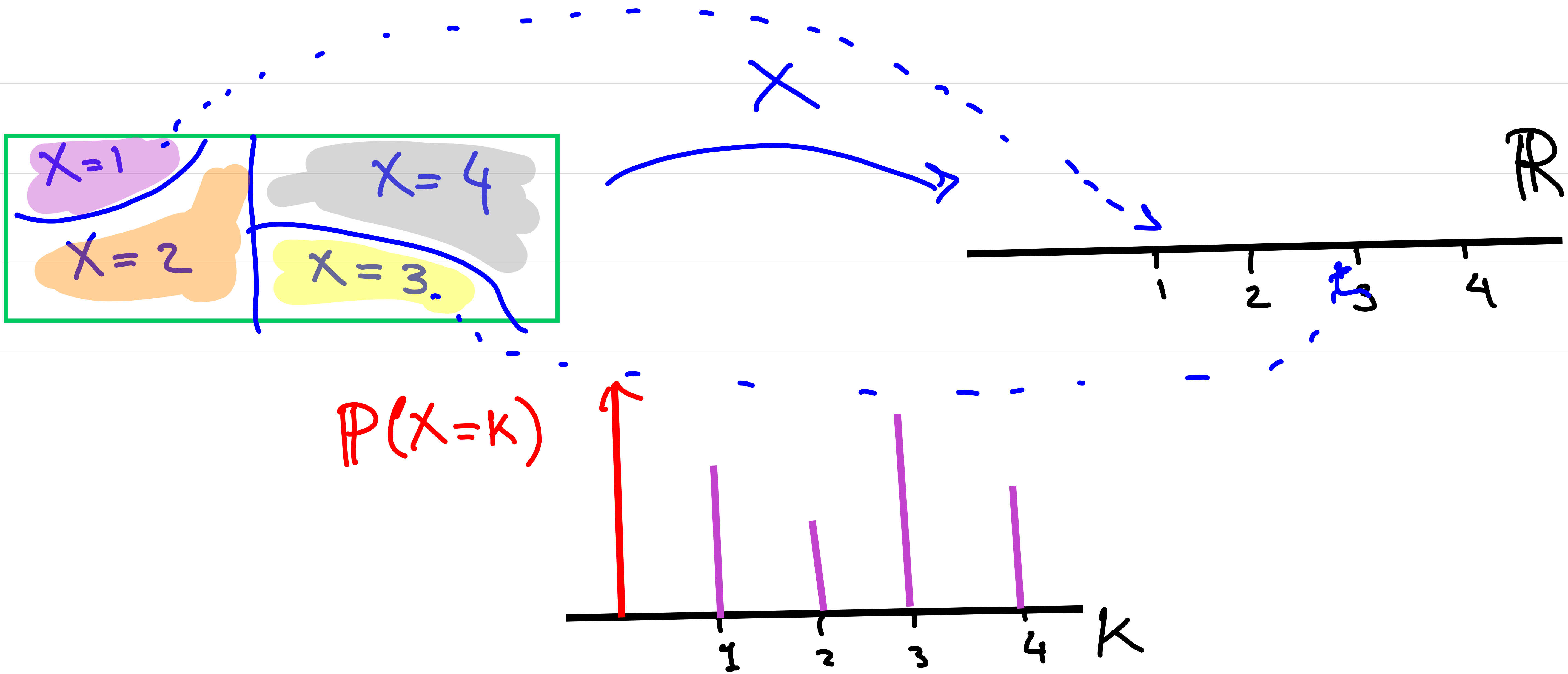
$$\mathbb{E}X = \sum_{s \in S} P(s)\,X(s) = \sum_{r \in \mathrm{range}(X)} \mathbb{P}(X=r) \cdot r$$

- **Linearity of Expectation**

$$\mathbb{E}(X_1 + \dots + X_n) = \sum_{i=1}^{n} \mathbb{E}X_i$$

Distribution

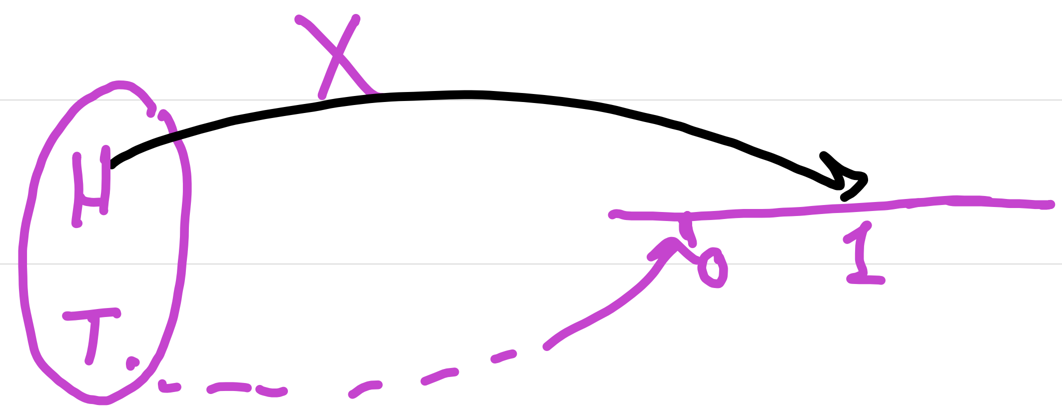
Defn: Given a probability space $(S, \mathbb{P})$ and a random variable $X: S \to \mathbb{R}$, the distribution of X is the set $\{(k, \mathbb{P}(X=k)) : k \in \text{range}(X)\}$



EX: Bernoulli Distribution

You are flipping a biased coin with probability $q \in [0,1]$ of head (success/yes/true/one)

$S = \{H, T\}$ $\quad P(H) = 1 - P(T) = q$

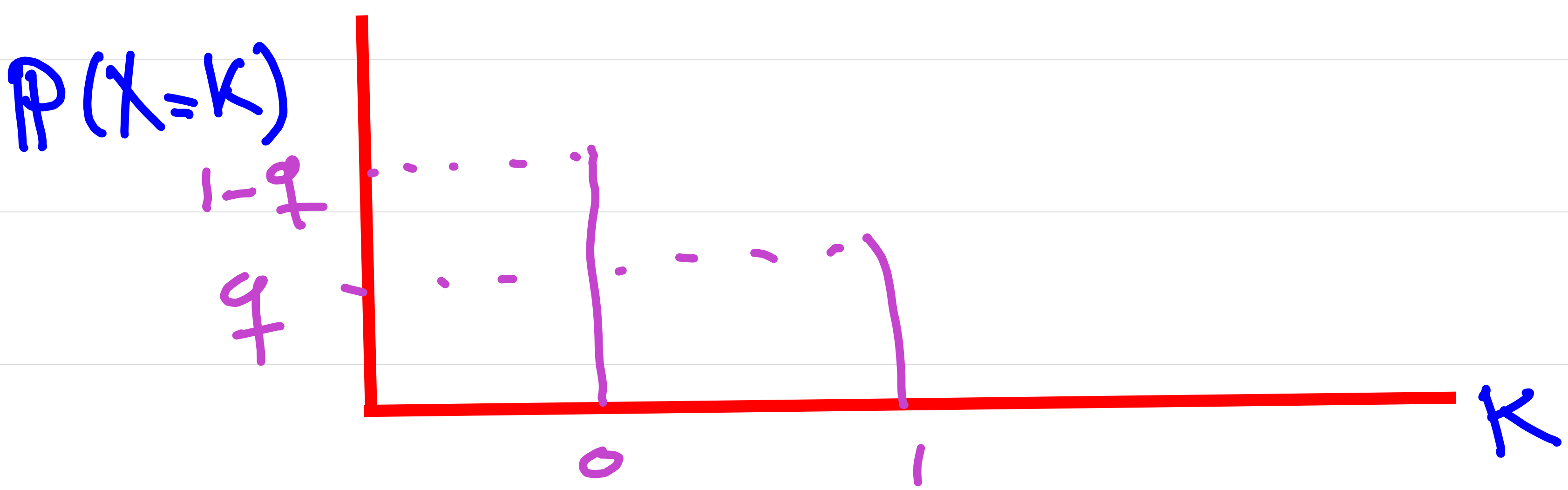


$X(s) = \begin{cases} 1 & \text{if } s = \text{heads} \\ 0 & \text{otherwise} \end{cases}$

$\mathbb{P}(X=1) = 1 - \mathbb{P}(X=0) = q$

$EX = 1 \times q + 0 \times (1-q) = q$

Bernoulli distribution : $\{(0, 1-q), (1, q)\}$



EX: Binomial Distribution.

flip n coins with bias q and count the number of heads.

$S = \{H, T\}^n$ $\qquad P(s) = q^{H(s)} (1-q)^{n-H(s)}$, where $H(s)$ is the # of heads.

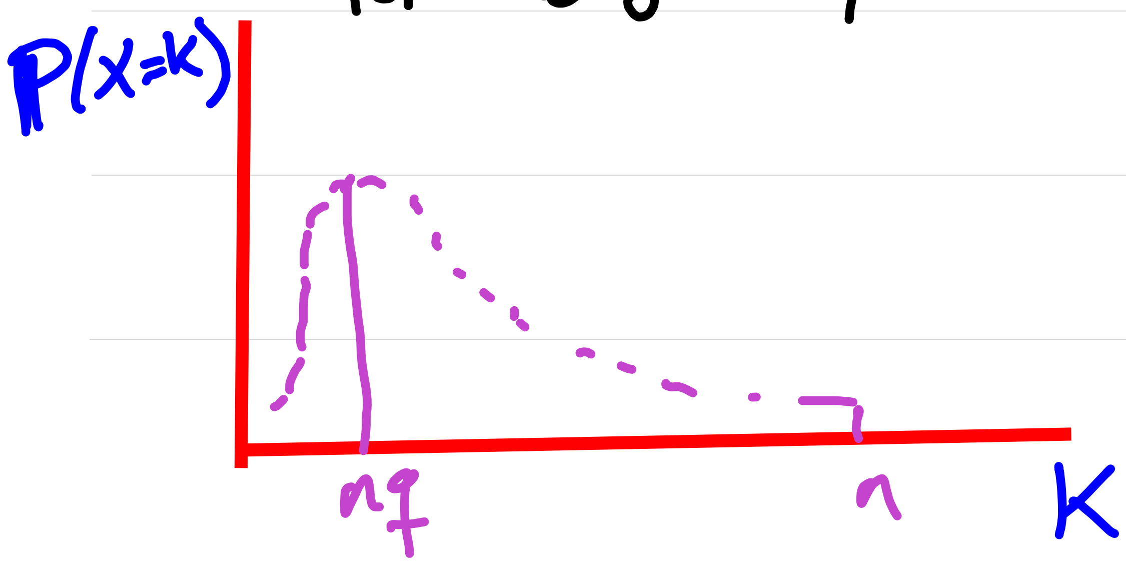
$X(s)$ represent the # heads in S.

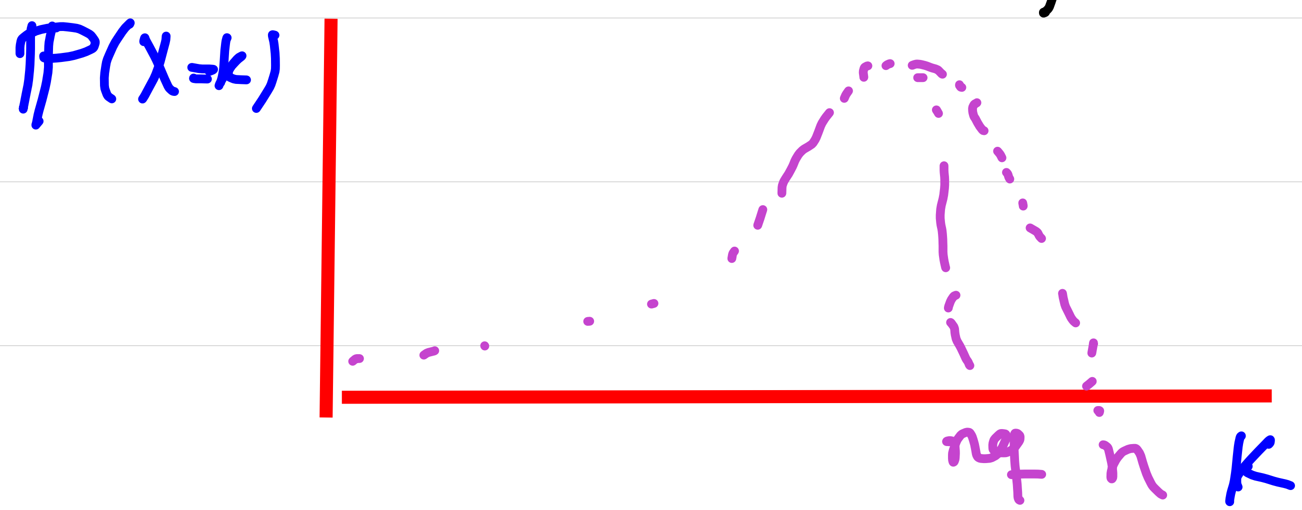
$$\mathbb{P}(X=k) = \binom{n}{k} q^k (1-q)^{n-k}$$

$$\mathbb{E}X = \sum_{k=1}^{n} k \binom{n}{k} q^k (1-q)^{n-k} = nq$$

Binomial Distribution: $\left\{ \left(k, \binom{n}{k} q^k (1-q)^{n-k} \right) \right\}$

for large q $\qquad$ for small q





EX: Geometric Distribution.

$q \in (0,1)$

flip a coin of biased q repeatedly until I see H.
independently
Count the number of flips.

$$S = \{H, TH, TTH, TTTH, \dots\} \qquad P(\overbrace{TT\cdots T}^{m}H) = (1-q)^m\, q$$

(before we move on, let's do some sanity check. Is $\sum_{s \in S} P(s) = 1$?

Let's check. $$\sum_{s \in S} P(s) = \sum_{m=0}^{\infty} P(\overbrace{TT\cdots T}^{m}H) = \sum_{m=0}^{\infty} q \cdot (1-q)^m = q \sum_{m=0}^{\infty} (1-q)^m$$
$$= q\,\frac{1-0}{1-(1-q)}$$
$$= q\,\tfrac{1}{q} = 1.)$$

for any $k \in \mathbb{Z}_{\geq 0}$, $X(\overbrace{TT\cdots T}^{k}H) = k+1$

$$\mathbb{P}(X = k) = (1-q)^{k-1}\, q$$

$$\mathbb{E}X = ?$$

Let's find the expectation of a Geometric random variable with parameter q (i.e. Probability of success).

$$\mathbb{E}X = \sum_{r \in \text{range}(X)} r \cdot \mathbb{P}(X=r) = \sum_{r=1}^{\infty} r \cdot (1-q)^{r-1} \cdot q = q \sum_{r=1}^{\infty} r(1-q)^{r-1}$$

Observe that $\displaystyle\sum_{r=0}^{\infty} (1-q)^r = \frac{1}{1-(1-q)} = \frac{1}{q}$. Now, let's take the derivative:

$$\frac{d}{dq} \sum_{r=0}^{\infty} (1-q)^r = \frac{d}{dq} \frac{1}{q}.$$ Hence, $$\sum_{r=1}^{\infty} (-1) r \cdot (1-q)^{r-1} = \frac{-1}{q^2}$$

Thus, $\displaystyle\sum_{r=1}^{\infty} r \cdot (1-q)^{r-1} = 1/q^2$.

So, $$\mathbb{E}X = q \sum_{r=1}^{\infty} r \cdot (1-q)^{r-1} = q \cdot 1/q^2 = 1/q$$

Algebra of Random Variables

If X, Y are random variables over the same probability space, we can define

(sum) $(X+Y)(s) = X(s) + Y(s) \quad \forall s \in S$

$\mathbb{E}(X+Y) = \mathbb{E}X + \mathbb{E}Y$

(scalar Multiple) $(cX)(s) = cX(s) \quad \forall s \in S, \; c \in \mathbb{R}$

$\mathbb{E}(cX) = c\,\mathbb{E}X$

$\sum_{s \in S} (cX)(s)\, p(s) = \sum_{s \in S} c \cdot X(s) P(s)$

$= c \sum_{s \in S} X(s) p(s)$

$= c \cdot \mathbb{E}X$

(product) $(XY)(s) = X(s)\, Y(s) \quad \forall s \in S$

in general $\leftarrow$ $\mathbb{E}(XY) \neq \mathbb{E}X \cdot \mathbb{E}Y$

Say X is a Bernoulli R.V. with parameter q and $Y = 1 - X$. Observe that $\mathbb{E}(XY) = 0 \neq q(1-q)$

Consider two events E and F. define 1_E and 1_F as indicator R.V. for E and F, i.e.,

$$1_E(s) = \begin{cases} 1 & s \in E \\ 0 & s \notin E \end{cases}, \quad 1_F(s) = \begin{cases} 1 & s \in F \\ 0 & s \notin F \end{cases}.$$

$$1_E 1_F(s) = \begin{cases} 1 & s \in E \cap F \\ 0 & s \notin E \cap F \end{cases}$$

$$\mathbb{E}\, 1_E 1_F = \mathbb{P}(E \cap F)$$

$$\mathbb{E}\, 1_E = \mathbb{P}(E) \qquad \mathbb{E}\, 1_F = \mathbb{P}(F)$$

observe that $\mathbb{P}(E \cap F) \neq \mathbb{P}(F)\mathbb{P}(E)$, in general.

Defn: Two random variables defined on the same sample space $X, Y : S \to \mathbb{R}$ are independent if for all $r_1, r_2 \in \mathbb{R}$

$$\mathbb{P}(X = r_1 \underbrace{\text{ and }}_{\cap} Y = r_2) = \mathbb{P}(X = r_1)\,\mathbb{P}(Y = r_2)$$

these events are independent